Energy Metrics and Their Ricci Flows

***

Sergio Benenti, Accademia delle Scienze di Torino, Via Maria Vittoria 3, I-10123 Torino.

Fabio Cardone, CNR, Piazzale Aldo Moro 7, I-00185, Roma.

March 5, 2024

We dedicate this paper to the memory of deceased colleagues, of Roberto Mignani who saw the need for the physical and extended fifth dimension, and of Eliano Pessa who proposed an appropriate dimensional homogeneity constant for the fifth dimension.

Abstract. The framework of the Deformed Space Time theory has been extended in the past from four to five dimensions (Ref. [1]) where the fifth coordinate is the energy exchanged by the interaction. In this theory each fundamental interaction is described by an energy depended metric. This picture has been exploited in order to take care of the interaction behaviour both when Lorentz invariance holds and the spacetime is Minkowskian and when Lorentz is violated and must be recovered in a non-Minkowskian spacetime. It has been successfully attempted to complete the pentadimensional metric of the four fundamental interaction calculating the fifth element of the metric corresponding to the fifth coordinate energy. The mathematical tool exploited is the method of the Ricci flow which gave the complete explicit form of the fifth element of the metric, answering in this way to the question of how the energy measure the energy for each interaction, setting the electromagnetic interaction as the reference for the energy measure. In this sense it has been given meaning to the problem of the energy gauge for interaction, identifying the gauge with the fifth metric element. The consequences for the nuclear metamorphosis have been also examined for reaching the technological goal of a device producing this metamorphosis in a stable way under the hadronic metric. The most valuable consequence is that in this pentadimensional picture the old Einsteinian dream of a complete geometrization of the interactions is reached. The results achieved in the present work have allowed to design, build and test devices capable of exploiting the behavior of the fifth element of the metrics to obtain the production of electric charges directly from the nuclear metamorphosis of the matter.

Contents

1 Introduction

In order to progress beyond the results presented in [1] in the present work we want to explicitly determine the **fifth element** of the pentadimensional metrics associated with the fundamental interactions — dependent on the energy coordinate — by means of the technique of the **Ricci flow**.

The pentadimensional metrics studied so far in [1] derive their origin from four-dimensional metrics on a space-time of Cartesian coordinates (x_0, x_1, x_2, x_3), where energy E plays the role of parameter, not of coordinate.

Turning to the $5D$ representation, energy E also takes on the role of coordinate. Energy E is an additional measurable and extended real physical dimension, thus endowed with measurable physical dimensions.

The four pentadimensional metrics associated with the four fundamental interactions

(1)

strong (hadronic)
gravitational
electromagnetic
weak (leptonic)

are defined in a **space-time-energy** manifold endowed with global *length-dimensional* coordinates $(x_0, x_1, x_2, x_3, x_4)$ so that the g_{ij} components of the metric tensor turn out to be *dimensionless*.

The first coordinate x_0 represents time t through the product $x_0 = u\,t$, that is, the product of time by the velocity u which is the maximum relativistically invariant causal velocity corresponding to each interaction, see [1].

The fifth coordinate $x_4 \in [0, +\infty)$ represents the energy E through the product

$$x_4 = k\,E \tag{2}$$

where k is a positive constant having the dimensions $\mathsf{L} \times \text{energy}^{-1}$, so that the coordinate x_4 has the dimensions of a length.

The intermediate spatial coordinates (x_1, x_2, x_3) have the dimension of a length.

2 Classification of the pentadimensional metrics

The components of the four pentadimensional metrics we are going to examine are taken from [1], §19.3. *For all of them the metric tensor is diagonalized:* $g_{ij} = 0$ *for* $i \neq j$. A careful comparison of these metrics reveals that they can be classified into *two types*:

$$\text{type 1:} \quad \begin{cases} g_{00} = G(x_4) & \text{positive dimensionless function,} \\ g_{11} = -\,\alpha, & \alpha \text{ positive dimensionless constant,} \\ g_{22} = -\,\beta, & \beta \text{ positive dimensionless constant,} \\ g_{33} = -\,G(x_4), & \\ g_{44} = \pm F(x_4), & F(x_4) \text{ positive dimensionless function.} \end{cases} \tag{3}$$

$$(4) \quad \text{type 2:} \begin{cases} g_{00} = 1 \quad \text{dimensionless,} \\ g_{11} = g_{22} = g_{33} = -\,G(x_4), \quad G(x_4) \text{ positive dimensionless function,} \\ g_{44} = \pm F(x_4), \quad F(x_4) \text{ positive dimensionless function.} \end{cases}$$

In both types are involved two dimensionless positive functions $F(x_4)$ and $G(x_4)$ of the one coordinate x_4 on which the metric depends.[1] We call $G(x_4)$ the **characteristic function** of the metric. The function $F(x_4)$ that defines the fifth component g_{44}, that is, the 'fifth element' mentioned at the beginning, is preceded by the double sign $\pm$. So each type splits into two subtypes. The choice of the sign $\pm$ is equivalent to the choice of the genus of the energy axis x_4:

$$(5) \quad \begin{cases} \text{upper sign } + \iff \text{the } x_4\text{−axis is timelike} \\ \text{lower sign } \;\, - \iff \text{the } x_4\text{−axis is spacelike} \end{cases}$$

The distinction of the metrics into these two types (four sub-types) allows us to highlight some of their peculiar properties valid for any characteristic function $G(x_4)$. As will be seen, this results in a valuable simplification of the calculations as well as efficient checking of their correctness.

Each of these metrics has a discontinuity at a particular value x_{4int} of the x_4-coordinate, which is called **threshold energy**.

We will use the symbol int to label any of the four interactions:

$$int = (em, grav, weak, strong) = (\text{electromagnetic,gravitational,weak,strong})$$

Or, more simply,

$$int = (e, g, w, s).$$

Each threshold divides the axis $x_4 \geq 0$ into two separated intervals. In one of these (before or after the threshold) the geometry is flat with sign $(+ - - - \pm)$ depending on the sign $\pm$ of g_{44}, while in the complementary interval the geometry undergoes a deformation and may therefore exhibit curvature. This discontinuity is represented by means of the **Heaviside step function**.[2]

2.1 Unitary Heaviside step function

The **unitary Heaviside step function** is defined as follows

$$H(x) = \begin{cases} 0 & for\ x < 0 \\ 1 & for\ x \geq 0 \end{cases}$$

[1] The other coordinates are said to be *ignorable*.

[2] Before proceeding further we want to note here that the Heaviside step function can also be considered as a limit of continuous functions or even series of functions. We leave this topic as further study to be developed in later work.

Its graph is shown in Figure 1.

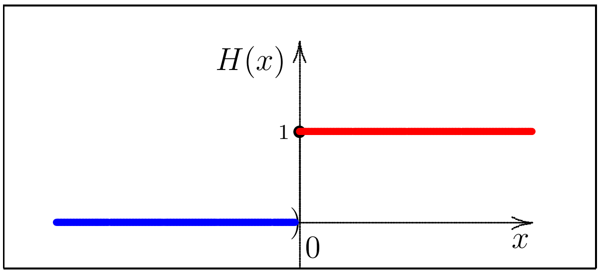


Figure 1: Unitary Heaviside step function.

From this definition derive two other types of step functions which we will use below:

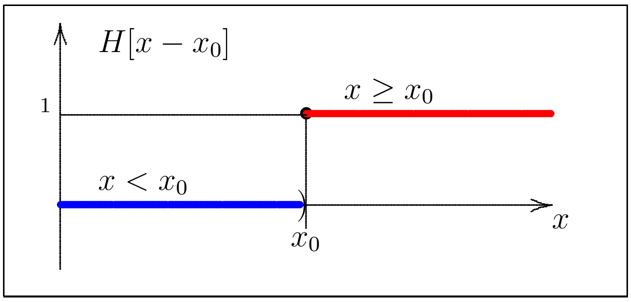


Figure 2: Translated Heaviside step (right).

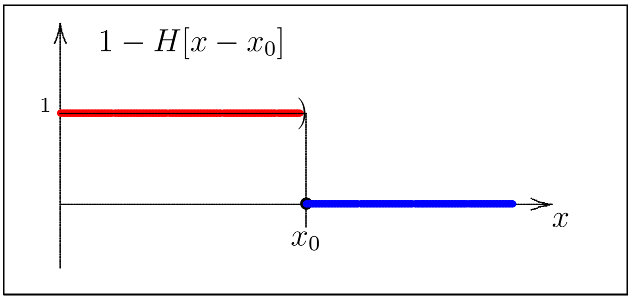


Figure 3: Translated and inverted Heaviside step.

In a physical context we can also adopt this definition: *the Heaviside step function can represent a signal activated in a physical system for a given value of the variable x that remains constant for successive values, without regard to the order of variability* (*increasing or decreasing*). Given this definition of a Heaviside step in physical context, we do not wish to go into the reversibility of the physical system described by this function here.

2.2 Hadronic (strong) metric

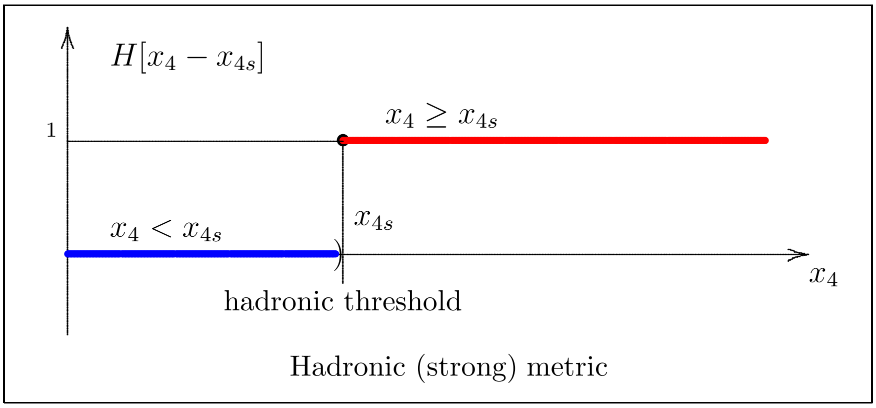


Figure 4: Hadronic Heaviside step of axis x_4 with threshold x_{4s}.

The components of the metric are:

$$
(6) \quad \begin{cases} g_{00} = 1 + H[x_4 - x_{4s}] \left(\dfrac{x_4^2}{x_{4s}^2} - 1 \right) , \\ g_{11} = -\,\alpha, \quad \alpha > 0 \quad \text{dimensionless constant,} \\ g_{22} = -\,\beta, \quad \beta > 0 \quad \text{dimensionless constant,} \\ g_{33} = -\,g_{00}, \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0, \quad F(x_4) \;\; \text{dimensionless function.} \end{cases}
$$

• Before the threshold we have $H[x_4 - x_{4s}] = 0$ and metric (6) becomes

$$
(7) \quad \begin{cases} g_{00} = 1 \\ g_{11} = -\,\alpha \\ g_{22} = -\,\beta \end{cases} \qquad \begin{cases} g_{33} = -\,1 \\ g_{44} = \pm F(x_4) \end{cases}
$$

The comparison with (3) shows that this metric is type 1 with characteristic function $G_s = 1$. Thus this metric is flat with signature $(+ - - - \pm)$, see Figure 5.

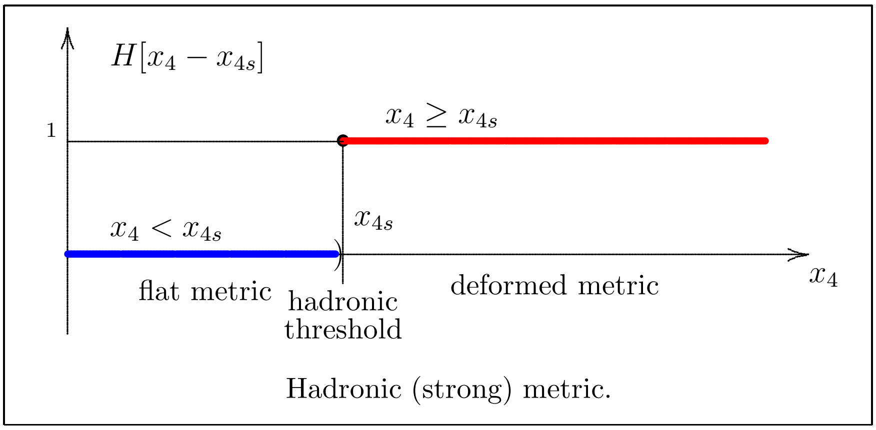


Figure 5: Once the threshold is reached, we move from a flat metric to a deformed one.

•• After the threshold we have $H[x_4 - x_{4s}] = 1$ and the metric becomes

$$(8) \qquad \begin{cases} g_{00} = \dfrac{x_4^2}{x_{4s}^2} \\ g_{11} = -\alpha \\ g_{22} = -\beta \end{cases} \qquad \begin{cases} g_{33} = -\dfrac{x_4^2}{x_{4s}^2} \\ g_{44} = \pm F(x_4) \end{cases}$$

The comparison with (3) shows that this metric is type 1 with characteristic function

$$(9) \qquad \boxed{G_s = \left(\frac{x_4}{x_{4s}}\right)^2}$$

and signature $(+ - - - \pm)$. As will be seen later (Theorem 6.1) its Ricci tensor cannot cancel after the threshold: *after the threshold the metric is* *deformed*.

2.3 Gravitational metric

The situation is quite similar to that of hadronic interaction:

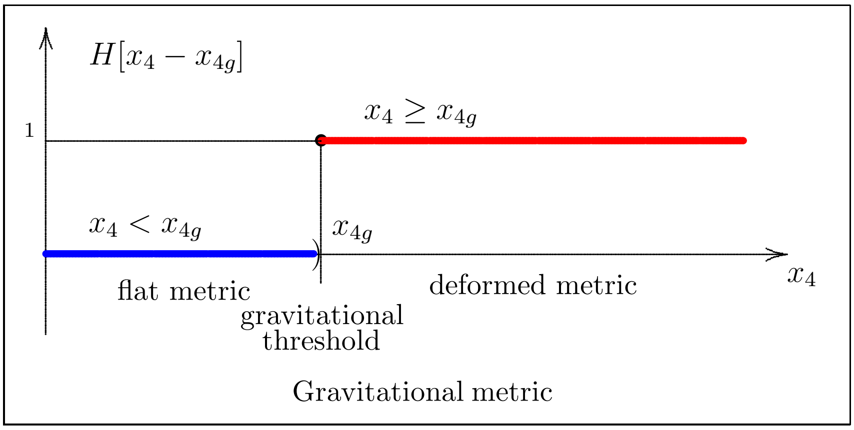


Figure 6: Once the threshold is reached, we move from a flat metric to a deformed one.

The metric components are:

$$(10) \qquad \begin{cases} g_{00} = 1 + H[x_4 - x_{4g}] \left[\frac{1}{4} \left(1 + \dfrac{x_4}{x_{4g}} \right)^2 - 1 \right] , \\ g_{11} = -\alpha, \quad \alpha > 0 \quad \text{dimensionless constant,} \\ g_{22} = -\beta, \quad \beta > 0 \quad \text{dimensionless constant,} \\ g_{33} = -g_{00}, \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0, \quad F(x_4), \ \text{dimensionless function.} \end{cases}$$

• Before the threshold we have $H[x_4 - x_{4g}] = 0$ and the metric becomes

$$(11) \qquad x_4 < x_{4g} \quad \begin{cases} g_{00} = 1, \\ g_{11} = -\,\alpha, \quad \alpha > 0 \text{ dimensionless constant}, \\ g_{22} = -\,\beta, \quad \beta > 0 \text{ dimensionless constant}, \\ g_{33} = -\,g_{00} = -1, \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0, \quad F(x_4) \text{ dimensionless function}. \end{cases}$$

The comparison with (3) shows that this metric is type 1 with characteristic function $G_g = 1$. Thus this metric is flat[3] with signature $(+ - - - \pm)$, see Figure 6.

•• After the threshold we have $H[x_4 - x_{4g}] = 1$ and the metric becomes

$$(12) \qquad x_4 \geq x_{4g} \quad \begin{cases} g_{00} = \frac{1}{4}\left(1 + \dfrac{x_4}{x_{4g}}\right)^2 \\ g_{11} = -\,\alpha, \quad \alpha > 0 \text{ dimensionless constant} \\ g_{22} = -\,\beta, \quad \beta > 0 \text{ dimensionless constant} \\ g_{33} = -\,g_{00} \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0, \quad F(x_4) \text{ dimensionless function} \end{cases}$$

The comparison with (3) shows that this metric is type 1 with characteristic function

$$(13) \qquad \boxed{G_g = \frac{1}{4}\left(1 + \frac{x_4}{x_{4g}}\right)^2 = \frac{(x_{4g} + x_4)^2}{4\,x_{4g}^2}}$$

and signature $(+ - - - \pm)$. As will be seen later (Theorem 6.2) its Ricci tensor cannot cancel after the threshold: *after the threshold the metric is deformed*.

2.4 Electromagnetic metric

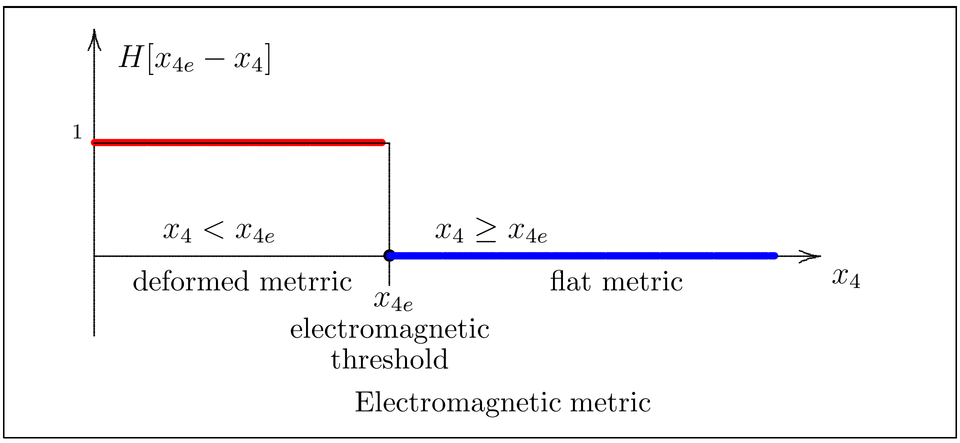


Figure 7: Electromagnetic Heaviside step over x_4-axis with threshold x_{4e}.

[3] Since $F(x_4) > 0$ we can transform the coordinate x_4 into a new coordinate for which the new component g_{44} of the metric is constant.

The metric components are

$$
(14) \qquad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left\{1 + H[x_{4e} - x_4]\left[\left(\dfrac{x_4}{x_{4e}}\right)^{1/3} - 1\right]\right\} \quad \text{spatial isotropy} \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0. \end{cases}
$$

• Before the threshold we have $H[x_{4e} - x_4] = 1$ and the metric becomes

$$
(15) \qquad x_4 < x_{4e} \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left(\dfrac{x_4}{x_{4e}}\right)^{1/3} \quad \text{spatial isotropy} \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0. \end{cases}
$$

The comparison with (4) shows that this metric is type 2 with characteristic function

$$
(16) \qquad \boxed{G_e = \left(\frac{x_4}{x_{4e}}\right)^{1/3}}
$$

and signature $(+ - - - \pm)$, see Figure 7.

•• After the threshold we have $H[x_{4e} - x_4] = 0$ and the metric becomes

$$
(17) \qquad x_4 \geq x_{4e} \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -1, \quad \text{spatial isotropy} \\ g_{44} = \pm F(x_4), \quad F(x_4) > 0. \end{cases}
$$

The comparison with (4) shows that this metric is type 2 with characteristic function $G_e = 1$. It is flat with signature $(+ - - - \pm)$ (see the previous footnote).

2.5 Leptonic (weak) metric

The metric components are

$$
(18) \qquad \begin{cases} g_{00} = 1, \\ g_{11} = g_{22} = g_{33} = -\left\{1 + H[x_{4w} - x_4]\left[\left(\dfrac{x_4}{x_{4w}}\right)^{1/3} - 1\right]\right\} \text{ spatial isotropy,} \\ g_{44} = \pm F(x_4), \quad F > 0. \end{cases}
$$

• Before the threshold we have $H[x_{4w} - x_4] = 1$ and the metric becomes

$$
(19) \qquad \boxed{x_4 < x_{4w} \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left(\dfrac{x_4}{x_{4w}}\right)^{1/3} \\ g_{44} = \pm F(x_4) \end{cases}}
$$

The comparison with (4) shows that this metric is type 2 with characteristic function

(20)
$$\boxed{G_w = \left(\frac{x_4}{x_{4w}}\right)^{1/3}}$$

and signature $(+ - - - \pm)$, see Figure 8.

•• After the threshold we have $H[x_{4w} - x_4] = 0$ and the metric becomes

(21)
$$x_4 \geq x_{4w} \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -1 \\ g_{44} = \pm F(x_4) \end{cases}$$

The comparison with (4) shows that this metric is type 2 with characteristic function $G_w = 1$. It is flat with signature $(+ - - - \pm)$ (see the previous footnote).

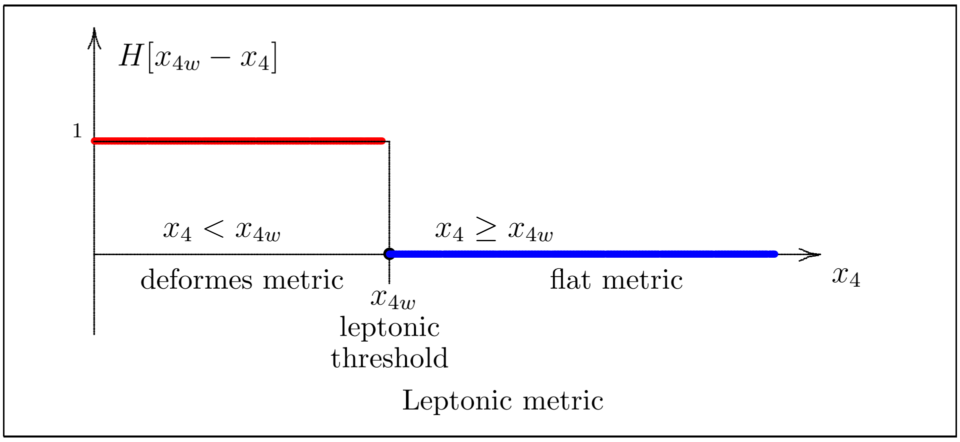


Figure 8: Leptonic Heaviside step over x_4-axis with threshold x_{4w}.

Note that the leptonic Heaviside step is similar to the electromagnetic step, except that the leptonic threshold is $2 \cdot 10^{16}$ times the electromagnetic threshold.[4]

2.6 Summary of characteristic functions

(22)

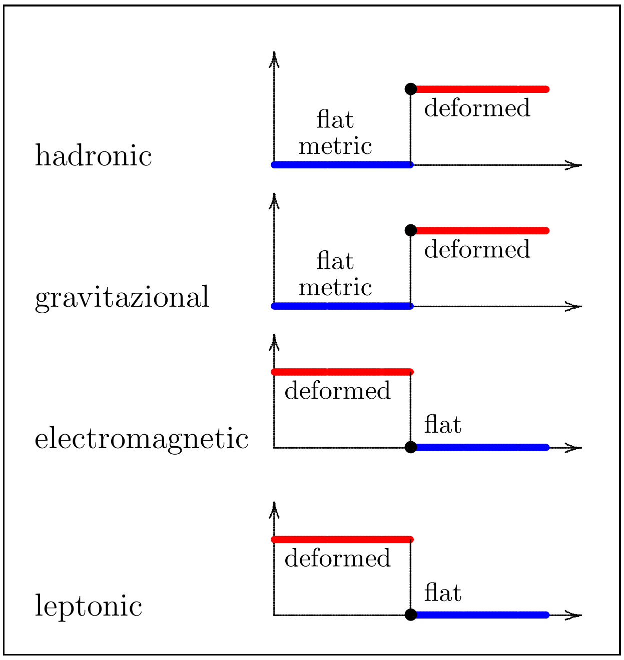
(9) $G_s = \left(\frac{x_4}{x_{4s}}\right)^2$	hadronic, after the threshold x_{4s}
(13) $G_g = \frac{1}{4}\left(1 + \frac{x_4}{x_{4g}}\right)^2$	gravitational, after the threshold x_{4g}
(16) $G_e = \left(\frac{x_4}{x_{4e}}\right)^{1/3}$	electromagnetic, before the threshold x_{4e}
(20) $G_w = \left(\frac{x_4}{x_{4w}}\right)^{1/3}$	leptonic, before the threshold x_{4w}

[4] [1], Cap. 4, §4.2, p. 61, fig. 4.2.

2.7 Summary of Heaviside's steps

We give here the graphical translation of what is expressed in equations (22) from a qualitative point of view consistent with the definition of the Heaviside step and the variants that we adopted at §2.1.

(23)



3 Metric flows and volume conservation

From here up to Section 8, with the addition of an Appendix concerning the calculation of the Ricci tensor, purely mathematical topics focused on the notion of Ricci flow will be covered.

We work on an n-dimensional manifold M_n with generic coordinates $(x) = (x_1, x_2, ..., x_n)$ and on this manifold we consider a **coordinated domain** $D \subset M_n$, i.e., a deformed hyperparallelepipedon whose edges are segments of coordinate lines. To such a domain we can apply the derivation theorem under the integral sign.[5].

