Mathematical Models in Isotropic Cosmology

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" ... and there was light."

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Preface

Mathematical models must be based on a well stated set of postulates which should obey a sort of *principle of aesthetic simplicity*. For this reason we cannot start from the celebrated *Weyl principle*, which imposes from the very beginning the existence of a Lorentzian metric on space-time in which the world lines of the galaxies are time-like curves. In my opinion this principle lies at a too advanced position. In mathematical terms, the principle Weyl is at the level of the category of Riemannian (or semi-Riemannian) manifolds, while it seems more appropriate to begin from a lower-level category, as that of the differentiable manifolds (in the present theory the category of Riemannian manifolds operates in the fourth postulate). In doing so, we acquire two advantages: we can locate primary concepts and their relationships in the right order and at the right place and we do not lose secondary but noteworthy concepts, which otherwise would remain unknown. The proof is that we are able to bring much forward our analysis, without any assumptions about the dynamical laws. These topics occupy the first three chapters.

In Chapter 1 (Cosmic kinematics) we compose the basic geometrical structures of the **cosmic space-time** (Fig. 1). The four-dimensional manifold M of the (non singular) **cosmic events**[1] is endowed with two foliations. The first foliation is made of three-dimensional manifolds S_t, parametrized by a **cosmic time** t. They represent the sets of simultaneous events and are called **spatial sections**. Each S_t is endowed with a Riemannian structure with metric tensor g_t. The isotropic principle implies that each metric g_t has a constant curvature, varying with t. The second foliation is made of the **world-lines** of the galaxies, which are transversal to the spatial sections. Since the world-lines do not intersect each other, this theory does not contemplate fragmentations or collisions of galaxies.

[1] Actually, the differentiable manifold M does not contain the singular events of birth and death of the universe.

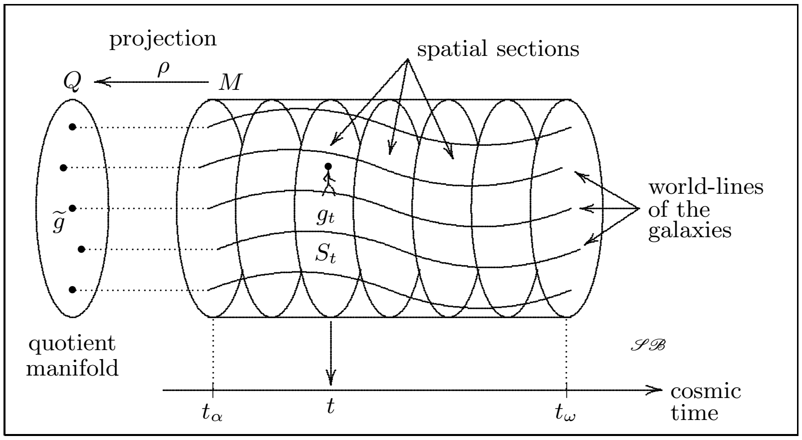

Figure 1: The basic geometrical structures of the cosmic space-time.

We postulate that the set Q of the galactic world-lines (which is nothing but the set of all the galaxies) is endowed with the structure of a three-dimensional manifold: it will be called the **quotient manifold** and it will play a basic role in this theory. A theorem establishes that the quotient manifold Q is endowed with a Riemannian structure with metric tensor $\widetilde{g}$, called **quotient metric** in such a way that any spatial section (S_t, g_t) is isomorphic to $(Q, \widetilde{g})$, so that any spatial section can be taken as a representative of the quotient manifold.

In any cosmological theory the so-called **scale factor** plays a key role. It is a function of the cosmic time t, commonly denoted as $a(t)$, that appears as a conformal factor of the spatial metrics. In our theory, however, the notion of scale factor arises as a conformal factor linking two spatial metrics through the formula

$$\boxed{g(t_1) = a^2(t_1, t_2)\, g(t_2)}$$

therefore as a two-variable function in the cosmic time: $a(t_1, t_2)$. This fact offers significant opportunities and advantages in the mathematical analysis of cosmological phenomena, which instead escaped in other approaches to Cosmology. For example, if we fix a value of the second variable t_2 (which we can call **reference time**) and leave $t_1 = t$ to run as the only independent variable, then a theorem shows that two scale factors obtained in this way, say $a(t, t_2)$ and $a(t, t_2')$, with different values of the reference time differs by a constant factor. As a consequence, we can always impose to the scale factor the **normalization condition**

$$a(t, t_\sharp) = 1$$

to be satisfied under the free choice of a **normalization time** $t_\sharp$. This allows us to establish an effective check on the physical validity of equations involving the scaling factor, because such equations must be invariant under the change of the normalization time. Once the normalization time is chosen, the scale factor $a(t, t_\sharp)$ becomes the **principal cosmic function** from which, in principle, we should derive the evolution of other observational quantities (like the Hubble parameter, the energy density, the matter density, etc.). For this reason the evolution in time of the scale factor $a(t, t_\sharp)$ will be called **profile of the universe**.

In Chapter 2 (Cosmic connections) we prove the existence on the cosmic space-time of a family of linear symmetric connections that are the natural prelude to the formulation of a dynamics. These connections depend on an indeterminate (but not arbitrary) function of the cosmic time. The assignment of such a function through a **bridge-postulate** will mark the passage from kinematics to dynamics. We will examine two possible bridge-postulates. The first one leads towards a generalization of the Newtonian space-time, where we can build-up a **Newtonian cosmic dynamics**. The second one consists in assuming the existence of **special particles** (*photons*) wandering in the cosmos with a constant 'peculiar velocity'. The surprising fact is that (i) *the resulting cosmic connection is the Levi-Civita connection of a space-time metric*, (ii) *this metric has necessarily a Lorentzian signature*, (iii) *the galactic world-lines are time-like geodesics and the world-lines of the photons are null geodesics*. This makes us to move towards a **relativistic cosmic dynamics** founded on the Einstein field equations (Chapter 4).

Chapter 3 is dedicated to the preparation of the essential elements that we need for the formulation of the dynamics (Ricci tensor, Einstein tensor, etc.). It is first noted a remarkable fact which greatly simplifies the calculations: due to the isotropy principle any symmetric two-tensor $T^{\alpha\beta}$ having a geometrical or physical meaning is fully determined by two functions $\phi(t)$ and $\psi(t)$ only, which we call **characteristic functions**. In generic co-moving coordinates the components of such a tensor are

$$\boxed{\begin{cases} T^{oo} = \phi(t) = \text{a function of } t \text{ only} \\ T^{oa} = 0 \\ T^{ab} = \psi(t)\, \widetilde{g}^{ab}(\widetilde{q}) = \text{a function of } t \text{ times the quotient metric } \widetilde{g}^{ab} \end{cases}}$$

Then it can be proved that the conservation law and the Einstein field equations result in ordinary differential equations involving the two characteristic functions of the momentum-energy tensor and the scale factor. Many things can be said in this context without specifying the form the energy-momentum tensor.

Chapter 4. The choice of this tensor is the first topic of this chapter, devoted to the relativistic cosmic dynamics. We will consider the standard energy-momentum tensor of a **perfect fluid**

$$\boxed{T^{\alpha\beta} = (\epsilon + p)\, U^\alpha\, U^\beta + p\, g^{\alpha\beta}}$$

where $\epsilon(t)$ is the energy density, $p(t)$ is the intergalactic pressure and U^α is the unitary four-velocity of the galactic fluid

$$U^\alpha \overset{\text{def}}{=} c^{-1}\, \frac{d\gamma^\alpha}{dt} : \begin{cases} U^0 = 1, \\ U^a = 0, \end{cases} \qquad g_{\alpha\beta}\, U^\alpha\, U^\beta = -1.$$

It follows that the conservation law $\nabla_\alpha T^{\alpha\beta} = 0$ and the Einstein field equations are respectively equivalent to the two **dynamical equations**

$$\boxed{a\,\dot{\epsilon} + 3\,(\epsilon + p)\,\dot{a} = 0} \tag{1}$$

$$\boxed{\frac{\dot{a}^2}{c^2} = \tfrac{1}{3}\, a^2\, (\Lambda + \chi\,\epsilon) - \widetilde{K}} \tag{2}$$

where $\widetilde{K}$ is the curvature of the quotient manifold.

Chapter 5. A second crucial step is the choice of a state equation that ties the three unknown functions $\epsilon(t)$, $p(t)$ and $a(t)$. In this chapter we deal with the simplest possible case, namely that of a **linear barotropic fluid**, whose equation of state is

$$p = w\,\epsilon,$$

w being a constant parameter. Thus, we are faced with various models to examine, depending on the sign of the constant curvature $\widetilde{K}$. However, the models with negative spatial curvature are 'a priori' excluded from this theory because, as it is easily shown, their radial speed expansion turns out to be always greater than light speed. On the other hand, models with positive spatial curvature seems to be incopatible with the astrophysical observations. As a consequence, we confine ourselves to study the models with zero spatial curvature only. These models are in fact very simple and do not take into account of all the complex quantum-physical phenomenology occurring in the evolution of the universe, especially in the vicinity of its creation. Nevertheless they reveal the typical features of the isotropic cosmology and can serve as a starting point for the creation of finer models. For example, in the gallery of the possible profiles of the universe we find that of Fig. 2, which is perfect agreement with that appearing in the Nobel Lecture by G. Riess [20] (Fig. 3).[2]

[2] The Nobel Prize in Physics 2011 was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess "for the discovery of the accelerating expansion of the Universe through observations of distant supernovae".

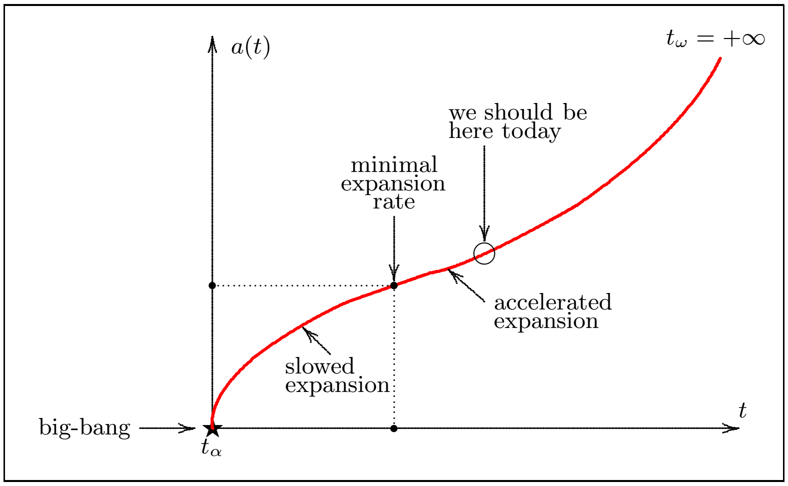

Figure 2: One of the eligible universe profiles

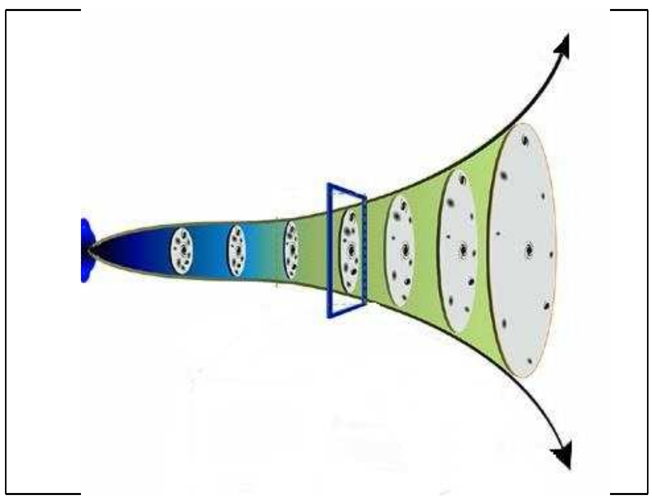

Figure 3: From Riess Nobel Lecture,

The fortunate circumstance which makes the difference with respect the other mathematical approaches to Cosmology is that the dynamical equations of a flat barotropic model are solvable in terms of elementary functions (exponential and hyperbolic functions). This allows us to get the exact expression of all the observational variables related to the scale factor. The only approximation is then due to the numerical evaluation of the cosmological parameters involved.

Another feature of our approach is that there is no refrence to the Robertson-Walker metric. In fact, when the use of coordinates is necessary, we will refer to quite generic length-dimensional co-moving coordinates, so that the metric tensor components are dimensionless. Particular attention is paid to the dimensionality of the physical objects. This is a way for checking the correctness of the formulas.

Chapter 6. In order to avoid the common misconceptions on the various notions of horizons, pointed out for example in [6], we afford the so-called 'horizon problems' on the basis of what we have learned from the previous chapter and with a significant graphic method. When applied to the flat dust-matter model, this method gives numerical results in very good agreement with the current observational data.

The postulates on which we relied are very simple and intuitive. They do not take into account the complex evolutionary physical processes of the universe and in particular the presence of dark matter, whose nature is still unknown. It is therefore quite natural that the results obtained here do not fully meet the expectations and experiences of cosmologists. However, it is surprising that, despite the extreme simplification of our model, the resulting estimates of the age of the universe, as well as of other data, are quite close to that obtained by astronomers. Nanyway, like for any axiomatic theory, the cosmological theory presented here must be subjected to criticisms, adjustments and extensions.

***

This work stops without covering other charming topics, which require more time for their study —and perhaps some modification to postulates. For example, models with positive spatial curvature (even if these are deemed incompatible by the astrophysical observations), with multi-components, with dark matter, with anisotropy, and so on.

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Chapter 1

Cosmic kinematics

1.1 The postulates of the cosmic space-time

In an axiomatic formulation of cosmology the concept of **event** is a primary concept as that of **point** in Euclidean geometry.

$\mathbf{1}^{st}$ **Postulate**. The history of the universe is made of events, whose set will be called **cosmic space-time** and denoted by $\mathcal{E}$. There are two **critical events**, denoted by α and ω, representing the **beginning** and the **end** of the universe, respectively. All other events are called **regular** and form a four-dimensional manifold M:

$$\mathcal{E} = \alpha \cup M \cup \omega.$$

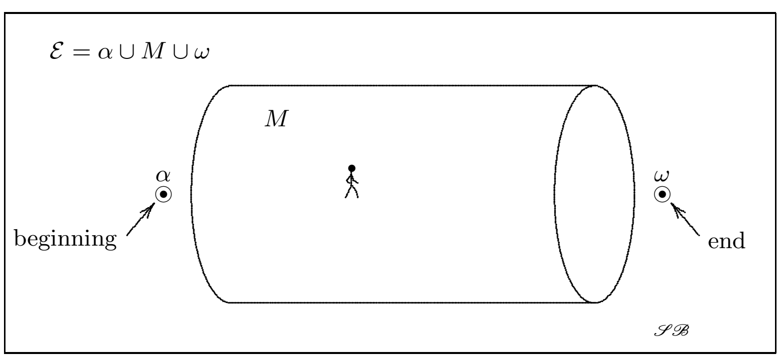


Figure 1.1: The cosmic space-time.

A regular event occurs in a certain 'place' and at a certain 'date'. So, in order to locate an event we need to allocate three coordinates for the place and one coordinate for the date. This is the reason why we require M to have four dimensions. With the next two postulates we will see how this localization is achievable.

2nd Postulate. (i) The manifold M of the regular events is endowed with a fibration over an open real interval $t\colon M \to (t_\alpha, t_\omega) \subseteq \mathbb{R}$, which we call **cosmic time**. The value $t(e)$ of t on an event e will be called the **date** of e. (ii) The cosmic time t is defined up to an affine transformation

$$t \mapsto \bar{t} = \lambda\, t + \mu, \quad \lambda, \mu \in \mathbb{R}, \quad \mu > 0.$$

(iii) The fibers of t form a foliation of M made of three-dimensional manifolds which we call **spatial sections**; we will denote by S_t the spatial section of the events occurring at the date t. (iv) The mapping t is extended to the critical events by setting

$$t(\alpha) = t_\alpha, \quad t(\omega) = t_\omega.$$

The attribute 'spatial' will be justified below.

Remark 1.1 – By a reversible smooth transformation $t \mapsto \bar{t}$ any mapping $t\colon M \to (t_\alpha, t_\omega)$ can be transformed into a mapping $\bar{t}\colon M \to \mathbb{R} = (-\infty, +\infty)$. Note that any reversible smooth transformation is compatible with items (iii) and (iv). However, if we want to give meaning to the concept of duration of the universe then such a transformation must not be allowed. This is the meaning of item (ii). The choice of an affine gauge for the cosmic time will be further justified in dealing with the existence of cosmic connections on space-time, Section 2.4. As a consequence of these observations, the 'life-interval' of the universe is of four types:

$$\begin{cases} (-\infty, +\infty) \\ (-\infty, t_\omega] \\ [t_\alpha, t_\omega] \\ [t_\alpha, +\infty) \quad \bullet \end{cases}$$

The cosmic time provides a **cosmic chronology** (Fig. 1.2): given two events e_1 and e_2 we say that

e_1 and e_2 are **simultaneous**	if	$t(e_1) = t(e_2)$
e_1 occurs **before** e_2	if	$t(e_1) < t(e_2)$
e_1 occurs **after** e_2	if	$t(e_1) > t(e_2)$

In particular the event α (beginning of the universe) occurs before all other events and ω (end of the universe) occurs after all other events.

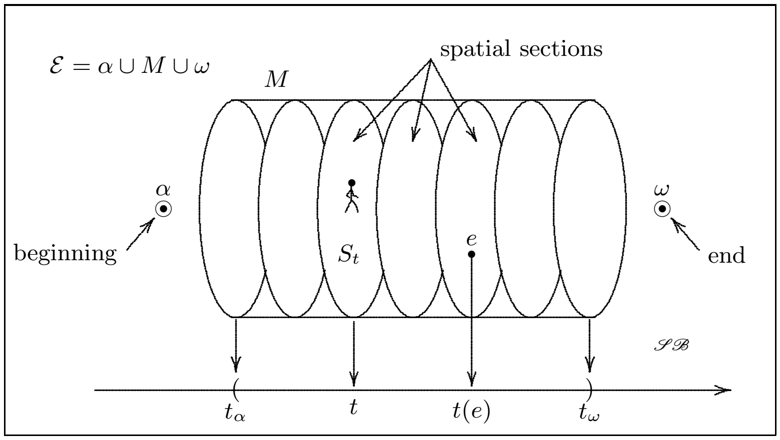


Figure 1.2: The cosmic chronology.

Remark 1.2 – This second postulate is the result of a reasoning based on the notion of simultaneity of events. In Newtonian mechanics the postulate of existence of an absolute time implies the notions of absolute chronology and absolute simultaneity. In special relativity both these notions depend on the choice of an inertial reference frame. In general relativity the simultaneity of events cannot be easily defined. The modern cosmology, at least in its simplest formulation, is based on the assumptions of isotropy and homogeneity of the universe: the distribution of the celestial bodies is uniform and there are no privileged directions. As a consequence, we can assume by definition that two events e_A and e_B are simultaneous when in a small neighborhood of e_A and e_B the densities of matter are equal. This definition is of course the result of an ideal experiment, but it makes plausible the concept of simultaneity, although not currently experienced through some physical method. •

The galaxies are considered as particles forming the **galactic fluid**. The **history of a galaxy** A is a (smooth) sequence of events parametrized by the cosmic time t. So, it can be represented by a (smooth) curve $\gamma_A \colon I \to M$, where I is a life-interval contained in (t_α, t_ω), called the **world-line** of A. With the next postulate, we consider the simplest situation in which the life duration of all galaxies coincides with the life duration of the universe: $I = (t_\alpha, t_\omega)$.