We call **metric flow** a family of metric tensors $g_{ij}(x,t)$ defined over D, depending on an **evolution parameter** t such that it satisfies **flow equations** of the type

$$\boxed{\partial_t g_{ij}(x,t) = -\, S_{ij}(x,t) + \frac{1}{n}\, \bar{S}(t)\, g_{ij}(x,t)} \tag{24}$$

where $S_{ij}(x,t)$ is a symmetric tensor defined on D and

$$\boxed{\bar{S} \stackrel{\text{def}}{=} \frac{1}{V_D} \int_D S\, dV = \frac{1}{V_D} \int_D S\, \sqrt{|g|}\, dx} \tag{25}$$

[5] In fact, the results obtained below are also valid in the more general case in which the domain D can be covered by several coordinate domains

is the mean value over D of the scalar

$$\boxed{S \stackrel{\text{def}}{=} g^{ij}\, S_{ij}} \tag{26}$$

In (25) the volume V_D of the domain D is defined by

$$V_D = \int_D dV = \int_D \sqrt{|g|}\, dx \qquad \begin{cases} g \stackrel{\text{def}}{=} \det[g_{ij}] \\ dx \stackrel{\text{def}}{=} dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n, \end{cases} \tag{27}$$

where the n-differential form $dV = \sqrt{|g|}, dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ is the **volume form** associated with the metric $g_{ij}(x,t)$.

Remark 3.1 – The equations of the **normalized Ricci flow**, which we will discuss later, are of the type (24)

$$\boxed{\partial_t g_{ij}(x,t) = -R_{ij}(x,t) + \frac{1}{n}\bar{R}(t)\, g_{ij}(x,t)} \tag{28}$$

where $R_{ij}(x,t)$ is the Ricci tensor of the metric $g_{ij}(x,t)$ and $\bar{R}$ is the mean value of the Ricci scalar R in the domain D:

$$\boxed{\bar{R} \stackrel{\text{def}}{=} \frac{1}{V_D}\int_D R\, dV = \frac{1}{V_D}\int_D R\sqrt{|g|}\, dx} \tag{29}$$

$$\boxed{R \stackrel{\text{def}}{=} g^{ij}\, R_{ij}} \tag{30}$$

However, equations (24) differ conceptually from (28) because, unlike the latter, in equations (24) no functional link is specified between the tensors $g_{ij}(x,t)$ and $S_{ij}(x,t)$. For this very reason, we need the following theorem •

Theorem 3.1 – *If in a metric flow* (24) *the mean value $\bar{S}$ remains constant with respect to t then the volume V_D also remains constant.*

Proof. By multiplying both members of (24) by g^{ij} and summing we get the equation

$$g^{ij}\partial_t g_{ij} = -S + \bar{S}.$$

By virtue of **Jacobi's formula**

$$\boxed{g^{ij}\,\partial_t g_{ij} = \partial_t \log|g|} \tag{31}$$

this equation becomes

$$\boxed{\partial_t \log|g| = -S + \bar{S}} \tag{32}$$

We then proceed to calculate the derivative with respect to t of the volume (27):

$$\frac{dV_D}{dt} = \frac{d}{dt}\int_D dV = \frac{d}{dt}\int_D \sqrt{|g|}\, dx.$$

As mentioned above, for a coordinated domain D the theorem of differentiation under the sign of integral applies, so that

$$\frac{dV_D}{dt} = \int_D \frac{d\sqrt{|g|}}{dt}\, dx = ...$$

Since $dx = \dfrac{1}{\sqrt{|g|}}, dV$ it follows that

$$... = \int_D \frac{d\sqrt{|g|}}{dt} \frac{1}{\sqrt{|g|}}\, dV = \int_D \frac{d\log\sqrt{|g|}}{dt}\, dV = \tfrac{1}{2} \int_D \frac{d\log|g|}{dt}\, dV = ...$$

Finally, by virtue of (32), we find

$$... = \tfrac{1}{2} \int_D (\bar{S} - S)\, dV = \tfrac{1}{2}\, \bar{S} \int_D dV - \tfrac{1}{2} \int_D S\, dV$$

because $\bar{S}$ is a constant (it is a number). We have then shown that

$$\frac{dV_D}{dt} = \tfrac{1}{2}\, \bar{S}\, V_D - \tfrac{1}{2} \int_D S\, dV.$$

Dividing both members by the volume $V_D = \int_D dV$, and multiplying by 2, we find

$$\frac{2}{V_D} \frac{dV_D}{dt} = \bar{S} - \frac{1}{V_D} \int_D S\, dV = \bar{S} - \bar{S} = 0.$$

Thus $\dfrac{dV_D}{dt} = 0$. ■

Remark 3.2 – We will see later (Theorem 7.1) that the existence of a normalized Ricci flow necessarily implies $\bar{R} = 0$. So in this case we can definitely apply Theorem 3.1 with that additional assumption. •

4 Dimensional homogeneity

Any equation of the type (24) must satisfy the **dimensional homogeneity principle** according to which both members of an equation must have the same physical dimension. If not, the equation is meaningless.[6] In our case, in which *the components of the metric tensor* g_{ij} *are dimensionless*, the flow equations (24) are dimensionally homogeneous if and only if the parameter t obeys the dimensional equality

$$\boxed{\mathrm{Dim}[\,t\,] = \frac{1}{\mathrm{Dim}[S_{ij}]}} \tag{33}$$

Then with regard to the first member of (24) we have

$$\mathrm{Dim}\big[\partial_t g_{ij}\big] = \mathrm{Dim}\left[\frac{1}{t}\right] = \frac{1}{\mathrm{Dim}\big[t\big]}.$$

On the other hand, as far as the second member is concerned, from $S \overset{\text{def}}{=} g^{ij}\, S_{ij}$ it follows that $\mathrm{Dim}\big[S\big] = \mathrm{Dim}\big[S_{ij}\big]$. This means that the second member is homogeneous. Therefore, the dimensional equation to be taken into account is (33):

[6] Especially in a physical-mathematical context, but not only, this principle should also be given due consideration because it constitutes a check on the correctness of calculations.

5 Ricci tensors

> S.M. Carroll, [16], p.75: *... there is a convention that needs to be chosen for the ordering of the indices. There is no agreement at all on what this convention should be, so be careful.*

In the aim to analyze the Ricci flow properties of the metrics associated with the four fundamental interactions, it should be preliminarily noted that:

(i) There are properties of the Ricci flows which change seriously if we change the sign of the Ricci tensor.

(ii) As Carroll warns, although in the literature the defininions of Riemann and Ricci tensors may vary from author to author, the Ricci tensor may at most change in sign.

(iii) It is therefore necessary to conduct a comparative study of the definitions or conventions adopted by a sufficiently significant number of authors. A small number of them are examined in Appendix 15.1, sufficient, however, to highlight the fact that:

(34) Regardless of the conventions adopted for Riemann and Ricci tensors, all the Ricci tensors have the opposite sign to that adopted by L.P. Eisenhart..[a] Thus, *the definitions according to Eisenhart of Ricci tensors come to assume an important comparative role.*

[a] In his time professor of differential geometry at Princeton.

On the other hand, to ensure the maximum reliability of the results we are going to achieve, it is a must to adopt for Riemann and Ricci the conventions of R. Hamilton or Cao-Zhu,[7] *because on them these authors built their fundamental approach to Ricci's flow theory.* Given the property (34) we conclude that:

(35) *The components of the Ricci tensors on which to base the study of Ricci flows of type 1 and 2 metrics are those of Eisenhart with opposite sign.*

In Appendices 15.3 and 15.4 it is shown that the Ricci-Eisenhart components are

$$\text{(36)} \qquad \boxed{\begin{array}{c}\text{type 1}\\\text{metrics}\end{array}} \; \boxed{\pm F} \left[\begin{array}{l} \overset{E}{R}_{00} = \pm \dfrac{2\,G''\,F - G'\,F'}{4\,F^2}, \\[2mm] \overset{E}{R}_{11} = R_{22} = 0, \quad \overset{E}{R}_{33} = -\overset{E}{R}_{00} \\[2mm] \overset{E}{R}_{44} = \dfrac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F} \end{array} \right.$$

[7] As we shall see they turn out to be equivalent.

$$(37) \qquad \boxed{\begin{array}{c}\text{type 2}\\ \text{metrics}\end{array}} \; \boxed{\pm F} \left[\begin{array}{l} \overset{E}{R}_{00} = 0 \\ \overset{E}{R}_{11} = \overset{E}{R}_{22} = \overset{E}{R}_{33} = \mp \dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G} \\ \overset{E}{R}_{44} = 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F} \end{array} \right.$$

So, according to these guidelines, the components of the Ricci tensors whose flow we have to analyze turn out to be:

$$(38) \qquad \boxed{\begin{array}{c}\text{type 1}\\ \text{metrics}\end{array}} \; \boxed{\pm F} \left[\begin{array}{l} R_{00} = \mp \dfrac{2\,G''\,F - G'\,F'}{4\,F^2}, \\ R_{11} = R_{22} = 0, \quad R_{33} = -R_{00} \\ R_{44} = -\dfrac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F} \end{array} \right.$$

$$(39) \qquad \boxed{\begin{array}{c}\text{type 2}\\ \text{metrics}\end{array}} \; \boxed{\pm F} \left[\begin{array}{l} R_{00} = 0 \\ R_{11} = R_{22} = R_{33} = \pm \dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G} \\ R_{44} = -\,3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F} \end{array} \right.$$

The following properties apply to both types of metrics.

(i) The Ricci tensor is diagonalized.

(ii) The component R_{44} does not change sign in the transition from sign $+$ to sign $-$.

(iii) The constants α and β disappear.

(iv) F' is present but not F''.

(v) Both G' and G'' derivatives of G are present.

6 Peculiar properties of hadronic and gravitational metrics

Theorem 6.1 – *After the threshold x_{4s} the Ricci tensor of the hadronic metric cannot cancel.*

Proof. After the threshold x_{4s} this metric is of type 1 with characteristic function $G_s = x_4^2/x_{4s}^2$. Suppose $R_{00} = 0$. From the first of (38) we derive the equivalence

$$R_{00} = 0 \iff 2\,G''\,F = G'\,F'.$$

Furthermore we have $G = \dfrac{x_4^2}{x_{4s}^2}$, $\quad \dfrac{G''}{G'} = \dfrac{2}{2\,x_4} = \dfrac{1}{x_4}, \quad \dfrac{2}{x_4} = \dfrac{F'}{F} = \dfrac{dF}{dx_4}\,\dfrac{1}{F}, \quad 2\,\dfrac{dx_4}{x_4} = \dfrac{dF}{F}$

$d\log x_4^2 = d\log F, \quad \log x_4^2 = \text{cost.} + \log F, \quad x_4^2 = e^{\text{cost.}}F$. Therefor:

$$R_{00} = 0 \iff F = C_s\, x_4^2$$

(40) where C_s is an arbitrary positive constant with dimension L^{-2}

Now suppose also $R_{44} = 0$:

$$R_{44} = -3\,\frac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F} = 0 \iff 2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G = 0$$

$$\iff 2\,G''\,G - (G')^2 - \frac{F'}{F}\,G'\,G = 0 \iff 2*2*x_4^2 - (2\,x_4)^2 - \frac{2}{x_4}*2\,x_4*x_4^2 = 0$$

$$\iff x_4^2 - (x_4)^2 - x_4^2 = 0\text{: absurd.} \ \blacksquare$$

The same property also holds for the gravitational metric:

Theorem 6.2 – *After the threshold x_{4g} the Ricci tensor of the gravitational metric cannot cancel.*

Proof. After the threshold this metric is type 1 with characteristic function

$$G_g = \tfrac{1}{4}\left(1 + \frac{x_4}{x_{4g}}\right)^2 = \tfrac{1}{4}\left(\frac{x_4 + x_{4g}}{x_{4g}}\right)^2.$$

Also in this case we start by assuming $R_{00} = 0$ and therefore again from the equivalence

$$R_{00} = 0 \iff 2\,G''\,F = G'\,F'.$$

$$G' = \tfrac{1}{2}\,\frac{x_4 + x_{4g}}{x_{4g}^2}, \qquad G'' = \tfrac{1}{2}\,\frac{1}{x_{4g}^2}, \qquad \frac{G''}{G'} = \frac{\tfrac{1}{2}\,\frac{1}{x_{4g}^2}}{\tfrac{1}{2}\,\frac{x_4 + x_{4g}}{x_{4g}^2}} = \frac{1}{x_4 + x_{4g}}.$$

$$2\,G''\,F = G'\,F' \iff \frac{2}{x_4 + x_{4g}} = \frac{F'}{F} = \frac{dF}{dx_4}\,\frac{1}{F} \iff \frac{2\,dx_4}{x_4 + x_{4g}} = \frac{dF}{F}$$

$\iff d\log(x_4 + x_{4g})^2 = d\log F \iff \log(x_4 + x_{4g})^2 = \text{cost.} + \log F \iff (x_4 + x_{4g})^2 = e^{\text{cost.}}F$. Therefore:

$$R_{00} = 0 \iff F = C_g\,(x_4 + x_{4g})^2$$

(41) where C_g is an arbitrary positive constant with dimension L^{-2}

Now suppose also $R_{44} = 0$:

$$R_{44} = 0 \iff 2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G = 0 \iff 2\,G''\,G - (G')^2 - \frac{F'}{F}\,G'\,G = 0$$

$$\iff 2\,\tfrac{1}{2}\,\frac{1}{x_{4g}^2}\,\tfrac{1}{4}\left(\frac{x_4 + x_{4g}}{x_{4g}}\right)^2 - \left(\tfrac{1}{2}\,\frac{x_4 + x_{4g}}{x_{4g}^2}\right)^2 - \frac{2}{x_4 + x_{4g}}\,\tfrac{1}{2}\,\frac{x_4 + x_{4g}}{x_{4g}^2}\,\tfrac{1}{4}\left(\frac{x_4 + x_{4g}}{x_{4g}}\right)^2 = 0$$

$$\iff \frac{1}{x_{4g}^2}\left(\frac{x_4 + x_{4g}}{x_{4g}}\right)^2 - \left(\frac{x_4 + x_{4g}}{x_{4g}^2}\right)^2 - \frac{1}{x_{4g}^2}\left(\frac{x_4 + x_{4g}}{x_{4g}}\right)^2 = 0$$

$$\implies x_4 + x_{4g} = 0\text{: absurd.} \ \blacksquare$$

Remark 6.1 – For the remaining two metrics, leptonic and electromagnetic, one can repeat the calculation on the Ricci tensor before the threshold, concluding that before the thresholds x_{4e} and x_{4w} the Ricci tensor does not cancel. •

7 Normalized Ricci flows

The definition of *normalized Ricci flow* has already been introduced in Remark 3.1 of §3: it is a family of metrics $g_{ij}(x,t)$ parametrized by an *independent evolution variable* t and defined over a domain D of a Riemannian manifold M_n such as to satisfy the **normalized flow equations**[8]

$$\boxed{\partial_t g_{ij}(x,t) = -R_{ij}(x,t) + \frac{1}{n}\bar{R}(t)\, g_{ij}(x,t)} \tag{42}$$

where $R_{ij}(x,t)$ is the Ricci tensor of the metric $g_{ij}(x,t)$ and $\bar{R}$ is the mean value of the Ricci scalar R in the domain D:

$$\boxed{\bar{R} \overset{\text{def}}{=} \frac{1}{V_D}\int_D R\, dV = \frac{1}{V_D}\int_D R\sqrt{|g|}\, dx} \tag{43}$$

$$\boxed{R \overset{\text{def}}{=} g^{ij} R_{ij}} \tag{44}$$

Remark 7.1 – We are working in L-dimensional x_i coordinates with dimensionless metric components. It follows that the Ricci components have the inverse dimension of a squared length:

$$\text{Dim}[R_{ij}] = \text{Dim}[R] = \text{Dim}[\bar{R}] = \frac{1}{\mathsf{L}^2}.$$

Theorem 3.1 and formula (33) regarding metric flows are valid *mutatis mutandis* for a Ricci flow. So for dimensional homogeneity of the flow equations (42) must be

$$\text{Dim}[\,t\,] = \frac{1}{\text{Dim}[R_{ij}]} = \mathsf{L}^2. \quad \bullet \tag{45}$$

Theorem 7.1 – *The existence of a normalized Ricci flow for type 1 or 2 metrics necessarily implies* $\bar{R} = 0$.

Proof. Type 1. By virtue of equations (3)

$$\text{type 1 metric} \begin{cases} g_{00} = G(x_4) \\ g_{11} = -\,\alpha, \quad \alpha > 0 \\ g_{22} = -\,\beta, \quad \beta > 0 \end{cases} \qquad \begin{cases} g_{33} = -\,G(x_4) \\ g_{44} = \pm F(x_4), \end{cases}$$

[8] In the literature these equations also appear in the form

$$\partial_t g_{ij} = -2\, R_{ij}(x,t) + \frac{2}{n}\bar{R}\, g_{ij}.$$

However, the presence of factor 2 is inessential because it can delete by changing parameter. Let us remember that n is the number of dimensions that we have set equal to 5 following the physical indication of the hadronic metric, where the anisotropy is linked to the coordinates that have parameters of the metric α and β which are constant fractions having the number 5 in the denominator, and are the result of the phenomenological study of the hadronic metric (see [1], Chap. 19, §19.1, p. 280, and related footnote).

the Ricci flow equations (42) for $n = 5$

$$\partial_t g_{ij} = -\,R_{ij} + \tfrac{1}{5}\,\bar{R}\,g_{ij} \;:\; \begin{cases} [00] & \partial_t g_{00} = -\,R_{00} + \frac{1}{5}\,\bar{R}\,g_{00} \\ [11] & \partial_t g_{11} = -\,R_{11} + \frac{1}{5}\,\bar{R}\,g_{11} \\ [22] & \partial_t g_{22} = -\,R_{22} + \frac{1}{5}\,\bar{R}\,g_{22} \\ [33] & \partial_t g_{33} = -\,R_{33} + \frac{1}{5}\,\bar{R}\,g_{33} \\ [44] & \partial_t g_{44} = -\,R_{44} + \frac{1}{5}\,\bar{R}\,g_{44} \end{cases}$$

become, also taking into account (38) $R_{11} = R_{22} = 0$ and $R_{33} = -R_{00}$,

$$(46) \qquad \begin{cases} [00] & \partial_t G = -\,R_{00} + \frac{1}{5}\,\bar{R}\,G \\ [11] & 0 = \frac{1}{5}\,\bar{R}\,\alpha \\ [22] & 0 = \frac{1}{5}\,\bar{R}\,\beta \\ [44] & \pm\,\partial_t F = -\,R_{44} \pm \frac{1}{5}\,\bar{R}\,F \end{cases}$$

From [11] and [22] it follows that $\bar{R} = 0$.

Type 2. By virtue of equations (4)

$$\text{type 1 metric} \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\,G(x_4) \\ g_{44} = \pm F(x_4), \end{cases}$$

the Ricci flow equations (42) with $n = 5$

$$(47) \qquad \begin{cases} [00] & 0 = -\,R_{00} + \frac{1}{5}\,\bar{R}, \\ [11] & -\,\partial_t G = -\,R_{11} - \frac{1}{5}\,\bar{R}\,G, \\ [22] & -\,\partial_t G = -\,R_{22} - \frac{1}{5}\,\bar{R}\,G, \\ [33] & -\,\partial_t G = -\,R_{33} - \frac{1}{5}\,\bar{R}\,G, \\ [44] & \pm\,\partial_t F = -\,R_{44} \pm \frac{1}{5}\,\bar{R}\,F, \end{cases}$$

also taking into account that $R_{00} = 0$ and $R_{11} = R_{22} = R_{33}$, reduce to the three equations

$$(48) \qquad \begin{cases} [00] & 0 = \frac{1}{5}\,\bar{R}, \\ [11] & \partial_t G = R_{11} + \frac{1}{5}\,\bar{R}\,G, \\ [44] & \pm\,\partial_t F = -\,R_{44} \pm \frac{1}{5}\,\bar{R}\,F, \end{cases}$$

i.e. to

$$(49) \qquad \begin{cases} [00] & \bar{R} = 0, \\ [11] & \partial_t G = R_{11}, \\ [44] & \pm\,\partial_t F = -\,R_{44}. \qquad \blacksquare \end{cases}$$

We must underline that the Theorem 7.1 puts us in front of a rather paradoxical situation:

From the hypothesis that the metrics of type 1 and 2 admit a normalized Ricci flow it necessarily follows $\bar{R} = 0$, i.e. that the Ricci flow is in fact <u>*not*</u> *normalized.*

In the next section we therefore move on to the study of non-normalized Ricci flows in order to establish their conditions of existence.

8 Non-normalized Ricci flows

With $\bar{R} = 0$ the equations of the normalized flow (42) reduce to those of a **non-normalized Ricci flow**

$$\boxed{\partial_t g_{ij} = -\,R_{ij}} \tag{50}$$

It is known that the existence of a normalized Ricci flow is a sufficient condition for the conservation of the volume of the definition domain D, but it is not necessarily a necessary condition.

However, we observe that the conservation of the volume V_D is still guaranteed by Theorem 3.1 according to which if in a flow of metrics (24) the mean value $\bar{S}$ is constant in t then also the volume V_D remains constant. In the present case it is $\bar{R} = 0$, so this condition is satisfied.

From (46) and (48) it follows that, for the metrics of the two types, the system of equations (50) reduces respectively to:

$$\text{non-normalized Ricci flow of type 1} \begin{cases} [00] & \partial_t G = -\,R_{00} \\ [44] & \pm\,\partial_t F = -\,R_{44} \end{cases}$$

$$\text{non-normalized Ricci flow of type 2} \begin{cases} [11] & \partial_t G = R_{11} \\ [44] & \pm\,\partial_t F = -\,R_{44} \end{cases}$$

Since the components of the metric are dimensionless and the components of the Ricci tensors have dimension L^{-2}, then in equations (50) the evolution parameter t must have dimension L^2, see (45).

If in equations (50) we replace the parameter t with the coordinate x_4 thought of as a function of t then they take the form:

$$\text{tipo 1} \begin{cases} [00] & G'\,\dot{x}_4 = -\,R_{00} \\ [44] & \pm\,F'\,\dot{x}_4 = -\,R_{44} \end{cases} \qquad \text{tipo 2} \begin{cases} [11] & G'\,\dot{x}_4 = R_{11} \\ [44] & \pm\,F'\,\dot{x}_4 = -\,R_{44} \end{cases}$$

Since

$$\mathrm{Dim}(G'\,\dot{x}_4) = \frac{1}{\mathsf{L}}\,\frac{\mathsf{L}}{\mathsf{L}^2} = \frac{1}{\mathsf{L}^2}, \quad \mathrm{Dim}(F'\,\dot{x}_4) = \frac{1}{\mathsf{L}}\,\frac{\mathsf{L}}{\mathsf{L}^2} = \frac{1}{\mathsf{L}^2}, \quad \mathrm{Dim}(R_{ij}) = \frac{1}{\mathsf{L}^2}$$

these equations are dimensionally homogeneous.