3^{rd} **Postulate**. (i) The life of a galaxy is represented by a curve on M called **world-line**. The world-lines of all galaxies form a congruence of non-intersecting curves transversal to the spatial foliation. (ii) The set Q of all the galactic world-lines has the structure of a three-dimensional differentiable manifold such that the canonical projection $\rho\colon M \to Q$, which associates with any event $e \in M$ the galaxy $\rho(e) \in Q$ where this event occurs, is a surjective submersion (Fig. 1.3).

We call Q the **quotient manifold** . Note that, being the set of all the galactic world-lines, Q is nothing but the set of all galaxies.

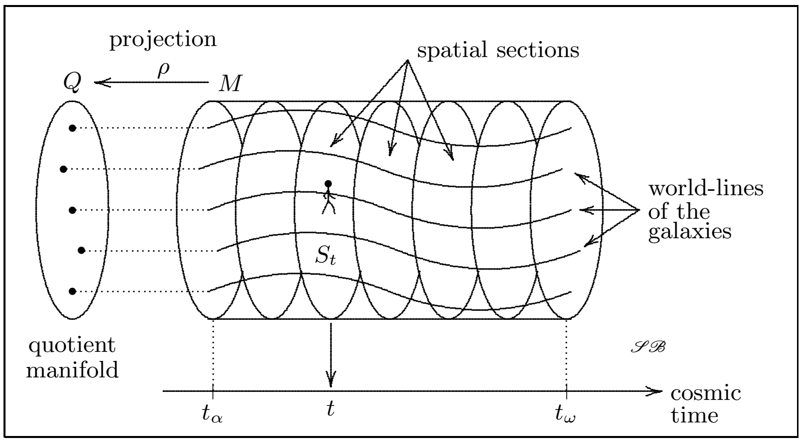


Figure 1.3: The galactic world-lines and the quotient manifold.

The transversality condition means that a world-line is nowhere tangent to a spatial section. By virtue of well-known arguments of differential geometry this third postulate implies that

Theorem 1.1 – *The restriction of the projection ρ to any spatial section S_t is a diffeomorphism.*

Thus, all S_t and Q are diffeomorphic manifolds. Another implication is

Theorem 1.2 – *Any coordinate system $\widetilde{q} = (q^a) = (q^1, q^2, q^3)$ on an open domain $U \subseteq Q$ generates a coordinate system on (t, q^a) on the open subset of M made of the world-lines determined by U* (Fig. 1.4).

Coordinates on M of this type are called **co-moving coordinates**.[1]

[1] Since t is constant on each spatial section, co-moving coordinates are also called **synchronous coordinates**.

The coordinates $\widetilde{q} = (q^a)$ are Lagrangian coordinates of the galactic fluid: they have a constant value on each world-line of U. In this way the coordinates $\widetilde{q}$ can be interpreted as coordinates on each spatial section S_t. So, they will be called **spatial coordinates**.[2]

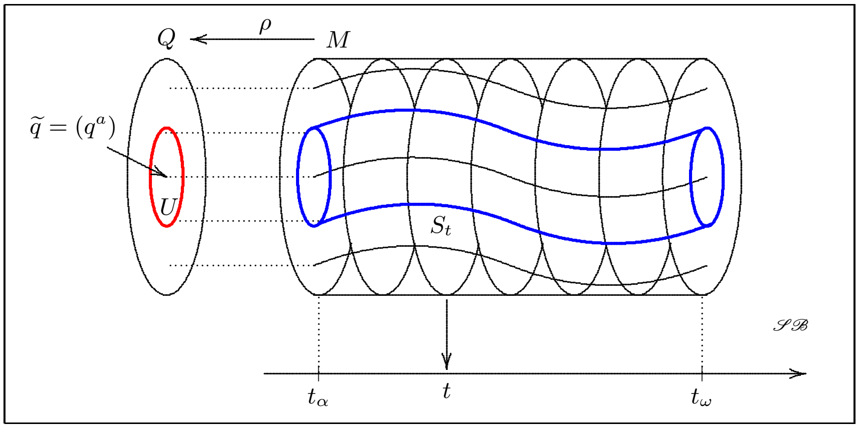


Figure 1.4: Co-moving coordinates.

> **4th Postulate**. Each spatial section S_t of simultaneous events is the representation of the three-dimensional 'physical world' at the date t and is endowed with a positive-definite metric tensor g_t smoothly depending on t.

In other words, we think of each S_t as a three-dimensional Riemannian manifold where the metric tensor components $g_{ab}(t, \widetilde{q})$ in co-moving coordinates are smooth functions of t.

The **Copernican principle** assumes that neither the Sun nor the Earth are in a central, specially favored position in the universe. This principle is extended to cosmology with the following **isotropy principle**.

> **5th Postulate**. On each spatial section S_t there is no privileged vector field having a geometrical or physical meaning.

Theorem 1.3 – *Any scalar field on M having a geometrical or physical meaning is a function of the cosmic time t only i.e., it is constant on each S_t.*

Proof – By means of the metric g_t we can define the gradient of such a scalar field which is then a distinguished vector field on S_t. This is in contrast with the isotropy principle. ■

[2] Greek indices $\alpha, \beta, \ldots$ will run from 0 to 3. Latin indices $a, b, \ldots$ will run from 1 to 3.

Theorem 1.4 – *Each spatial section (S_t, g_t) is a manifold with constant curvature $K(t)$.*[3]

Proof – The Ricci tensor R_t of the metric g_t must is proportional to the metric tensor itself, $R_t = \lambda_t\, g_t$, otherwise the existence of distinguished Ricci directions would be in contrast with the isotropy principle. In turn, the factor λ_t must be constant on S_t because of Theorem 1.3. Thus, (S_t, g_t) is an Einstein manifold. It is known that an Einstein manifold of dimension 3 is of constant curvature. ∎

1.2 Manifolds with constant curvature

In this section we recall the main features of the manifolds with constant curvature.[4] For the Riemann curvature tensor and the Ricci tensor of a linear symmetric connection Γ we will refer to the following definitions:[5]

$$\boxed{R^{\nu}{}_{\alpha\mu\beta} \overset{\text{def}}{=} \partial_\mu \Gamma^{\nu}_{\alpha\beta} - \partial_\beta \Gamma^{\nu}_{\mu\alpha} + \Gamma^{\ell}_{\alpha\beta}\, \Gamma^{\nu}_{\mu\ell} - \Gamma^{\ell}_{\mu\alpha}\, \Gamma^{\nu}_{\beta\ell}} \tag{1.1}$$

$$\boxed{R_{\alpha\beta} \overset{\text{def}}{=} R^{\mu}{}_{\alpha\mu\beta} = \partial_\mu \Gamma^{\mu}_{\alpha\beta} - \partial_\beta \Gamma^{\mu}_{\alpha\mu} + \Gamma^{\sigma}_{\alpha\beta}\, \Gamma^{\mu}_{\sigma\mu} - \Gamma^{\sigma}_{\alpha\mu}\, \Gamma^{\mu}_{\sigma\beta}} \tag{1.2}$$

where $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$ are the symbols of Γ in any coordinate system.

A Riemannian manifold with metric tensor $g_{\alpha\beta}$ is said to be of **constant curvature** K when the totally covariant Riemann tensor of the Levi-Civita connection

$$\boxed{R_{\lambda\alpha\mu\beta} \overset{\text{def}}{=} g_{\lambda\nu}\, R^{\nu}{}_{\alpha\mu\beta}} \tag{1.3}$$

satisfies the equation[6]

$$\boxed{R_{\alpha\beta\gamma\delta} = K\, (g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma})} \tag{1.4}$$

equivalent to

$$R^{\alpha}_{\beta\gamma\delta} = K\, (\delta^{\alpha}_{\gamma}\, g_{\beta\delta} - \delta^{\alpha}_{\delta}\, g_{\beta\gamma}), \tag{1.5}$$

[3] In general, the constant of curvature will depend on the cosmic time.

[4] A high level reference to this topic is the book [1]. A classical reference for a simpler approach, sufficient for our needs, is the book [7].

[5] Our definition of the Riemann tensor is that of [15] and [7]. On the contrary, our Ricci tensor is the opposite of that of [7].

[6] [7] Section 26.

It follows that

$$\boxed{R_{\alpha\beta} = (n-1)\, K\, g_{\alpha\beta}} \quad \boxed{R = n\,(n-1)\, K} \tag{1.6}$$

where n is the dimension of the manifold and R is the **Ricci scalar** (or **scalar curvature**)

$$\boxed{R \overset{\text{def}}{=} g^{\alpha\beta}\, R_{\alpha\beta}} \tag{1.7}$$

For a conformal transformation $\widetilde{g}_{\alpha\beta} = c\, g_{\alpha\beta}$ with a *constant* factor c the Riemann tensor components are invariant: $\widetilde{R}^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta}$. Hence, from (1.5),

$$\widetilde{K}\,(\delta^{\alpha}_{\gamma}\,\bar{g}_{\beta\delta} - \delta^{\alpha}_{\delta}\,\bar{g}_{\beta\gamma}) = K\,(\delta^{\alpha}_{\gamma}\, g_{\beta\delta} - \delta^{\alpha}_{\delta}\, g_{\beta\gamma}),$$

and consequently $c\,\widetilde{K} = K$. This shows that

$$\boxed{\widetilde{g}_{\alpha\beta} = c\, g_{\alpha\beta} \;\; (c = \text{constant}) \implies \widetilde{K} = \frac{K}{c}} \tag{1.8}$$

With a similar argument one can show that: *two metric tensors on a same manifold $\widetilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$, with the same signature and with constant curvatures of the same sign are conformal $\widetilde{g}_{\alpha\beta} = c\, g_{\alpha\beta}$ with a <u>positive</u> constant conformal factor c.*

1.3 Isotropic vectors and tensors

Definition 1.1 – A tensor on space-time is **isotropic** if it meets the isotropy principle: it does not generates distinguished vector fields tangent to the spatial sections.

Since the isotropy principle is one of the main postulates, only isotropic tensors are **admissible** in the present cosmological theory. The **isotropic scalar** have been considered in Theorem 1.3. Now we characterize the isotropic vectors and two-tensors.

Theorem 1.5 – *A vector field V^{α} is isotropic if and only if its components in a co-moving coordinate system (q^0, q^a) are of the following type:*

$$\begin{cases} V^0 = \text{function of } q^0 \text{ only}, \\ V^a = 0. \end{cases}$$

Proof – If V^0 is also a function of the coordinates $\widetilde{q}$ then its gradient defines a distinguished direction on each spatial section. If $V^a \neq 0$ then a distinguished direction field is defined on each spatial section. ∎

Theorem 1.6 – *A contravariant symmetric two-tensor $T^{\alpha\beta}$ is isotropic if and only if its components are of the following type:*

$$\boxed{\begin{cases} T^{00} = \Phi(q^0) = \text{a function of } q^0 \text{ only}, \\ T^{0a} = 0, \\ T^{ab} = \Psi(q^0)\, \widetilde{g}^{ab}(\widetilde{q}) = \text{a function of } q^0 \text{ times } \widetilde{g}^{ab}, \end{cases}} \tag{1.9}$$

where $[\widetilde{g}^{ab}]$ is the inverse matrix of $[\widetilde{g}_{ab}]$. A similar result holds for an admissible covariant symmetric tensor:

$$\boxed{\begin{cases} T_{00} = \Phi(q^0) = \text{a function of } q^0 \text{ only}, \\ T_{0a} = 0, \\ T_{ab} = \Psi(q^0)\, \widetilde{g}_{ab}(\widetilde{q}) = \text{a function of } q^0 \text{ only times } \widetilde{g}_{ab} \end{cases}} \tag{1.10}$$

Proof – $T^{\alpha\beta} = J^{\alpha}_{\alpha'}\, J^{\beta}_{\beta'}\, T^{\alpha'\beta'}, \quad J^{\alpha'}_{\alpha} \stackrel{\text{def}}{=} \dfrac{\partial q^{\alpha'}}{\partial q^{\alpha}}, \quad J^{\alpha}_{\alpha'} \stackrel{\text{def}}{=} \dfrac{\partial q^{\alpha}}{\partial q^{\alpha'}}.$

For a transformation of co-moving coordinates leaving q^0 invariant we have $J^{0'}_{0} = 1, \quad J^{0'}_{a} = 0, \quad J^{a'}_{0} = 0, \quad J^{0}_{c'} = 0, \quad J^{c}_{0'} = 0.$ Thus

$$\left[\begin{array}{l} T^{00} = J^{0}_{\alpha'}\, J^{0}_{\beta'}\, T^{\alpha'\beta'} = (J^{0}_{0'})^2\, T^{0'0'} = T^{0'0'}. \\ T^{0b} = J^{0}_{\alpha'}\, J^{b}_{\beta'}\, T^{\alpha'\beta'} = J^{0}_{0'}\, J^{b}_{b'}\, T^{0'b'} = J^{b}_{b'}\, T^{0'b'}. \\ T^{ab} = J^{a}_{\alpha'}\, J^{b}_{\beta'}\, T^{\alpha'\beta'} = J^{a}_{a'}\, J^{b}_{b'}\, T^{a'b'}. \end{array}\right.$$

This shows that: T^{00} is a scalar, so it must be a function of q^0 only; T^{0b} is a vector, so it must vanish; T^{ab} is a symmetric tensor on each spatial section generating eigenfields, so it must be proportional to the spatial metric $g_{ab}(q^0, \widetilde{q}) = A^2(q^0)\, \widetilde{g}_{ab}(\widetilde{q})$. ∎

Remark 1.3 – According to this theorem any admissible symmetric two-tensor $T^{\alpha\beta}$ is fully determined by two functions Φ and Ψ of q^0 only. •

Theorem 1.7 – *No skew-symmetric isotropic two-tensor is admissible in the cosmic space-time.*[7]

Proof – A skew-symmetric two-tensor $A^{\alpha\beta}$ gives rise to a spatial vector field A^{0a} and to a spatial skew-symmetric tensor field A^{ab}. The isotropy principle implies $A^{0a} = 0$. Any antisymmetric tensor field A^{ab} on a three-dimensional Riemannian space has a real eigenvector. This is in contrast with the isotropy principle. Thus $A^{ab} = 0$. ∎

[7] In other words: *only symmetric two-tensors are admissible in the isotropic models of the universe.*

Remark 1.4 – Theorem 1.7 shows that the existence in space-time of a single or a finite number of electro-magnetic fields is incompatible with the isotropic cosmology. However, a continuous distribution of electro-magnetic fields such as those emitted from galaxies, may not generate any particular direction. Such fields are then admissible. •

Remark 1.5 – Since the torsion $\Gamma^{\gamma}_{\alpha\beta} - \Gamma^{\gamma}_{\beta\alpha}$ of a linear connection is a skew-symmetric tensor in the lower indices, only symmetric connections are allowed in space-time. •

1.4 Dimensional analysis

To test the correctness of the formulas that we will write, it is important to consider the physical dimension of the involved objects. We will denote the physical dimension of an object X by the symbol $\mathrm{Dim}\,(X)$.[8] The basic physical dimensions are

$$\begin{cases} \mathrm{Dim}\,(\text{time}) = T \\ \mathrm{Dim}\,(\text{length}) = L \\ \mathrm{Dim}\,(\text{mass}) = M \\ \mathrm{Dim}\,(\text{dimensionless quantity}) = 1 \end{cases}$$

Then the dimension of any object X will be expressed by the product of positive or negative integer powers of these simbols,

$$\mathrm{Dim}\,(X) = T^a\, L^b\, M^c, \quad a, b, c \in \mathbb{Z}.$$

For instance:

Object	Dim
area	L^2
volume	L^3
velocity	$L\,T^{-1}$
acceleration	$L\,T^{-2}$
angle	1
angular velocity	T^{-1}

Table 1.4

[8] The symbol $[X]$ is more commonly used.

Object	Symbol	Dim	Note
force	F	$M\,L\,T^{-2}$	mass $\times$ acceleration
pressure	P	$M\,L^{-1}\,T^{-2}$	force/area
energy (work)	E	$M\,L^{2}\,T^{-2}$	work = force $\times$ length
energy density	ϵ	$M\,L^{-1}\,T^{-2}$	energy/volume
mass density	ρ	$M\,L^{-3}$	mass/volume

Table 1.4

The coordinates of a manifold can be dimensionless (e.g. angles) or with a physical dimension (time, length,...). Concerning the co-moving coordinates introduced by Theorem 1.2 we assume that

> the cosmic time t is time-dimensional: $\mathrm{Dim}\,(t) = T$.
> the spatial coordinates q^a are length-dimensional: $\mathrm{Dim}\,(q^a) = L$.

Then it will be convenient to replace the time coordinate t with a length-dimensional coordinate q^0 via the simple relationship

$$\boxed{q^0 = \kappa\, t} \tag{1.11}$$

where κ is a constant with the dimension of a velocity: $\mathrm{Dim}\,(\kappa) = LT^{-1}$. It is immaterial the numerical value of this constant.[9] In this way we get **length-dimensional co-moving coordinates**.

> Henceforth we will always refer to
> length-dimensional co-moving coordinates.

Consequently:

$$\boxed{\begin{aligned}&\mathrm{Dim}\,(g_{\alpha\beta}) = \mathrm{Dim}\,(g^{\alpha\beta}) = 1.\\ &\mathrm{Dim}\,(\Gamma_{\alpha\beta,\gamma}) = \mathrm{Dim}\,(\Gamma^{\gamma}_{\alpha\beta}) = L^{-1}.\\ &\mathrm{Dim}\,(R^{\nu}{}_{\alpha\mu\beta}) = L^{-2}.\\ &\mathrm{Dim}\,(R_{\alpha\beta}) = \mathrm{Dim}\,(R) = L^{-2}.\end{aligned}} \tag{1.12}$$

[9] Later, when dealing with cosmic dynamics, we shall be led to consider $\kappa = c$, the light speed.

Remark 1.6 – Do not pay attention to the dimension of the coordinates is unfortunately a widespread harmful habit, which produces confusion and mistakes in the writing and interpreting of the tensorial-type formulas. For example, if you use angular coordinates (for instance on a sphere or hypersphere with fixed radius) which are dimensionless, then $\mathrm{Dim}\,(g_{\alpha\beta}) = L^2$ and consequently,

$$\begin{cases} \mathrm{Dim}\,(g^{\alpha\beta}) = L^{-2}, \\ \mathrm{Dim}\,(\Gamma_{\alpha\beta,\gamma}) = L^2, \\ \mathrm{Dim}\,(\Gamma^{\gamma}_{\alpha\beta}) = 1, \end{cases} \qquad \begin{cases} \mathrm{Dim}\,(R^{\nu}{}_{\alpha\mu\beta}) = 1, \\ \mathrm{Dim}\,(R_{\alpha\beta}) = 1, \\ \mathrm{Dim}\,(R) = L^{-2}. \end{cases}$$

(Note that the Ricci scalar still maintains its dimension L^{-2}, as should be.) But if you use coordinates of mixed type (dimensionless, length-dimensional, time-dimensional,...) then the dimension of the objects listed above depends on the indices. This creates a big mess. •

1.5 Scale factor and quotient metric

Theorem 1.8 – *There exists a two-variable function $a(t_1, t_2)$ such that*

$$\boxed{g(t_1) = a^2(t_1, t_2)\, g(t_2)} \tag{1.13}$$

where $g(t_1)$ and $g(t_2)$ are the metric tensors on the spatial sections S_{t_1} and S_{t_2}.