Now, recalling the expressions of the components of the Ricci tensors (38) and (39) we obtain the following two couples of equations:

$$
\text{(51)} \qquad \text{type 1} \begin{cases} [00] \quad G'\,\dot{x}_4 = -R_{00} = \pm \dfrac{2\,G''\,F - G'\,F'}{4\,F^2} \\ [44] \quad \pm\,F'\,\dot{x}_4 = -R_{44} = \dfrac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F} \end{cases}
$$

$$
\text{(52)} \qquad \text{type 2} \begin{cases} [11] \quad G'\,\dot{x}_4 = R_{11} = \pm \dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G} \\ [44] \quad \pm\,F'\,\dot{x}_4 = -R_{44} = 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F} \end{cases}
$$

Theorem 8.1 – *A* ***type 1*** *metric admits a non-normalized Ricci flow if its characteristic function G satisfies the equation*

$$
\text{(53)} \qquad \boxed{\frac{2\,G''\,F - G'\,F'}{2\,F\,G'} = \frac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{G^2\,F'}}
$$

which is equivalent to

$$
\text{(54)} \qquad \boxed{\frac{G''}{G'} - \frac{F'}{2\,F} = \frac{2\,G''}{G}\,\frac{F}{F'} - \frac{G'}{G} - \frac{(G')^2}{G^2}\,\frac{F}{F'}}
$$

Proof. Equations (51) are equivalent to

$$
\begin{cases} \dot{x}_4 = \pm \dfrac{2\,G''\,F - G'\,F'}{4\,F^2\,G'} \\ \pm\,\dot{x}_4 = \dfrac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F\,F'} \end{cases}
$$

whose combination produces (53). ■

Theorem 8.2 – *A* ***type 2*** *metric admits a non-normalized Ricci flow if its characteristic function G satisfies the equation*

$$
\text{(55)} \qquad \frac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{F\,G'} = 3\,\frac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{G\,F'}
$$

which is equivalent to

$$
\text{(56)} \qquad \frac{F'}{F}\,G + 3\left(2\,G'' - \frac{(G')^2}{G}\right)\frac{F}{F'} = 2\left(\frac{G''\,G}{G'} + 2\,G'\right).
$$

Proof. Equations (52) are equivalent to

$$
\begin{cases} \dot{x}_4 = \pm \dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G\,G'} \\ \pm\dot{x}_4 = 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F\,F'} \end{cases}
$$

$$\text{Upper sign:} \left\{ \begin{array}{l} \dot{x}_4 = \dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G\,G'} \\ \dot{x}_4 = 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F\,F'} \end{array} \right.$$

$$\Longrightarrow \quad \frac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G\,G'} = 3\,\frac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F\,F'}$$

Multiplying by $4\,F\,G$ we get equation (55).

$$\text{Lower sign:} \left\{ \begin{array}{l} \dot{x}_4 = -\dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G\,G'} \\ -\dot{x}_4 = 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F\,F'} \end{array} \right.$$

Simplifying, we still find equation (55). Development of equation (55):

$$\text{Left hand side} \quad \left[\; \frac{2\,G''\,F\,G}{F\,G'} + \frac{(G')^2\,F}{F\,G'} - \frac{F'\,G'\,G}{F\,G'} = \frac{2\,G''\,G}{G'} + G' - \frac{F'\,G}{F} \right.$$

$$\text{Right hand side} \quad \left[\begin{array}{l} 3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{G\,F'} \\ = 3\,\dfrac{2\,F\,G''\,G}{G\,F'} - 3\,\dfrac{(G')^2\,F}{G\,F'} - 3\,\dfrac{F'\,G'\,G}{G\,F'} = 6\,\dfrac{F\,G''}{F'} - 3\,\dfrac{(G')^2\,F}{G\,F'} - 3\,G' \end{array} \right.$$

Equation:

$$\frac{2\,G''\,G}{G'} + G' - \frac{F'\,G}{F} = 6\,\frac{F\,G''}{F'} - 3\,\frac{(G')^2\,F}{G\,F'} - 3\,G'$$

$$\Longrightarrow \quad \frac{2\,G''\,G}{G'} + 4\,G' - \frac{F'}{F}\,G = \left(6\,G'' - 3\,\frac{(G')^2}{G} \right) \frac{F}{F'} \quad \Longrightarrow (56). \blacksquare$$

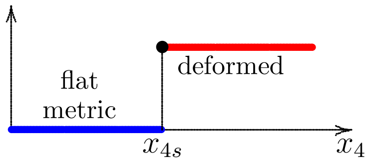
9 The 'fifth element'

By inserting into equation (54) or into equation (56) the expressions of the characteristic function G_{int} and its derivatives G'_{int} and G''_{int}, we obtain a first order differential equation in the unknown function $F_{int}(x_4)$ whose integration provides the '*fifth element*' i.e. the fifth component of the metric g_{44}. Furthermore, once the functions G_{int} and F_{int} are given, we can write down the explicit form of the components and of the eigenvalues of the Ricci tensor R^{int}_{ij} for each interaction. These will be functions of the coordinate x_4 dependent on the corresponding threshold x_{4int} and on a positive constant K_{int}.

In the coordinates $(x_0, x_1, x_2, x_3, x_4)$ to which we refer, the metric tensor and the Ricci tensor are both diagonalized. Then the main directions of curvature are identified by the coordinate axes. Consequently, the eigenvalues (principal curvatures) are defined by $\rho_i = g^{ii}\,R_{ii}$ and have dimension L^{-2}.

9.1 Hadronic metric after the threshold

Recall that before the threshold x_{4s} the hadronic metric is flat (§ 2.2) and therefore admits the trivial flow g_{ij} = constant.



Theorem 9.1 – *After the threshold x_{4s} the hadronic interaction metric admits a non-normalized Ricci flow as long as the function F satisfies the differential equation*

$$\boxed{x_4\,F' - 6\,F = 0} \tag{57}$$

whose complete integral is

$$\boxed{F_s = K_s\,x_4^6, \quad K_s > 0 \text{ constant}} \tag{58}$$

where K_s is an arbitrary positive constant. The components of the hadronic Ricci tensor are:

$$\begin{aligned} &\overset{s}{R}_{00} = \pm\frac{2}{K_s\,x_4^6\,x_{4s}^2} \\ &\overset{s}{R}_{11} = \overset{s}{R}_{22} = 0, \quad \overset{s}{R}_{33} = -\overset{s}{R}_{00} \\ &\overset{s}{R}_{44} = \frac{6}{x_4^2} \end{aligned} \tag{59}$$

Its eigenvalues are:

$$\begin{aligned} &\rho_0 = \pm\frac{2}{K_s\,x_4^8}, \quad \rho_1 = 0, \quad \rho_2 = 0, \\ &\rho_3 = \pm\frac{2}{K_s\,x_4^8} = \rho_0, \quad \rho_4 = \pm\frac{6}{K_s\,x_4^8} = 3\,\rho_0 \end{aligned} \tag{60}$$

Remark 9.1 – (i) The hadronic F_s function does not explicitly depend on the value of the threshold x_{4s}. (ii) The constant K_s has dimension L^{-6}. •

Proof. Starting from (9) for the hadronic metric we have

$$G = \frac{x_4^2}{x_{4s}^2} \Longrightarrow G' = \frac{2\,x_4}{x_{4s}^2} \Longrightarrow G'' = \frac{2}{x_{4s}^2},$$

$$\frac{G'}{G} = \frac{2\,x_4}{x_{4s}^2}\cdot\frac{x_{4s}^2}{x_4^2} = \frac{2}{x_4}, \quad \frac{G''}{G} = \frac{2}{x_{4s}^2}\cdot\frac{x_{4s}^2}{x_4^2} = \frac{2}{x_4^2}, \quad \frac{G''}{G'} = \frac{2}{x_{4s}^2}\cdot\frac{x_{4s}^2}{2\,x_4} = \frac{1}{x_4}.$$

We have to insert these expressions into equation (54):

$$\frac{G''}{G'} - \frac{F'}{2\,F} = \frac{2\,G''}{G}\frac{F}{F'} - \frac{G'}{G} - \frac{(G')^2}{G^2}\frac{F}{F'},$$

$$\frac{1}{x_4} - \frac{F'}{2\,F} = \frac{4}{\cancel{x_4^2}}\frac{\cancel{F}}{F'} - \frac{2}{x_4} - \frac{4\,x_4^2}{\cancel{x_{4s}^2}}\frac{\cancel{F}}{F'} \implies \frac{3}{x_4} - \frac{F'}{2\,F} = 0 \implies (57)$$

We get equation (58) with $K_s > 0$ since the function F is assumed to be positive. In the components (38) of the Ricci tensors of type 1,

$$\left[\begin{array}{l} R_{00} = \mp\,\dfrac{2\,G''\,F - G'\,F'}{4\,F^2}, \\ R_{11} = R_{22} = 0, \quad R_{33} = -R_{00}, \\ R_{44} = -\,\dfrac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F}, \end{array}\right.$$

we substitute the expressions

$$(9) \quad G = \frac{x_4^2}{x_{4s}^2} \Longrightarrow G' = \frac{2\,x_4}{x_{4s}^2} \Longrightarrow G'' = \frac{2}{x_{4s}^2},$$

and

$$(58) \quad F = K_s\,x_4^6 \Longrightarrow F' = 6\,K_s\,x_4^5.$$

We obtain:

$$R_{00} = \mp\,\frac{2\,\dfrac{2}{x_{4s}^2}\,K_s\,x_4^6 - \dfrac{2\,x_4}{x_{4s}^2}\,6\,K_s\,x_4^5}{4\,K_s^2\,x_4^{12}} = \mp\,\frac{\dfrac{1}{x_{4s}^2}\,x_4^6 - \dfrac{3}{x_{4s}^2}\,x_4^6}{K_s\,x_4^{12}} = \boxed{\pm\,\frac{2}{K_s\,x_4^6\,x_{4s}^2}}$$

$$\left[\begin{array}{l} R_{44} = -\,\dfrac{2\,\dfrac{2}{x_{4s}^2}\,K_s\,x_4^6\,\dfrac{x_4^2}{x_{4s}^2} - \dfrac{2\,x_4}{x_{4s}^2}\,6\,K_s\,x_4^5\,\dfrac{x_4^2}{x_{4s}^2} - \dfrac{4\,x_4^2}{x_{4s}^4}\,K_s\,x_4^6}{2\,\dfrac{x_4^4}{x_{4s}^4}\,K_s\,x_4^6} = -\,\dfrac{2\,\dfrac{2}{x_{4s}^2}\,\dfrac{x_4^2}{x_{4s}^2} - \dfrac{2}{x_{4s}^2}\,6\,\dfrac{x_4^2}{x_{4s}^2} - \dfrac{4\,x_4^2}{x_{4s}^4}}{2\,\dfrac{x_4^4}{x_{4s}^4}} \\ = -\,\dfrac{\dfrac{4}{\cancel{x_{4s}^2}}\,\dfrac{\cancel{1}}{x_{4s}^2} - \dfrac{12}{x_{4s}^2}\,\dfrac{1}{x_{4s}^2} - \dfrac{4}{\cancel{x_{4s}^4}}}{2\,\dfrac{x_4^2}{x_{4s}^4}} = \boxed{\dfrac{6}{x_4^2}} \end{array}\right.$$

Recall (59) and (9):

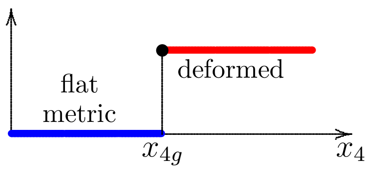
$$\left\{\begin{array}{l} R_{00}^s = \pm\,\dfrac{2}{K_s\,x_4^6\,x_{4s}^2}, \\ R_{11}^s = R_{22}^s = 0, \quad R_{33}^s = -R_{00}^s, \\ R_{44}^s = \dfrac{6}{x_4^2}, \end{array}\right. \qquad G_s = \left(\frac{x_4}{x_{4s}}\right)^2.$$

It follows that

$$\begin{cases} \rho_0 = g^{00}\, R_{00} = \dfrac{1}{G}\, R_{00} = \pm\, \dfrac{x_{4s}^2}{x_4^2}\, \dfrac{2}{K_s\, x_4^6\, x_{4s}^2} = \pm\, \dfrac{2}{K_s\, x_4^8} \\ \rho_1 = g^{11}\, R_{11} = 0, \quad \rho_2 = g^{22}\, R_{22} = 0 \\ \rho_3 = g^{33}\, R_{33} = -\dfrac{1}{G}\, R_{33} = \dfrac{1}{G}\, R_{00} = \rho_0 \\ \rho_4 = g^{44}\, R_{44} = \pm \dfrac{1}{F}\, \dfrac{6}{x_4^2} = \pm \dfrac{1}{K_s\, x_4^6}\, \dfrac{6}{x_4^2} = \pm \dfrac{6}{K_s\, x_4^8} = 3\, \rho_0 \qquad \blacksquare \end{cases}$$

9.2 Gravitational metric after the threshold

Recall that before the threshold x_{4g} the metric of the gravitational interaction is flat (§ 2.3) and therefore admits the trivial flow $g_{ij} = \text{constant}$.



Theorem 9.2 – *After the threshold x_{4g} the gravitational interaction metric admits a non-normalized Ricci flow as long as F satisfies the differential equation*

$$\boxed{(x_{4g} + x_4)\, F' - 6\, F = 0} \tag{61}$$

whose complete integral is

$$\boxed{F_g = K_g\, (x_{4g} + x_4)^6, \quad K_g > 0} \tag{62}$$

where K_g is an arbitrary positive constant. The components of the gravitational Ricci tensor are:

$$\boxed{\begin{aligned} &\overset{g}{R}_{00} = \pm\, \frac{1}{4\, K_g\, (x_{4g} + x_4)^6\, x_{4g}^2} \\ &\overset{g}{R}_{11} = \overset{g}{R}_{22} = 0, \quad \overset{g}{R}_{33} = -\overset{g}{R}_{00} \\ &\overset{g}{R}_{44} = \frac{6}{(x_{4g} + x_4)^2} \end{aligned}} \tag{63}$$

Its eigenvalues are:

$$\boxed{\begin{aligned} &\rho_0 = \pm\, \frac{1}{K_g\, (x_{4g} + x_4)^8} \\ &\rho_1 = \rho_2 = 0, \quad \rho_3 = \rho_0 \\ &\rho_4 = \pm \frac{6}{K_g\, (x_{4g} + x_4)^8} = 6\, \rho_0 \end{aligned}} \tag{64}$$

Remark 9.2 – (i) The gravitational function FG explicitly depends on the value of the threshold x_{4g}. (ii) The constant K_g has dimension L^{-6}. •

Proof. Starting from (13) for the gravitational metric we have

$$G = \frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2} \implies G' = \frac{x_{4g}+x_4}{2\,x_{4g}^2} \implies G'' = \frac{1}{2\,x_{4g}^2}.$$

$$\frac{G'}{G} = \frac{x_{4g}+x_4}{2\,x_{4g}^2}\cdot\frac{4\,x_{4g}^2}{(x_{4g}+x_4)^2} = \frac{2}{x_{4g}+x_4}, \qquad \frac{G''}{G'} = \frac{1}{2\,x_{4g}^2}\cdot\frac{2\,x_{4g}^2}{x_{4g}+x_4} = \frac{1}{x_{4g}+x_4}.$$

$$\frac{G''}{G} = \frac{1}{2\,x_{4g}^2}\cdot\frac{4\,x_{4g}^2}{(x_{4g}+x_4)^2} = \frac{2}{(x_{4g}+x_4)^2}.$$

We have to insert these expressions into equation (54):

$$\frac{1}{x_{4g}+x_4} - \frac{F'}{2\,F} = \cancel{\frac{4}{(x_{4g}+x_4)^2}\frac{F}{F'}} - \frac{2}{x_{4g}+x_4} - \cancel{\frac{4}{(x_{4g}+x_4)^2}\frac{F}{F'}}$$

$$\implies \frac{3}{x_{4g}+x_4} - \frac{F'}{2\,F} = 0 \implies \frac{6}{x_{4g}+x_4} - \frac{F'}{F} = 0 \implies (61)\ (x_{4g}+x_4)\,F' - 6\,F = 0.$$

In the complete integral (62) the constant K_g must be positive since F is a positive function. In the components (38) of type 1 Ricci tensors we replace the expressions

$$(13)\quad G = \frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2} \implies G' = \frac{x_{4g}+x_4}{2\,x_{4g}^2} \implies G'' = \frac{1}{2\,x_{4g}^2},$$

and

$$(62)\quad F = K_g\,(x_{4g}+x_4)^6 \implies F' = 6\,K_g\,(x_{4g}+x_4)^5 \implies \frac{F'}{F} = \frac{6}{x_{4g}+x_4}.$$

We obtain:

$$\begin{aligned} R_{00} &= \mp\frac{2\,\frac{1}{2\,x_{4g}^2}\,K_g\,(x_{4g}+x_4)^6 - \frac{x_{4g}+x_4}{2\,x_{4g}^2}\,6\,K_g\,(x_{4g}+x_4)^5}{4\,K_g^2\,(x_{4g}+x_4)^{12}} \\ &= \mp\frac{\frac{1}{x_{4g}^2}\,(x_{4g}+x_4)^6 - \frac{x_{4g}+x_4}{x_{4g}^2}\,3\,(x_{4g}+x_4)^5}{4\,K_g\,(x_{4g}+x_4)^{12}} = \mp\frac{-\frac{1}{x_{4g}^2}\,(x_{4g}+x_4)^6}{4\,K_g\,(x_{4g}+x_4)^{12}} \\ &= \boxed{\pm\frac{1}{4\,K_g\,(x_{4g}+x_4)^6\,x_{4g}^2}} \end{aligned}$$

$$\begin{aligned} R_{44} &= -\frac{\cancel{2\,\frac{1}{2\,x_{4g}^2}\,\frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2}} - \frac{\cancel{x_{4g}+x_4}}{2\,x_{4g}^2}\,\frac{6}{\cancel{x_{4g}+x_4}}\,\frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2} - \cancel{\left(\frac{x_{4g}+x_4}{2\,x_{4g}^2}\right)^2}}{2\left(\frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2}\right)^2} \\ &= \frac{6}{8}\,\frac{\frac{1}{x_{4g}^2}\,\frac{(x_{4g}+x_4)^2}{x_{4g}^2}}{\frac{1}{8}\,\frac{(x_{4g}+x_4)^4}{x_{4g}^4}} = \boxed{\frac{6}{(x_{4g}+x_4)^2}} \end{aligned}$$

Recall equations (63)

$$\begin{cases} \overset{g}{R}_{00} = \pm \dfrac{1}{4\,K_g\,(x_{4g}+x_4)^6\,x_{4g}^2} \\ \overset{g}{R}_{11} = \overset{g}{R}_{22} = 0, \quad \overset{g}{R}_{33} = -\overset{g}{R}_{00} \\ \overset{g}{R}_{44} = \dfrac{6}{(x_{4g}+x_4)^2} \end{cases}$$

and (13)

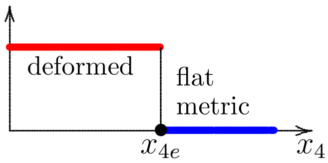
$$G_g = \tfrac{1}{4}\left(1+\frac{x_4}{x_{4g}}\right)^2 = \frac{(x_{4g}+x_4)^2}{4\,x_{4g}^2}.$$

It follows that

$$\begin{cases} \rho_0 = g^{00}\,\overset{g}{R}_{00} = \dfrac{1}{G_g}\,\overset{g}{R}_{00} = \pm\dfrac{4\,x_{4g}^2}{(x_{4g}+x_4)^2}\,\dfrac{1}{4\,K_g\,(x_{4g}+x_4)^6\,x_{4g}^2} = \pm\dfrac{1}{K_g\,(x_{4g}+x_4)^8} \\ \rho_1 = g^{11}\,\overset{g}{R}_{11} = 0, \quad \rho_2 = g^{22}\,\overset{g}{R}_{22} = 0 \\ \rho_3 = g^{33}\,\overset{g}{R}_{33} = -\dfrac{1}{G_g}\,\overset{g}{R}_{33} = \dfrac{1}{G_g}\,\overset{g}{R}_{00} = \rho_0 \\ \rho_4 = g^{44}\,\overset{g}{R}_{44} = \pm\dfrac{1}{F_g}\,\dfrac{6}{(x_{4g}+x_4)^2} = \pm\dfrac{6}{K_g\,(x_{4g}+x_4)^8} \end{cases} \quad \blacksquare$$

9.3 Electromagnetic metric before the threshold

The metric of the electromagnetic interaction is of type 2. Starting from zero energy up to the threshold energy the metric is deformed and becomes flat after the threshold x_{4e} (§ 2.4).



Theorem 9.3 – *The pre-threshold electromagnetic interaction metric admits a non-normalized Ricci flux as long as F is constant:*

(65) $$\boxed{F = K_e,\ K_e > 0 \ \textit{dimensionless constant}}$$

Proof. Let us consider a characteristic function that is a power of x_4/x_{4e}:

(66) $$\boxed{G = \frac{x_4^p}{x_{4e}^p}} \implies G' = p\,\frac{x_4^{p-1}}{x_{4e}^p} \implies G'' = p\,(p-1)\,\frac{x_4^{p-2}}{x_{4e}^p}$$

Insert these expressions into the equation (56) which characterizes the existence of a non-normalized Ricci flow

$$\frac{F'}{F}\,G + 3\left(2\,G'' - \frac{(G')^2}{G}\right)\frac{F}{F'} = 2\left(\frac{G''\,G}{G'} + 2\,G'\right)$$

and develop the terms A and B in parentheses:

$$A \stackrel{\text{def}}{=} \frac{G''\,G}{G'} + 2\,G', \quad B \stackrel{\text{def}}{=} 2\,G'' - \frac{(G')^2}{G}.$$

$$\left[\begin{array}{l} A = \dfrac{G''\,G}{G'} + 2\,G' = \dfrac{p\,(p-1)\,\dfrac{x_4^{p-2}}{x_{4e}^p}\,\dfrac{x_4^p}{x_{4e}^p}}{p\,\dfrac{x_4^{p-1}}{x_{4e}^p}} + 2\,p\,\dfrac{x_4^{p-1}}{x_{4e}^p} = (p-1)\,\dfrac{x_4^{p-2}}{x_{4e}^p}\,\dfrac{x_4^p}{x_{4e}^p}\,\dfrac{x_{4e}^p}{x_4^{p-1}} + 2\,p\,\dfrac{x_4^{p-1}}{x_{4e}^p} \\ = (p-1)\,\dfrac{x_4^{p-2}}{x_{4e}^p}\,\dfrac{x_4^p}{x_4^{p-1}} + 2\,p\,\dfrac{x_4^{p-1}}{x_{4e}^p} = (p-1)\,\dfrac{x_4^{p-1}}{x_{4e}^p} + 2\,p\,\dfrac{x_4^{p-1}}{x_{4e}^p} = \Big((p-1) + 2\,p\Big)\,\dfrac{x_4^{p-1}}{x_{4e}^p} \\ = (3\,p-1)\,\dfrac{x_4^{p-1}}{x_{4e}^p}. \end{array}\right.$$

From here we see that $A = 0$ if and only if $\boxed{p = \frac{1}{3}}$ In this case the characteristic function (16) and its derivatives become

$$G = \frac{x_4^{1/3}}{x_{4e}^{1/3}} \implies G' = \tfrac{1}{3}\,\frac{x_4^{-2/3}}{x_{4e}^{1/3}} \implies G'' = -\tfrac{2}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} \tag{67}$$

As A vanishes, equation (56) reduces to

$$\frac{F'}{F} = 2\left(\frac{G''}{G'} + 2\,\frac{G'}{G}\right). \tag{68}$$

Taking into account equations (67) we find

$$\left[\begin{array}{l} (68) \implies \dfrac{F'}{F} = 2\left(\dfrac{G''}{G'} + 2\,\dfrac{G'}{G}\right) = 2\,\left(-\tfrac{2}{9}\,\dfrac{x_4^{-5/3}}{x_{4e}^{1/3}}\cdot 3\,\dfrac{x_{4e}^{1/3}}{x_4^{-2/3}} + 2\,\tfrac{1}{3}\,\dfrac{x_4^{-2/3}}{x_{4e}^{1/3}}\cdot\dfrac{x_{4e}^{1/3}}{x_4^{1/3}}\right) \\ = 2\left(-\tfrac{2}{3}\,\dfrac{1}{x_4} + \tfrac{2}{3}\,\dfrac{1}{x_4}\right) = 0 \end{array}\right.$$

so that $F' = 0$. ■

Alternative proof. Starting from its characteristic function(16)

$$G_e = \left(\frac{x_4}{x_{4e}}\right)^{1/3}$$

and substituting p for $\frac{1}{3}$ we find the equalities

$$G = \frac{x_4^p}{x_{4e}^p}, \quad G' = p\,\frac{x_4^{p-1}}{x_{4e}^p}, \quad G'' = p\,(p-1)\,\frac{x_4^{p-2}}{x_{4e}^p}.$$

By inserting these expressions into equation (56) which characterizes the existence of a non-normalized Ricci flow

$$\frac{F'}{F}\,G + 3\left(2\,G'' - \frac{(G')^2}{G}\right)\frac{F}{F'} = 2\left(\frac{G''\,G}{G'} + 2\,G'\right)$$

we find that the term in brackets on the left hand side vanishes for $p = \frac{1}{3}$. As a result, this equation simplifies in order to allow its integration by separation of variables:

$$\frac{F'}{F} = 2\left(\frac{G''}{G'} + 2\,\frac{G'}{G}\right).$$

Since

$$\left[\begin{array}{l} \dfrac{G''}{G'} + 2\,\dfrac{G'}{G} = p\,(p-1)\,\dfrac{x_4^{p-2}}{x_{4e}^{p}}\cdot p^{-1}\,\dfrac{x_{4e}^{p}}{x_4^{p-1}} + 2\,p\,\dfrac{x_4^{p-1}}{x_{4e}^{p}}\cdot\dfrac{x_{4e}^{p}}{x_4^{p}} \\ = (p-1)\,x_4^{-1} + 2\,p\,x_4^{-1} = (3p-1)\,x_4^{-1} = 0 \quad \text{per } p = \frac{1}{3}, \end{array}\right.$$

we get $F' = 0$. ■

Theorem 9.4 – *The components of the electromagnetic Ricci tensor are*

$$\boxed{\begin{array}{l} \overset{e}{R}_{00} = 0 \\ \overset{e}{R}_{11} = \overset{e}{R}_{22} = \overset{e}{R}_{33} = \mp\frac{1}{12}\,\dfrac{1}{K_e}\,\dfrac{x_4^{-5/3}}{x_{4e}^{1/3}} \\ \overset{e}{R}_{44} = \frac{5}{12}\,\dfrac{1}{x_4^2} \end{array}} \tag{69}$$

Its eigenvalues are

$$\boxed{\begin{array}{l} \rho_0 = 0 \\ \rho_1 = \rho_2 = \rho_3 = \pm\frac{1}{12}\,\dfrac{1}{K_e}\,\dfrac{1}{x_4^2} \\ \rho_4 = \pm\frac{5}{12}\,\dfrac{1}{K_e}\,\dfrac{1}{x_4^2} = 5\,\rho_1 \end{array}} \tag{70}$$