Proof – Each spatial section S_t has a constant curvature $K(t)$. Since two spatial sections S_{t_1} and S_{t_2} are diffeomorphic, the constant curvatures $K(t_1)$ and $K(t_2)$ have the same sign (or are both equal to 0). In accordance with what has been said at the end of Section 1.2, the metrics $g(t_1)$ and $g(t_2)$ are conformal with a positive constant conformal factor a^2: $g(t_1) = a^2\, g(t_2)$. This 'constant' is actually a dimensionless positive function $a^2(t_1, t_2)$ of the two dates t_1 and t_2. ■

We assume, without loss of generality, that this function is positive. From the definition (1.13) it follows that it obeys the following rules:[10]

$$\boxed{\begin{aligned} & a(t,t) = 1 \\ & a(t_1, t_2)\, a(t_2, t_3) = a(t_1, t_3) \quad \text{(composition rule)} \\ & a(t_1, t_2) = \frac{1}{a(t_2, t_1)} \end{aligned}} \tag{1.14}$$

[10] Note that it is not commutative in (t_1, t_2). The use of the scale factor must be attentive to the location of the two variables (t_1, t_2).

If we fix a **reference time** $t_\sharp$ then by virtue of Theorem 1.1 (page 4) we can take the spatial section $S_{t_\sharp}$ as a representative of the quotient manifold Q. Then Q is endowed with the **quotient metric**

$$\boxed{g^\sharp \overset{\text{def}}{=} g(t_\sharp)} \tag{1.15}$$

By applying (1.13) we get

$$\boxed{g(t) = a^2(t, t_\sharp)\, g^\sharp} \tag{1.16}$$

This **factorization equation** will be widely applied in the following. An equivalent form of this equation is

$$\boxed{ds_t = a(t, t_\sharp)\, ds} \tag{1.17}$$

where ds_t and ds are the arc elements of $g(t)$ and of $g^\sharp$, respectively.

We call **scale factor** the positive function $a(t, t_\sharp)$.[11] It has to be regarded as a function of t, but determined by the preset value of $t_\sharp$. When there is no danger of confusion, we can simply denote it by $a(t)$,

$$\boxed{a(t) \overset{\text{def}}{=} a(t, t_\sharp)} \tag{1.18}$$

Note that $a(t_\sharp) = a(t_\sharp, t_\sharp) = 1$. For this reason $t_\sharp$ will be called **normalization time**.

Remark 1.7 – The composition rule in (1.14) implies that if $a(t_1, t_2)$ is constant then this constant is necessarily equal to 1. For $a(t) = \text{constant} = 1$ we have the so-called **static universe**: $g(t) = g^\sharp$ for all t. •

Remark 1.8 – The scale factor plays a central role in cosmology because, as a function of the cosmic time, it contains all the information concerning the **evolution of the universe**. It is determined by dynamical equations established in the next chapter. •

Remark 1.9 – In the current literature it is not sufficiently emphasized the fact that the scale factor $a(t)$ is defined up to the choice of a reference time. This oversight precludes the recognition of some important facts. It should be borne in mind that, in order to have a physical meaning, any formula involving $a(t)$ and its derivatives must be invariant under the change of a reference time. •

[11] It is also called **scale parameter** or **expansion factor**. It is also denoted by the symbol $R(t)$, here used for the radius of the universe.

Remark 1.10 – Let $a(t, t_\sharp)$ and $a(t, t_\flat)$ be the scale parameters associated with two different reference times $t_\sharp$ and $t_\flat$. By applying the composition rule (1.14) we get

$$\boxed{a(t, t_\sharp) = a(t, t_\flat)\, a(t_\flat, t_\sharp)} \tag{1.19}$$

This shows that the scale factors associated with two different reference dates differ by a constant factor (depending on the reference dates). •

Remark 1.11 – The metric tensor $g(t) = a^2(t, t_\sharp)\, g(t_\sharp)$ does not depend, by definition, on the choice of the reference time $t_\sharp$, so that

$$a^2(t, t_\sharp)\, g(t_\sharp) = a^2(t, t_\flat)\, g(t_\flat) \tag{1.20}$$

For $t = t_\sharp$ we get the relation

$$\boxed{g(t_\sharp) = a^2(t_\sharp, t_\flat)\, g(t_\flat)} \tag{1.21}$$

which is in agreement with (1.16). •

Remark 1.12 – We can write the factorization equation (1.16) in terms of the coordinate $q^0 = \kappa\, t$ by setting

$$\boxed{a(t) = A(q^0)} \tag{1.22}$$

$$\boxed{g_{ab}(q^0, \widetilde{q}) = A^2(q^0)\; g^\sharp_{ab}(\widetilde{q})} \tag{1.23}$$

As a rule, we will use a small letter to denote a scalar function of t and the corresponding capital letter to denote the same scalar as a function of q^0: $f(t) = F(q^0)$. •

Remark 1.13 – The quotient manifold is endowed with the Levi-Civita connection coming from the quotient metric, henceforth denoted by $\widetilde{\Gamma}$, whose Christoffel symbols are

$$\widetilde{\Gamma}^c_{ab} \stackrel{\text{def}}{=} \tfrac{1}{2}\, g^{\sharp cd}\,(\partial_a g^\sharp_{bd} + \partial_b g^\sharp_{da} - \partial_d g^\sharp_{ab}).$$

It is easy to check that these symbols are invariant under the change of the reference date. It follows that the $\widetilde{\Gamma}$-geodesics, as unparametrized curves, are not affected by the change of the reference time. •

Remark 1.14 – Since the co-moving coordinates q^a are L-dimensional (Section 1.4) the metric tensor components $g^\sharp_{ab}$ are dimensionless:

$$\begin{cases} \mathrm{Dim}\,(g^{\sharp\, ab}) = 1. \\ \mathrm{Dim}\,(\widetilde{\Gamma}^c_{ab}) = L^{-1}. \end{cases} \quad \bullet$$

Remark 1.15 – On the quotient manifold the **arc-element** ds of any curve $q^a = \gamma^a(\xi)$ (with generic parameter ξ) is defined by

$$ds \stackrel{\text{def}}{=} \sqrt{g^\sharp_{ab}\, dq^a\, dq^b} = \sqrt{g^\sharp_{ab}\, \frac{d\gamma^a}{d\xi}\, \frac{d\gamma^b}{d\xi}}\; d\xi. \tag{1.24}$$

In turn, the **arc-length** s is defined by

$$s(\xi_1) - s(\xi_0) \stackrel{\text{def}}{=} \int_{\xi_0}^{\xi_1} ds = \int_{\xi_0}^{\xi_1} \sqrt{g^\sharp_{ab}\, \frac{d\gamma^a}{d\xi}\, \frac{d\gamma^b}{d\xi}}\; d\xi. \tag{1.25}$$

It is a privileged parameter for any curve on Q, since

$$\boxed{g^\sharp_{ab}\, \frac{d\gamma^a}{ds}\, \frac{d\gamma^b}{ds} = 1} \tag{1.26}$$

This follows from (1.24)

$$\frac{ds}{d\xi} = \sqrt{g^\sharp_{ab}\, \frac{d\gamma^a}{d\xi}\, \frac{d\gamma^b}{d\xi}}$$

and

$$g^\sharp_{ab}\, \frac{d\gamma^a}{ds}\, \frac{d\gamma^b}{ds} = g^\sharp_{ab}\, \frac{d\gamma^a}{d\xi}\, \frac{d\gamma^b}{d\xi} \left(\frac{d\xi}{ds}\right)^2 = 1.$$

However, the arc-element and the arc-length are not invariant under the change of the reference date. If we denote by $ds_\sharp$ and ds_* the arc-elements defined in the quotient metrics $g^\sharp = g(t_\sharp)$ and $g^\flat = g(t_\flat)$, then equation (1.21) is equivalent to

$$\boxed{ds_\sharp = \frac{1}{a(t_\sharp, t_\flat)}\, ds_*} \quad \bullet \tag{1.27}$$

1.6 The Hubble law

Any curve in the quotient manifold, of length $\ell_\sharp$, is carried along the galactic world-lines and generates curves of length $\ell(t)$ on each spatial section S_t. By virtue of (1.17), this length is given by

$$\boxed{\ell(t) = a(t, t_\sharp)\, \ell_\sharp} \tag{1.28}$$

It follows that

$$\dot{\ell}_t = \dot{a}(t, t_\sharp)\, \ell_\sharp. \tag{1.29}$$

Theorem 1.9 – *The derivative $\dot{\ell}(t)$*[12] *obeys the rule*

$$\boxed{\dot{\ell}(t) = h(t)\, \ell(t)} \tag{1.30}$$

where the ratio

$$\boxed{h(t) \overset{\text{def}}{=} \frac{\dot{a}(t, t_\sharp)}{a(t, t_\sharp)}} \tag{1.31}$$

does not depend on the reference time $t_\sharp$.

Proof – From (1.29) and (1.28) we get

$$\dot{\ell}(t) = \frac{\dot{a}(t, t_\sharp)}{a(t, t_\sharp)}\, \ell(t). \tag{1.32}$$

The scale factors relative to different normalization times differ by a multiplicative constant. So, the ratio (1.31) remains unchanged. ■

This argument can be applied to the case of geodesic curves, where the 'lengths' are 'distances'. In particular, we can consider the distance between galaxies.

We can define two distances between two galaxies A and B:

(i) The **co–moving distance** $\ell^\sharp_{AB}$: this is the length of the geodesic joining A to B in the quotient metric $g^\sharp$. This distance is a constant depending on A and B only (Fig. 1.5).

(ii) The **synchronous distance** $\ell_{AB}(t)$ at the time t: this is the length of the shortest geodesic joining A to B in the spatial section S_t with metric $g(t)$. The derivative $\dot{\ell}_{AB}(t)$ with respect to t of the synchronous distance $\ell_{AB}(t)$ is called the **recession velocity** of the galaxies A and B.

By applying equations (1.28) and (1.30) to these distances we find the two relationships

$$\boxed{\ell_{AB}(t) = a(t_\sharp)\, \ell^\sharp_{AB}} \tag{1.33}$$

$$\boxed{\dot{\ell}_{AB}(t) = h(t)\, \ell_{AB}(t)} \tag{1.34}$$

In the second of these we recognize the famous **Hubble law**, and in $h(t)$ the **Hubble parameter** (not affected by the choice of $t_\sharp$).

[12] A dot over a symbol will denote the derivative with respect to t.

Remark 1.16 – In our theory the Hubble law is a matter of pure Kinematics: it is a consequence of the postulates so far stated for the structure of the cosmic space-time. In other words, *it does not depend on whatsoever dynamical equations*, like Einstein equations, Newton equations... •

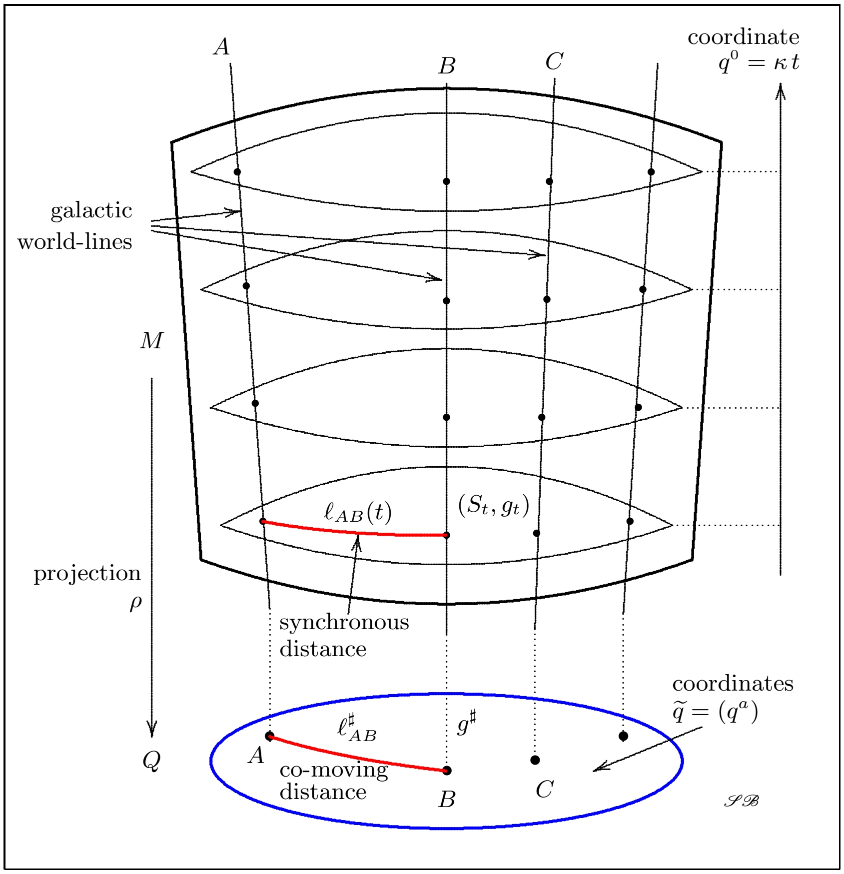


Figure 1.5: The synchronous distance $\ell_{AB}(t)$
and the co-moving distance $\ell^\sharp_{AB}$.

1.7 The cosmic radius and the angular distance

For cosmological models with non-flat spatial sections ($K(t) \neq 0$) we define the **cosmic radius** $R(t) > 0$ by

$$\boxed{K(t) = \frac{\varepsilon}{R^2(t)}, \quad \varepsilon = \pm 1} \tag{1.35}$$

with $\varepsilon = \pm 1$ according to the sign of the curvature. If we set

$$K_\sharp \overset{\text{def}}{=} K(t_\sharp), \quad R_\sharp \overset{\text{def}}{=} R(t_\sharp), \tag{1.36}$$

then we have[13]

$$\boxed{\varepsilon\, K_\sharp = \frac{1}{R_\sharp^2}} \tag{1.37}$$

and, due to the general formula (1.28), we have

$$\boxed{R(t) = a(t, t_\sharp)\, R_\sharp} \tag{1.38}$$

$$\boxed{K(t) = \frac{K_\sharp}{a^2(t, t_\sharp)}, \quad \forall\, t} \tag{1.39}$$

By putting in particular $t = t_\flat$ in these equations we get

$$\boxed{R(t_\flat) = a(t_\flat, t_\sharp)\, R_\sharp} \tag{1.40}$$

$$\boxed{K(t_\flat) = \frac{K_\sharp}{a^2(t_\flat, t_\sharp)}} \tag{1.41}$$

Furthermore, from $h = \dot{a}/a$ we get

$$\boxed{h(t) = \frac{\dot{R}(t)}{R(t)} = \frac{d \log R(t)}{dt}} \tag{1.42}$$

This formula gives the Hubble parameter in terms of the cosmic radius. We call $\dot{R}$ the **radial velocity** of the universe. Finally, from (1.33) and (1.38) it follows that for any pair of galaxies

$$\boxed{\psi_{AB} \overset{\text{def}}{=} \frac{\ell_{AB}^\sharp}{R_\sharp} = \frac{\ell_{AB}(t)}{R(t)} = \text{constant}} \tag{1.43}$$

We call **angular distance** of two galaxies the dimensionless constant ψ_{AB}. Its geometrical meaning is explained in the next section. As a consequence, the Hubble law can be written as

$$\boxed{\dot{\ell}_{AB}(t) = \psi_{AB}\, \dot{R}(t)} \tag{1.44}$$

Note that the angular distance ψ_{AB} does not depend on the choice of the reference date $t_\sharp$.

[13] Note that the product $\varepsilon\, \widetilde{K}$ is always positive.

1.8 Topological types of cosmological models

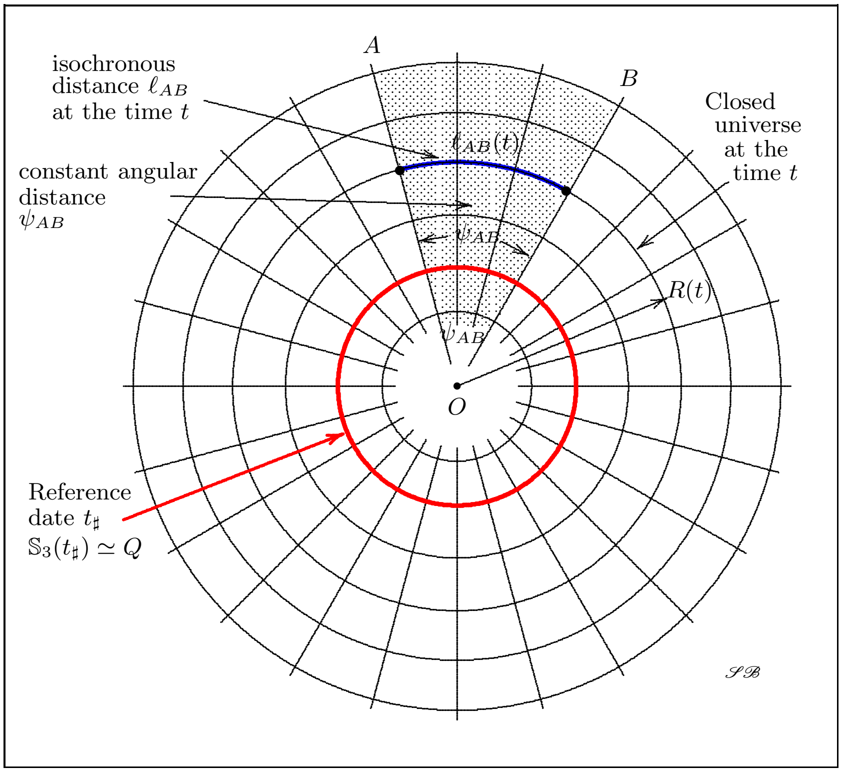


Figure 1.6: Radial diagram of the closed universe.

The isotropic cosmological models that can be constructed on the basis of the postulates so far stated have two characteristic elements: (i) the scale parameter $a(t)$, a function of the cosmic time t defined on a real (bounded or unbounded) interval (t_α, t_ω) (beginning and end of the universe) which contains all the information about the evolution of the cosmos (expansion, contraction, etc.) and (ii) the sign of the constant curvature $\widetilde{K}$ of the quotient manifold. However, it must be observed that this description is incomplete because it lacks a third characteristic element: the topology of the quotient manifold (that is the topology of the spatial sections).

There are several topological types of three-dimensional manifolds with constant curvature. The types that are commonly considered are the following three ones:

(i) $Q \simeq \mathbb{S}_3$, the three-dimensional sphere, positive curvature,

(ii) $Q \simeq \mathbb{H}_3$, the three-dimensional pseudo-sphere , negative curvature,

(iii) $Q \simeq \mathbb{E}_3$, the three-dimensional Euclidean space, zero curvature.

The corresponding models are called (i) **closed universe**, (ii) **open universe** and (iii) **flat universe**.

1.8.1 Closed universe

In a closed universe the spatial sections S_t are three-dimensional sphere of radius $R(t)$ immersed in the Euclidean affine space $\mathbb{R}^4 = (w, x, y, z)$ and centered at the origin O of the coordinates; they represented by the equation

$$w^2 + x^2 + y^2 + z^2 = R^2(t).$$

Their curvature is $K(t) = R^{-2}(t) > 0$. This geometrical vision differs from that of the space-time where the submanifolds S_t form a foliation. Here they contract and expand in time according to the function $R(t)$. What we get is a sort of movie which we call **radial diagram** (Fig. 1.6). One of the spheres, corresponding to a reference date $t_\sharp$, can be identified with the quotient manifold Q. Any galaxy is represented by a point moving along a straight line crossing the origin O. At any date t two galaxies A and B stay on the sphere of radius $R(t)$ and are separated by a circular arc of maximal radius (i.e. by a geodesic arc) of length $\ell_{AB}(t)$. Then the straight lines joining A and B to the center O form, in agreement with (1.43), an angle ψ_{AB} such that $\ell_{AB}(t) = \psi_{AB}\, R(t)$, which remains constant in time. The maximal distance between two galaxies is $\pi\, R$, the half of the length $2\,\pi\, R$ of a maximal (geodesic) circle. Then the maximal angular distance is $\psi_{\max} = \pi$.

1.8.2 Open universe

In this model the spatial sections S_t are three-dimensional hyperboloid $\mathbb{H}_3$ of radius R_t

$$w^2 - x^2 - y^2 - z^2 = R^2(t)$$

immersed in the Minkowski affine space $M_4 = (w, x, y, z)$ with signature $(-+$ $++)$. The hyperboloid $\mathbb{H}_3$ is the set of points P such that the vector OP is time-like and with positive component with respect the time-like coordinate w. It results to be a space-like three-dimensional surface whose curvature at the date t is $K(t) = -R^{-2}(t) < 0$ (Fig. 1.7). Remarks similar to those concerning the closed universe hold for the open universe. The only difference is that in the open universe the angular distance is not bounded.

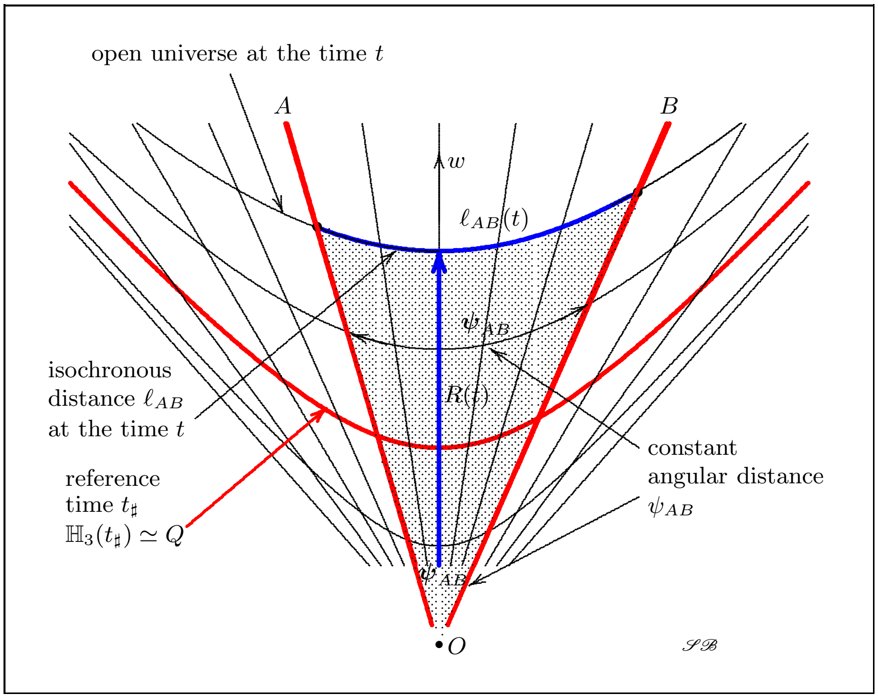


Figure 1.7: Radial diagram of the open universe.

1.8.3 Flat universe

In this model the concept of radius of the universe does not make sense. So we must refer to the scale parameter $a(t)$ only. The spatial sections are the three-dimensional planes $w =$ constant immersed in the Euclidean affine space $\mathbb{R}^4 = (w, x, y, z)$.

1.9 Co-moving volumes and conserved densities

Let U be a domain in the quotient manifold Q with a finite volume

$$\widetilde{\mathcal{V}}(U) = \int_U \sqrt{\det[g^\sharp_{ab}]}\ dq^1 \wedge dq^2 \wedge dq^3.$$

Carried along the world-lines, U generates domains $U_t \subset S_t$ with finite volumes

$$\mathcal{V}(U, t) = \int_{U_t} \sqrt{\det[g_{ab}(t, \widetilde{q})]}\ dq^1 \wedge dq^2 \wedge dq^3.$$

Theorem 1.10 – *The ratio $\mathcal{V}(U,t)/a^3(t)$ does not depend on t:*

$$\boxed{\frac{\mathcal{V}(U,t)}{a^3(t)} = \widetilde{\mathcal{V}}(U) = \text{constant} > 0} \tag{1.45}$$

Proof – The factorization $g_{ab}(t,\widetilde{q}) = a^2(t)\, g^\sharp_{ab}(\widetilde{q})$ implies

$$\begin{aligned}\mathcal{V}(U,t) &= \int_{U_t} \sqrt{\det[g_{ab}(t,\widetilde{q})]}\; dq^1 \wedge dq^2 \wedge dq^3 \\ &= a^3(t) \int_{U_t} \sqrt{\det[g^\sharp_{ab}(\widetilde{q})]}\; dq^1 \wedge dq^2 \wedge dq^3 = a^3(t)\, \widetilde{\mathcal{V}}(U). \quad \blacksquare\end{aligned}$$

Theorem 1.11 – *For any scalar function of the cosmic time $\mu(t)$ the following equations are equivalent,*

$$\begin{gathered}\boxed{\mu(t)\,\mathcal{V}^u(U,t) = \text{const.}\;\; \forall\, U} \iff \boxed{\mu(t)\; a^{3u}(t) = \text{const.}} \\ \iff \boxed{a\,\dot{\mu} + 3\,u\,\mu\,\dot{a} = 0} \iff \boxed{h = -\tfrac{1}{3u}\,\frac{\dot{\mu}}{\mu}}\end{gathered} \tag{1.46}$$

When these equations are satisfied we say that $\mu(t)$ is a **conserved density of order** u.

Proof – Write equation (1.45) as $\mathcal{V}(U,t) = \widetilde{\mathcal{V}}(U)\, a^3(t)$. Then the condition $\mu(t)\,\mathcal{V}^u(U,t) = \text{const.}$ is equivalent to $\mu(t)\,\widetilde{\mathcal{V}}^u(U)\, a^{3u}(t) = \text{const.}$, hence to $\mu(t)\, a^{3u}(t) = \text{const.}$ By differentiation we get the second equivalence. The last equivalence is due to the definition of the Hubble factor: $h = \dot{a}/a$. $\blacksquare$

The typical case is the mass (or matter) density for which $u = 1$. But there are other density of physical interest for which $u \neq 1$ (see Chapter 5).

1.10 Cosmic monitor and free particles

The quotient manifold will play a crucial role in the sequel. In order to make such an abstract concept more accessible we can think of it as a **cosmic monitor** whose *pixels*, which are bright fixed points, represent the galaxies. Of course we need an effort of imagination because such a monitor is neither flat nor two-dimensional: it is a (possibly curved) three-dimensional screen. On the cosmic monitor there is also a clock showing the cosmic time t.

Imagine a very special person, the **cosmic watcher**, sitting in front of (or better, sitting inside) the monitor. Since the cosmic monitor is endowed with a metric (the quotient metric $g^\sharp$) the cosmic watcher is able to recognize distinguished curves, called **geodesics**, connecting any pair of galaxies with a minimal (or stationary) distance. Then the cosmic watcher is able to measure the co-moving distance $\ell^{\sharp}_{AB}$ of two galaxies (Fig. 1.8).

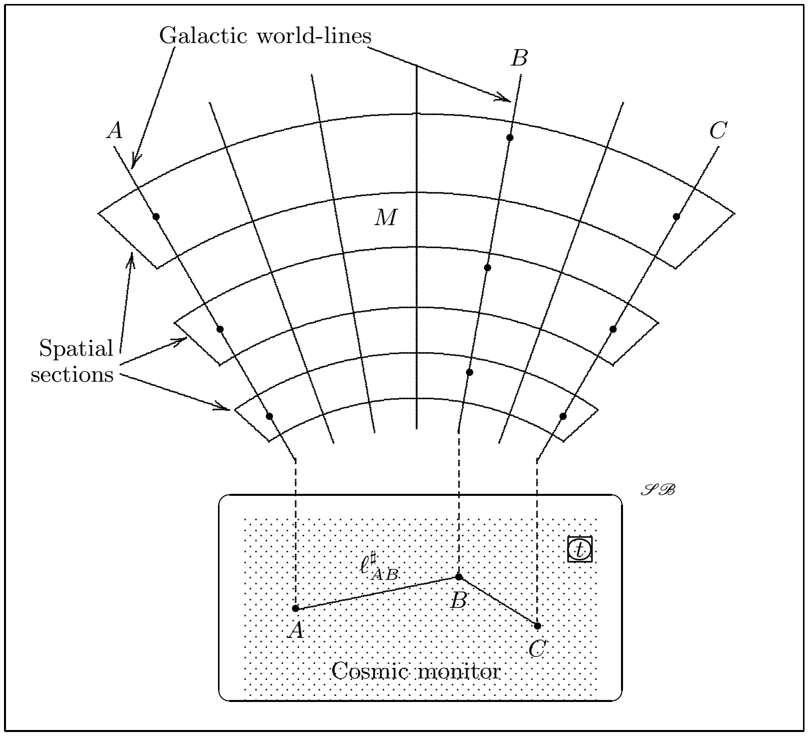


Figure 1.8: The cosmic space-time and the cosmic monitor.

> **6^{th} Postulate**. There are bodies, other than galaxies, running in the universe. We call them **particles**. The life of a particle is represented by a world-line in space-time transversal to the spatial sections.

The cosmic watcher cannot see the world-line $\gamma(t)$ of a particle. What he can see on the monitor is the projection $\widetilde{\gamma}(t)$ of $\gamma(t)$. He looks at $\widetilde{\gamma}(t)$ as the **motion of a point**. Then he can measure the **monitor speed** $v(t)$

$$\boxed{v(t) = \frac{ds}{dt}} \tag{1.47}$$

where s is the arc-length along $\widetilde{\gamma}(t)$. The concept of monitor speed will play a very important role in the following.

Among the various curves $\widetilde{\gamma}(t)$ that the cosmic watcher can see there are also geodesics. He argues that they represent distinguished particles whose world lines in space-time have some special features. But he cannot claim that these world-lines are geodesics because he does not know if in space-time there is any special metric or a connection. Anyway, he proposes the following formal definition:

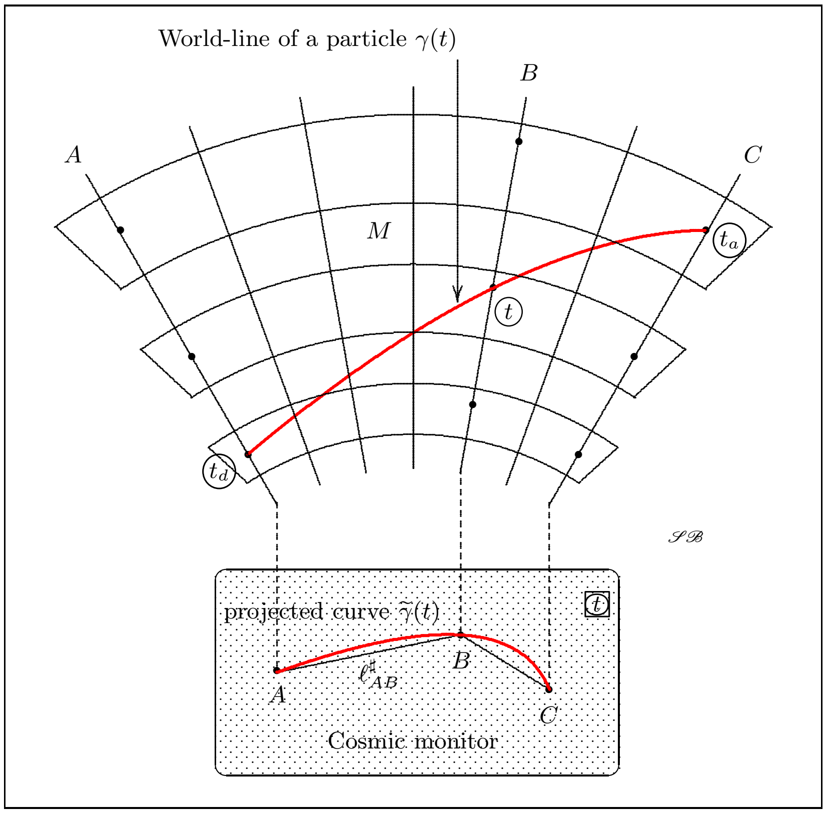


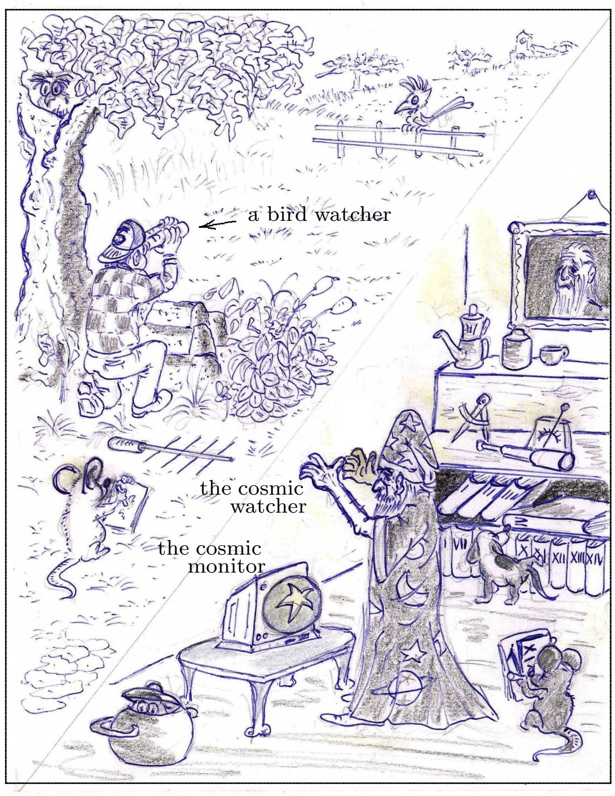
Figure 1.9: Particle observed on the cosmic monitor.

Definition 1.2 – A **free particle** (or **free-falling particle**) is a particle whose monitor motion $\widetilde{\gamma}(t)$ is a geodesic. No matter if the cosmic time t is an affine parameter or not.[14]

Thus, we are encouraged to investigate on the existence of linear connections on space-time that are, in a sense, *adapted* to the geometric structures so far determined by the postulates – the spatial foliation, the spatial metrics, the congruence of the galactic world-lines, the quotient manifold and the quotient metric. This will be done in Chapter 2. At the end of our analysis we will

[14] In fact this distinction will be subject to a postulate (page 37).

be faced with two types of connections (Sections 2.5 and 2.6) which open two distinct ways for developing the cosmic dynamics: the Newtonian way and the relativistic way.



The cosmic monitor and the cosmic watcher
(interpreted by my schoolmate Tony Magala)

1.11 Local reference frames and peculiar velocity

In order to locate an event in space-time we need to know where it occurs and when. To do this we need to assign a **reference frame** made of the congruence of world-lines of particles of an ideal 'body' and of a transversal foliation S_t of simultaneous events parametrized by a 'time' t. Then we can say that a certain event occurs in a point of a certain 'body' and at a certain 'date' t.

In cosmology we have a **privileged reference frame**: the galactic world-lines and the cosmic time t. Note that this is similar to what happens in

Newtonian space-time (which is an affine space): there is an absolute time t, a foliation S_t made of three-dimensional affine subspaces, and a class of equivalence of reference frames, whose world-lines are parallel straight lines, called **inertial frames**. Instead, in Special Relativity i.e., in the Minkowski space-time (which is still an affine space) there is not a privileged time but a class of equivalence of reference frames, whose world-lines are parallel straight lines, still called **inertial frames**. Each one of them determines an orthogonal foliation of three-dimensional affine subspaces, hence a time t which is 'relative' to the frame.

Consider now the intergalactic journey of a particle P from a galaxy A to a galaxy C. Let t_d and t_a be the dates of departure and arrival (see Fig. 1.9 on page 23). At a date $t \in (t_d, t_a)$ the particle P crosses a galaxy B_t. Then we are faced with the following data:

$$\begin{cases} \ell_{AB_t}(t) & = & \text{the isochronous distance from } A \text{ to } B_t \text{ measured} \\ & & \text{at the actual time } t. \\ \ell^\sharp_{AB_t}(t) & = & \text{the co-moving distance from } A \text{ to } B_t \text{ measured} \\ & & \text{by the cosmic watcher.} \\ a(t) & = & \text{the scale parameter, unknown to the cosmic watcher.} \end{cases}$$

They are linked by equation (1.33)

$$\ell_{AB_t}(t) = a(t)\,\ell^\sharp_{AB_t}(t).$$

It follows that $\dot{\ell}_{AB_t}(t) = \dot{a}(t)\,\ell^\sharp_{AB_t}(t) + a(t)\,\dot{\ell^\sharp}_{AB_t}(t) \implies$

$$\text{(1.48)} \qquad \dot{\ell}_{AB_t}(t) = h(t)\,\ell_{AB_t}(t) + a(t)\,\dot{\ell^\sharp}_{AB_t}(t), \quad h(t) \stackrel{\text{def}}{=} \frac{\dot{a}(t)}{a(t)}.$$

These formulas are quite similar to the composition law of velocities in classical mechanics. The galaxy B_t can be interpreted as a local reference frame that moves with respect to the main (fixed) reference frame A. Then the first term $\dot{\ell}_{AB_t}$ is the **absolute velocity** of the particle P (the velocity with respect to the main frame A). The second term $\dot{a}\,\ell^\sharp_{AB_t} = h\,\ell_{AB_t}$ plays the role of **dragging velocity**. The third term $a\,\dot{\ell^\sharp}_{AB_t}$ is the **relative velocity**, that is the velocity of the particle with respect to the moving frame B_t. Hence equation (1.48) can be read as

$$\begin{cases} \text{absolute velocity of } P \text{ w. r. to the frame } A \\ = \text{dragging velocity of the galaxy } B_t \\ + \text{ relative velocity of } P \text{ w. r.}\widehat{\text{to}} \text{ the frame } B_t. \end{cases} \iff \begin{cases} \dot{\ell}_{AB_t} \\ = \dot{a}\,\ell^\sharp_{AB_t} \\ + a\,\dot{\ell^\sharp}_{AB_t}. \end{cases}$$

In cosmology these three velocities are called **total velocity**, **recession ve-**

locity and **peculiar velocity**, respectively:

$$\boxed{\left\{\begin{array}{l}\text{total velocity } v_{\mathrm{tot}}(P/A)\\ = \text{recession velocity } v_{\mathrm{rec}}(B_t/A)\\ + \text{peculiar velocity } v_{\mathrm{pec}}(P/B_t)\end{array}\right. \iff \left\{\begin{array}{l}\dot{\ell}_{AB_t}\\ = \dot{a}\,\ell^{\sharp}_{AB_t}\\ + a\,\dot{\ell^{\sharp}}_{AB_t}\end{array}\right.} \tag{1.49}$$

Here the slash symbol / stands for *with respect to.* The total velocity pertains the galaxy A and the particle P when it crosses the galaxy B_t at the date t. The recession velocity pertains the two galaxies A and B_t only. The peculiar velocity pertains the particle P and the galaxy B_t only. The cosmic watcher is able to measure $\ell^{\sharp}_{AB_t} = s(t) - s(t_d)$ and $\dot{\ell^{\sharp}}_{AB_t} = \dot{s}(t)$ only.

$$\boxed{v_{\mathrm{rec}}(B_t/A) = h(t)\,\ell_{AB_t} = \dot{a}(t)\,\big[s(t) - s(t_d)\big]} \tag{1.50}$$

$\dot{a}(t)$ unknown to the cosmic watcher

$s(t)$ known to the cosmic watcher

$$\boxed{v_{\mathrm{pec}}(P/B_t) = a(t)\,\dot{\ell^{\sharp}}_{AB_t} = a(t)\,\frac{ds}{dt}} \tag{1.51}$$

$a(t)$ unknown to the cosmic watcher

$\dot{s}(t)$ known to the cosmic watcher

This last formula provides the definition of **peculiar velocity** of a particle.