Proof. We combine (39), which provide the general form of the Ricci components for a type 2 metric

$$\boxed{\begin{array}{l}\text{type 2}\\ \text{metrics}\end{array}} \quad \boxed{\pm F} \left[\begin{array}{l} R_{00} = 0 \\ R_{11} = R_{22} = R_{33} = \pm\,\dfrac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G} \\ R_{44} = -\,3\,\dfrac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F} \end{array}\right.$$

with (67) which provide the electromagnetic characteristic function and its derivatives:

$$G = \frac{x_4^{1/3}}{x_{4e}^{1/3}} \implies G' = \frac{1}{3}\,\frac{x_4^{-2/3}}{x_{4e}^{1/3}} \implies G'' = -\frac{2}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}}$$

Taking into account that $F = K_e$ (constant) we find

$$\left[\begin{aligned} & R_{11} = \pm\,\frac{2\,G''\,F\,G + (G')^2\,F - \cancel{F'}\,G'\,G}{4\,F^2\,G} = \pm\,\frac{2\,G''\,G + (G')^2}{4\,F\,G} = \pm\frac{1}{4\,K_e}\left(2\,G'' + \frac{(G')^2}{G}\right) \\ & = \pm\frac{1}{4\,K_e}\left[-\tfrac{4}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} + \left(\tfrac{1}{3}\,\frac{x_4^{-2/3}}{x_{4e}^{1/3}}\right)^2 \cdot \frac{x_{4e}^{1/3}}{x_4^{1/3}}\right] = \pm\frac{1}{4\,K_e}\left[-\tfrac{4}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} + \tfrac{1}{9}\,\frac{x_4^{-4/3}}{x_{4e}^{2/3}} \cdot \frac{x_{4e}^{1/3}}{x_4^{1/3}}\right] \\ & = \pm\frac{1}{4\,K_e}\left[-\tfrac{4}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} + \tfrac{1}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}}\right] = \boxed{\mp\frac{1}{12\,K_e}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}}} \end{aligned}\right.$$

$$\left[\begin{aligned} & R_{44} = -\,3\,\frac{2\,F\,G''\,G - (G')^2\,F - \cancel{F'}\,G'\,G}{4\,G^2\,F} = -\,\tfrac{3}{4}\,\frac{2\,G''\,G - (G')^2}{G^2} \\ & = -\,\tfrac{3}{4}\left(2\,G'' \cdot G^{-1} - (G')^2 \cdot G^{-2}\right) = -\,\tfrac{3}{4}\left(-\tfrac{4}{9}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} \cdot \frac{x_{4e}^{1/3}}{x_4^{1/3}} - \tfrac{1}{9}\,\frac{x_4^{-4/3}}{x_{4e}^{2/3}} \cdot \frac{x_{4e}^{2/3}}{x_4^{2/3}}\right) \\ & = -\,\tfrac{3}{4}\left(-\,\tfrac{4}{9}\,x_4^{-2} - \tfrac{1}{9}\,x_4^{-2}\right) = \tfrac{3}{4}\,\tfrac{5}{9}\,x_4^{-2} = \boxed{\tfrac{5}{12}\,\frac{1}{x_4^2}} \end{aligned}\right.$$

Calculation of the eigenvalues. From (15) we obtain the contravariant components of the metric

$$x_4 < x_{4e} \qquad \begin{cases} g^{00} = 1 \\ g^{11} = g^{22} = g^{33} = -\left(\dfrac{x_{4e}}{x_4}\right)^{1/3} & \text{spatial isotropy} \\ g^{44} = \pm\dfrac{1}{F} = \pm\dfrac{1}{K_e}. \end{cases}$$

Recalling (69) we find:

$$\left[\begin{aligned} & \rho_0 = g^{00}\,\overset{e}{R}_{00} = 0 \\ & \rho_1 = g^{11}\,\overset{e}{R}_{11} = \pm\left(\frac{x_{4e}}{x_4}\right)^{1/3}\,\tfrac{1}{12}\,\frac{1}{K_e}\,\frac{x_4^{-5/3}}{x_{4e}^{1/3}} = \pm\tfrac{1}{12}\,\frac{1}{K_e}\,\frac{1}{x_4^2} \\ & \rho_4 = g^{44}\,\overset{e}{R}_{44} = \pm\tfrac{5}{12}\,\frac{1}{K_e}\,\frac{1}{x_4^2} = 5\,\rho_1. \quad \blacksquare \end{aligned}\right.$$

9.4 Leptonic metric before the threshold

Recall that as the energy increases the type 2 leptonic metric becomes flat after the threshold (see §2.5 and Figure 8) and that before the threshold its characteristic function is the same as that of the electromagnetic metric

$$G_w = \left(\frac{x_4}{x_{4w}}\right)^{1/3}$$

so that Theorem 9.3 also holds for the leptonic metric. Consequently, for the leptonic metric the results obtained in the previous section for the electromagnetic metric hold true.

10 The fifth element of the metric according to Pessa convention

10.1 Pessa's constant and convention

So far we have considered and used the characteristic functions G dependent on the dimensionless variable given by the ratio

$$\frac{x_4}{x_{4int}}$$

between the energy coordinate x_4 and the energy threshold x_{4int} characteristic for each interaction.[9]

Now we redefine the variable x_4, which is our energy coordinate, introducing the Pessa constant ℓ, which has the dimensions of a length, preserving the dimensions of a length for x_4 as we have already introduced in (2). At the same time we introduce the new variable

$$\bar{x}_4 \stackrel{\text{def}}{=} \ell \cdot \frac{x_4}{x_{4int}} \tag{71}$$

so that the old dimensionless variable is expressed by

$$\frac{x_4}{x_{4int}} = \frac{\bar{x}_4}{\ell} \ .$$

We solve what has been said for the equation (2) with the following definition (**Pessa convention**):

$$k \equiv \frac{\ell}{x_{4int}}$$

where k has the dimensions of the inverse of a linear energy density. Its importance consists in explicitly making the fifth energy coordinate x_4 dimensionally homogeneous to the others through the introduction of the Pessa constant ℓ, whose meaning, physical identification and numerical value are reported below .

The value of ℓ is in the range $4 - 8\,\mu m$ as it was originally calculated theoretically and reported in [1], §16.3, page 250. This ℓ is the characteristic linear dimension for all interactions and whose volume ℓ^3 allows us to calculate the **critical energy density** D_{Cint} which gives rise to the **metamorphosis of matter**. For each interaction this critical energy density is given by

$$D_{Cint} = \frac{x_{4int}}{\ell^3} \tag{72}$$

where x_{4int} has the dimension of an energy.

From the phenomenology of experiments concerning both space-time deformation emissions [1] [3]–[5] and DST transformations, nuclear metamorphoses, [6] –[11], we can set $\ell = 10\,\mu m$ which corresponds to the characteristic diameter measured for the so-called **Ridolfi cavities** [5].

[9] Recall that the symbol int (interaction) stands for $(em, grav, weak, strong) = (e, g, w, s)$.

Pessa convention definitively replaces the convention and the related Kostro constant [2], already used in [1], p. 282.

It is not necessary to define different ℓ for different interactions since the distinction is already inherent in $\bar{x}_4$ for each metric of each interaction. This can be seen from the (71)

$$\boxed{\bar{x}_4} \stackrel{\text{def}}{=} \ell \cdot \frac{x_4}{\boxed{x_{4int}}}$$

where on the right hand side the interaction dependence is in $\boxed{x_{4int}}$

We must strongly underline that the Pessa constant $\ell = 10\,\mu m$ is not a universal constant but a phenomenological constant useful for defining the critical energy density for each interaction, via (72), when it acts on matter in condition of space-time deformation, as we have already said above.

10.2 Leptonic metric

As an example we deal with the leptonic metric according to Pessa convention using the coordinate $\bar{x}_4$ and Pessa constant ℓ. What we do here for the leptonic metric can be retrospectively repeated for all metrics of other interactions. We wanted to follow this path so as not to make the whole discussion too heavy.

As already said at the end of §2.5, the leptonic Heaviside step is similar to the electromagnetic one, with the difference that the leptonic threshold is $2 \cdot 10^{16}$ times electromagnetic threshold.

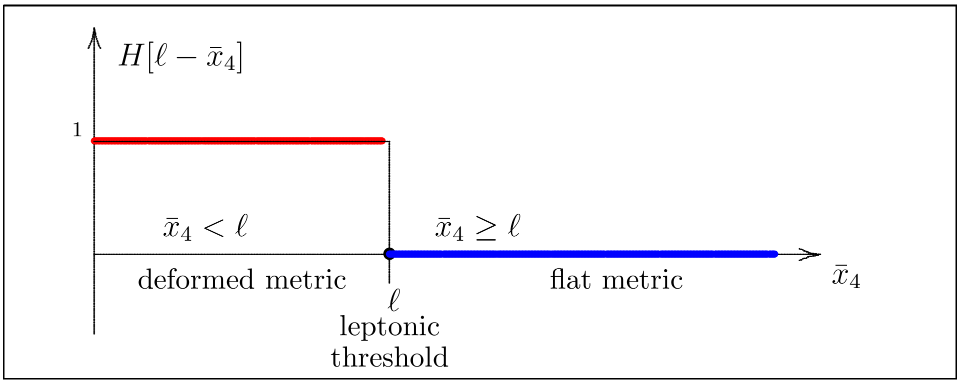


Figure 9: Leptonic Heaviside step of axis $\bar{x}_4$ with threshold ℓ.

Figure 9 is obtained from Figure 8 by replacing x_4 with its expression within Pessa convention, i.e.

$$\bar{x}_4 \stackrel{\text{def}}{=} \ell \cdot \frac{x_4}{x_{4w}}$$

whose inverse is

$$x_4 = \bar{x}_4 \cdot \frac{x_{4w}}{\ell}.$$

The leptonic metric is

$$(73)\qquad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left\{1 + H[\ell - \bar{x}_4]\left[\left(\dfrac{\bar{x}_4}{\ell}\right)^{1/3} - 1\right]\right\} \text{ spatial isotropy} \\ g_{44} = \pm F(\bar{x}_4), \quad F > 0. \end{cases}$$

• Before the threshold we have $H[\ell - \bar{x}_4] = 1$ and the metric becomes

$$(74)\qquad \bar{x}_4 < \ell \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left(\dfrac{\bar{x}_4}{\ell}\right)^{1/3} \quad \text{spatial isotropy} \\ g_{44} = \pm F(\bar{x}_4), \quad F(\bar{x}_4) > 0. \end{cases}$$

The comparison with (4) shows that this metric is type 2 with characteristic function

$$(75)\qquad \boxed{G = \left(\frac{\bar{x}_4}{\ell}\right)^{1/3}}$$

(the same as the electromagnetic metric) and signature $(+ - - - \pm)$.

•• After the threshold we have $H[\ell - \bar{x}_4] = 0$ and the metric becomes

$$(76)\qquad \bar{x}_4 \geq \ell \quad \begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -1, \quad \text{spatial isotropy} \\ g_{44} = \pm F(\bar{x}_4), \quad F(\bar{x}_4) > 0. \end{cases}$$

The comparison with (4) shows that this metric is type 2 with $G = 1$. It is flat with signature $(+ - - - \pm)$.

10.3 Remarks on metrics

It is worth highlighting the analogy between the strong and gravitational metrics. In both cases a deformation of the temporal coordinate occurs. Furthermore, one of the spatial parameters (which we have conventionally assumed as the third parameter) varies with energy like the temporal one in a **over–Minkowskian** way, that is, it approaches the Minkowskian limit for energy values greater than the energy of interaction threshold. The other two spatial parameters are constant, but of different value for the hadronic case (i.e., the three-space is anisotropic for the hadronic interaction even in derived forms, see its behavior inside the atomic nucleus).

The threshold energy, generally indicated by E_0, or x_{4int}, so that $E_0 \equiv x_{4int}$, is the energy value at which the metric parameters of the interactions reach a constant value, i.e. the metric becomes Minkowskian.

Note that for both electromagnetic and leptonic interactions the metric is isochronous, i.e. the time parameter does not change as the energy varies, furthermore it is spatially isotropic and **sub–Minkowskian**, i.e. it approaches the Minkowski limit for values increasing in energy but less than the threshold energy.

10.4 Physical-phenomenological identification of the conserved volume

Keep in mind that the proof of Theorem 3.1 has a general character, it is independent of the normalized Ricci flow and does not depend on the metrics of the interactions but is applied to them to verify that the interaction conserves the volume. Remember what has already been said in §8 (non-normalized Ricci flows): for type 1 and 2 metrics the conservation of volumes is in any case guaranteed by the Theorem 3.1 which it does not involve the Ricci tensor.

In particular, its application to the hadronic metric allows us to conclude that it conserves volume. Likewise, the interaction represented by this metric conserves the volume, therefore one of the main characteristics of the nuclear interaction (hadronic interaction in the nucleus), which is the constancy of the density in the nucleus, is respected even if the nucleus is subjected to deformation beyond of those already known for ellipsoidal nuclei. In fact, the conservation of the volume, regardless of the deformation, allows the nuclear density to be constant.

In order to identify the conserved volume we evaluate the physical volume of the deformed hyperparallelepiped referred to in equation (27). In general we estimate V_D at the energy thresholds E_0 counting on making a useful estimate not only for the metric and the hadronic interaction but also for the other interactions and related metrics.

Let us now present our proposals regarding the physical volume of the deformed hyperparallelepiped whose edges are segments of coordinated lines.

First three-dimensional proposal. Setting ℓ equal to Pessa's constant, already introduced in §10.1, we evaluate the volume of this hyperparallelepiped through the following identification with the critical volume of nuclear metamorphosis.

Let us remember that this identification is valid in a three-dimensional Euclidean space with only three spatial coordinates, i.e. for the spatial part of the Minkowskian metric representation of an interaction: $V_D \equiv V_C \equiv \ell^3$, where V_D is the volume of the domain D, V_C is the critical volume and ℓ is the aforementioned Pessa constant. It is clear that this estimate, based on this identification, can be valid for any interaction metric around the threshold energy E_0 which, remember, is a point of discontinuity in the representation by the Heaviside function.

Second four-dimensional proposal. Remember that $x_0 = u(E)\,t$, $[x_0] = \mathsf{L}$.

• **First chance**. Setting $E \to E_0$, let's set $x_0 \propto c\,(h/E_0)$, i.e. for $E \to E_0$, $x_0 = u(E)\,t \to c\,(h/E_0)$. Therefore we estimate that $V_D \equiv \ell^3\,c\,(h/E_0)$, ℓ Pessa constant, $c = u(E_0)$, Plank constant h, from which $[V_D] = [\ell^3\,c\,(h/E_0)] = \mathsf{L}^4$. This estimate is based on the fact that Plank's constant can be identified with the constant of the first integral of the geodesic motion referred to the coordinates <u>time</u> and <u>energy</u>.[10]

•• **Second chance**. Setting $E \to E_0$, let's set $x_0 \propto (e^2/E_0)$, i.e. for $E \to E_0$, x_0 tends to (e^2/E_0), so we estimate that $V_D \equiv \ell^3\,(e^2/E_0)$, ℓ Pessa constant, e^2 square of the elementary electric charge (again remember that e is constant and relativistically invariant in Minkowskian space) from which $[V_D] = [\ell^3\,(e^2/E_0)] = \mathsf{L}^4$. This estimate is based on the

[10] See [1], Chap. 24, §24.4, p. 377, eq. 24.91.

fact that the square of the elementary electric charge could be identifiable with a constant of the first integral of the geodesic motion referred to the space and energy coordinates.

Third five-dimensional proposal. Remember that $x_0 = u(E)\,t$, $[x_0] = \mathsf{L}$, $x_4 = \ell\,(E/E_0)$, $[x_4] = \mathsf{L}$.

• **First chance**. Setting $E \to E_0$, let's set $x_0 \propto c\,(h/E_0)$, i.e. for $E \to E_0$, $x_0 = u(E)\,t$ tends to c , (h/E_0), furthermore for E which tends to E_0 we have $x_4 = \ell\,(E/E_0)$ tends to ℓ, where h is Planck constant, ℓ is Pessa constant, $c = u(E_0)$. Therefore we estimate that $V_D \equiv \ell^3\,c\,(h/E_0) = \ell^4\,c\,(h/E_0)$. Dimensions: $[V_D] = [\ell^3\,c\,(h/E_0)] = [\ell^4\,c\,(h/E_0)] = \mathsf{L}^5$.

Also this estimate is based on the fact that Plank constant can be identified with the constant of the first integral of the geodesic motion referred to the time and energy coordinates.

•• **Second chance**. Setting $E \to E_0$, let's set $x_0 \propto (e^2/E_0)$, i.e. for $E \to E_0$, x_0 tends to (e^2/E_0). Furthermore for E which tends to E_0 we have $x_4 = \ell\,(E/E_0)$ tends to ℓ, Pessa constant, e^2 squared of the elementary electric charge (remember that e is constant and relativistically invariant in Minkowskian space). Therefore we estimate that $V_D \equiv \ell^4\,(e^2/E_0)$, from which $[V_D] = [\ell^4\,(e^2/E_0)] = \mathsf{L}^5$.

Also this estimate is based on the fact that the square of the elementary electric charge could be identifiable with a constant of the first integral of the geodesic motion referred to the space and energy coordinates.

It should be noted that all these three-dimensional, four-dimensional and five-dimensional estimates were carried out using the dimensional analysis method as proposed by P. Dirac. In this sense we can also interpret the presence of $\ell^{-6} = K_s$ in equation (58) as the inverse of the square of a volume; similarly for K_g in equation (62).

10.5 Physical meaning of Ricci eigenvalues

With the eigenvalues of the Ricci tensor for the interaction metrics, interpreted as **principal curvatures**, we can describe the deformation not only in space but also in time and, novelty, in energy. In fact, the eigenvalue of the energy ρ_4 in each interaction explains how the interaction itself measures the energy it has. In this sense we have further information on the **calibration** (gauge) of the energy for each interaction. The union of the information coming for each interaction, both from the eigenvalue and from the metric element corresponding to the energy coordinate, gives us the complete picture of the calibration, thus overcoming the arbitrariness and ambiguities that can arise in other physical-mathematical forms of representation of interactions.

Since these eigenvalues have the dimension of an area, they give us the area of comparison within which the deformation of a surface occurs. In other words they identify the minimum area where the deformation is effective and generates phenomena unrelated to a flat and Minkowskian area. In this sense, the Fermi-Walker theorem cannot be applied in general in this area identified by these eigenvalues.

11 Pentadimensional metrics

Here we summarize the $5D$ metrics where (i) *the fifth element is reported explicitly* and (ii) *for each interaction the natural unit of measurement of energy is its own threshold energy.*

Adronic metric §2.2. Formulas (7) and (8) are reported, where $F(x_4)$ takes on the value $F_s(x_4)$ calculated in (58):

• Before the threshold (7): $$\begin{cases} g_{00} = 1, \\ g_{11} = -\alpha \\ g_{22} = -\beta \\ g_{33} = -g_{00} = -1 \\ g_{44} = \pm F_s(x_4) = \text{dimensionless constant} \stackrel{\text{def}}{\equiv} +1 \end{cases}$$

•• After the threshold (8): $$\begin{cases} g_{00} = \dfrac{x_4^2}{x_{4s}^2} \\ g_{11} = -\alpha \\ g_{22} = -\beta \\ g_{33} = -g_{00} = -\dfrac{x_4^2}{x_{4s}^2} \\ g_{44} = \pm F_s(x_4),\ F_s(x_4) > 0,\ F_s(x_4) \text{ dimensionless.} \end{cases}$$

• • • Fifth element (58):

$$F_s = K_s\, x_4^6, \quad K_s > 0 \text{ constant, dimension } \mathsf{L}^{-6}.$$

In this way we realize both g_{44} being dimensionless and K_s being a positive constant, but we also measure the energy in natural units with the basic unit of reference being the threshold energy, as mentioned above. beginning of this paragraph. We specify that this method of energy measurement is proposed here as a general paradigm valid for every interaction where this is necessary.

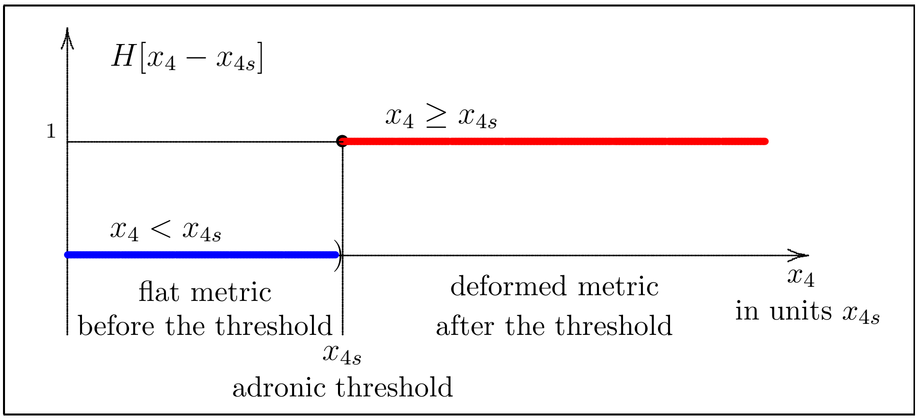


Figure 10: Hadronic Heaviside step of axis x_4 in threshold units x_{4s}.

Gravitazional metric §2.3. Formulas (11) and (12) are reported, where $F(x_4)$ takes on the value $F_g(x_4)$ calculated in (62):

$$\bullet \text{ Before the threshold } \begin{cases} g_{00} = 1 \\ g_{11} = -\alpha, \quad \alpha > 0 \text{ dimensionless constant} \\ g_{22} = -\beta, \quad \beta > 0 \text{ dimensionless constant} \\ g_{33} = -g_{00} = -1 \\ g_{44} = \pm F_g(x_4), \; F_g(x_4) > 0, \; F_g(x_4) \text{ dimensionless} \end{cases}$$

$$\bullet\bullet \text{ After the threshold } \quad (12) \begin{cases} g_{00} = \frac{1}{4}\left(1 + \dfrac{x_4}{x_{4g}}\right)^2 \\ g_{11} = -\alpha, \quad \alpha > 0 \text{ dimensionless constant} \\ g_{22} = -\beta, \quad \beta > 0 \text{ dimensionless constant} \\ g_{33} = -g_{00} \\ g_{44} = \pm F_g(x_4), \quad F_g(x_4) > 0, \quad F_g(x_4) \text{ dimensionless.} \end{cases}$$

••• Fifth element (62): $F_g = K_g\,(x_{4g} + x_4)^6$, $K_g > 0$ constant, dimension L^{-6}.

The measurement of energy in natural units via its threshold energy also applies to gravity. The situation is therefore similar to that of hadronic interaction:

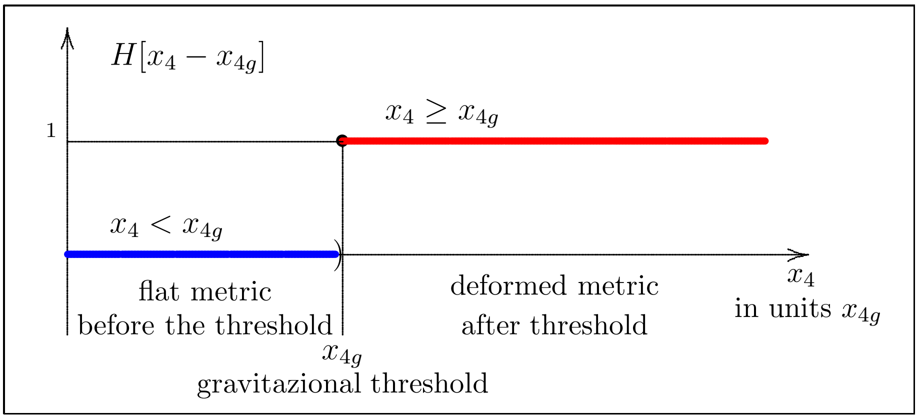


Figure 11: Gravitational Heaviside step of axis x_4 in threshold units x_{4g}.

Elettromagnetic metric. Formulas (15) and (17) are reported, where $F(x_4)$ takes the value $F_e(x_4)$ calculated in (65):

• Before the threshold (15) $\begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\left(\dfrac{x_4}{x_{4e}}\right)^{1/3} \quad \text{spatial isotropy} \\ g_{44} = \pm F_e(x_4), \quad F_e(x_4) > 0 \text{ dimesionless constant.} \end{cases}$

•• After the threshold (17) $\begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -1, \quad \text{spatial isotropy} \\ g_{44} = \pm F_e(x_4), \quad F_e(x_4) > 0 \ \text{ dimensionless.} \end{cases}$

••• Fifth element (65): $F_e = K_e,\ K_e > 0$ dimensionless constant, which we set $\equiv +1$.