Chapter 2

Cosmic connections

2.1 Preamble

A linear connection Γ on space-time is locally determined by its **symbols** $\Gamma^{\gamma}_{\alpha\beta}$ in any given coordinate system. Due to the isotropy principle only symmetric connections, for which $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$, are admissible.[1] In homogeneous co-moving coordinates (q^0, q^a) we classify the symbols according to the number of the 0–indices:

$$
\begin{cases}
\Gamma^{0}_{00}, \\
\Gamma^{c}_{00}, \quad \Gamma^{0}_{a0} = \Gamma^{0}_{0a}, \\
\Gamma^{c}_{a0} = \Gamma^{c}_{0a}, \quad \Gamma^{0}_{ab} = \Gamma^{0}_{ba}, \\
\Gamma^{c}_{ab} = \Gamma^{c}_{ba}.
\end{cases}
\tag{2.1}
$$

Then the parallel transport equations of a vector $v^{\alpha}(\xi)$ along any curve $q^{\alpha} = \gamma^{\alpha}(\xi)$,

$$
\frac{dv^{\gamma}}{d\xi} + \Gamma^{\gamma}_{\alpha\beta}\, v^{\alpha}\, \frac{d\gamma^{\beta}}{d\xi} = 0,
$$

separate into the system

$$
\begin{cases}
\dfrac{dv^{0}}{d\xi} + \Gamma^{0}_{00}\, v^{0}\, \dfrac{d\gamma^{0}}{d\xi} + \Gamma^{0}_{a0} \left(v^{a}\, \dfrac{d\gamma^{0}}{d\xi} + v^{0}\, \dfrac{d\gamma^{a}}{d\xi} \right) + \Gamma^{0}_{ab}\, v^{a}\, \dfrac{d\gamma^{b}}{d\xi} = 0, \\
\dfrac{dv^{c}}{d\xi} + \Gamma^{c}_{00}\, v^{0}\, \dfrac{d\gamma^{0}}{d\xi} + \Gamma^{c}_{a0} \left(v^{a}\, \dfrac{d\gamma^{0}}{d\xi} + v^{0}\, \dfrac{d\gamma^{a}}{d\xi} \right) + \Gamma^{c}_{ab}\, v^{a}\, \dfrac{d\gamma^{b}}{d\xi} = 0.
\end{cases}
\tag{2.2}
$$

Along any curve $q^{\alpha} = \gamma^{\alpha}(\xi)$ we can define the **acceleration vector** with respect to the parameter ξ:

$$
a^{\gamma}(\xi) \stackrel{\text{def}}{=} \frac{d^2\gamma^{\gamma}}{d\xi^2} + \Gamma^{\gamma}_{\alpha\beta}\, \frac{d\gamma^{\alpha}}{d\xi}\, \frac{d\gamma^{\beta}}{d\xi}.
\tag{2.3}
$$

[1] See Remark 1.5 below.

The curve is a **Γ-geodesic** when the acceleration is parallel to the **velocity** $d\gamma^\alpha/d\xi$,

$$\boxed{a^\gamma(\xi) = \lambda(\xi)\,\frac{d\gamma^\gamma}{d\xi}} \tag{2.4}$$

i.e.,

$$\boxed{\frac{d^2\gamma^\gamma}{d\xi^2} + \Gamma^\gamma_{\alpha\beta}\,\frac{d\gamma^\alpha}{d\xi}\,\frac{d\gamma^\beta}{d\xi} = \lambda(\xi)\,\frac{d\gamma^\gamma}{d\xi}} \tag{2.5}$$

These equations separate into the system

$$\begin{cases} \dfrac{d^2\gamma^0}{d\xi^2} + \Gamma^0_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} + 2\,\Gamma^0_{a0}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^0}{d\xi} + \Gamma^0_{00}\,\dfrac{d\gamma^0}{d\xi}\,\dfrac{d\gamma^0}{d\xi} = \lambda\,\dfrac{d\gamma^0}{d\xi}, \\ \dfrac{d^2\gamma^c}{d\xi^2} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} + 2\,\Gamma^c_{a0}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^0}{d\xi} + \Gamma^c_{00}\,\dfrac{d\gamma^0}{d\xi}\,\dfrac{d\gamma^0}{d\xi} = \lambda\,\dfrac{d\gamma^c}{d\xi}. \end{cases} \tag{2.6}$$

The parameter ξ is said to be **affine** when the acceleration vanishes: $a^\gamma(\xi) = 0$ ($\iff \lambda = 0$).

Remark 2.1 – If a curve is transversal to the spatial sections, then the coordinate q^0 can be taken as a parameter. So, $\gamma^0(q^0) = q^0$ and consequently $d\gamma^0/dq^0 = 1$ and for such a curve the transport equations (2.2) and the geodesic equations (2.6) reduce respectively to

$$\begin{cases} \dfrac{dv^0}{dq^0} + \Gamma^0_{00}\,v^0 + \Gamma^0_{a0}\left(v^a + v^0\,\dfrac{d\gamma^a}{dq^0}\right) + \Gamma^0_{ab}\,v^a\,\dfrac{d\gamma^b}{dq^0} = 0, \\ \dfrac{dv^c}{dq^0} + \Gamma^c_{00}\,v^0 + \Gamma^c_{a0}\left(v^a + v^0\,\dfrac{d\gamma^a}{dq^0}\right) + \Gamma^c_{ab}\,v^a\,\dfrac{d\gamma^b}{dq^0} = 0. \end{cases} \tag{2.7}$$

$$\begin{cases} \Gamma^0_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} + 2\,\Gamma^0_{a0}\,\dfrac{d\gamma^a}{dq^0} + \Gamma^0_{00} = \lambda, \\ \dfrac{d}{dq^0}\,\dfrac{d\gamma^c}{dq^0} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} + 2\,\Gamma^c_{a0}\,\dfrac{d\gamma^a}{dq^0} + \Gamma^c_{00} = \lambda\,\dfrac{d\gamma^c}{dq^0}. \end{cases} \tag{2.8}$$

•

2.2 The basic requirements

We look for a linear connection $\Gamma = (\Gamma^\gamma_{\alpha\beta})$ that meet the postulates of the cosmic kinematics or, in other words, that is adapted to the geometrical structures introduced on the space-time: the congruence of the galactic world-lines and the spatial foliation. We will translate our aims into precise requirements.

> 1^{st} **Requirement**. The galactic world-lines are geodesics of Γ with affine parameter q^0.

> 2^{nd} **Requirement**. The property of a vector to be tangent to the spatial foliation is preserved by the Γ-transport along the galactic world-lines.

Henceforth we will call **spatial** every vector tangent to a spatial section. In co-moving coordinates a spatial vector is characterized by the condition $v^0 = 0$.

Theorem 2.1 – *The 1^{st} requirement is satisfied if and only if* $\Gamma^0_{00} = 0$ *and* $\Gamma^c_{00} = 0$.

Proof – A galactic world-line is parametrized by q^0 and characterized by the condition $q^a =$ constant. In this case the geodesic equations (2.8) give

$$\begin{cases} \Gamma^0_{00} = \lambda, \\ \Gamma^c_{00} = 0. \end{cases}$$

The parameter q^0 is affine if and only if $\lambda = 0$. ■

Theorem 2.2 – *The 2^{nd} requirement is satisfied if and only if* (i) $\Gamma^0_{a0} = 0$ *and* (ii) *the transport of a spatial vector along a galactic world-line is represented by the equations*

$$\frac{dv^c}{dq^0} + \Gamma^c_{a0}\, v^a = 0. \tag{2.9}$$

Proof – For a spatial vector $v^0 = 0$ and the transport equations (2.7) with $\gamma^a =$ constant reduce to

$$\begin{cases} \Gamma^0_{a0}\, v^a = 0, \\ \dfrac{dv^c}{dq^0} + \Gamma^c_{a0}\, v^a = 0. \quad ■ \end{cases}$$

By virtue of the 3^{rd} postulate each spatial section S_t is endowed with a positive-definite metric tensor g_t. Hence, a further 'natural' requirement is the following.

> 3^{rd} **Requirement**. The norm of the spatial vectors is preserved by the Γ-transport along the galactic world-lines.

The **norm** of a spatial vector $v^a(\xi)$ along a curve $q^\alpha(\xi)$ is defined by[2]

$$\|v(\xi)\| \overset{\text{def}}{=} g_{ab}(q^0, \widetilde{q})\, v^a(\xi)\, v^b(\xi) = A^2(q^0)\, \widetilde{g}_{ab}(\widetilde{q})\, v^a(\xi)\, v^b(\xi). \tag{2.10}$$

Theorem 2.3 – *If the 2^{nd} requirement is satisfied then the connection meets the 3^{rd} requirement if and only if $\Gamma^b_{a0} = H\,\delta^b_a$, where $H(q^0)$ is the Hubble parameter.*

Proof – A galactic world-line can be parametrized by q^0. From the definition (2.10) it follows that ($' = d/dq^0$)

$$\left[\begin{array}{l} \dfrac{d\|v\|}{dq^0} = 2\,A\,A'\,\widetilde{g}_{ab}\,v^a\,v^b + 2\,A^2\,\widetilde{g}_{ab}\,v^a\,\dfrac{dv^b}{dq^0} \\ \text{use the transport equations (2.9) } \dfrac{dv^b}{dq^0} = -\Gamma^b_{c0}\,v^c \\ = 2\,A\,A'\,\widetilde{g}_{ab}\,v^a\,v^b - 2\,A^2\,\widetilde{g}_{ab}\,v^a\,\Gamma^b_{c0}\,v^c \\ = 2\,A\,A\,\widetilde{g}_{ab}\,v^a\left(A'\,v^b - H^{-1}\Gamma^b_{c0}\,v^c\right). \end{array}\right.$$

The 3^{rd} requirement is equivalent to

$$\frac{d\|v\|}{dq^0} = 0 \iff \widetilde{g}_{ab}\,v^a\,(v^b - H^{-1}\Gamma^b_{c0}\,v^c) = 0.$$

According to this last equations the spatial vectors v^a and $v^a + x^a$, with $x^a \overset{\text{def}}{=} -H^{-1}\Gamma^a_{c0}\,v^c$, must be orthogonal. This is absurd unless $v^a + x^a = 0$ i.e., $v^b = H^{-1}\Gamma^b_{c0}\,v^c \iff \Gamma^b_{a0} = H\,\delta^b_a$. ∎

At this point, the table of the symbols of the connection Γ is the following:

$$\boxed{\begin{array}{l} \Gamma^0_{00} = 0, \quad \Gamma^0_{a0} = 0, \quad \Gamma^c_{00} = 0 \\ \Gamma^b_{a0} = H(q^0)\,\delta^b_a \\ \Gamma^0_{ab} \text{ and } \Gamma^c_{ab} \text{ to be determined} \end{array}} \tag{2.11}$$

The geodesic equations (2.6) in a generic parameter ξ reduce to

$$\left\{\begin{array}{l} \dfrac{d^2\gamma^0}{d\xi^2} + \Gamma^0_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} = \lambda\,\dfrac{d\gamma^0}{d\xi}, \\ \dfrac{d^2\gamma^c}{d\xi^2} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} + 2\,H\,\dfrac{d\gamma^c}{d\xi}\,\dfrac{d\gamma^0}{d\xi} = \lambda\,\dfrac{d\gamma^c}{d\xi}. \end{array}\right. \tag{2.12}$$

For $\xi = q^0$,

$$\left\{\begin{array}{l} \Gamma^0_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} = \lambda, \\ \dfrac{d}{dq^0}\,\dfrac{d\gamma^c}{dq^0} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} + 2\,H\,\dfrac{d\gamma^c}{dq^0} = \lambda\,\dfrac{d\gamma^c}{dq^0}. \end{array}\right. \tag{2.13}$$

[2] Recall (1.22), $a(t) = A(q^0)$.

Assuming that the above three requirements are met, now we ask the connection Γ to meet the 4^{th} postulate: the isotropy principle.

> $\mathbf{4^{th}}$ **Requirement**. The connection Γ is **isotropic** in the sense that it does not give rise to distinguished vector fields tangent to the spatial sections.

Theorem 2.4 – *If the connection is isotropic then*

$$\Gamma^{0}_{ab} = F(q^0)\,\widetilde{g}_{ab}(\widetilde{q}) \tag{2.14}$$

where $F(q^0)$ is a function of q^0 only.

Proof – Going back to the definition (2.4) of geodesic, $a^\alpha = \lambda\, d\gamma^\alpha/d\xi$, we observe that the multiplier λ is a scalar since a^α and $d\gamma^\alpha/d\xi$ are vectors. Then the left hand side of the first equation (2.13) is a scalar. It follows that $\Gamma^0_{ab}(q^0,\widetilde{q})$ are the components of a spatial covariant symmetric tensor for each fixed q^0. It generates distinguished eigenvectors unless it is of the form (2.14) (see also Section 1.3 below). ∎

Now the table of the Γ-symbols is

$$\boxed{\begin{array}{l} \Gamma^0_{00} = 0, \quad \Gamma^0_{a0} = 0, \quad \Gamma^c_{00} = 0 \\[1ex] \Gamma^b_{a0} = H(q^0)\,\delta^b_a, \quad \Gamma^0_{ab} = F(q^0)\,\widetilde{g}_{ab} \\[1ex] F(q^0) \text{ and } \Gamma^c_{ab} \text{ to be determined} \end{array}} \tag{2.15}$$

The Γ-geodesic equations (2.12) in a generic parameter ξ reduce to

$$\left\{ \begin{array}{l} \dfrac{d^2\gamma^0}{d\xi^2} + F\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} = \lambda\,\dfrac{d\gamma^0}{d\xi}, \\[2ex] \dfrac{d^2\gamma^c}{d\xi^2} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} + 2\,H\,\dfrac{d\gamma^c}{d\xi}\,\dfrac{d\gamma^0}{d\xi} = \lambda\,\dfrac{d\gamma^c}{d\xi}. \end{array} \right. \tag{2.16}$$

and for $\xi = q^0$

$$\left\{ \begin{array}{l} F\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} = \lambda, \\[2ex] \dfrac{d}{dq^0}\dfrac{d\gamma^c}{dq^0} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} + 2\,H\,\dfrac{d\gamma^c}{dq^0} = \lambda\,\dfrac{d\gamma^c}{dq^0}. \end{array} \right. \tag{2.17}$$

2.3 Projection of geodesics

Let us return to the end of Section 1.10 where the cosmic watcher noted the presence of geodesics on its monitor. Arguing that these $\widetilde{\Gamma}$-geodesics may result from the projection of geodesics of a connection in space-time we are led to the following

> 5^{th} **Requirement**. Γ-geodesics project onto $\widetilde{\Gamma}$-geodesics.

This means that if $\gamma(\xi)$ is a geodesic for the connection Γ then the projected curve $\widetilde{\gamma}(\xi)$ must be a geodesic for the Levi-Civita connection $\widetilde{\Gamma}$ of the quotient metric $\widetilde{g}$. Note that this requirement is already satisfied by the galactic world-lines. In fact they are Γ-geodesics which projects onto single points of the quotient manifold which may be interpreted as **singular geodesics**.[3]

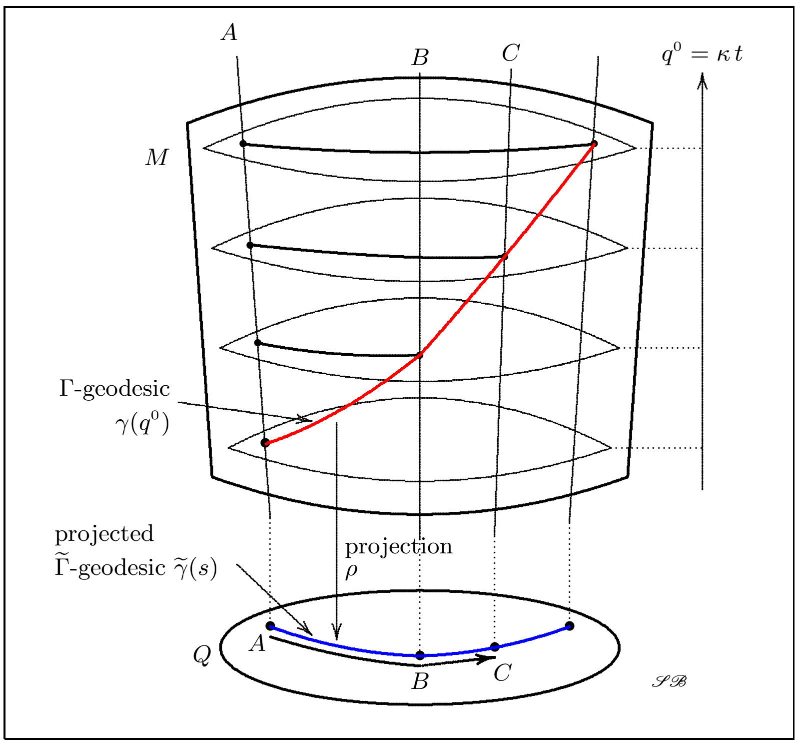


Figure 2.1: Regular Γ-geodesics project onto $\widetilde{\Gamma}$-geodesics.