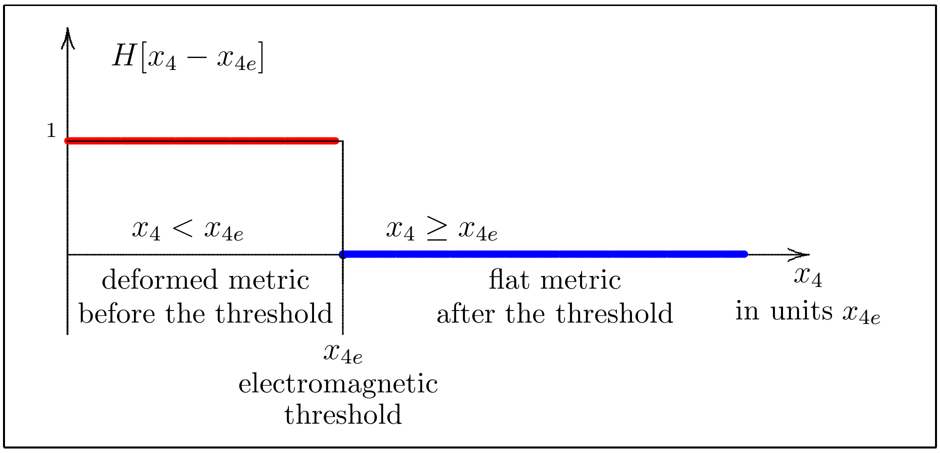


Figure 12: Electromagnetic Heaviside step of axis x_4 in threshold units x_{4e}.

Leptonic metric. The situation is similar to that of the electromagnetic interaction (x_{4e} should be replaced by x_{4w}).

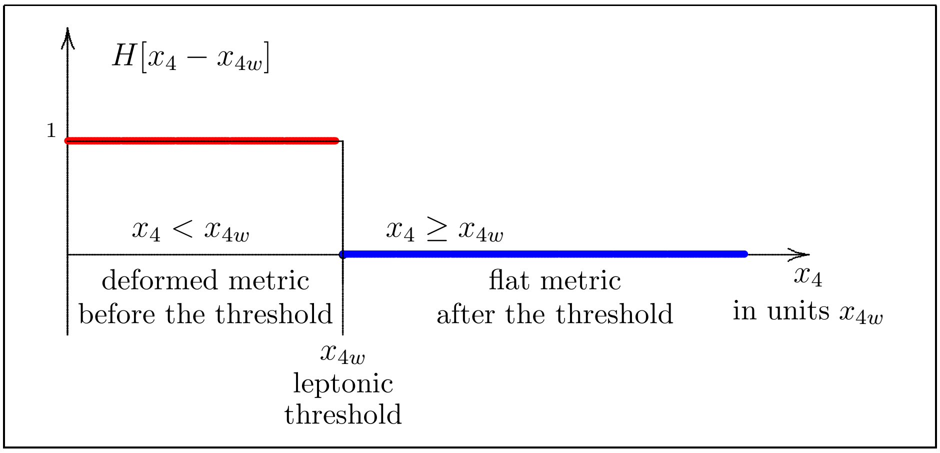


Figure 13: Leptonic Heaviside step of axis x_4 in threshold units x_{4w}.

As seen from their pentadimensional expressions, for the metrics of the four interactions the fifth elements of the metrics are the following:

(i) for hadronic and gravitational interactions it is a power of energy measured in units of its threshold energy E_{0int} and does not respect any type of threshold, so this element of the metric acts even when the metric is flat;

(ii) for leptonic and electromagnetic interactions it is a constant and therefore indifferent to whether the metric is flat or not flat (deformed).

Here we can hazard the hypothesis that the fifth parameter of the metric g_{44} for each interaction is the calibration of the energy for that interaction, in fact it describes for the energy coordinate how it is modified in the metric of the interaction itself, i.e., in simple words, how *in each interaction energy measures energy.* In conclusion, we remind that the reference energy for each phenomenon is that measured with instruments that use electromagnetic interaction in conditions of flat space-time and the validity of Hamilton's theorem for the conservation of total energy. In fact we can ignore the g_{44} of the electromagnetic interaction ([1] Chap. 1) by setting $K_e \equiv 1$. Therefore:

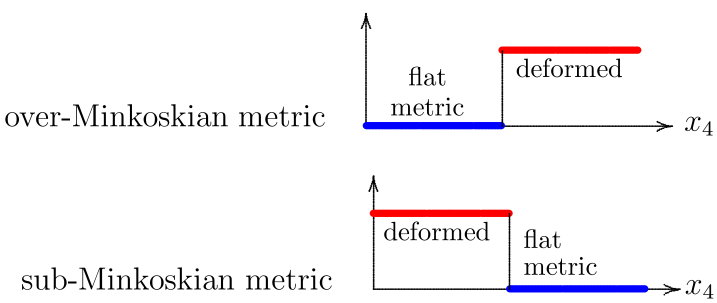
The electromagnetic metric and the electromagnetic interaction constitute the fixed point of reference in all measurements of phenomena also governed by other interactions with their relative metrics.

The result that the fifth element of the metric, corresponding to the coordinate energy has a functional dependence on a power of the energy, had already been hypothesized previously as we will summarize at the beginning of §13.

12 Over- and sub-Minkowskian metrics

As already mentioned at §10.3 a pentadimensional metric is called:

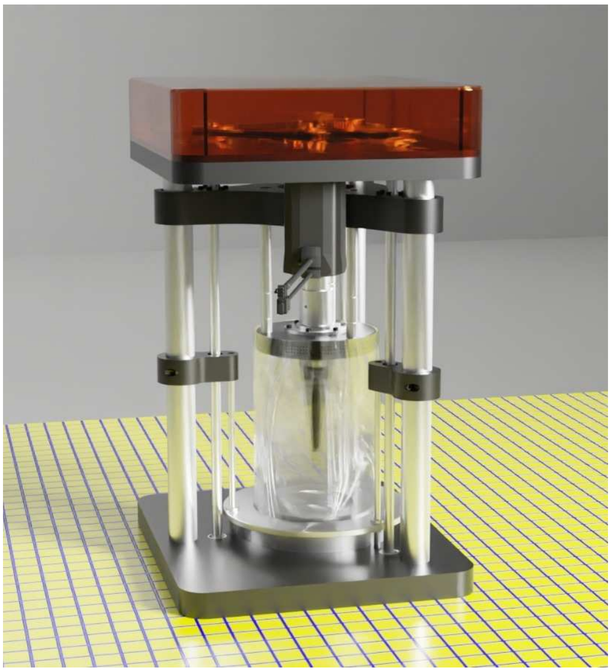
over-Minkowskian if the deformed metric becomes flat while x_4 decreases,
sub-Minkowskian if the deformed metric becomes flat while x_4 increases.



Ultimately we can say that *over-Minkoskian metric interactions*, such as hadronic and gravitational, have a variable energy calibration with the energy itself, regardless of whether the metric is flat or deformed. On the other hand, *sub-Minkowskian metric* *interactions*, such as electromagnetic and leptonic, always have the same energy calibration regardless of the value of the energy itself. We have conventionally set this calibration equal to 1, i.e. $K_e = 1$ for the electromagnetic interaction as a convention and for convenience. Despite our choice to set $K_e = K_w = 1$ in the next section, it is clear that we can leave the value for the leptonic interaction, i.e. the value of K_w, undefined for an appropriate phenomenological, experimental and theoretical verification. In simple words we want to maintain the possibility of checking whether $K_e = K_w$ or whether $K_e \neq K_w$.

The results achieved in the present work have allowed us to design, build and test devices capable of exploiting the behavior of the fifth element g_{44} of the metrics, in particular hadronic and leptonic, to obtain the production of electric charges directly from the nuclear metamorphosis of the matter (Ref. [3]– [11]).
One of these devices is being designed and built in the laboratories of High Sonic Technology (HST) in Rome as a *reactor-generator* which exploits in particular the results obtained here relating to the F function in order to determine the dimensions and operating conditions of its components.[a]

[a] Private communication from the HST owner regarding patents pending.



An experimental sample of the core
of a reactor-generator (courtesy of Eng. D. Bassani).

13 Overview of the five-dimensional metrics

As announced at the end of § 11 we underline that the results relating to the fifth element of the metric can be summarized for all interactions with the following expression of the functional dependence on energy:

$$b_5^2(E) = E^r, \quad r \in \mathbb{Q} + \{0\}.$$

This expression, which provides the functional dependency form from energy, had already been hypothesized previously in the context of the 12 classes of solutions of Einstein's field equations in vacuum for penta-dimensional metrics in deformed space-time. The result was also presented in 2004 as an assumption in [18] Chap. 15, §15.3, p. 135–135, eq. (15.11). It is therefore a further result to have verified with the Ricci Flow method that this previous hypothesis is correct.

Finally, for the convenience of the reader, we summarize in tabs the results already exposed for each interaction where, however, the metrics are written with the convention and symbols used in [18] Chap. 11, p. 93–95 and [1] Chap. 4, p. 53–60. In practice we replace the elements of the diagonal metric expressed with the symbols g_{ii} used in this work with the symbols $b_i^2(E)$ used in the cited references, in order to obtain a graphical representation of their evolution (in function of E) before and after the relevant threshold $E_{0\,int}$. This is a purely nominalistic and conventional fact that we wish to do in order to reconnect with the fundamental works from which this one derives and constitutes further progress.

Below, for each interaction, we report the forms filled out during the experiments. These forms contain data and sketchy graphs of the various $b^2(E)$. However, giving the values of the thresholds $E_{0\,int}$, it is possible to reprocess these data in order to display the trend of $b^2(E)$ in numerically reliable graphs.

13.1 Electromagnetic interaction

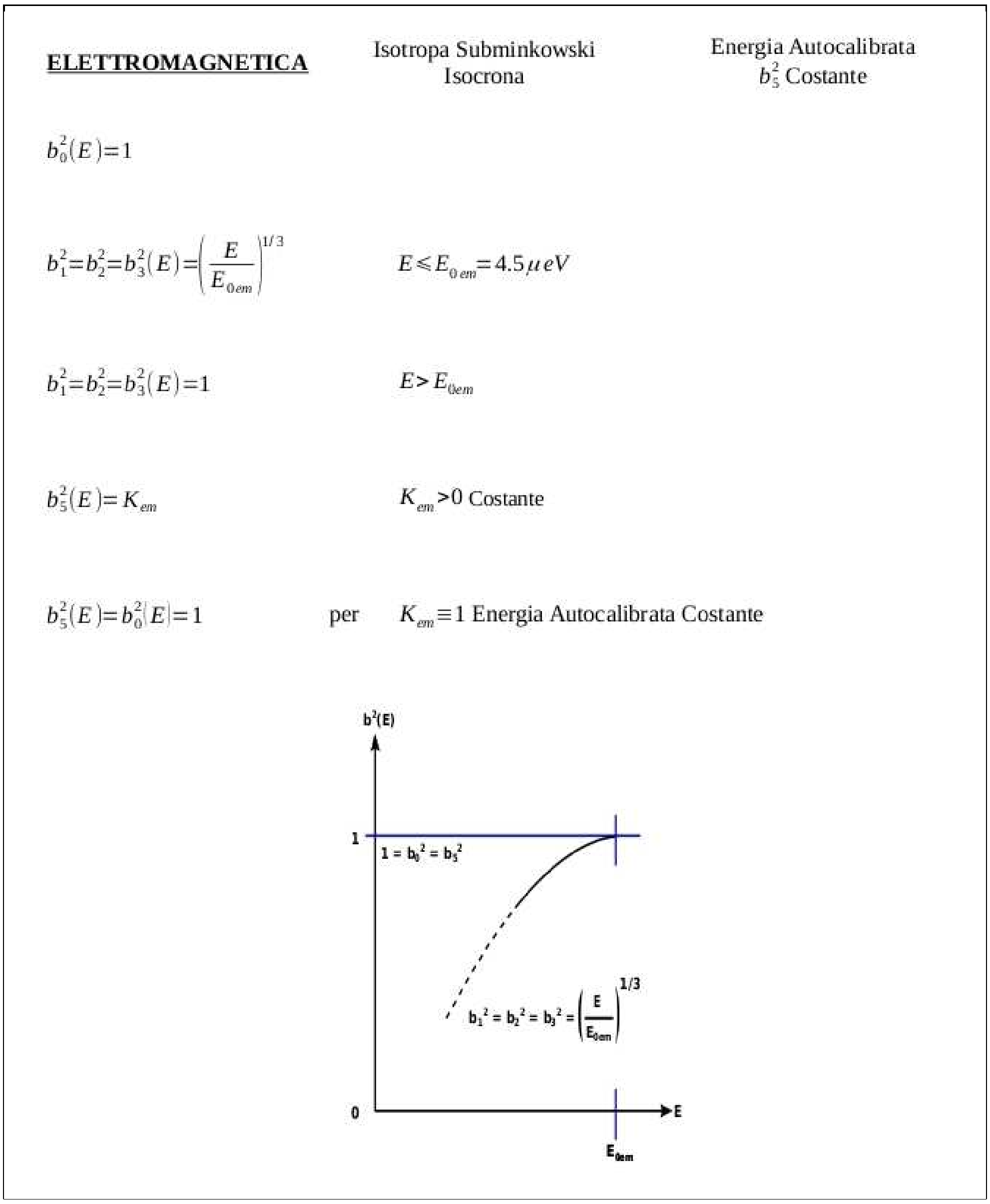


Figure 14: Electromagnetic lab-form.

Transcription of the electromagnetic lab-form:

$$
\begin{array}{l}
[0]\ \ \text{Threshold energy } E_{0\,em} = 4.5\,\mu eV \\
[1]\ \ b_0^2(E) = 1 \\
[2]\ \ b_1^2 = b_2^2 = b_3^2(E) = \left(\dfrac{E}{E_{0\,em}}\right)^{1/3}, \quad E < E_{0\,em} \\
[3]\ \ b_1^2 = b_2^2 = b_3^2(E) = 1, \quad E > E_{0\,em} \\
[4]\ \ b_5^2(E) = K_{em}, \quad K_{em} > 0 \text{ constant} \\
[5]\ \ b_5^2(E) = b_0^2(E) = 1 \quad \text{per } K_{em} \equiv 1 \text{ constant self-calibrated energy}
\end{array}
\tag{77}
$$

For convention and convenience we have chosen K_{em} coinciding with 1 since the electromagnetic interaction is the paradigm of all our phenomenological and experimental measurements as all the instruments at our disposal to date work with it. So it is in this sense that energy is considered self-calibrated for electromagnetic interaction.

From table (77) we get the values of the $b^2(E)$ before the threshold, on the threshold and after the threshold:

$$E < E_{0\,em} \begin{cases} b_0^2(E) = 1 \\ b_1^2(E) = b_2^2(E) = b_3^2(E) = \left(\dfrac{E}{E_{0\,em}}\right)^{1/3} \\ b_5^2(E) = K_{em} \end{cases}$$

$$E = E_{0\,em} \begin{cases} b_0^2(E_{0\,em}) = 1 \\ b_1^2(E_{0\,em}) = b_2^2(E_{0\,em}) = b_3^2(E_{0\,em}) = 1 \\ b_5^2(E) = K_{em} \end{cases}$$

$$E > E_{0\,em} \begin{cases} b_0^2(E) = 1 \\ b_1^2(E) = b_2^2(E) = b_3^2(E) = 1 \\ b_5^2(E) = K_{em} \end{cases}$$

Figure 15 provides a graphical representation of these results.

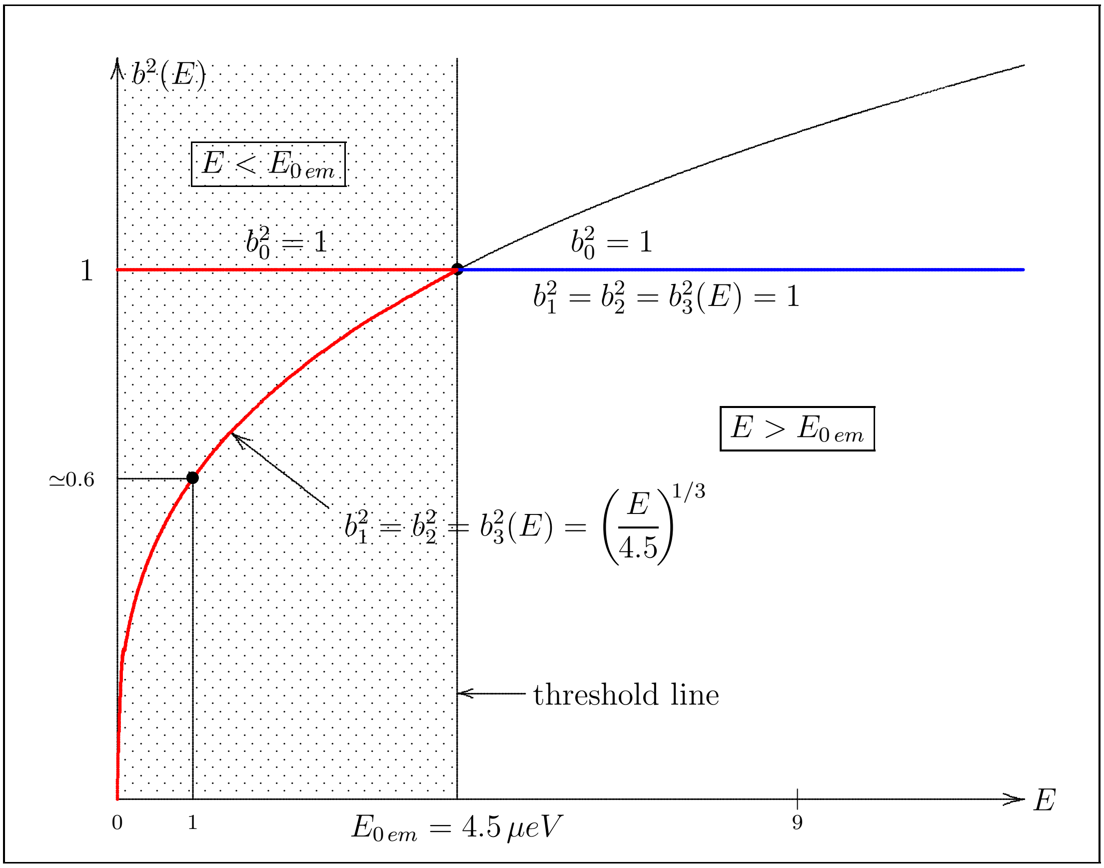


Figure 15: Electromagnetic interaction around the threshold.

13.2 Leptonic interaction

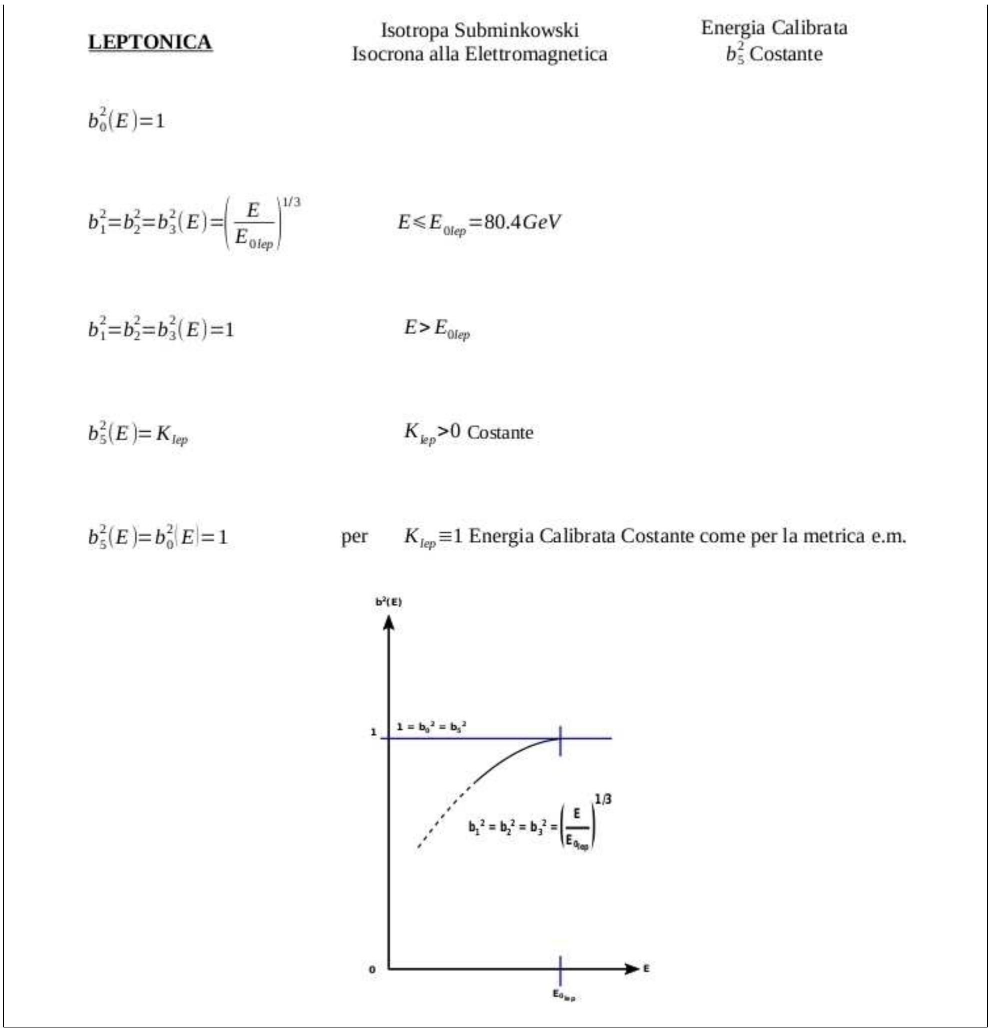


Figure 16: Leptonic lab-form.

Transcription of the leptonic lab-form:

(78)

[0] Threshold energy $E_{0\,lep} = 80.4\,GeV$

[1] $b_0^2(E) = 1$

[2] $b_1^2(E) = b_2^2(E) = b_3^2(E) = (E/E_{0\,lep})^{1/3}, \quad E \leq E_{0\,lep}$

[3] $b_1^2(E) = b_2^2(E) = b_3^2(E) = 1, \quad E > E_{0\,lep}$

[4] $b_5^2(E) = K_{lep} > 0$ constant

[5] $b_5^2(E) = b_0^2(E) = 1$ valid for $K_{lep} = 1$ constant calibrated energy as for the electromagnetic metric.

Graphic representation, Figure 17. From [2] and [3] we observe that $b_1^2(E) = b_2^2(E) = b_3^2(E)$ follow the bold curve. The remaining conditions [1], [4] and [5] in the case $K_{lep} = 1$ give $b_5^2(E) = b_0^2(E) = 1$: the area where this condition is valid it is the entire energy axis.

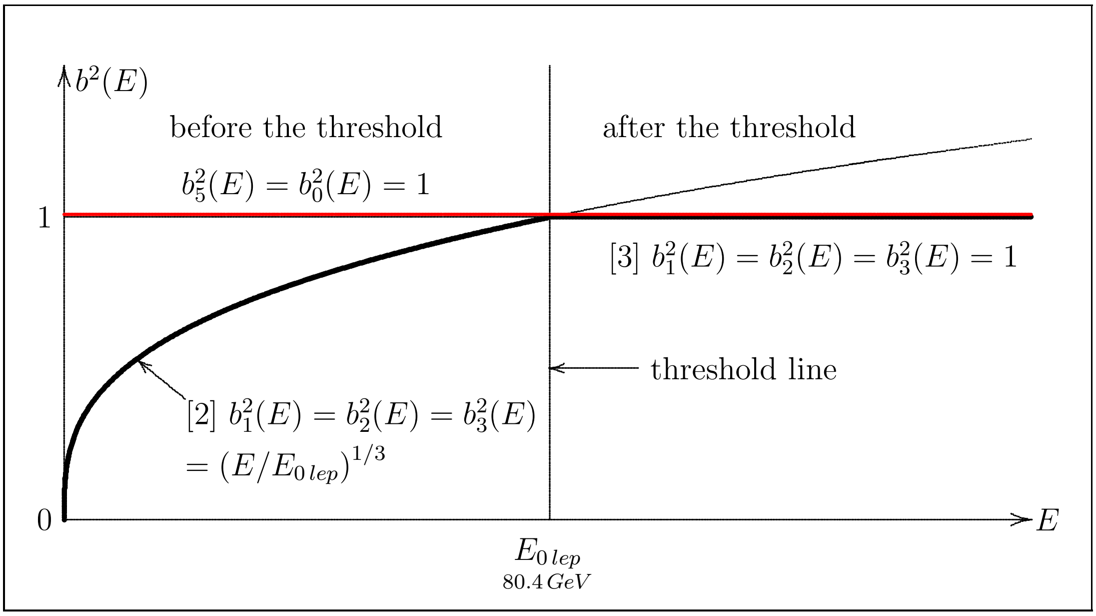


Figure 17: Leptonic interaction around the threshold.

13.3 Gravitational interaction

GRAVITAZIONALE

Anisotropa Sovraminkowski Anisocrona

Energia Eterocalibrata

$b_0^2(E)=1$ $\quad E \leqslant E_{0grav}=20.2\,\mu eV$

$b_1^2(E)=Costante=1$

Con questa scelta Isotropia per $E<E_{0grav}$

$b_2^2(E)=Costante=1$

$b_3^2(E)=1$ $\quad E<E_{0grav}$

$b_5^2(E)=K_{grav}$ $\quad E<E_{0grav}$ $\quad K_{grav}>0$ Costante

$b_5^2(E)=b_0^2(E)=1$ $\quad$ per $K_{grav}=1$ Energia Calibrata Costante come per la metrica e.m.

$b_0^2(E)=\left(1+\frac{E}{E_{0grav}}\right)^2$ $\quad E \geqslant E_{0grav}=20.2\,\mu eV$

$b_1^2(E)=1$

$b_2^2(E)=1$

$b_3^2(E)=b_0^2(E)=\left(1+\frac{E}{E_{0grav}}\right)^2$ $\quad E \geqslant E_{0grav}$

$b_5^2(E)=\left(1+\frac{E}{E_{0grav}}\right)^6$ $\quad E \geqslant E_{0grav}$

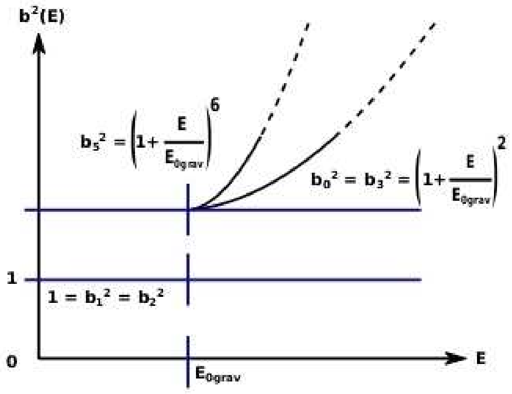


Figure 18: Gravitational lab-form.

Transcription of the gravitational lab-form:

(79)

[0]	Threshold enery $E_{0\,grav} = 20.2\ \mu eV$
[1]	$b_0^2(E) = 1, \quad E \leq E_{0\,grav}$
[2]	$b_1^2(E) = b_2^2(E) = b_3^2(E) = 1, \quad E < E_{0\,grav}$
[3]	$b_5^2(E) = K_{grav} > 0$ constant, $\quad E < E_{0\,grav}$
[4]	$b_5^2(E) = b_0^2(E) = 1,$ per $K_{grav} = 1$ constant calibrated energy as for electromagnetic metric.
[5]	$b_3^2(E) = b_0^2(E) = \left(1 + \frac{E}{E_{0\,grav}}\right)^2, \quad E \geq E_{0\,grav}$
[6]	$b_1^2(E) = b_2^2(E) = 1$
[7]	$b_5^2(E) = \left(1 + \frac{E}{E_{0\,grav}}\right)^6, \quad E \geq E_{0\,grav}$

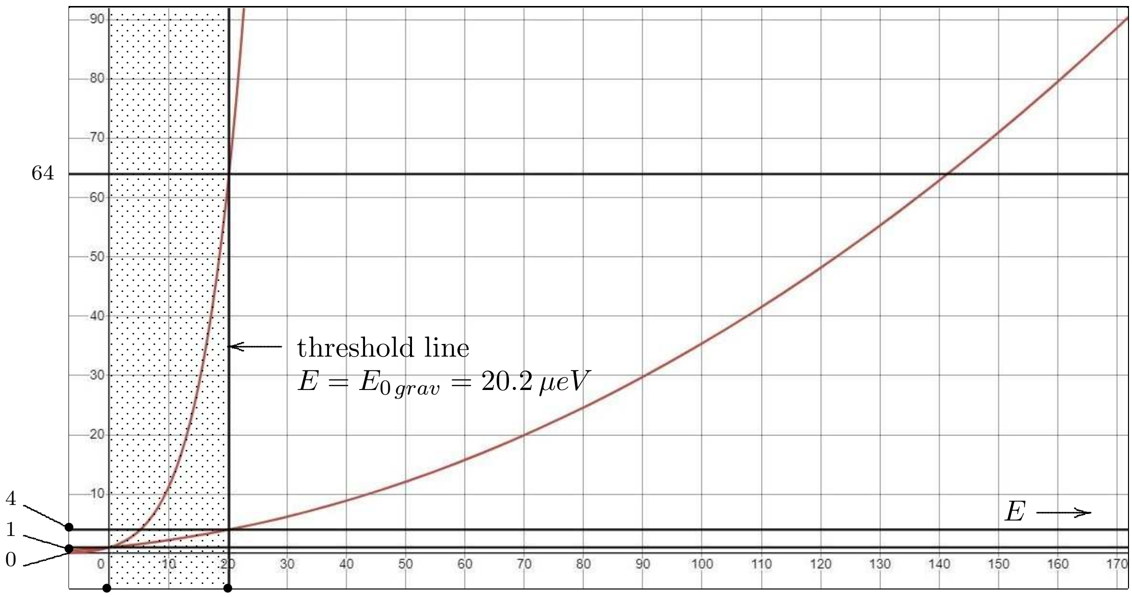


Figure 19: Graphic of the garavitational interaction.

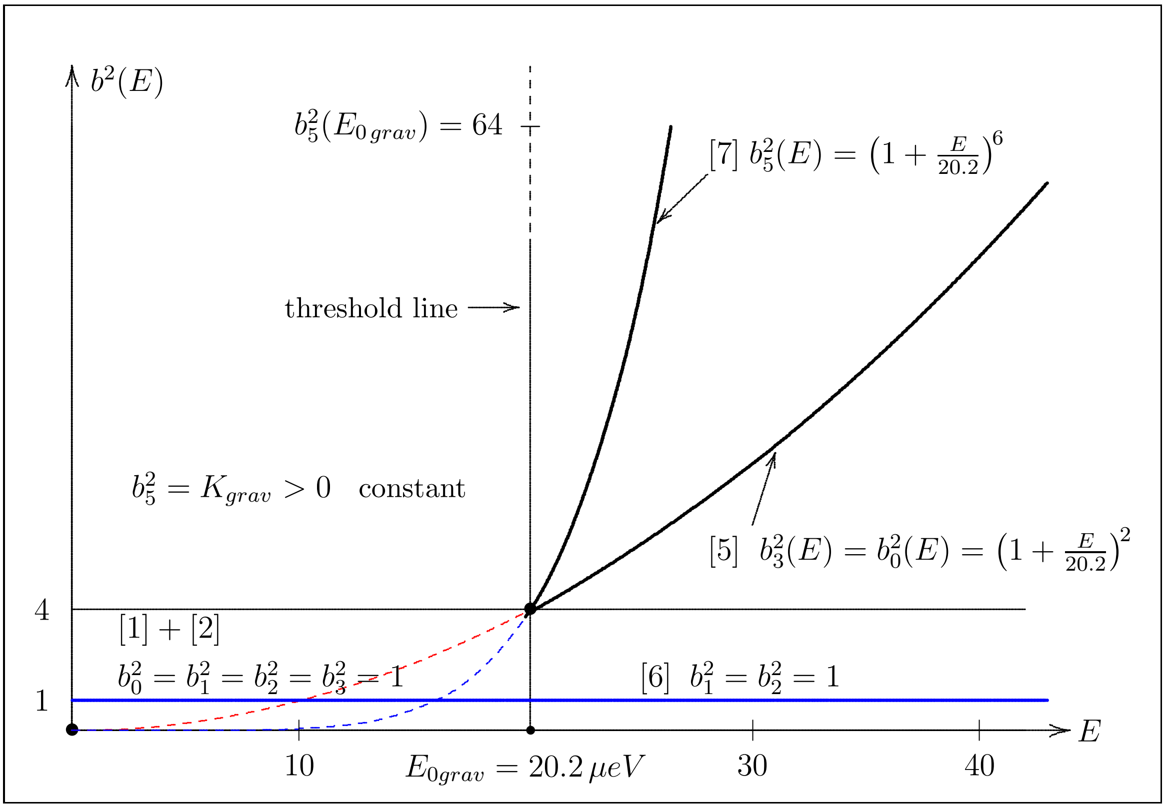


Figure 20: Gravitational interaction around the threshold.

13.4 Adronic interaction

ADRONICA	Anisotropa Sovraminkowski Anisocrona	Energia Eterocalibrata
$b_0^2(E)=1$	$E \leqslant E_{0Strong}=367.5\,GeV$	
$b_1^2(E)=\left(\frac{\sqrt{2}}{5}\right)^2$	Anisotropa	
$b_2^2(E)=\left(\frac{2}{5}\right)^2$		
$b_3^2(E)=1$	$E<E_{0\,Strong}$	
$b_5^2(E)=K_{Strong}$	$K_{Strong}>0$ Costante	
$b_5^2(E)=b_0^2(E)=1$	per $K_{strong}=1$ Energia Calibrata Costante come per la metrica e.m.	
$b_0^2(E)=\left(\frac{E}{E_{0\,Strong}}\right)^2$	$E \geqslant E_{0Strong}=367.5\,GeV$	
$b_1^2(E)=\left(\frac{\sqrt{2}}{5}\right)^2$	Anisotropa	
$b_2^2(E)=\left(\frac{2}{5}\right)^2$		
$b_3^2(E)=b_0^2(E)=\left(\frac{E}{E_{0\,Strong}}\right)^2$	$E \geqslant E_{0Strong}$	
$b_5^2(E)=\left(\frac{E}{E_{0\,Strong}}\right)^6$	$E \geqslant E_{0Strong}$	

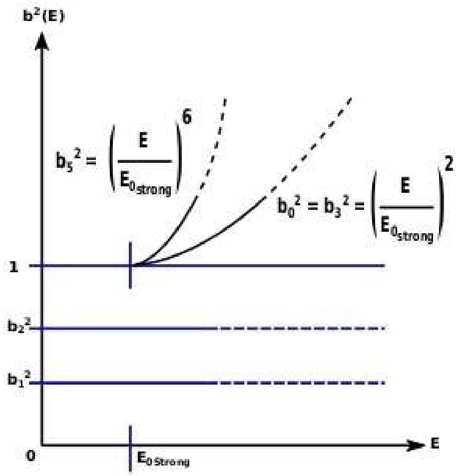


Figure 21: Hadronic lab-form.

Transcription of the hadronic lab-form:

$$
\begin{array}{|l|}
\hline
[0]\ \text{Threshold energy } E_{0\,strong} = 367.5\,GeV \\
[1]\ b_0^2(E) = 1, \quad E \le E_{0\,strong} \\
[2]\ b_1^2(E) = \left(\frac{\sqrt{2}}{5}\right)^2 = 0.08, \quad b_2^2(E) = \left(\frac{2}{5}\right)^2 = 0.16 \quad \text{anisotropy} \\
[3]\ b_3^2(E) = 1, \quad E < E_{0\,strong} \\
[4]\ b_5^2(E) = K_{strong} > 0 \ \text{constant} \\
[5]\ b_5^2(E) = b_0^2(E) = 1 \text{ per } K_{strong} = 1 \text{ constant calibrated energy} \\
\qquad\qquad\qquad\text{as for the elettromagnetic metric} \\
[6]\ b_0^2(E) = \left(\frac{E}{E_{0\,strong}}\right)^2, \quad E \ge E_{0\,strong} \\
[7]\ b_1^2(E) = \left(\frac{\sqrt{2}}{5}\right)^2, \quad b_2^2(E) = \left(\frac{2}{5}\right)^2 \quad \text{anisotropy} \\
[8]\ b_3^2(E) = b_0^2(E) = \left(\frac{E}{E_{0\,strong}}\right)^2, \quad E \ge E_{0\,strong} \\
[9]\ b_5^2(E) = \left(\frac{E}{E_{0\,strong}}\right)^6, \quad E \ge E_{0\,strong} \\
\hline
\end{array}
\tag{80}
$$

Graphic representation Figure 22.

$\boxed{b_0^2(E)}$ Due to [1] and [6] it is equal to 1 up to the threshold where it continues with $(E/E_{0\,strong})^2$.

$\boxed{b_1^2(E).\ b_2^2(E)}$ Their constant values are given by [2] and [7], regardless of the threshold and therefore for every value of the energy E.

$\boxed{b_3^2(E)}$ Due to [3] it is equal to 1 before the threshold. By [8] it is equal to 1 at the threshold and $(E/E_{0\,strong})^2$ after the threshold. Ultimately it is $b_0^2(E) = b_3^2(E)$.

$\boxed{b_5^2(E)}$ Due to [8] it is equal to $(E/E_{0\,strong})^6$ on the threshold and for energy values greater than the threshold. For energy values lower than the threshold we assumed a value equal to 1, in accordance with [4] and [5].

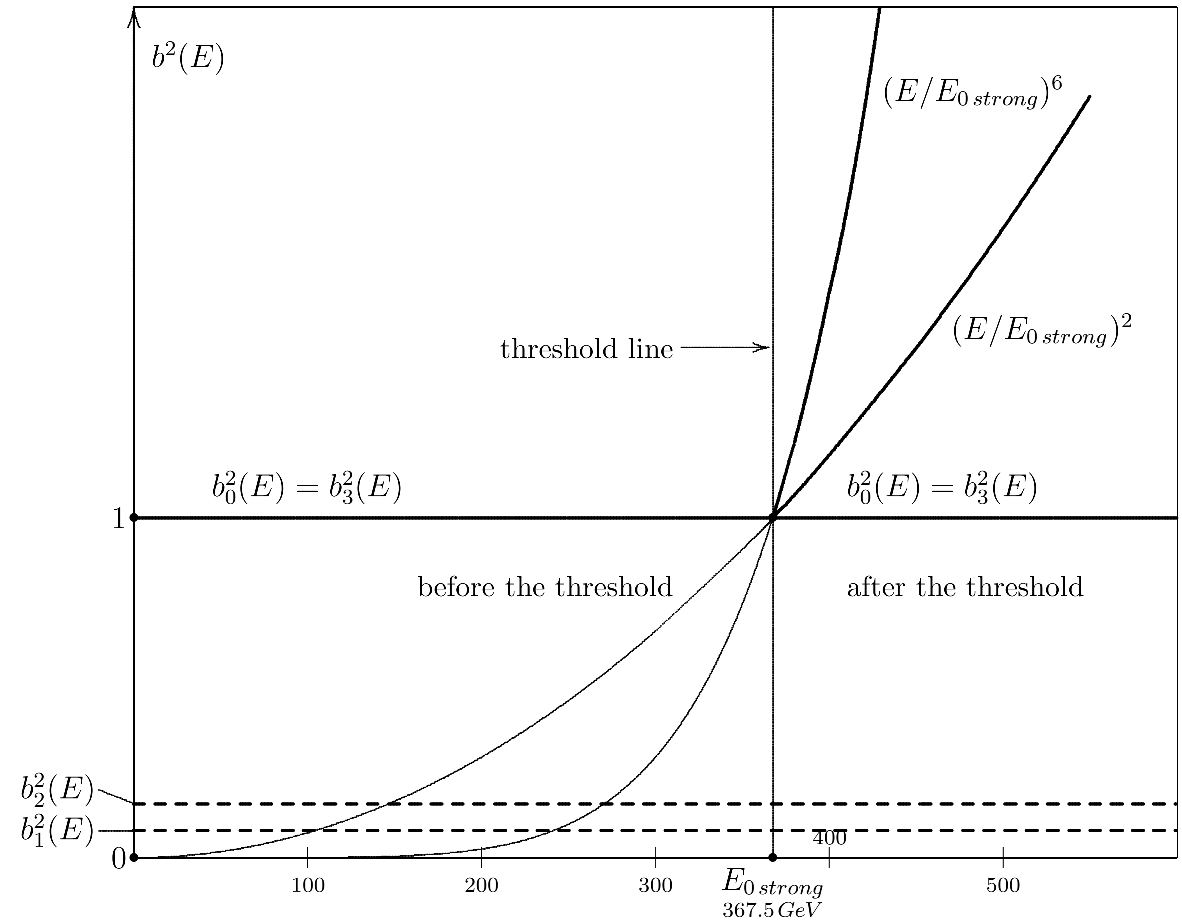


Figure 22: Hadronic interaction around the threshold.

14 Hadronics, astrophysics and asymmetry

We wish here to make some interpretative remarks on the fifth element of hadronic and gravitational metrics. We then want to add an observation on the asymmetry that occurs in various electromagnetic and nuclear phenomena studied by the method of energy-dependent deformed metrics.

14.1 Confinement and asymptotic freedom

The phenomenon of confinement and asymptotic freedom in a composite hadronic system has already been interpreted for hadronic metrics with considerations of the proper (hadronic) time and of the observer's coordinated (electromagnetic) time.[11]

In summary we have that as energy varies the time element for hadronic and electromagnetic metrics is different. Thus as energy increases the reaction time interval of the hadronic system is much less than that of the electromagnetic action by which the hadronic system is energized and observed and appears to be bound. Conversely as the

[11] See [1] §4.1.3, p. 57, eq. 4.15–4.16. For a detailed discussion see [18] §10.4–10.5, p. 89–92.

energy decreases the hadronic and electromagnetic time intervals tend to equalize and the hadronic system appears pseudo–free.

We now wish to make some interpretive remarks here regarding the fifth element of hadronic and electromagnetic metrics with reference to the same phenomenon.

Again we have that as the energy changes, the element of the energy coordinate for the hadronic and electromagnetic metrics is different. The hadronic metric calibrates the energy differently from the electromagnetic metric also the scales of the energies are different.

In summary, if the energy varies by one-tenth on the hadronic scale it is calibrated as one-millionth relative to the electromagnetic scale, that is, it is depotentiated and the system appears to the observer as weakly bound. Conversely, if the energy varies ten times on the hadronic scale it is calibrated as one millionth relative to the electromagnetic scale, that is, it is amplified and the hadronic system appears to the observer as strongly bound.

We do not want to push the level of interpretation further, we only observe that the two elements of hadronic metrics referring to time and energy as coordinates give consistent and coincident conclusions regarding the phenomenon of confinement and asymptotic freedom in a hadronic system.

We leave the reader the freedom and opportunity to consider the proper and coordinated interval also for the energy coordinate in analogy to what is commonly proposed in the literature for the time coordinate.

14.2 Dark energy and superluminal galaxies

Dark energy, like its precursor **dark matter**, were introduced into astrophysics and cosmology as ill-defined and delineated concepts in an uncertain attempt to search for a way that could absolve them from the condemnation of having to consider seemingly paradoxical phenomena, such as an expanding universe with positive acceleration and visible galaxies with a seemingly superluminal Doppler effect.

In Chap. 5 *Signal Transmission and Visibility* of the Memoir [26] it is shown that within an isotropic model of the Universe, the phenomenon of superluminal velocity is closely related to the recession velocity of galaxies, i.e., Hubble's law. It is shown, for example, that paradoxically, *if the current distance of two galaxies A and B is greater than the Hubble radius,* $d_{AB}(t_0) > r_H$*, then A and B have superluminal recessional velocity even though they are mutually visible* (§ 5.5).

The problem of superluminality, which in gravitational systems is related to dark energy understood as a kind of extra energy, is further examined in the next section.

14.3 Interpretation of the fifth element of gravitational and electromagnetic metrics

We now wish to make some interpretative remarks about the fifth element of the gravitational and electromagnetic metrics. Regarding the existence of both luminal and superluminal velocities in gravitation we refer the reader to Chap. 15 of [1] where the problem is extensively examined both experimentally and mathematically, as well as historically.[12]

We note that as the energy varies, the element of the energy coordinate for the gravitational and electromagnetic metrics is different. Let us also remember that all measurements, whether astronomical observational with various and different telescopes or experimental with various devices either in the laboratory or in orbit, still occur with electromagnetic interaction. Thus one observes gravitational phenomena with the fatally distorted view of "electromagnetic glasses". In fact, gravitational phenomena occur with a calibration of energy that exponentially expands energy to the sixth power. No surprise if to an "electromagnetic observer" the gravitational systems appear to behave "as if" there is an "additional energy" that accelerates them in a paradoxical way while they remain visible and thus measurable electromagnetically at any frequency of electromagnetic energy itself.

Here again, we do not want to push the level of interpretation any further, except to mention that even the balance for weighing objects and even the Cavendish balance are instruments that use electromagnetic interaction to have the measurement of gravitational phenomena. The balance uses coulombic electric repulsion between the atoms of matter on its plate and those of weight. The Cavendish balance likewise uses the coulombic electric repulsion between the atoms of the cable holding the dumbbell when that cable twists according to its torsion constant while the dumbbell twisting it undergoes a gravitational action

14.4 Asymmetry and Heaviside function

It has been found (see [20], [21], [22], [23], [24], [25]) that everything goes as if there is a fundamental asymmetry underlying all physical phenomena and conditioning all interactions governing them. It has been proposed and verified that the preferred direction with which to compare asymmetric phenomena is the *cold spot* of the cosmic background radiation. Finally, it was necessary to recognize from the comparison of several electromagnetic experiments with nuclear experiments that the Lorentz violation is not kinematic in nature but appears to be geometric in nature, depending on the angles of direction of the phenomenon but also on the angles of torsion of the phenomenon. In fact, it was found in the experiments that the coincidence of the privileged direction of the phenomenon with the projection of the direction of the cold spot of cosmic radiation referred to the geographical position on Earth and the astronomical position of the Earth in space, where and when the measurements were made.

The proposal for future work is as follows: modulate the pentadimensional metric by introducing an angle-dependent Heaviside function in each element of the metric to account for asymmetry. For this purpose, we use the direction of the *cold spot* of the background radiation as the reference direction for calculating the angle (as in [20]— [25]).

[12] See the fifth volume of Laplace's *Celestial Mechanics* [27] translated into English and annotated by N. Bowditch (1829), how the evaluation of the speed of gravitational action in the Sun-Earth-Moon system is inferred from the study of lunar libration motions.

15 Appendix 1. Calculation of the Ricci tensor

15.1 Conventions on Riemann and Ricci tensors

To demonstrate what has already been stated in Box (34), p. 14, we review the conventions concerning Riemann and Ricci tensors adopted by eminent authors (Hamilton, Cao and Zhu, Carroll, Wald, Misner, Thorne and Wheeler) and then compare them with those of Eisenhart.

L.P. Eisenhart.[13]

(81) $$\overset{E}{R}{}^{i}{}_{\ell mn} \overset{\text{def}}{=} \partial_m \Gamma^i_{\ell n} - \partial_n \Gamma^i_{\ell m} + \Gamma^i_{km}\,\Gamma^k_{\ell n} - \Gamma^i_{kn}\,\Gamma^k_{\ell m}$$

(82) $$\overset{E}{R}{}_{\ell m} \overset{\text{def}}{=} \overset{E}{R}{}^{i}{}_{\ell mi} = \partial_m \Gamma^i_{\ell i} - \partial_i \Gamma^i_{\ell m} + \Gamma^i_{km}\,\Gamma^k_{\ell i} - \Gamma^i_{ki}\,\Gamma^k_{\ell m}$$

Ricci is defined by summing the upper index and the last one at the bottom.

R. S. Hamilton.[14]

(83) $$\overset{H}{R}{}^{h}_{ijk} \overset{\text{def}}{=} \partial_i \Gamma^h_{jk} - \partial_j \Gamma^h_{ik} + \Gamma^h_{ip}\,\Gamma^p_{jk} - \Gamma^h_{jp}\,\Gamma^p_{ik}.$$

(84) $$\overset{H}{R}{}_{ijk\ell} \overset{\text{def}}{=} g_{hk}\,\overset{H}{R}{}^{h}_{ij\ell}.$$

(85) $$\overset{H}{R}{}_{ik} \overset{\text{def}}{=} g^{j\ell}\,\overset{H}{R}{}_{ijk\ell} = g^{j\ell}\,g_{hk}\,\overset{H}{R}{}^{h}_{ij\ell}.$$

Comparison of $\overset{H}{R}$ with $\overset{E}{R}$:

$$\begin{cases} \overset{H}{R}{}^{h}_{ijk} \overset{\text{def}}{=} \partial_i \Gamma^h_{jk} - \partial_j \Gamma^h_{ik} + \Gamma^h_{ip}\,\Gamma^p_{jk} - \Gamma^h_{jp}\,\Gamma^p_{ik} \\ \overset{E}{R}{}^{h}{}_{ijk} \overset{\text{def}}{=} \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\,\Gamma^h_{mj} - \Gamma^m_{ij}\,\Gamma^h_{mk} \end{cases}$$

Exchange i with j: $\overset{E}{R}{}^{h}{}_{jik} = \partial_i \Gamma^h_{jk} - \partial_k \Gamma^h_{ji} + \Gamma^m_{jk}\,\Gamma^h_{mi} - \Gamma^m_{ji}.\,\Gamma^h_{mk}$

Exchange k with j: $\overset{E}{R}{}^{h}{}_{kij} = \partial_i \Gamma^h_{kj} - \partial_j \Gamma^h_{ki} + \Gamma^m_{kj}\,\Gamma^h_{mi} - \Gamma^m_{ki}\,\Gamma^h_{mj}.$

Put $m = p$: $\overset{E}{R}{}^{h}{}_{kij} = \partial_i \Gamma^h_{kj} - \partial_j \Gamma^h_{ki} + \Gamma^p_{kj}\,\Gamma^h_{pi} - \Gamma^p_{ki}\,\Gamma^h_{pj}.$

From $\overset{H}{R}{}^{h}_{ijk} \overset{\text{def}}{=} \partial_i \Gamma^h_{jk} - \partial_j \Gamma^h_{ik} + \Gamma^h_{ip}\,\Gamma^p_{jk} - \Gamma^h_{jp}\,\Gamma^p_{ik}$ we get

(86) $$\boxed{\overset{H}{R}{}^{h}_{ijk} = \overset{E}{R}{}^{h}{}_{kij}}$$

According to Eisenhart, the Ricci tensor is obtained by summing the top index with the last index at the bottom:

$$\overset{E}{R}{}_{ki} \overset{\text{def}}{=} \overset{E}{R}{}^{j}{}_{kij}.$$

[13] [13] Formulas (8.3), (8.5), (8.12), (8.14). See also [14], p. 55, formula (21.1):

$$B^i_{jk\ell} = \partial_k \Gamma^i_{j\ell} - \partial_\ell \Gamma^i_{jk} + \Gamma^h_{j\ell}\,\Gamma^i_{hk} - \Gamma^h_{jk}\,\Gamma^i_{h\ell}.$$

[14] [12] p. 258.