Impose this 5^{th} requirement to those geodesics γ which can be parametrized by q^0 and by the arc-length s of their projected curves $\widetilde{\gamma}$ (see Fig. 2.1). We will call them **regular Γ-geodesics**. The arc-length s is a distinguished

[3] A parametrized curve is interpreted as a motion. So, a point represents the motion of a point at rest.

parameter for any curve on Q since

$$\widetilde{g}_{ab}\,\frac{d\gamma^a}{ds}\,\frac{d\gamma^b}{ds}=1 \tag{2.18}$$

and is an affine parameter for all $\widetilde{\Gamma}$-geodesics,

$$q^a=\gamma^a(s) \text{ is a } \widetilde{\Gamma}\text{-geodesic} \iff \frac{d^2\gamma^c}{ds^2}+\widetilde{\Gamma}^c_{ab}\,\frac{d\gamma^a}{ds}\,\frac{d\gamma^b}{ds}=0. \tag{2.19}$$

The reversible relationship between the parameters q^0 and s is represented by a function[4]

$$\boxed{V(q^0)\overset{\text{def}}{=}\frac{ds}{dq^0}>0} \tag{2.20}$$

Theorem 2.5 – *The* 5^{th} *requirement implies that the symbols* Γ^c_{ab} *coincide with the Christoffel symbols* $\widetilde{\Gamma}^c_{ab}$ *of the quotient metric*

$$\boxed{\Gamma^c_{ab}(q^0,\widetilde{q})=\widetilde{\Gamma}^c_{ab}(\widetilde{q})} \tag{2.21}$$

and that the function $V(q^0)>0$ (2.20) *satisfies the equation*

$$\boxed{\frac{d\log V}{dq^0}-F\,V^2+2\,H=0} \tag{2.22}$$

Proof – Taking into account equation (2.18) the geodesic equations (2.17) are equivalent to

$$\begin{cases} V^2\,F=\lambda, \\ V^2\left(\dfrac{d}{ds}\dfrac{d\gamma^c}{ds}+\Gamma^c_{ab}\,\dfrac{d\gamma^a}{ds}\,\dfrac{d\gamma^b}{ds}\right)=\left(V^3\,F-V'-2\,H\,V\right)\dfrac{d\gamma^c}{ds}. \end{cases} \tag{2.23}$$

By replacing the expression of λ given by the first equation in the second set we get the **characteristic equations of the regular Γ-geodesics** in the parameter s:

$$\frac{d}{ds}\frac{d\gamma^c}{ds}+\Gamma^c_{ab}\,\frac{d\gamma^a}{ds}\,\frac{d\gamma^b}{ds}=V^{-1}\left(V^2\,F-(\log V)'-2\,H\right)\frac{d\gamma^c}{ds}. \tag{2.24}$$

These equations involve only the parametric equations $q^a=\gamma^a(q^0)$ and their first and second derivatives. Consequently, they are satisfied by the projected

[4] Actually, the reversibility condition is $\dfrac{dq^0}{ds}\neq 0$. The condition $\dfrac{dq^0}{ds}>0$ is not restrictive: it simply means that the two parameters q^0 and s are assumed to be with the same orientation.

curve $\widetilde{\gamma}(s)$. But, in acoordance with the 5^{th} requirement, this projected curve must be a $\widetilde{\Gamma}$-geodesic. As a consequence, by the comparison with the $\widetilde{\Gamma}$-geodesic equations (2.19) we get

$$\left(\Gamma^c_{ab} - \widetilde{\Gamma}^c_{ab}\right) \frac{d\gamma^a}{ds} \frac{d\gamma^b}{ds} = V^{-1} \left(V^2\, F - (\log V)' - 2\, H\right) \frac{d\gamma^c}{ds}.$$

These equations must hold for all geodesics. Since the left sides are quadratic in the s-velocities and the right sides are linear, both sides must vanish. ∎

The following is in a sense the inverse of the previous theorem.

Theorem 2.6 – *Let* γ *be a* **regular curve** *on space-time, that is a curve which can be parametrized by* q^0 *as well as by the arc-length* s *of the projected curve* $\widetilde{\gamma}$*. Assume that* (i) *equations* (2.21) *and* (2.22) *are satisfied and that* (ii) *the projected curve* $\widetilde{\gamma}$ *is a* $\widetilde{\Gamma}$*-geodesic. Then* γ *is a* Γ*-geodesic.*

Proof – Under the assumptions (i) and (ii)

$$\begin{cases} \Gamma^c_{ab} = \widetilde{\Gamma}^c_{ab} \\ \dfrac{d}{ds}\dfrac{d\gamma^c}{ds} + \widetilde{\Gamma}^c_{ab}\, \dfrac{d\gamma^a}{ds}\, \dfrac{d\gamma^b}{ds} = 0 \end{cases}$$

and equations (2.23) reduce to

$$\begin{cases} V^2\, F = \lambda, \\ V^2\, F - (\log V)' - 2\, H\, V = 0 \end{cases}$$

The second equation is satisfied because of the assumption (i). The first equation gives the expression of λ and in this context is irrelevant. Hence, equations (2.23) are satisfied. On the other hand, ss we have seen in the previous proof, equations (2.23) are equivalent to equations (2.17) which in turn are the Γ-geodesic equations in the parameter q^0. ∎

2.4 Cosmic connections

The arguments of this chapter can be summarized in a definition and a theorem.

Definition 2.1 – A **cosmic connection** is a linear symmetric connection on the cosmic space-time satisfying the five requirements listed in this chapter, which are compatible with the postulates of the cosmic kinematics stated in the first chapter.

Theorem 2.7 – *In any co-moving coordinate system the symbols of a cosmic connection are*

$$\boxed{\begin{aligned}&\Gamma^0_{00} = 0, \quad \Gamma^0_{a0} = 0, \quad \Gamma^c_{00} = 0\\ &\Gamma^b_{a0} = H(q^0)\,\delta^b_a, \quad \Gamma^0_{ab} = F(q^0)\,\widetilde{g}_{ab}\\ &\Gamma^c_{ab} = \widetilde{\Gamma}^c_{ab}\end{aligned}} \tag{2.25}$$

where $H(q^0)$ is the Hubble parameter, $\widetilde{\Gamma}^c_{ab}$ are the Christoffel symbols of the quotient metric $\widetilde{g}$, and $F(q^0)$ is a function satisfying the equation

$$\boxed{\frac{d\log V}{dq^0} + 2\,H = F\,V^2} \tag{2.26}$$

$$\boxed{V(q^0) \overset{\text{def}}{=} \frac{ds}{dq^0}} \tag{2.27}$$

along any regular Γ-geodesic.

Despite the complexity of the above discussion, the result (2.25) is very simple.

Remark 2.2 – Having in mind Remark 1.9 we observe that the five requirements of a cosmic connection are expressed in a geometrical way which is manifestly invariant with respect to the choice of a reference date (take also into account Remark 1.13). Nevertheless, we can observe this invariance directly from the expressions (2.25) of the symbols. (i) The symbols $\Gamma^b_{a0} = H(q^0)\,\delta^b_a$ are invariant since the Hubble factor is invariant (Theorem 1.9). (ii) The coefficients $\Gamma^c_{ab} = \widetilde{\Gamma}^c_{ab}$ are invariant (Remark 1.13). (iii) About the undetermined function F entering the symbols $\Gamma^0_{ab} = F(q^0)\,\widetilde{g}_{ab}$, we observe that due to equation (1.27)

$$ds(t_0) = \frac{1}{a(t_0, t_*)}\,ds(t_*)$$

from the definition (2.27) it follows that

$$V(t_0, q^0) = \frac{ds(t_0)}{dq^0} = \frac{1}{a(t_0, t_*)}\,\frac{ds(t_*)}{dq^0} = \frac{1}{a(t_0, t_*)}\,V(t_*, q^0), \tag{2.28}$$

thus

$$d\log V(t_0, q^0) = d\log V(t_*, q^0).$$

Hence, the left-hand side of equation (2.26) is invariant, so that $F\,V^2$ must be invariant:

$$F(t_0, q^0)\,V^2(t_0, q^0) = F(t_*, q^0)\,V^2(t_*, q^0).$$

Due to (2.28), this equation is equivalent to

$$\boxed{F(t_0, q^0) = a^2(t_0, t_*)\, F(t_*, q^0)} \tag{2.29}$$

In turn, due to (1.21), this last equation is equivalent to the invariance of $\Gamma^0_{ab} = F(q^0)\, \widetilde{g}_{ab}$. •

Remark 2.3 – The role of the undetermined function $F(q^0)$ raises a subtle argument. In fact equation (2.26) involve not only the functions F and H, which participate in the definition of the connection, but also the function V which instead, by definition, is linked to the structure of the regular geodesics. This is a paradox that may cast doubt on the correctness of this equation which in a sense is 'hybrid'. This paradox will be clarified in the following. •

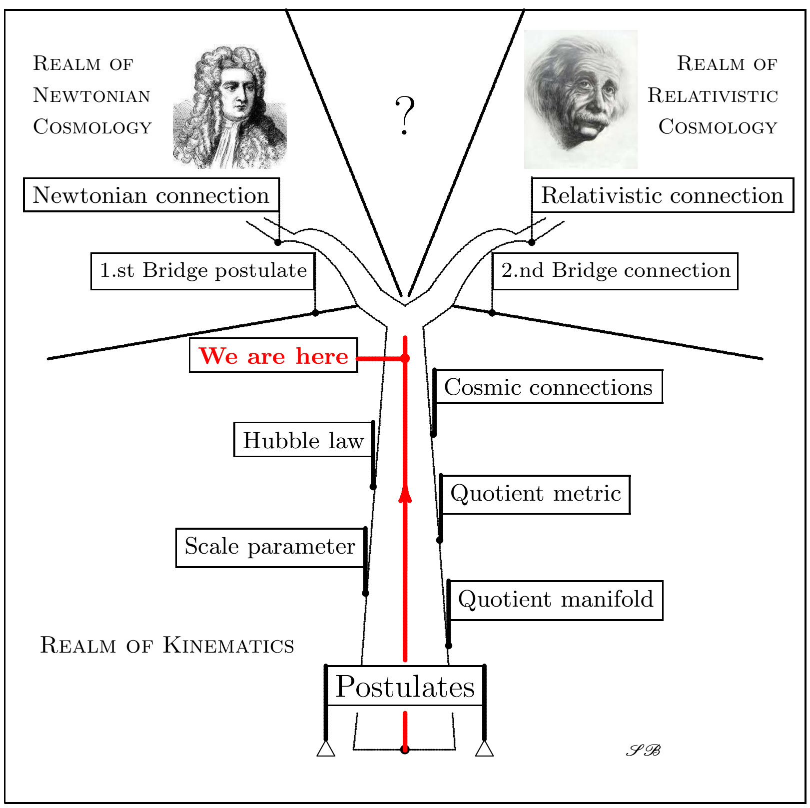


Figure 2.2: The cosmic road.

The determination of a cosmic connection through the choice of the function $F(q^0)$ must be the consequence of a postulate. In this regard we observe that a connection provides the basis of a dynamics. In other words, the kind of postulate we are arguing will form a 'bridge' between the Cosmic Kinematics and the Cosmic Dynamics.

We will consider two of these **bridge-postulates** that open the way to two different dynamics (Fig. 2.2).

2.5 The Newtonian cosmic connection

1st Bridge−postulate. The cosmic time t is an affine parameter for the world-lines of the free particles.

Remind that the world-line of a particle is (by definition) transversal to the spatial foliation (paragraph 2 of Section 1.10).

Theorem 2.8 – *The parameter $q^0 = \kappa\, t$ is affine for all transversal geodesics of a cosmic connection if and only if $F(q^0) = 0$.*

Proof – Write the first geodesic equation (2.16) for $\xi = q^0$ and put $\lambda = 0$:

$$F\, \widetilde{g}_{ab}\, \frac{d\gamma^a}{dq^0}\, \frac{d\gamma^b}{dq^0} = 0. \quad [*]$$

$$\widetilde{g}_{ab}\, \frac{d\gamma^a}{dq^0}\, \frac{d\gamma^b}{dq^0} = 0 \iff \frac{d\gamma^a}{dq^0} = 0 \implies \text{absurd}$$

because of the transversality assumption. Then $[*] \Longrightarrow F = 0$. ∎

There is a unique cosmic connection meeting the above postulate. We call it **Newtonian** since a cosmic space-time equipped with this connection is a generalization of the Newtonian space-time of classical mechanics (Fig. 2.3), where:

1. The manifold M is an affine four-dimensional space.

2. The spatial sections are Euclidean three-dimensional affine spaces.

3. The galactic world-lines are parallel straight lines and represents the motion of the so-called **fixed stars**. The congruence of these lines is an **inertial reference frame**, as well as any other congruence of parallel lines transversal to the foliation $\mathcal{S}$.

4. The world-lines of the free-falling particles are transversal straight lines (law of inertia).

5. The cosmic time t is the **absolute time**.

6. The expansion factor is constant and equal to 1, and the Hubble parameter is $h = 0$; thus $H = 0$. Consequently if the coordinates $\widetilde{q}$ are Cartesian, then all symbols $\Gamma^{\gamma}_{\alpha\beta}$ vanishes. The cosmic connection is flat and coincides with the canonical connection of an affine space.

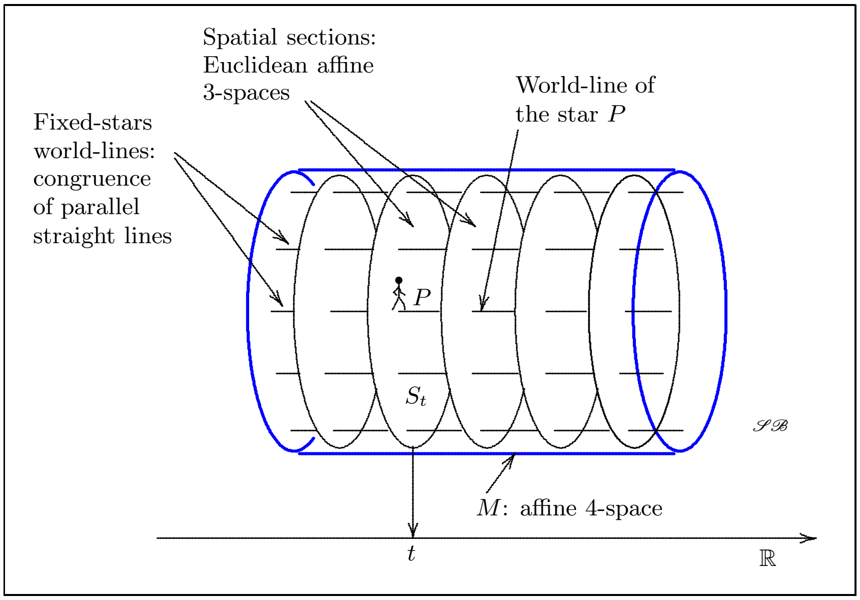


Figure 2.3: Newtonian space-time.

2.6 The relativistic cosmic connection

2nd Bridge−postulate. There exist **special particles** whose peculiar velocity (1.51) is constant

(2.30)
$$\boxed{a(t)\,\frac{ds}{dt} = c = \text{constant}}$$

Since the peculiar velocity is in fact the velocity with respect to a local frame of reference, this postulate clearly falls within the relativistic vision. Thus, the connection we are going to define will be called **relativistic cosmic connection**.

Theorem 2.9 – *There is a unique cosmic connection compatible with the* *existence of special particles. The function $F(q^0)$ is given by*

$$\boxed{F = \frac{\kappa^2}{c^2}\, A^2\, H} \tag{2.31}$$

Proof – Due to the condition (2.30) the monitor speed (1.47) of a special particle is

$$\boxed{v(t) \stackrel{\text{def}}{=} \frac{ds}{dt} = \frac{c}{a(t)}} \tag{2.32}$$

Then in the parameter q^0 the monitor speed is expressed by the function

$$V(q^0) \stackrel{\text{def}}{=} \frac{ds}{dq^0} = \frac{c}{\kappa\, A(q^0)}. \tag{2.33}$$

Note that this 'speed' coincides with the function $V(q^0)$ defined in (2.20) for which equation (2.22) holds,

$$\frac{d\log V}{dq^0} - F\,V^2 + 2\,H = 0.$$

Due to (2.33) this equation is equivalent to

$$\frac{d\log A^{-1}}{dq^0} + 2\,H - F\,\frac{c^2}{\kappa^2}\,A^{-2} = 0$$

Since $H = (\log A)'$, we have

$$H - F\,\frac{c^2}{\kappa^2}\,A^{-2} = 0$$

and we find equation (2.31). ■

Remark 2.4 – The definition (2.31) of F shows that the cosmic connection depends explicitly on c (the functions A and H do not depend on c). Then the existence of two (or more) special particles with different peculiar velocities is incompatible with the existence of a unique cosmic connection. If we think of a unique cosmic connection then c becomes a **universal constant**. •

Remark 2.5 – Remind that the constant κ has been introduced at the beginning of our discussion for a dimensional consistency in the correlation between the cosmic time t and the coordinate q^0: $q^0 = \kappa\, t$. Its numerical value has been left arbitrary, but fixed. This constant has been present throughout our discussion, and it is also present in the definition (2.31) of F. Then there is no loss of generality in considering

$$\boxed{\kappa = c}$$

It follows from (2.33) that for a special particle

$$\boxed{\frac{ds}{dq^0} = V = A^{-1}} \quad \bullet \tag{2.34}$$

The symbols of the relativistic cosmic connection are

$$\boxed{\begin{cases} \Gamma^0_{a0} = 0 \\ \Gamma^c_{00} = 0 \\ \Gamma^0_{00} = 0 \end{cases} \begin{cases} \Gamma^b_{a0} = H\,\delta^b_a \\ \Gamma^0_{ab} = A^2\,H\,\widetilde{g}_{ab}(\widetilde{q}) \\ \Gamma^c_{ab} = \widetilde{\Gamma}^c_{ab} \end{cases}} \tag{2.35}$$

Theorem 2.10 – *The world-line of a special particle is a geodesic of the relativistic cosmic connection.*[5]

Proof – The world-line of a special particle is a curve transversal to the spatial sections. The characteristic equations (2.24) of the Γ-geodesics in the parameter s read

$$\frac{d}{ds}\frac{d\gamma^c}{ds} + \Gamma^c_{ab}\,\frac{d\gamma^a}{ds}\,\frac{d\gamma^b}{ds} = V^{-1}\left(V^2\,F - (\log V)' - 2\,H\right)\frac{d\gamma^c}{ds}$$

Due to (2.34) $V = A^{-1}$, the coefficient at the right hand side vanishes: $V^2\,F - (\log V)' - 2\,H = H + (\log A)' - 2\,H = 0$. It follows that $\dfrac{d^2\gamma^c}{ds^2} + \Gamma^c_{ab}\,\dfrac{d\gamma^a}{ds}\,\dfrac{d\gamma^b}{ds} = 0$ and we can apply Theorem 2.6. ■

If we accept the existence of 'special particles' then at the bifurcation of Fig. 2.2 we turn right. On this way we are going towards a relativistic formulation of the isotropic cosmology. In fact at the first station we will find a surprise: as a consequence of our postulates the space-time admits in a canonical way a metric of Lorentzian signature.