It follows that

$$\overset{E}{R}_{ki} = \partial_i\Gamma^j_{kj} - \partial_j\Gamma^j_{ki} + \Gamma^p_{kj}\,\Gamma^j_{pi} - \Gamma^p_{ki}\,\Gamma^j_{pj}.$$

Recall

$$\overset{H}{R}{}^h_{ijk} \stackrel{\text{def}}{=} \partial_i\Gamma^h_{jk} - \partial_j\Gamma^h_{ik} + \Gamma^h_{ip}\,\Gamma^p_{jk} - \Gamma^h_{jp}\,\Gamma^p_{ik}$$

and sum over $h = j$:

$$\overset{H}{R}{}^j_{ijk} = \partial_i\Gamma^j_{jk} - \partial_j\Gamma^j_{ik} + \Gamma^j_{ip}\,\Gamma^p_{jk} - \Gamma^j_{jp}\,\Gamma^p_{ik}.$$

We find equation (83) again. From here it can be deduced that according to Hamilton the definition of the Ricci tensor *could* be

$$\overset{H}{R}_{ik} \stackrel{\text{def}}{=} \overset{H}{R}{}^j_{ijk}.$$

In fact, Hamilton goes from Riemann to Ricci in a somewhat tortuous manner. He lowers the upper Riemann index by posing

$$\overset{H}{R}_{ijk\ell} \stackrel{\text{def}}{=} g_{hk}\,\overset{H}{R}{}^h_{ij\ell}.$$

He then defines Ricci by posing

$$\overset{H}{R}_{ik} \stackrel{\text{def}}{=} g^{j\ell}\,\overset{H}{R}_{ijk\ell}.$$

Recalling (86) we find $\overset{H}{R}_{ijk\ell} \stackrel{\text{def}}{=} g_{hk}\,\overset{H}{R}{}^h_{ij\ell} = g_{hk}\,\overset{E}{R}{}^h{}_{\ell ij} = \overset{E}{R}_{k\ell ij}$. Therefore, $\overset{H}{R}_{ik} \stackrel{\text{def}}{=} g^{j\ell}\,\overset{H}{R}_{ijk\ell} = g^{j\ell}\,\overset{E}{R}_{k\ell ij} = -\,g^{j\ell}\,\overset{E}{R}_{\ell kij} = -\,\overset{E}{R}{}^j{}_{kij}$. Since $\overset{E}{R}_{ki} \stackrel{\text{def}}{=} \overset{E}{R}{}^j{}_{kij}$ we find

(87) $$\boxed{\overset{H}{R}_{ik} = -\,\overset{E}{R}_{ik}}$$

The Ricci tensors of Hamilton and Eisenhart are opposite in sign.

Cao, Zhu.[15]

The definition of the Riemann tensor is

$$\overset{C}{R}{}^k_{ij\ell} \stackrel{\text{def}}{=} \partial_i\Gamma^k_{j\ell} - \partial_j\Gamma^k_{i\ell} + \Gamma^k_{ip}\,\Gamma^p_{j\ell} - \Gamma^k_{jp}\,\Gamma^p_{i\ell}.$$

Let us recall the Hamilton definition (83) with a change of indices:

$$\overset{H}{R}{}^k_{ij\ell} \stackrel{\text{def}}{=} \partial_i\Gamma^k_{j\ell} - \partial_j\Gamma^k_{i\ell} + \Gamma^k_{ip}\,\Gamma^p_{j\ell} - \Gamma^k_{jp}\,\Gamma^p_{i\ell}.$$

The comparison of these two expressions shows that

$$\boxed{\overset{C}{R}{}^k_{ij\ell} = \overset{H}{R}{}^k_{ij\ell}}$$

Then the Riemann tensors are the same. Furthermore, we have[16] $\overset{C}{R}_{ijk\ell} \stackrel{\text{def}}{=} g_{kp}\,\overset{C}{R}{}^p_{ij\ell}$ and $\overset{C}{R}_{ik} \stackrel{\text{def}}{=} g^{j\ell}\overset{C}{R}_{ijk\ell}$.

[15] [15] p. 152.
[16] [15] p. 173.

By virtue of the identities $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}$ from (81) we get

$$\overset{E}{R}{}^{i}{}_{\ell mn} \overset{\text{def}}{=} \partial_m \Gamma^i_{\ell n} - \partial_n \Gamma^i_{\ell m} + \Gamma^i_{km}\,\Gamma^k_{\ell n} - \Gamma^i_{kn}\,\Gamma^k_{\ell m}$$

i.e.

$$\overset{E}{R}{}^{k}{}_{\ell mn} \overset{\text{def}}{=} \partial_m \Gamma^k_{\ell n} - \partial_n \Gamma^k_{\ell m} + \Gamma^k_{pm}\,\Gamma^p_{\ell n} - \Gamma^k_{pn}\,\Gamma^p_{\ell m}$$

Comparison with (8.3) $\overset{E}{R}{}^{h}{}_{ijk} \overset{\text{def}}{=} \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\,\Gamma^h_{mj} - \Gamma^m_{ij}\,\Gamma^h_{mk}$

i.e.

$$\overset{E}{R}{}^{h}{}_{ijk} \overset{\text{def}}{=} \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^p_{ik}\,\Gamma^h_{pj} - \Gamma^p_{ij}\,\Gamma^h_{pk}.$$

$\overset{C}{R}{}^{k}_{ij\ell} \overset{\text{def}}{=} \partial_i \Gamma^k_{j\ell} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{ip}\,\Gamma^p_{j\ell} - \Gamma^k_{jp}\,\Gamma^p_{i\ell}$ p.152

S.M. Carroll.[17]

$$\overset{Cl}{R}{}^{\rho}{}_{\sigma\mu\nu} \overset{\text{def}}{=} \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\,\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\,\Gamma^\lambda_{\mu\sigma}$$

$$\overset{Cl}{R}{}_{\rho\sigma\mu\nu} \overset{\text{def}}{=} g_{\rho\lambda}\,\overset{Cl}{R}{}^{\rho}{}_{\sigma\mu\nu}$$

$$\overset{Cl}{R}{}_{\mu\nu} \overset{\text{def}}{=} R^{\lambda}{}_{\mu\lambda\nu}$$

Check: for a 2-sphere $da^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ we have

$$\begin{cases} R_{\theta\theta} = 1 \\ R_{\theta\phi} = R_{\phi\theta} = 0 \\ R_{\phi\phi} = \sin^2\theta \end{cases} \qquad \boxed{R = \frac{2}{a^2}} \quad \text{Gauss} > 0.\ a = \text{radius, correct.}$$

Comparison with Eisenhart:
$$\begin{cases} (3.67) & \overset{Cl}{R}{}^{\rho}{}_{\sigma\mu\nu} \overset{\text{def}}{=} \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\,\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\,\Gamma^\lambda_{\mu\sigma}, \\ (8.3) & \overset{E}{R}{}^{\ell}{}_{ijk} \overset{\text{def}}{=} \partial_j \Gamma^\ell_{ik} - \partial_k \Gamma^\ell_{ij} + \Gamma^m_{ik}\,\Gamma^\ell_{mj} - \Gamma^m_{ij}\,\Gamma^\ell_{mk}, \end{cases}$$

$$\overset{Cl}{R}{}^{a}{}_{bcd} \overset{\text{def}}{=} \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{c\lambda}\,\Gamma^\lambda_{db} - \Gamma^a_{d\lambda}\,\Gamma^\lambda_{cb}.$$

$$\overset{E}{R}{}^{a}{}_{bcd} \overset{\text{def}}{=} \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^m_{bd}\,\Gamma^a_{mc} - \Gamma^m_{bc}\,\Gamma^a_{md}.$$

Symmetry of $\Gamma \implies$ Riemann is the same:

(88) $$\overset{E}{R}{}^{a}{}_{bcd} = \overset{Cl}{R}{}^{a}{}_{bcd},$$

but Ricci is opposite in sign:

$$(3.90)\ \overset{Cl}{R}{}_{\mu\nu} \overset{\text{def}}{=} \overset{Cl}{R}{}^{\lambda}{}_{\mu\lambda\nu}.\ (8.14)\quad \overset{E}{R}{}_{ij} \overset{\text{def}}{=} \overset{E}{R}{}^{k}{}_{ijk} \implies \overset{Cl}{R}{}_{\mu\nu} = -\overset{E}{R}{}_{\mu\nu} \implies$$

(89) $$\boxed{\overset{Cl}{R}{}_{ab} = -\overset{E}{R}{}_{ab}}$$

---

[17] [16] Formulas (3.67), (3.76), (3.90).

R.M. Wald.[18]

$$\overset{W}{R}{}_{\mu\nu\rho}{}^{\sigma} \overset{\text{def}}{=} \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\alpha_{\mu\rho}\,\Gamma^\sigma_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\,\Gamma^\sigma_{\alpha\mu}.$$

$$\overset{W}{R}{}_{\mu\rho} \overset{\text{def}}{=} \overset{W}{R}{}_{\mu\nu\rho}{}^{\nu}.$$

Comparison with Eisenhart $\begin{cases} \overset{W}{R}{}_{bcd}{}^{a} \overset{\text{def}}{=} \partial_c \Gamma^a_{bd} - \partial_b \Gamma^a_{cd} + \Gamma^\alpha_{bd}\,\Gamma^a_{\alpha c} - \Gamma^\alpha_{cd}\,\Gamma^a_{\alpha b} \\ \overset{E}{R}{}^{a}{}_{dcb} \overset{\text{def}}{=} \partial_c \Gamma^a_{db} - \partial_b \Gamma^a_{dc} + \Gamma^m_{db}\,\Gamma^a_{mc} - \Gamma^m_{dc}\,\Gamma^a_{mb} \end{cases}$

shows that

(90) $$\overset{W}{R}{}_{bcd}{}^{a} = \overset{E}{R}{}^{a}{}_{dcb}.$$

Hence, $\overset{W}{R}{}_{bd} = \overset{W}{R}{}_{bad}{}^{a} = \overset{E}{R}{}^{a}{}_{dab} = -\overset{E}{R}{}_{db} = -\overset{E}{R}{}_{bd} \implies$ Ricci changes the sign:

(91) $$\boxed{\overset{W}{R}{}_{ab} = -\overset{E}{R}{}_{ab}}$$

Misner, Thorne, Wheeler.[19]

$$\overset{M}{R}{}^{\mu}{}_{\nu\alpha\beta} \overset{\text{def}}{=} \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\rho\alpha}\,\Gamma^\rho_{\nu\beta} - \Gamma^\mu_{\rho\beta}\,\Gamma^\rho_{\nu\alpha}.$$

$$\overset{M}{R}{}_{\mu\nu} \overset{\text{def}}{=} \overset{M}{R}{}^{\alpha}{}_{\mu\alpha\nu}.$$

Check: sphere of radius a, $ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$

$$R^\theta_\theta = R^\phi_\phi = \frac{1}{a^2}. \quad R^\theta_\phi = 0. \quad R = \frac{2}{a^2}, \quad \text{correct.}$$

Comparison with Eisenhart $\begin{cases} (340) \quad \overset{M}{R}{}^{\mu}{}_{\nu\alpha\beta} \overset{\text{def}}{=} \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\rho\alpha}\,\Gamma^\rho_{\nu\beta} - \Gamma^\mu_{\rho\beta}\,\Gamma^\rho_{\nu\alpha}. \\ (8.3) \quad \overset{E}{R}{}^{h}{}_{ijk} \overset{\text{def}}{=} \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\,\Gamma^h_{mj} - \Gamma^m_{ij}\,\Gamma^h_{mk}. \end{cases}$

$$\begin{cases} \overset{M}{R}{}^{a}{}_{bcd} \overset{\text{def}}{=} \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{\rho c}\,\Gamma^\rho_{bd} - \Gamma^a_{\rho d}\,\Gamma^\rho_{bc} \\ \overset{E}{R}{}^{a}{}_{bcd} \overset{\text{def}}{=} \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^m_{bd}\,\Gamma^a_{mc} - \Gamma^m_{bc}\,\Gamma^a_{md} \end{cases}$$

Riemann is the same

(92) $$\overset{E}{R}{}^{a}{}_{bcd} = \overset{M}{R}{}^{a}{}_{bcd}$$

but Ricci Changes the sign: (8.14) $\overset{E}{R}{}_{ij} \overset{\text{def}}{=} \overset{E}{R}{}^{k}{}_{ijk}$. (p. 343) $\overset{M}{R}{}_{\mu\nu} \overset{\text{def}}{=} \overset{M}{R}{}^{\alpha}{}_{\mu\alpha\nu} \implies$

(93) $$\boxed{\overset{M}{R}{}_{ab} = -\overset{E}{R}{}_{ab}}$$

In conclusion, what is stated in the box (34) on page 14 is confirmed, i.e. that ***the Ricci tensor according to Eisenhart has the opposite sign to all the other examined conventions***:

$$\boxed{\overset{*}{R}{}_{ij} = -\overset{E}{R}{}_{ij}}$$

[18] [17] p. 48.
[19] [19] pp. 340 e 343.

15.2 'Lagrangian' algorithm for calculating Christoffel symbols

To prove formulas (36) and (37), which provide the components of the Ricci-Eisenhart tensors of type 1 and type 2 metrics, respectively, we begin by computing their Christoffel symbols. For this purpose we can make use of the quick and safe algorithm consisting of the following three steps:

$\boxed{1}$ Given a metric tensor with components $g_{ij}(x)$ in coordinates $x = (x^i)$ we write the **kinetic energy**

$$\boxed{T = \tfrac{1}{2}\, g_{ij}\, v^i\, v^j}$$

as a second-degree homogeneous polynomial in the **Lagrangian velocities** v^i. By setting

$$\boxed{\frac{dx^i}{dt} = v^i}$$

where t is a generic evolution parameter, we calculate the **Lagrangian binomials**

$$\boxed{L_i \stackrel{\text{def}}{=} \frac{d}{dt}\frac{\partial T}{\partial v^i} - \frac{\partial T}{\partial x^i}}$$

$\boxed{2}$ We calculate the contravariant components $g^{ij}(x)$ of the metric and raise the indices of the Lagrangian binomials

$$\boxed{L^i \stackrel{\text{def}}{=} g^{ij}\, L_j}$$

Each L^i turns out to have the form

$$L^i = \frac{dv^i}{dt} + \Gamma^i_{hk}\, v^h\, v^k \tag{94}$$

where the three-index quantities Γ^i_{hk}, which are functions of the coordinates x alone, are precisely the **Christoffel symbols** that we want to calculate.

$\boxed{3}$ From the expressions (94) we extract the quadratic forms

$$\boxed{Q^i \stackrel{\text{def}}{=} L^i - \frac{dv^i}{dt} = \Gamma^i_{hk}\, v^h\, v^k} \tag{95}$$

from which the expressions of the Christoffel symbols Γ^i_{hk} can be derived.

15.3 Ricci-Eisenhart tensor of a type 1 metric

$$\text{Type 1 metric}\quad \begin{cases} g_{00} = G(x_4) & \text{dimensionless positive function} \\ g_{11} = -\,\alpha, & \alpha \text{ dimensionless positive constant} \\ g_{22} = -\,\beta, & \beta \text{ dimensionless positive constant} \\ g_{33} = -\,G(x_4) & \\ g_{44} = \pm F(x_4), & F(x_4) \text{ dimensionless positive function} \end{cases}$$

Step $\boxed{1}$

$$T = \tfrac{1}{2}\, g_{ij}\, v^i\, v^j = \tfrac{1}{2}\left[G\,(v^0)^2 - \alpha\,(v^1)^2 - \beta\,(v^2)^2 - G\,(v^3)^2 \pm F\,(v^4)^2\right]$$
$$= \tfrac{1}{2}\left[G\left((v^0)^2 - (v^3)^2\right) - \alpha\,(v^1)^2 - \beta\,(v^2)^2 \pm F\,(v^4)^2\right]$$

$$\begin{cases} \dfrac{\partial T}{\partial v^0} = G\,v^0 \\ \dfrac{\partial T}{\partial v^1} = -\alpha\,v^1 \\ \dfrac{\partial T}{\partial v^2} = -\beta\,v^2 \\ \dfrac{\partial T}{\partial v^3} = -G\,v^3 \\ \dfrac{\partial T}{\partial v^4} = \pm F\,v^4 \end{cases} \qquad \begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial v^0} = G'\,v^4\,v^0 + G\,\dfrac{dv^0}{dt} \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial v^1} = -\alpha\,\dfrac{dv^1}{dt} \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial v^2} = -\beta\,\dfrac{dv^2}{dt} \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial v^3} = -G'\,v^4\,v^3 - G\,\dfrac{dv^3}{dt} \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial v^4} = \pm F'\,(v^4)^2 \pm F\,\dfrac{dv^4}{dt} \end{cases}$$

Single non-ignorable coordinate x_4:

$$\frac{\partial T}{\partial x^4} = \tfrac{1}{2}\,\left[G'\,(v^0)^2 - G'\,(v^3)^2 \pm F'\,(v^4)^2\right]$$

Lagrangian binomials $\quad L_i \overset{\text{def}}{=} \dfrac{d}{dt}\dfrac{\partial T}{\partial v^i} - \dfrac{\partial T}{\partial x^i}$:

$$\begin{cases} L_0 = G'\,v^4\,v^0 + G\,\dfrac{dv^0}{dt}, \quad L_3 = -G'\,v^4\,v^3 - G\,\dfrac{dv^3}{dt}, \\ L_1 = -\alpha\,\dfrac{dv^1}{dt}, \quad L_2 = -\beta\,\dfrac{dv^2}{dt}, \\ L_4 = \pm F'\,(v^4)^2 \pm F\,\dfrac{dv^4}{dt} - \tfrac{1}{2}\,[G'\,(v^0)^2 - G'\,(v^3)^2 \pm F'\,(v^4)^2] \end{cases}$$

Step $\boxed{2}$ Lagrangian binomials with raised indices $L^i = g^{ii}\,L_i$ (orthogonal metric)

$$\begin{cases} g^{00} = -g^{33} = G^{-1} \\ g^{11} = -\,\alpha^{-1} \\ g^{22} = -\,\beta^{-1} \\ g^{44} = \pm F^{-1} \end{cases}$$

$$\begin{cases} L^0 = \dfrac{dv^0}{dt} + G'G^{-1}\,v^4\,v^0, \quad L^3 = \dfrac{dv^3}{dt} + G'G^{-1}\,v^4\,v^3. \\ L^1 = \dfrac{dv^1}{dt}, \quad L^2 = \dfrac{dv^2}{dt} \\ L^4 = \dfrac{dv^4}{dt} + F^{-1}F'\,(v^4)^2 \mp \tfrac{1}{2}\,F^{-1}\,\left[G'\,(v^0)^2 - G'\,(v^3)^2 \pm F'\,(v^4)^2\right] \end{cases}$$

Antisymmetry with respect to v^0 and v^3 in L^4 only.

Step $\boxed{3}$ From the above expressions of L^i we extract the quadratic forms $Q^i \stackrel{\text{def}}{=} \Gamma^i_{hk}\, v^h\, v^k$:

$$(96)\quad \begin{cases} Q^0 = G'G^{-1}\, v^4\, v^0, \quad Q^3 = G'G^{-1}\, v^4\, v^3, \quad Q^1 = 0, \quad Q^2 = 0. \\ Q^4 = F^{-1}F'\,(v^4)^2 \mp \tfrac{1}{2}\, F^{-1}\left[G'\,(v^0)^2 - G'\,(v^3)^2 \pm F'\,(v^4)^2\right] \\ \quad = \tfrac{1}{2}\, F^{-1}F'\,(v^4)^2 \mp \tfrac{1}{2}\, F^{-1}\, G'\left[(v^0)^2 - (v^3)^2\right] \end{cases}$$

and from these we derive the non-identically null Christoffel symbols:

$$(97)\quad \boxed{\begin{cases} \Gamma^0_{04} = \tfrac{1}{2}\, G'G^{-1} \\ \Gamma^3_{34} = \tfrac{1}{2}\, G'G^{-1} \end{cases} \qquad \begin{cases} \Gamma^4_{44} = \tfrac{1}{2}\, F^{-1}F' \\ \Gamma^4_{00} = \mp\tfrac{1}{2}\, F^{-1}\, G' \\ \Gamma^4_{33} = \pm\tfrac{1}{2}\, F^{-1}\, G' \end{cases}}$$

Note that symbols with at least one lower index equal to 1 or 2 are null and that the following equalities hold:

$$\left[\begin{array}{l} \Gamma^i_{0i} = \Gamma^0_{00} + \Gamma^1_{01} + \Gamma^2_{02} + \Gamma^3_{03} + \Gamma^4_{04} = 0 \\ \Gamma^i_{1i} = \Gamma^0_{10} + \Gamma^3_{13} + \Gamma^4_{14} = 0 \\ \Gamma^i_{2i} = \Gamma^0_{20} + \Gamma^3_{23} + \Gamma^4_{24} = 0 \\ \Gamma^i_{3i} = \Gamma^0_{30} + \Gamma^3_{33} + \Gamma^4_{34} = 0 \\ \Gamma^i_{4i} = \Gamma^0_{40} + \Gamma^3_{43} + \Gamma^4_{44} = G'G^{-1} + \tfrac{1}{2}\, F^{-1}F' \end{array}\right.$$

End of the algorithm.

Now we recall the definition of the Ricci tensor according to Eisenhart (82)

$$\overset{E}{R}_{\ell m} \stackrel{\text{def}}{=} \partial_m \Gamma^i_{\ell i} - \partial_i \Gamma^i_{\ell m} + \Gamma^i_{km}\, \Gamma^k_{\ell i} - \Gamma^i_{ki}\, \Gamma^k_{\ell m}$$

where the terms $\partial_m \Gamma^i_{\ell i}$ cancel for $m \neq 4$ because x_4 is the only coordinate that cannot be ignored.

$\overset{E}{R}_{00}$ ---

$$\overset{E}{R}_{00} = \cancel{\partial_0 \Gamma^i_{0i}} - \partial_i \Gamma^i_{00} + \Gamma^i_{k0}\, \Gamma^k_{0i} - \Gamma^i_{ki}\, \Gamma^k_{00} = -\partial_4 \Gamma^4_{00} + \boxed{\Gamma^i_{k0}\, \Gamma^k_{0i}} - \boxed{\Gamma^i_{ki}\, \Gamma^k_{00}}$$

$$\left[\begin{array}{l} \boxed{\Gamma^i_{0k}\, \Gamma^k_{i0}} = \Gamma^i_{00}\, \Gamma^0_{i0} + \Gamma^i_{03}\, \Gamma^3_{i0} + \Gamma^i_{04}\, \Gamma^4_{i0} = \Gamma^4_{00}\, \Gamma^0_{40} + \Gamma^0_{04}\, \Gamma^4_{00} = 2\, \Gamma^4_{00}\, \Gamma^0_{40} \\ = \mp 2\, \tfrac{1}{2}\, G'G^{-1}\, \tfrac{1}{2}\, F^{-1}\, G' = \mp\tfrac{1}{2}\, (G')^2\, G^{-1}\, F^{-1} \end{array}\right.$$

$$\boxed{\Gamma^i_{ki}\, \Gamma^k_{00}} = \Gamma^i_{4i}\, \Gamma^4_{00} = \left(G'G^{-1} + \tfrac{1}{2}\, F^{-1}F'\right)\left(\mp\tfrac{1}{2}\, F^{-1}\, G'\right)$$

$$\begin{aligned}
\overset{E}{R}_{00} &= -\partial_4\Gamma^4_{00} + \boxed{\Gamma^i_{k0}\,\Gamma^k_{0i}} - \boxed{\Gamma^i_{ki}\,\Gamma^k_{00}} \\
&= -\left(\mp\tfrac{1}{2}\,F^{-1}\,G'\right)' \mp \tfrac{1}{2}\,(G')^2\,G^{-1}\,F^{-1} - \left(G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'\right)\left(\mp\tfrac{1}{2}\,F^{-1}\,G'\right) \\
&= \pm\left(\tfrac{1}{2}\,F^{-1}\,G'\right)' \mp \tfrac{1}{2}\,(G')^2\,G^{-1}\,F^{-1} \pm \left(G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'\right)\left(\tfrac{1}{2}\,F^{-1}\,G'\right) \\
&= \pm\tfrac{1}{2}\left\{(F^{-1}\,G')' - (G')^2\,G^{-1}\,F^{-1} + \left(G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'\right)(F^{-1}\,G')\right\} \\
&= \pm\tfrac{1}{2}\left\{(F^{-1}\,G')' - (G')^2\,G^{-1}\,F^{-1} + G'G^{-1}(F^{-1}\,G') + \tfrac{1}{2}\,F^{-1}F'(F^{-1}\,G')\right\} \\
&= \pm\tfrac{1}{2}\left\{(F^{-1}\,G')' + \tfrac{1}{2}\,F^{-2}F'\,G'\right\} \\
&= \pm\tfrac{1}{2}\left\{\left(\frac{G'}{F}\right)' + \frac{1}{2}\,\frac{F'\,G'}{F^2}\right\} = \pm\tfrac{1}{2}\left\{\frac{G''\,F - G'\,F'}{F^2} + \frac{1}{2}\,\frac{F'\,G'}{F^2}\right\} = \pm\frac{G''\,F - \frac{1}{2}\,G'\,F'}{2\,F^2} \implies
\end{aligned}$$

$$\boxed{\overset{E}{R}_{00} = \pm\frac{2\,G''\,F - G'\,F'}{4\,F^2}}$$

$\overset{E}{R}_{11}$, $\overset{E}{R}_{22}$

$$\begin{cases}
\overset{E}{R}_{11} = \cancel{\partial_1\Gamma^i_{1i}} - \partial_i\cancel{\Gamma^i_{11}} + \cancel{\Gamma^i_{k1}}\,\cancel{\Gamma^k_{1i}} - \Gamma^i_{ki}\,\cancel{\Gamma^k_{11}} = 0 \\
\overset{E}{R}_{22} = \cancel{\partial_2\Gamma^i_{2i}} - \partial_i\cancel{\Gamma^i_{22}} + \cancel{\Gamma^i_{k2}}\,\cancel{\Gamma^k_{2i}} - \Gamma^i_{ki}\,\cancel{\Gamma^k_{22}} = 0
\end{cases}$$

$\overset{E}{R}_{33}$

$$\begin{aligned}
&\Gamma^m_{3k}\,\Gamma^k_{m3} = \Gamma^m_{30}\,\Gamma^0_{m3} + \Gamma^m_{33}\,\Gamma^3_{m3} + \Gamma^m_{34}\,\Gamma^4_{m3} \\
&= 0 + \Gamma^4_{33}\,\Gamma^3_{43} + \Gamma^3_{34}\,\Gamma^4_{33} = 2\,\Gamma^4_{33}\,\Gamma^3_{43} = \pm\tfrac{1}{2}\,F^{-1}\,(G')^2\,G^{-1}
\end{aligned}$$

$$\begin{aligned}
\overset{E}{R}_{33} &= \partial_3\Gamma^k_{3k} - \partial_k\Gamma^k_{33} + \Gamma^m_{3k}\,\Gamma^k_{m3} - \Gamma^m_{33}\,\Gamma^k_{mk} = -\partial_4\Gamma^4_{33} + \Gamma^m_{3k}\,\Gamma^k_{m3} - \Gamma^4_{33}\,\Gamma^k_{4k} \\
&= \mp\tfrac{1}{2}\,(F^{-1}\,G)' \pm \tfrac{1}{2}\,F^{-1}\,(G')^2\,G^{-1} \mp \tfrac{1}{2}\,F^{-1}\,G'\,\left(G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'\right) \\
&= \mp\tfrac{1}{2}\,(F^{-1}\,G)' \mp \tfrac{1}{4}\,F^{-2}\,G'\,F' = \ldots \implies
\end{aligned}$$

$$\boxed{\overset{E}{R}_{33} = -\overset{E}{R}_{00} = \mp\,\frac{2\,G''\,F - G'\,F'}{4\,F^2}}$$

$\overset{E}{R}_{44}$

$R_{44} = \partial_4\Gamma^i_{4i} - \partial_i\Gamma^i_{44} + \Gamma^i_{k4}\,\Gamma^k_{4i} - \Gamma^i_{ki}\,\Gamma^k_{44}$. $\qquad \Gamma^i_{4i} = \Gamma^0_{40} + \Gamma^3_{43} + \Gamma^4_{44} = G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'$

$$\begin{aligned}
\overset{E}{R}_{44} &= \left(G'G^{-1} + \tfrac{1}{2}\,F^{-1}F'\right)' - \tfrac{1}{2}\,(F^{-1}F')' + \left(\Gamma^0_{k4}\,\Gamma^k_{40} + \Gamma^3_{k4}\,\Gamma^k_{43} + \Gamma^4_{k4}\,\Gamma^k_{44}\right) - \Gamma^i_{4i}\,\Gamma^4_{44} \\
&= \left(G'G^{-1}\right)' + \left(\Gamma^0_{04}\,\Gamma^0_{40} + \Gamma^3_{34}\,\Gamma^3_{43} + \Gamma^4_{44}\,\Gamma^4_{44}\right) - \Gamma^i_{4i}\,\Gamma^4_{44} \\
&= \left(G'G^{-1}\right)' + \left(\tfrac{1}{2}\,G'G^{-1}\right)^2 + \left(\tfrac{1}{2}\,G'G^{-1}\right)^2 + \left(\tfrac{1}{2}\,F^{-1}F'\right)^2 - (G'G^{-1} + \tfrac{1}{2}\,F^{-1}F')\,\tfrac{1}{2}\,F^{-1}F' \\
&= \left(G'G^{-1}\right)' + \tfrac{1}{2}\,(G'G^{-1})^2 - \tfrac{1}{2}\,G'G^{-1}\,F^{-1}F'
\end{aligned}$$

$$\left[\begin{array}{l} \overset{E}{R}_{44} = \left(\frac{G'}{G}\right)' + \frac{1}{2}\left(\frac{G'}{G}\right)^2 - \frac{1}{2}\,\frac{F'\,G'}{F\,G} = \frac{G''\,G-(G')^2}{G^2} + \frac{1}{2}\left(\frac{G'}{G}\right)^2 - \frac{F'G'}{2\,F\,G} \\ = \frac{G''\,G-\frac{1}{2}\,(G')^2}{G^2} - \frac{F'G'}{2\,F\,G} = \frac{2\,G''\,G-(G')^2}{2\,G^2} - \frac{F'G'}{2\,F\,G} = \frac{2\,F\,G''\,G-F\,(G')^2}{2\,F\,G^2} - \frac{F'G'G}{2\,F\,G^2} \end{array}\right.$$

$$\boxed{\overset{E}{R}_{44} = \frac{2\,G''\,F\,G - G'\,F'\,G - (G')^2\,F}{2\,G^2\,F}}$$

Thus all formulas (36) *are proved.*

15.4 Ricci-Eisenhart tensor of a type 2 metric

$$\text{Type 2 metric}\begin{cases} g_{00} = 1 \\ g_{11} = g_{22} = g_{33} = -\,G(x_4), \quad G(x_4)\ \text{dimensionless positive function} \\ g_{44} = \pm F(x_4), \quad F(x_4)\ \text{dimensionless positive function} \end{cases}$$

Step $\boxed{1}$

$$T = \tfrac{1}{2}\,g_{ij}\,v^i\,v^j = \tfrac{1}{2}\left[(v^0)^2 - G\,\left((v^1)^2+(v^2)^2+(v^3)^2\right) \pm F\,(v^4)^2\right]$$

$$\begin{cases} \frac{\partial T}{\partial v^0} = v^0 \\ \frac{\partial T}{\partial v^1} = -G\,v^1 \\ \frac{\partial T}{\partial v^2} = -G\,v^2 \\ \frac{\partial T}{\partial v^3} = -G\,v^3 \\ \frac{\partial T}{\partial v^4} = \pm F\,v^4 \end{cases} \qquad \begin{cases} \frac{d}{dt}\frac{\partial T}{\partial v^0} = \frac{dv^0}{dt} \\ \frac{d}{dt}\frac{\partial T}{\partial v^1} = -\,G'\,v^4\,v^1 - G\,\frac{dv^1}{dt} \\ \frac{d}{dt}\frac{\partial T}{\partial v^2} = -\,G'\,v^4\,v^2 - G\,\frac{dv^2}{dt} \\ \frac{d}{dt}\frac{\partial T}{\partial v^3} = -\,G'\,v^4\,v^3 - G\,\frac{dv^3}{dt} \\ \frac{d}{dt}\frac{\partial T}{\partial v^4} = \pm F'\,(v^4)^2 \pm F\,\frac{dv^4}{dt} \end{cases}$$

Single non-ignorable coordinate x_4:

$$\frac{\partial T}{\partial x^4} = \tfrac{1}{2}\left[-\,G'\,\left((v^1)^2+(v^2)^2+(v^3)^2\right) \pm F'\,(v^4)^2\right]$$

Lagrangian binomials $L_i \overset{\text{def}}{=} \dfrac{d}{dt}\dfrac{\partial T}{\partial v^i} - \dfrac{\partial T}{\partial x^i}$:

$$\begin{cases} L_0 = \frac{dv^0}{dt} \\ L_1 = -\,G'\,v^4\,v^1 - G\,\frac{dv^1}{dt} \\ L_2 = -\,G'\,v^4\,v^2 - G\,\frac{dv^2}{dt} \\ L_3 = -\,G'\,v^4\,v^3 - G\,\frac{dv^3}{dt} \\ L_4 = \pm F'\,(v^4)^2 \pm F\,\frac{dv^4}{dt} - \frac{1}{2}\left[-\,G'\,\left((v^1)^2+(v^2)^2+(v^3)^2\right) \pm F'\,(v^4)^2\right] \end{cases}$$

Step $\boxed{2}$ Lagrangian binomials with raised indices $L^i = g^{ii}\, L_i$ (orthogonal metric):

$$\begin{cases} g^{00} = 1 \\ g^{11} = -\,G^{-1} \\ g^{22} = -\,G^{-1} \\ g^{33} = -\,G^{-1} \\ g^{44} = \pm F^{-1} \end{cases} \qquad \begin{cases} L^0 = \dfrac{dv^0}{dt} \\ L^1 = G^{-1}G'\, v^4\, v^1 + \dfrac{dv^1}{dt} \\ L^2 = G^{-1}G'\, v^4\, v^2 + \dfrac{dv^2}{dt} \\ L^3 = G^{-1}G'\, v^4\, v^3 + \dfrac{dv^3}{dt} \\ L^4 = F^{-1}F'\,(v^4)^2 + \dfrac{dv^4}{dt} \\ \qquad \mp \frac{1}{2}\, F^{-1} \Big[-\,G' \Big((v^1)^2 + (v^2)^2 + (v^3)^2 \Big) \pm F'\,(v^4)^2 \Big] \end{cases}$$

Step $\boxed{3}$ From the above expressions of L^i we extract the quadratic forms $Q^i \stackrel{\text{def}}{=} \Gamma^i_{hk}\, v^h\, v^k$::

$$\text{(98)} \qquad \begin{cases} Q^0 = 0 \\ Q^1 = G^{-1}G'\, v^4\, v^1 \\ Q^2 = G^{-1}G'\, v^4\, v^2 \\ Q^3 = G^{-1}G'\, v^4\, v^3 \\ Q^4 = F^{-1}F'\,(v^4)^2 \mp \frac{1}{2}\, F^{-1} \Big[-\,G' \Big((v^1)^2 + (v^2)^2 + (v^3)^2 \Big) \pm F'\,(v^4)^2 \Big] \\ \quad = \frac{1}{2}\, F^{-1}F'\,(v^4)^2 \pm \frac{1}{2}\, F^{-1}G' \Big((v^1)^2 + (v^2)^2 + (v^3)^2 \Big) \end{cases}$$

and from these we derive the non-identically null Christoffel symbols:

$$\text{(99)} \qquad \boxed{ \begin{cases} \Gamma^1_{41} = \frac{1}{2}\, G^{-1}G' \\ \Gamma^2_{42} = \frac{1}{2}\, G^{-1}G' \\ \Gamma^3_{43} = \frac{1}{2}\, G^{-1}G' \end{cases} \qquad \begin{cases} \Gamma^4_{44} = \frac{1}{2}\, F^{-1}F' \\ \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \pm\frac{1}{2}\, F^{-1}G' \end{cases} }$$

Note that symbols with at least one lower index equal to 0 are null.

End of the algorithm.

Once more we recall the definition of the Ricci tensor according to Eisenhart (82)

$$\overset{E}{R}_{\ell m} \stackrel{\text{def}}{=} \partial_m \Gamma^i_{\ell i} - \partial_i \Gamma^i_{\ell m} + \Gamma^i_{km}\, \Gamma^k_{\ell i} - \Gamma^i_{ki}\, \Gamma^k_{\ell m}$$

where the terms $\partial_m \Gamma^i_{\ell i}$ and $\partial_i \Gamma^i_{\ell m}$ cancel for $m \neq 4$ and $i \neq 4$ because x_4 is the only coordinate that cannot be ignored.

$\overset{E}{R}_{00}$

$\overset{E}{R}_{00} = \partial_0 \Gamma^i_{0i} - \partial_i \Gamma^i_{00} + \Gamma^i_{k0}\, \Gamma^k_{0i} - \Gamma^i_{ki}\, \Gamma^k_{00}$. Since all symbols involved have at least one lower index equal to 0, we find

$$\boxed{\overset{E}{R}_{00} = 0}$$

$\overset{E}{R}_{11}$

$$\overset{E}{R}_{11} = \partial_1 \Gamma^i_{1i} - \partial_i \Gamma^i_{11} + \Gamma^i_{k1}\Gamma^k_{1i} - \Gamma^i_{ki}\Gamma^k_{11} = -\partial_4 \Gamma^4_{11} + \Gamma^i_{k1}\Gamma^k_{1i} - \Gamma^i_{4i}\Gamma^4_{11}$$
$$= -\partial_4 \Gamma^4_{11} + \Gamma^1_{k1}\Gamma^k_{11} + \Gamma^4_{k1}\Gamma^k_{14} - \Gamma^i_{4i}\Gamma^4_{11} = -\partial_4 \Gamma^4_{11} + \Gamma^1_{41}\Gamma^4_{11} + \Gamma^4_{11}\Gamma^1_{14} - \Gamma^i_{4i}\Gamma^4_{11}$$
$$= -\partial_4 \Gamma^4_{11} + \Gamma^4_{11}\left(\Gamma^1_{41} + \Gamma^1_{14} - \Gamma^i_{4i}\right)$$
$$= -\partial_4 \Gamma^4_{11} + \Gamma^4_{11}\left(\Gamma^1_{41} + \Gamma^1_{14} - \Gamma^0_{40} - \Gamma^1_{41} - \Gamma^2_{42} - \Gamma^3_{43} - \Gamma^4_{44}\right)$$
$$= -\partial_4 \Gamma^4_{11} + \Gamma^4_{11}\left(\Gamma^1_{41} - \Gamma^2_{42} - \Gamma^3_{43} - \Gamma^4_{44}\right) \text{....ma } \Gamma^1_{41} = \Gamma^2_{42}$$
$$= -\partial_4 \Gamma^4_{11} - \Gamma^4_{11}\left(\Gamma^3_{43} + \Gamma^4_{44}\right)$$
$$= \mp\tfrac{1}{2}\,(F^{-1}G')' \mp \tfrac{1}{2}\,F^{-1}G'\left(\tfrac{1}{2}\,G^{-1}G' + \tfrac{1}{2}\,F^{-1}F'\right)$$

$$(F^{-1}G')' = -\,F^{-2}F'G' + F^{-1}G''$$

$$\overset{E}{R}_{11} = \mp\tfrac{1}{4}\left[2\,(F^{-1}G')' + F^{-1}G'\left(G^{-1}G' + F^{-1}F'\right)\right]$$
$$= \mp\tfrac{1}{4}\left[2\left(F^{-1}G'' - F^{-2}F'G'\right) + F^{-1}G^{-1}(G')^2 + F^{-2}G'F'\right]$$
$$= \mp\tfrac{1}{4}\left[2\,F^{-1}G'' + F^{-1}G^{-1}(G')^2 - F^{-2}G'F'\right] = \mp\frac{2\,FG'' + FG^{-1}(G')^2 - G'F'}{4\,F^2}.$$

By virtue of the spatial isotropy of a type 2 metric we have

$$\boxed{\overset{E}{R}_{11} = \overset{E}{R}_{22} = \overset{E}{R}_{33} = \mp\frac{2\,FG'' + FG^{-1}(G')^2 - G'F'}{4\,F^2}}$$

i.e.

$$\boxed{\overset{E}{R}_{11} = \overset{E}{R}_{22} = \overset{E}{R}_{33} = \mp\,\frac{2\,G''\,F\,G + (G')^2\,F - F'\,G'\,G}{4\,F^2\,G}}$$

$\overset{E}{R}_{44}$

$$
\begin{aligned}
\overset{E}{R}_{44} &= \partial_4\Gamma^i_{4i} - \partial_i\Gamma^i_{44} + \Gamma^i_{k4}\,\Gamma^k_{4i} - \Gamma^i_{ki}\,\Gamma^k_{44} \\
&= \partial_4\Gamma^0_{40} + \partial_4\Gamma^1_{41} + \partial_4\Gamma^2_{42} + \partial_4\Gamma^3_{43} + \partial_4\Gamma^4_{44} - \partial_4\Gamma^4_{44} + \Gamma^i_{k4}\,\Gamma^k_{4i} - \Gamma^i_{4i}\,\Gamma^4_{44} \\
&= \partial_4\Gamma^1_{41} + \partial_4\Gamma^2_{42} + \partial_4\Gamma^3_{43} + \Gamma^i_{k4}\,\Gamma^k_{4i} - \Gamma^i_{4i}\,\Gamma^4_{44} \\
&= \tfrac{3}{2}\,(G^{-1}G')' + \Gamma^1_{k4}\,\Gamma^k_{41} + \Gamma^2_{k4}\,\Gamma^k_{42} + \Gamma^3_{k4}\,\Gamma^k_{43} + \Gamma^4_{k4}\,\Gamma^k_{44} - \left(\Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43} + \Gamma^4_{44}\right)\Gamma^4_{44} \\
&= \tfrac{3}{2}\,(G^{-1}G')' + \Gamma^1_{14}\,\Gamma^1_{41} + \Gamma^2_{24}\,\Gamma^2_{42} + \Gamma^3_{34}\,\Gamma^3_{43} + \Gamma^4_{44}\,\Gamma^4_{44} - \left(\Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43} + \Gamma^4_{44}\right)\Gamma^4_{44} \\
&= \tfrac{3}{2}\,(G^{-1}G')' + (\Gamma^1_{14})^2 + (\Gamma^2_{24})^2 + (\Gamma^3_{34})^2 + (\Gamma^4_{44})^2 - \left(\Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43} + \Gamma^4_{44}\right)\Gamma^4_{44} \\
&= \tfrac{3}{2}\,(G^{-1}G')' + (\Gamma^1_{14})^2 + (\Gamma^2_{24})^2 + (\Gamma^3_{34})^2 - \left(\Gamma^1_{41} + \Gamma^2_{42} + \Gamma^3_{43}\right)\Gamma^4_{44} \\
&= \tfrac{3}{2}\,(G^{-1}G')' + \tfrac{3}{4}\,(G^{-1}G')^2 - \tfrac{3}{4}\,G^{-1}G'\,F^{-1}F' \\
&= \tfrac{3}{2}\,(-G^{-2}(G')^2 + G^{-1}G'') + \tfrac{3}{4}\,(G^{-2}(G')^2 - \tfrac{3}{4}\,G^{-1}G'\,F^{-1}F' \\
&= \tfrac{3}{2}\,G^{-1}G'' - \tfrac{3}{4}\,G^{-2}(G')^2 - \tfrac{3}{4}\,G^{-1}G'\,F^{-1}F' \\
&= \tfrac{3}{4}\left(2\,G^{-1}G'' - G^{-2}(G')^2 - G^{-1}G'\,F^{-1}F'\right) \\
&= \tfrac{3}{4}\,G^{-2}\left(2\,GG'' - (G')^2 - GG'\,F^{-1}F'\right) \quad \Longrightarrow
\end{aligned}
$$

$$
\boxed{\overset{E}{R}_{44} = 3\,\frac{2\,GG'' - (G')^2 - GG'\,F^{-1}F'}{4\,G^2}}
$$

i.e.

$$
\boxed{\overset{E}{R}_{44} = 3\,\frac{2\,F\,G''\,G - (G')^2\,F - F'\,G'\,G}{4\,G^2\,F}}
$$

Thus all the formulas (36) *and* (37) *are proved.*

16 Acknowledgments

We are grateful to Dr. Giovanni Rastelli for his assistance in calculating the components of Ricci tensors. We also thank Eng. Domenico Bassani for his help in compiling the metric summary sheets of §13.

This is the original preprint published by Annals of Mathematics and Physics 2024, 7(1), pp. 024-053, same title.

References

[1] F. Cardone, R. Mignani, *Deformed Spacetime*, Fundamental Theories of Physics 157, Springer (2007).

[2] L. Kostro, P. Lange, Physick Essays **10** (1999).

[3] G. Albertini, V. Calbucci, F. Cardone, G. Fattorini, R. Mignani, A. Petrucci, F. Ridolfi, A. Rotili, *Evidence of Alpha Emission from Compressed Steel Bars*, International Journal of Modern Physics B, Vol. 27, No. 23, World Scientific Publishing Company (2013).

[4] F. Cardone, V. Calbucci, G. Albertini, *Possible Evidence of Piezonuclear Alpha Emission*, Journal of Advanced Physics, Vol. 2, No. 01, American Scientific Publishers (2013).

[5] F. Cardone, V. Calbucci, G. Albertini, *Deformed Space-Time of the Piezonuclear Emissions*, Modern Physics Letters B, Vol. 28, No. 02 (2014).

[6] F. Cardone, G. Albertini, D. Bassani, G. Cherubini, E. Guerriero, R. Mignani, M. Monti, A. Petrucci, F. Ridolfi, A. Rosada, F. Rosetto, V. Sala, E. Santoro, and G. Spera, *Nuclear metamorphosis in mercury*, International Journal of Modern Physics B, Vol. 30, No. 01, 1550239 (2016).

[7] F. Cardone, G. Albertini, D. Bassani, G. Cherubini, E. Guerriero, R. Mignani, M. Monti, A. Petrucci, F. Ridolfi, A. Rosada, F. Rosetto, V. Sala, E. Santoro, and G. Spera, *deformed Space-Time Transformations in Mercury*, International Journal of Modern Physics B, Vol. 31, No. 23, World Scientific Publishing Company (2017).

[8] G. Albertini, F. Cardone, G. Cherubini, E. Guerriero., A. Rosada, *Reduction of the Radiation in Radioactive Substances*, International Journal of Modern Physics B, Vol. 34, No. 04, World Scientific Publishing Company (2020).

[9] A. Rosada, F. Cardone, P. Avino, *The Astonishing 63 Ni Radioactivity Reduction in Radioactive Wastes by Means of Ultrasounds Application*, Springer Nature Applied Sciences, Vol. 1, No. 1319 (2019).

[10] F. Cardone, G. Albertini, D. Bassani, G. Cherubini, A. Petrucci. A. Rosada, *DST-deactivation of Nickel-63 Nitrate*. Radiochimica Acta, Vol. 107, No. 06 (2019).

[11] G. Albertini, D. Bassani, F. Cardone, G. Cherubini, E. Guerriero. and A. Rosada, *Neutralization of radionuclides*, International Journal of Modern Physics B, Vol. 35, No. 2, World Scientific Publishing Company (2021).

[12] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, Differential Geometry 17 (1982), 255-306.

[13] L.P. Eisenhart, *Riemannian Geometry*, Princeton University Press (1925).

[14] L.P. Eisenhart, *Non-Riemannian Geometry*, Am. Math. Society (1949).

[15] H.D. Cao, X.P. Zhu, *A complete proof of the Poincar and geometrization conjectures – Application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math., Vol. 10, No. 02, pp. 165-492, International Press (2006).

[16] S.M. Carroll, *Lectures Notes on General Relativity*, ArXiv 1997.

[17] R.M. Wald, *General Relativity*, University Chicago Press (1984).

[18] F. Cardone, R. Mignani, *Energy and Geometry*, World Scientific (2004).

[19] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*. W.H. Freeman & Company (1973).

[20] F. Cardone, W. Perconti, C. Laposta, *Cosmic Microwave Background Radiation Preferred Directions and Asymmetry in Lorentz Violation*, Journal of Advanced Physics, Vol. 4, pp. 1–5 (2015).

[21] F. Cardone, W. Perconti, A. Petrucci, *Deformed Space Time Asymmetric Emissions and Cosmic Microwave Background Radiation Preferred Directions*, Journal of Advanced Physics, Vol. 6, pp. 1–5 (2016).

[22] F. Cardone, W. Perconti, A. Petrucci, *Lorentz Violation, Cosmic Microwave Background Radiation Anisotropy and Critical Energy Density of the Universe*, Journal of Advanced Physics Vol. 7, pp. 1–7 (2018).

[23] G. Albertini, F. Cardone, S. Bellucci, *The Asymmetry of the Cosmic Microwave Background Radiation as Signature of Local Lorentz Invariance Violation*, Advances in High Energy Physics, Vol. 2022, Article ID 6271119, 10 pages.
https://doi.org/10.1155/2022/6271119.

[24] F. Cardone, G. Albertini, D. Bassani, *Lorentz Violation in Torsional Antenna*, Foundations of Science 27:43–55 (2022).
https://doi.org/10.1007/s10699-020-09710-z.

[25] G. Albertini, D. Bassani, *Deformed Space–Time Reactions and Their Phenomenology*, Physics Journal, Vol. 1, No. 3, 2015, pp. 382–387.

[26] S. Benenti, *Fondamenti Matematici e Analisi Numerica della Dinamica di un Universo Isotropo*, Memorie dell'Accademia delle Scienze di Torino, Classe di S.F.M.N., Vol. 42-43, 2018-2019.

[27] P.S.M. de LaPlace, *Mćanique Céleste, Traslated with a Commentary by N. Bodwitch*, Boston, 1829.