[5] In other words: *a special particle is a free particle of the relativistic cosmic connection.*

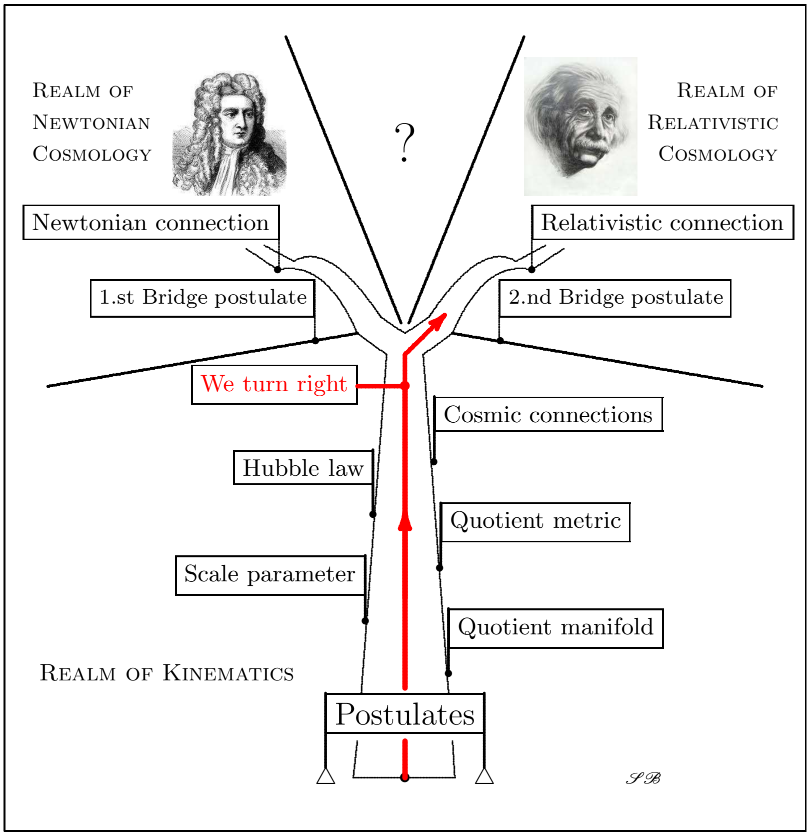


Figure 2.4: Towards the relativistic cosmology.

2.7 The canonical cosmic metric

Theorem 2.11 – *The relativistic cosmic connection is the Levi-Civita connection of the Lorentzian metric*[6]

$$(2.36)\qquad \boxed{g_{\alpha\beta}:\begin{cases} g_{oo} = \alpha = \text{const.} \\ g_{ao} = 0 \\ g_{ab} = -\alpha\, A^2\, \widetilde{g}_{ab} \end{cases}} \quad \boxed{g_{\alpha\beta}\, dq^\alpha\, dq^\beta = \alpha\,(dq^{o2} - A^2\, \widetilde{g}_{ab}\, dq^a\, dq^b)}$$

Proof – According to Theorem 1.6 the components of any metric have the form

[6] Note that the Newtonian connection, for which $F = 0$, cannot admit a cosmic metric.

$$g_{\alpha\beta}:\begin{cases} g_{oo}=\alpha(q^o),\\ g_{oa}=0,\\ g_{ab}=\beta(q^o)\,\widetilde{g}_{ab}\end{cases} \iff g^{\alpha\beta}:\begin{cases} g^{oo}=\alpha^{-1},\\ g^{oa}=0,\\ g^{ab}=\beta^{-1}\,\widetilde{g}^{ab}.\end{cases}$$

Computation of the first-kind Christoffel $\Gamma_{\alpha\beta,\gamma}=\frac{1}{2}\,(\partial_\alpha g_{\beta\gamma}+\partial_\beta g_{\gamma\alpha}-\partial_\gamma g_{\alpha\beta})$:

$$2\,\Gamma_{oo,\gamma}=\partial_o g_{o\gamma}+\partial_o g_{\gamma o}-\partial_\gamma g_{oo}=\begin{cases}2\,\Gamma_{oo,o}=\partial_o g_{oo}+\partial_o g_{oo}-\partial_o g_{oo}=\alpha'.\\ 2\,\Gamma_{oo,a}=\partial_o g_{oa}+\partial_o g_{ao}-\partial_a g_{oo}=0.\end{cases}$$

$$2\,\Gamma_{ob,\gamma}=\partial_o g_{b\gamma}+\partial_b g_{\gamma o}-\partial_\gamma g_{ob}=\begin{cases}2\,\Gamma_{ob,o}=\partial_o g_{bo}+\partial_b g_{oo}-\partial_o g_{ob}=0.\\ 2\,\Gamma_{ob,c}=\partial_o g_{bc}+\partial_b g_{co}-\partial_c g_{ob}=\beta'\,\widetilde{g}_{bc}.\end{cases}$$

$$2\,\Gamma_{ab,\gamma}=\partial_a g_{b\gamma}+\partial_b g_{\gamma a}-\partial_\gamma g_{ab}=$$

$$\begin{cases}2\,\Gamma_{ab,o}=\partial_a g_{bo}+\partial_b g_{oa}-\partial_o g_{ab}=-\beta'\,\widetilde{g}_{ab}.\\ 2\,\Gamma_{ab,c}=\partial_a g_{bc}+\partial_b g_{ca}-\partial_c g_{ab}=2\,\beta\,\widetilde{\Gamma}_{ab,c}.\end{cases}$$

Computation of the second-kind Christoffel $\Gamma^\gamma_{\alpha\beta}=g^{\gamma\delta}\,\Gamma_{\alpha\beta,\delta}$:

$$\Gamma^\gamma_{oo}=g^{\gamma\delta}\,\Gamma_{oo,\delta}=\begin{cases}\Gamma^o_{oo}=g^{o\delta}\,\Gamma_{oo,\delta}=g^{oo}\,\Gamma_{oo,o}=\frac{1}{2}\,\alpha^{-1}\,\alpha'=\frac{1}{2}\,(\log\alpha)'.\\ \Gamma^c_{oo}=g^{c\delta}\,\Gamma_{oo,\delta}=g^{cd}\,\Gamma_{oo,d}=0.\end{cases}$$

$$\Gamma^\gamma_{ao}=g^{\gamma\delta}\,\Gamma_{ao,\delta}=\begin{cases}\Gamma^o_{ao}=g^{o\delta}\,\Gamma_{ao,\delta}=g^{oo}\,\Gamma_{ao,o}=0.\\ \Gamma^c_{ao}=g^{c\delta}\,\Gamma_{ao,\delta}=g^{cd}\,\Gamma_{ao,d}=\frac{1}{2}\,\beta^{-1}\,\widetilde{g}^{cd}\,\beta'\,\widetilde{g}_{ad}\\ \qquad=\frac{1}{2}\,(\log\beta)'\,\delta^c_a.\end{cases}$$

$$\Gamma^\gamma_{ab}=g^{\gamma\delta}\,\Gamma_{ab,\delta}=\begin{cases}\Gamma^o_{ab}=g^{o\delta}\,\Gamma_{ab,\delta}=g^{oo}\,\Gamma_{ab,o}=-\frac{1}{2}\,\alpha^{-1}\,\beta'\,\widetilde{g}_{ab}.\\ =g^{c\delta}\,\Gamma_{ab,\delta}=g^{cd}\,\Gamma_{ab,d}=\beta^{-1}\,\widetilde{g}^{cd}\,\beta\,\widetilde{\Gamma}_{ab,d}=\widetilde{\Gamma}^c_{ab}.\end{cases}$$

$$\text{Summary:}\quad\begin{cases}\Gamma^o_{oo}=\frac{1}{2}\,(\log\alpha)',\\ \Gamma^c_{oo}=0,\\ \Gamma^o_{ao}=0,\end{cases}\qquad\begin{cases}\Gamma^c_{ao}=\frac{1}{2}\,(\log\beta)'\,\delta^c_a,\\ \Gamma^o_{ab}=-\frac{1}{2}\,\alpha^{-1}\,\beta'\,\widetilde{g}_{ab},\\ \Gamma^c_{ab}=\widetilde{\Gamma}^c_{ab}.\end{cases}$$

These symbols coincide with those of the relativistic cosmic connection (2.35)

$$\begin{cases}\Gamma^o_{oo}=0,\\ \Gamma^c_{oo}=0,\\ \Gamma^o_{ao}=0,\end{cases}\qquad\begin{cases}\Gamma^c_{ao}=H\,\delta^c_a,\\ \Gamma^o_{ab}=K\,\widetilde{g}_{ab},\\ \Gamma^c_{ab}=\widetilde{\Gamma}^c_{ab}.\end{cases}\qquad\begin{cases}H=A^{-1}A'=(\log A)'\\ \\ K=A\,A'=A^2\,H\end{cases}$$

if and only if

$$\begin{cases}\alpha=\text{constant},\\ \frac{1}{2}\,(\log\beta)'=(\log A)',\\ -\frac{1}{2}\,\alpha^{-1}\,\beta'=A\,A'\end{cases}\iff\begin{cases}\alpha=\text{constant},\\ A^{-2}\,\beta=\text{constant}=\gamma,\\ \beta'=-2\,\alpha\,A\,A'.\end{cases}$$

$$\iff \begin{cases} \alpha = \text{constant}, \\ \beta = \gamma\, A^2, \\ \beta' = -2\,\alpha\, A\, A'. \end{cases} \iff \begin{cases} \alpha = \text{constant}, \\ \beta' = 2\,\gamma\, A\, A', \\ \beta' = -2\,\alpha\, A\, A'. \end{cases}$$

$$\iff \begin{cases} \alpha = \text{constant}, \\ \gamma = -\alpha, \\ \beta = -\alpha\, A^2. \end{cases} \implies (2.36) \quad \blacksquare$$

Without loss of generality we can take $\alpha = -1$ and consider the metric

$$(2.37) \quad \boxed{g_{\alpha\beta} : \begin{cases} g_{00} = -1 \\ g_{0a} = 0 \\ g_{ab} = A^2(q^0)\, \widetilde{g}_{ab}(\widetilde{q}) \end{cases}} \quad \boxed{g_{\alpha\beta}\, dq^\alpha\, dq^\beta = -\, dq^{0\,2} + A^2\, \widetilde{g}_{ab}\, dq^a\, dq^b}$$

with signature $-+++$ as the **canonical cosmic metric**. The contravariant components are

$$(2.38) \quad \boxed{g^{\alpha\beta} : \begin{cases} g^{00} = -1 \\ g^{0a} = 0 \\ g^{ab} = A^{-2}(q^0)\, \widetilde{g}^{ab}(\widetilde{q}) \end{cases}}$$

Theorem 2.12 – *The galactic world-lines are time-like geodesics of the canonical cosmic metric orthogonal to the spatial sections.*

Proof – By virtue of the first requirement of a cosmic connection (p. 29) the galactic world-lines are geodesics of the relativistic connection, thus of the cosmic metric. Since in co-moving coordinates $g_{0a} = 0$, these world lines are orthogonal to the spatial sections, thus they are time-like. ■

2.8 Photons

Theorem 2.13 – *The world-line of a special particle is a light-like geodesic of the canonical cosmic metric.*

Proof – For any world-line we have

$$g_{\alpha\beta}\, \frac{d\gamma^\alpha}{dq^0}\, \frac{d\gamma^\alpha}{dq^0} = -\left(\frac{d\gamma^0}{dq^0}\right)^2 + g_{ab}\, \frac{d\gamma^a}{dq^0}\, \frac{d\gamma^b}{dq^0} = -\,1 + A^2\, \widetilde{g}_{ab}\, \frac{d\gamma^a}{ds}\, \frac{d\gamma^b}{ds}\, \left(\frac{ds}{dq^0}\right)^2.$$

Since

$$\widetilde{g}_{ab}\, \frac{d\gamma^a}{ds}\, \frac{d\gamma^b}{ds} = 1$$

we get

$$g_{\alpha\beta}\,\frac{d\gamma^\alpha}{dq^0}\,\frac{d\gamma^\alpha}{dq^0} = A^2\left(\frac{ds}{dq^0}\right)^2 - 1. \tag{2.39}$$

For a photon equation (2.34) $\dfrac{ds}{dq^0} = A^{-1}$ holds. Then equation (2.39) gives

$$g_{\alpha\beta}\,\frac{d\gamma^\alpha}{dq^0}\,\frac{d\gamma^\alpha}{dq^0} = A^2\,A^{-2} - 1 = 0.$$

This proves that the world-line of a special particle is a light-like curve. We already know that the world-line of a special particle is a geodesic (Theorem 2.10). ∎

Special particles are very strange particles: they are never at rest and have the same peculiar velocity in whatever local reference frame. This is in accordance with the theory of propagation of the electro-magnetic waves. Then by virtue of Theorem 2.13 the concept of special particle can be identified with that of **electro-magnetic signal** or that of **photon** in a broadest sense i.e., as a particle associated with visible or non-visible light.

2.9 Sub-luminal particles

Definition 2.2 – A **sub-luminal particle** is a particle (2$^{\text{nd}}$ postulate of Kinematics, p. 22) whose world-line $\gamma^\alpha(t)$ is a time-like curve admitting a parameter τ, called **proper time**, such that

$$\boxed{g_{\alpha\beta}\,\frac{d\gamma^\alpha}{d\tau}\,\frac{d\gamma^\beta}{d\tau} = -\,c^2}\quad \boxed{\frac{d\tau}{dt} > 0}\ ^7 \tag{2.40}$$

Theorem 2.14 – *Sub-luminal particles have peculiar velocity less than c and* $d\tau < dt$.[8]

PROOF –

$$(2.39) \iff g_{\alpha\beta}\,\frac{d\gamma^\alpha}{d\tau}\,\frac{d\gamma^\alpha}{d\tau}\left(\frac{d\tau}{dq^0}\right)^2 = A^2\left(\frac{ds}{dq^0}\right)^2 - 1.$$

$$(2.40) \Longrightarrow -c^2\left(\frac{d\tau}{dq^0}\right)^2 = A^2\left(\frac{ds}{dq^0}\right)^2 - 1 \Longrightarrow 1 - \left(\frac{d\tau}{dt}\right)^2 = c^{-2}\,a^2\left(\frac{ds}{dt}\right)^2.$$

[7] The proper time is oriented towards the future, as the cosmic time t.

[8] $d\tau < dt$: the proper time is runs slower than the cosmic time: twins paradox.

(1.51): $v_{\rm pec} \stackrel{\rm def}{=} a(t)\dfrac{ds}{dt} \Longrightarrow 1 - \left(\dfrac{d\tau}{dt}\right)^2 = c^{-2}\, v_{\rm pec}^2 \Longrightarrow$

$$\begin{cases} \text{if } v_{\rm pec} \neq 0 : 1 - \left(\dfrac{d\tau}{dt}\right)^2 > 0 \;\text{ i.e. }\; \left(\dfrac{d\tau}{dt}\right)^2 < 1. \\ \text{if } v_{\rm pec} = 0 : 1 - \left(\dfrac{d\tau}{dt}\right)^2 = 0 \;\text{ i.e. }\; \dfrac{d\tau}{dt} = 1. \\ v_{\rm pec}^2 = c^2 \left[\, 1 - \left(\dfrac{d\tau}{dt}\right)^2 \,\right] < c^2 \quad \blacksquare \end{cases}$$

Remark 2.6 – Galaxies have $v_{\rm pec} = 0$, then galaxies are subluminal particles with $\tau = t$. •

2.10 Geodesics in the cosmic metric

For later use we summarize here some fundamental equations concerning the geodesics in the relativistic cosmic metric.

The symbols of the relativistic cosmic connection are given in (2.35). The non-vanishing symbols are

$$\text{(2.41)} \qquad \boxed{\begin{cases} \Gamma^b_{a0}(q^0) = H(q^0)\,\delta^b_a = \dfrac{A'(q^0)}{A(q^0)}\,\delta^b_a \\ \Gamma^0_{ab}(q^0, \widetilde{q}) = A^2(q^0)\,H(q^0)\,\widetilde{g}_{ab}(\widetilde{q}) = A(q^0)\,A'(q^0)\,\widetilde{g}_{ab}(\widetilde{q}) \\ \Gamma^c_{ab}(\widetilde{q}) = \widetilde{\Gamma}^c_{ab}(\widetilde{q}) \end{cases}}$$

where $\widetilde{q}$ is any set of co-moving spatial coordinates. As a consequence, the geodesic equations (2.6) for a curve $\gamma^\alpha(\xi)$ with a generic parameter ξ read

$$\text{(2.42)} \qquad \begin{cases} \dfrac{d^2\gamma^0}{d\xi^2} + A^2\,H\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} = \lambda\,\dfrac{d\gamma^0}{d\xi}, \\ \dfrac{d^2\gamma^c}{d\xi^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{d\xi}\,\dfrac{d\gamma^b}{d\xi} + 2\,H\,\dfrac{d\gamma^c}{d\xi}\,\dfrac{d\gamma^0}{d\xi} = \lambda\,\dfrac{d\gamma^c}{d\xi}. \end{cases}$$

In the parameter q^0:

$$\text{(2.43)} \qquad \begin{cases} \gamma^0 = q^0, \\ A^2\,H\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} = \lambda, \\ \dfrac{d^2\gamma^c}{dq^{0\,2}} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dq^0}\,\dfrac{d\gamma^b}{dq^0} + 2\,H\,\dfrac{d\gamma^c}{dq^0} = \lambda\,\dfrac{d\gamma^c}{dq^0}. \end{cases}$$

In the parameter t:

$$(2.42) \Longrightarrow \begin{cases} \gamma^0 = c\,t, \\ A^2\,H\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} = \lambda\,c, \\ \dfrac{d^2\gamma^c}{dt^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} + 2\,c\,H\,\dfrac{d\gamma^c}{dt} = \lambda\,\dfrac{d\gamma^c}{dt}. \end{cases}$$

Take into account that $A' = \dfrac{da}{dt}\,\dfrac{dt}{dq^0} = \dfrac{\dot{a}}{c}$, so that $H = A'/A = \dfrac{\dot{a}}{c\,a} = \dfrac{h}{c}$:

$$\Longrightarrow \begin{cases} \gamma^0 = c\,t, \\ a^2\,h\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} = \lambda\,c^2, \\ \dfrac{d^2\gamma^c}{dt^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} + 2\,h\,\dfrac{d\gamma^c}{dt} = \lambda\,\dfrac{d\gamma^c}{dt}. \end{cases}$$

$$\Longrightarrow \begin{cases} \gamma^0 = c\,t, \\ \lambda = \dfrac{a^2}{c^2}\,h\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt}, \\ \dfrac{d^2\gamma^c}{dt^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} + 2\,h\,\dfrac{d\gamma^c}{dt} = \dfrac{a^2}{c^2}\,h\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt}\,\dfrac{d\gamma^c}{dt}. \end{cases}$$

$\Longrightarrow$

$$(2.44) \qquad \boxed{\begin{cases} \gamma^0 = c\,t, \\ \dfrac{d^2\gamma^c}{dt^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} = \dfrac{h}{c^2}\left(a^2\,\widetilde{g}_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} - 2\,c^2\right)\dfrac{d\gamma^c}{dt} \end{cases}}$$

These are the general cosmic geodesic equations in the cosmic time t. Observe that for any world-line (transversal to the spatial sections)

$$(2.45) \quad g_{\alpha\beta}\,\frac{d\gamma^\alpha}{dt}\,\frac{d\gamma^\alpha}{dt} = -\left(\frac{d\gamma^0}{dt}\right)^2 + g_{ab}\,\frac{d\gamma^a}{dt}\,\frac{d\gamma^b}{dt} = -c^2 + a^2\,\widetilde{g}_{ab}\,\frac{d\gamma^a}{dt}\,\frac{d\gamma^b}{dt}.$$

For the world-line of a photon

$$g_{\alpha\beta}\,\frac{d\gamma^\alpha}{dt}\,\frac{d\gamma^\alpha}{dt} = 0 \;\Longrightarrow\; a^2\,\widetilde{g}_{ab}\,\frac{d\gamma^a}{dt}\,\frac{d\gamma^b}{dt} = c^2$$

and equations (2.44) reduce to

$$(2.46) \qquad \boxed{\begin{cases} \gamma^0 = c\,t, \\ \dfrac{d^2\gamma^c}{dt^2} + \widetilde{\Gamma}^c_{ab}\,\dfrac{d\gamma^a}{dt}\,\dfrac{d\gamma^b}{dt} = -h\,\dfrac{d\gamma^c}{dt} \end{cases}}$$

These are the light-like geodesic equations.

2.11 Comments on the Weyl principle

The standard texts of cosmology refer to the postulate of Weyl and the cosmological principle as basic statements on which to build up models of the evolution of the universe. They are essentially formulated as follows:[9]

> **Weyl's postulate**: *In cosmic space-time the world-lines of the galaxies form a bundle of non-intersecting time-like geodesics orthogonal to a series of space-like hyper-surfaces.*

> **The cosmological principle**: *On large scales the universe is spatially homogeneous and spatially isotropic.*

1st comment: As mentioned in the Introduction, the Weyl postulate put cosmology in the framework of general relativity from the very beginning. In our approach the second part of this postulate is a theorem (Theorem 2.12) while the first part is contained in our third postulate. One might argue why not accept from the very beginning the Weyl postulate instead to spend so a long time starting from several postulates. The answer is that in this longer way we do not lose the knowledge of important facts which are not strictly pertinent to the theory of general relativity. For instance, the Hubble law as well as several other results,[10] are the subject of theorems valid regardless of any dynamical assumptions on the evolution of the galactic fluid.

2nd comment about the cosmological principle: in our approach isotropy implies homogeneity (Theorem 1.3). In fact this follows at once from the fourth postulate concerning the existence of a metric on each spatial section.

> At the end of these first two chapters it is worth emphasizing that so far, as well as in the following discussion, we did not use any special coordinate system like, for instance, one of those which are commonly used on manifolds with constant curvature.

[9] See e.g. [17].

[10] Concerning, for instance, the scale parameter and the quotient metric (Section 1.5), the cosmic connections (this chapter) and the symmetric tensors (Chapter 3).

Chapter 3

Fundamental symmetric tensors

3.1 Conservation equations for a symmetric tensor

Theorem 3.1 – *Let ∇_α be the covariant derivative with respect to a general cosmic connection Γ (2.25) and $T^{\alpha\beta}$ the components (1.9) of an isotropic symmetric tensor. The four conservation equations $\nabla_\alpha T^{\alpha\beta} = 0$ are equivalent to the single equation*

$$\boxed{\Phi' + 3\,(H\,\Phi + F\,\Psi) = 0} \tag{3.1}$$

Proof – It is sufficient to prove that

$$\boxed{\begin{aligned} &\nabla_\alpha T^{\alpha 0} = \Phi' + 3\,(H\,\Phi + F\,\Psi) \\ &\nabla_\alpha T^{\alpha b} = 0 \end{aligned}} \tag{3.2}$$

$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma\beta} + \Gamma^\beta_{\alpha\gamma} T^{\alpha\gamma} \implies$

$\nabla_\alpha T^{\alpha 0} = \partial_\alpha T^{\alpha 0} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma 0} + \Gamma^0_{\alpha\gamma} T^{\alpha\gamma} = \partial_0 T^{00} + \Gamma^\alpha_{\alpha 0}\, T^{00} + \Gamma^0_{00} T^{00} + \Gamma^0_{ab} T^{ab}$

$= \Phi' + (\Gamma^\alpha_{\alpha 0} + \Gamma^0_{00})\,\Phi + \Gamma^0_{ab}\,\Psi\,\widetilde{g}^{ab} = \nabla_\alpha T^{\alpha b} = \partial_\alpha T^{\alpha b} + \Gamma^\alpha_{\alpha\gamma} T^{\gamma b} + \Gamma^b_{\alpha\gamma} T^{\alpha\gamma}$

$= \partial_a T^{ab} + \Gamma^\alpha_{\alpha a} T^{ab} + \Gamma^b_{a\gamma} T^{a\gamma} + \Gamma^b_{0\gamma} T^{0\gamma} = \partial_a T^{ab} + \Gamma^\alpha_{\alpha a} T^{ab} + \Gamma^b_{ac}\, T^{ac} + \Gamma^b_{00}\, T^{00}$

$= \Psi\,(\partial_a \widetilde{g}^{ab} + \Gamma^\alpha_{\alpha a}\,\widetilde{g}^{ab} + \Gamma^b_{ac}\,\widetilde{g}^{ac}) + \Gamma^b_{00}\,\Phi.$

Use (2.25),

$$\begin{cases} \nabla_\alpha T^{\alpha 0} = \Phi' + 3\,(H\,\Phi + F\,\Psi), \\ \nabla_\alpha T^{\alpha b} = \Psi\,(\partial_a \widetilde{g}^{ab} + \widetilde{\Gamma}^c_{ca}\,\widetilde{g}^{ab} + \widetilde{\Gamma}^b_{ac}\,\widetilde{g}^{ac}) = \Psi\,\widetilde{\nabla}_a \widetilde{g}^{ab} = 0. \end{cases} \quad ■$$

3.2 Ricci tensor of a cosmic connection

Theorem 3.2 – *The components of the Ricci tensor of a general cosmic connection* (2.25) *are*

$$\text{(3.3)} \qquad \boxed{\begin{aligned} &R_{00} = -3\,(H' + H^2) \\ &R_{a0} = 0 \\ &R_{ab} = (F' + H\,F + 2\,\widetilde{K})\,\widetilde{g}_{ab} \end{aligned}}$$

where $\widetilde{K}$ is the curvature constant of the quotient metric $\widetilde{g}$.

Proof – The Ricci tensor components are defined by (see (1.2))

$$R_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\sigma\mu}\,\Gamma^\sigma_{\alpha\beta} - \Gamma^\mu_{\sigma\beta}\,\Gamma^\sigma_{\alpha\mu}.$$

1. Computation of R_{00}:

$R_{00} = \partial_\mu \Gamma^\mu_{00} - \partial_0 \Gamma^\mu_{0\mu} + \Gamma^\mu_{\sigma\mu}\,\Gamma^\sigma_{00} - \Gamma^\mu_{\sigma 0}\,\Gamma^\sigma_{0\mu} = -\,\partial_0 \Gamma^\mu_{0\mu} - \Gamma^\mu_{\sigma 0}\,\Gamma^\sigma_{0\mu}$

$= -\,\partial_0 \Gamma^a_{0a} - \Gamma^a_{b0}\,\Gamma^b_{0a} = -H'\,\delta^a_a - H\,\delta^a_b\,H\,\delta^b_a = -3\,(H' + H^2).$

2. Computation of R_{ab}:

$R_{ab} = \partial_\mu \Gamma^\mu_{ab} - \partial_b \Gamma^\mu_{a\mu} + \Gamma^\mu_{\sigma\mu}\,\Gamma^\sigma_{ab} - \Gamma^\mu_{\sigma b}\,\Gamma^\sigma_{a\mu}$

$= \partial_0 \Gamma^0_{ab} + \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{0c}\,\Gamma^0_{ab} + \Gamma^c_{dc}\,\Gamma^d_{ab} - \Gamma^0_{cb}\,\Gamma^c_{a0} - \Gamma^c_{0b}\,\Gamma^0_{ac} - \Gamma^c_{db}\,\Gamma^d_{ac}.$

Reordering:

$R_{ab} = \partial_c \widetilde{\Gamma}^c_{ab} - \partial_b \widetilde{\Gamma}^c_{ac} + \widetilde{\Gamma}^c_{dc}\,\widetilde{\Gamma}^d_{ab} - \widetilde{\Gamma}^c_{db}\,\widetilde{\Gamma}^d_{ac} + \partial_0 \Gamma^0_{ab} + \Gamma^c_{0c}\,\Gamma^0_{ab} - \Gamma^0_{cb}\,\Gamma^c_{a0} - \Gamma^c_{0b}\,\Gamma^0_{ac}$

(the first four terms give the Ricci-tensor components $\widetilde{R}_{ab}$ of $\widetilde{g}$)

$= \widetilde{R}_{ab} + F'\,\widetilde{g}_{ab} + 3\,H\,F\,\widetilde{g}_{ab} - F\,\widetilde{g}_{cb}\,H\,\delta^c_a - H\,\delta^c_b\,H\,\widetilde{g}_{ac}$

$= \widetilde{R}_{ab} + F'\,\widetilde{g}_{ab} + 3\,H\,F\,\widetilde{g}_{ab} - F\,\widetilde{g}_{ab}\,H - H\,F\,\widetilde{g}_{ab}$

$= \widetilde{R}_{ab} + (F' + H\,F)\,\widetilde{g}_{ab}$ (apply (1.6), $\widetilde{R}_{ab} = 2\,\widetilde{K}\,\widetilde{g}_{ab}$) $= (2\,\widetilde{\Gamma} + F' + H\,F)\,\widetilde{g}_{ab}$. ■

Theorem 3.3 – *The covariant components of the Ricci tensors of the cosmic connections take the form*

$$\text{(3.4)} \qquad \boxed{\text{Newtonian}: \begin{cases} R_{00} = -3\,A^{-1}\,A'' \\ R_{ab} = 2\,\widetilde{K}\,\widetilde{g}_{ab} \end{cases}}$$

$$\boxed{\text{Relativistic}: \begin{cases} R_{00} = -3\, A^{-1}\, A'' \\ R_{ab} = \left(2\, A'^2 + A\, A'' + 2\, \widetilde{K}\right)\, \widetilde{g}_{ab} \end{cases}} \tag{3.5}$$

Proof – (i) For $\Gamma = 0$ (Newtonian connection) equations (3.3) reduce to

$$\begin{cases} R_{00} = -3\, (H' + H^2), \\ R_{ab} = 2\, \widetilde{K}\, \widetilde{g}_{ab}. \end{cases}$$

Since $H = (\log A)' = A^{-1}\, A'$, we find

$$H^2 + H' = A^{-2}\, (A')^2 - A^{-2}\, (A')^2 + A^{-1}\, A'' = A^{-1}\, A'',$$

and (3.4) are proved. (ii) For $F = A^2\, H$ (relativistic connection)
$F' + H\, F = (A^2\, H)' + A^2\, H^2 = 2\, A\, A'\, H + A^2\, H' + A^2\, H^2$
(use again $H = A^{-1}\, A'$)
$= 2\, (A')^2 + A^2\, \left[A^{-1}\, A'' - A^{-2}\, (A')^2\right] + (A')^2 = 2\, (A')^2 + A\, A''$.
Enter this result in equations (3.3) and (3.5) are proved. ∎

3.3 Einstein tensor of the relativistic cosmic connection

For the relativistic cosmic connection we can compute the mixed and the contravariant components of the Ricci tensor by raising the indices of the components (3.5) by means of the contravariant components (2.38) of the metric. We find that

$$\boxed{\begin{cases} R^0_0 = 3\, A^{-1}\, A'' \\ R^b_a = A^{-2}\left(2\, (A')^2 + A\, A'' + 2\, \widetilde{K}\right)\, \delta^b_a \end{cases}} \tag{3.6}$$

$$\boxed{\begin{cases} R^{00} = -3\, A^{-1}\, A'' \\ R^{ab} = A^{-4}\left(2\, (A')^2 + A\, A'' + 2\, \widetilde{K}\right)\, \widetilde{g}^{ab} \end{cases}} \tag{3.7}$$

From (3.6) we derive the Ricci scalar curvature

$$R \overset{\text{def}}{=} R^\alpha_\alpha = 3\left[A^{-1}\, A'' + A^{-2}\left(2\, (A')^2 + A\, A'' + 2\, \widetilde{K}\right)\right] \implies$$

$$\boxed{R = 6\, A^{-2}\left(A'^2 + A\, A'' + \widetilde{K}\right)} \tag{3.8}$$

As a consequence we can compute the contravariant[1] components of the Einstein tensor $G^{\alpha\beta} \overset{\text{def}}{=} R^{\alpha\beta} - \frac{1}{2} R\, g^{\alpha\beta}$.

$$G^{00} = R^{00} - \tfrac{1}{2} R\, g^{00} = -3\, A^{-1} A'' + 3\, A^{-2} \left(A'^2 + A\, A'' + \widetilde{K} \right)$$
$$= 3\, A^{-2} \left(A'^2 + \widetilde{K} \right).$$

$$G^{ab} = R^{ab} - \tfrac{1}{2} R\, g^{ab} = A^{-4} \left(2\, A'^2 + A\, A'' + 2\, \widetilde{K} \right) \widetilde{g}^{ab}$$
$$- 3\, A^{-4} \left(A'^2 + A\, A'' + \widetilde{K} \right) \widetilde{g}^{ab}$$
$$= A^{-4} \left[2\, A'^2 + A\, A'' + 2\, \widetilde{K} - 3 \left(A'^2 + A\, A'' + \widetilde{K} \right) \right] \widetilde{g}^{ab}$$
$$= -A^{-4} \left[A'^2 + 2\, A\, A'' + \widetilde{K} \right] \widetilde{g}^{ab}. \implies$$

$$\boxed{\begin{cases} G^{00} = 3\, A^{-2} \left(A'^2 + \widetilde{K} \right) \\ G^{ab} = -A^{-4} \left(A'^2 + 2\, A\, A'' + \widetilde{K} \right) \widetilde{g}^{ab} \end{cases}} \tag{3.9}$$

3.4 The intervention of the cosmic time

So far we have used the coordinate q^0 as a parameter. However, in the perspective of applying in a dynamical context the formulas so far found, we should rewrite them using as a parameter the cosmic time t. For this purpose we observe that

$$\begin{cases} q^0 = c\, t \implies \dfrac{d}{dq^0} = \dfrac{1}{c} \dfrac{d}{dt} \implies \dfrac{dt}{dq^0} = \dfrac{1}{c}. \\ A(q^0) = a(t) \implies A' = \dfrac{\dot{a}}{c}, \quad A'' = \dfrac{\ddot{a}}{c^2}. \\ H(q^0) = A^{-1} A' \implies H(q^0) = \dfrac{\dot{a}}{c\, a}. \\ F(q^0) = A^2\, H = A\, A' \implies \dfrac{a\, \dot{a}}{c}. \end{cases} \qquad \begin{cases} {}' = \dfrac{d}{dq^0}, \\ {}^{\cdot} = \dfrac{d}{dt}. \end{cases} \tag{3.10}$$

The Ricci curvature (3.8) and the contravariant components (3.9) of the Einstein tensor become

$$\boxed{R_{\text{icci}} = 6\, c^{-2}\, a^{-2} \left(\dot{a}^2 + a\, \ddot{a} + c^2\, \widetilde{K} \right)} \tag{3.11}$$

$$\boxed{\begin{aligned} G^{00} &= \frac{3}{c^2\, a^2} \left(\dot{a}^2 + c^2\, \widetilde{K} \right) \\ G^{ab} &= -\frac{1}{c^2\, a^4} \left(2\, a\, \ddot{a} + \dot{a}^2 + c^2 \widetilde{K} \right) \widetilde{g}^{ab} \end{aligned}} \tag{3.12}$$

[1] As we will see, we are more interested in these components, rather than in the covariant or mixed ones.

The contravariant components (1.9) of an isotropic symmetric two-tensor take the form

$$\boxed{\begin{cases} T^{oo} = \phi(t) = \text{a function of } t \text{ only}, \\ T^{oa} = 0, \\ T^{ab} = \psi(t)\,\widetilde{g}^{ab}(\widetilde{q}) = \text{a function of } t \text{ times } \widetilde{g}^{ab}, \end{cases}} \tag{3.13}$$

We call $\phi(t)$ and $\psi(t)$ the **characteristic functions** of the symmetric tensor $T^{\alpha\beta}$.

Theorem 3.4 – *For an isotropic symmetric tensor* (3.13) *the conservation equations* $\nabla_\alpha T^{\alpha\beta} = 0$ *are equivalent to the single equation*

$$\boxed{a\,\dot{\phi} + 3\,\dot{a}\,\left(\phi + a^2\,\psi\right) = 0} \tag{3.14}$$

In turn, this equation is equivalent to

$$\boxed{(\phi\,a^3)^{\cdot} + 3\,a^4\,\dot{a}\,\psi = 0} \tag{3.15}$$

Proof – Apply the rules (3.10) to equation (3.1):

$$\Phi' + 3\,(H\,\Phi + F\,\Psi) = 0 \iff \frac{\dot{\phi}}{c} + 3\left(\frac{\dot{a}}{c\,a}\,\phi + \frac{a\,\dot{a}}{c}\,\psi\right) = 0 \iff (3.14). \quad \blacksquare$$

Chapter 4

Relativistic cosmic dynamics

4.1 The principles of the relativistic cosmic dynamics

In the previous chapters we have constructed the geometrical background we need for passing to the foundation of the COSMIC DYNAMICS. We have seen that the evolution of the universe can be described by a single function of the cosmic time, the scale parameter $a(t)$. Our goal is now to state physical laws governing the evolution of $a(t)$.

> 1^{st} **Postulate**. We found the cosmic dynamics on the principles of the cosmic kinematics (first chapter) and on the existence of photons (2^{nd} bridge-postulate).

The cosmic space-time is then equipped with the cosmic metric (2.37)

$$\boxed{\begin{cases} g_{00} = -1 \\ g_{a0} = 0 \\ g_{ab} = a^2(t)\,\widetilde{g}_{ab}(\widetilde{q}) \end{cases}} \quad \boxed{g_{\alpha\beta}\,dq^\alpha\,dq^\beta = -c^2\,dt^2 + a^2(t)\,\widetilde{g}_{ab}\,dq^a\,dq^b} \tag{4.1}$$

The contravariant components are

$$\boxed{\begin{cases} g^{00} = -1 \\ g^{a0} = 0 \\ g^{ab} = a^{-2}(t)\,\widetilde{g}^{ab}(\widetilde{q}) \end{cases}} \tag{4.2}$$

$\mathbf{2}^{nd}$ **Postulate**. The evolution of the scale parameter $a(t)$ is governed by the **Einstein field equations**

$$\boxed{R^{\alpha\beta} + \left(\Lambda - \tfrac{1}{2}\,R\right)\,g^{\alpha\beta} = \chi\,T^{\alpha\beta}} \tag{4.3}$$

equivalent to

$$\boxed{G^{\alpha\beta} = \chi\,T^{\alpha\beta} - \Lambda\,g^{\alpha\beta}} \tag{4.4}$$

where $T^{\alpha\beta}$ is the **energy tensor**, $\Lambda \geq 0$ is the **cosmological constant** and the costant χ is given by

$$\chi = \frac{8\,\pi\,G_N}{c^4},$$

where G_N is the **Newtonian gravitational constant**.

Remark 4.1 – Dimensional analysis of the Einstein equations:

Table III

Object	Dim	Note
Λ	L^{-2}	[a]
$\chi\,T^{\alpha\beta}$	L^{-2}	[a]
$T^{\alpha\beta}$	$M\,L^{-1}\,T^{-2}$ (energy density)	[b]
χ	$M^{-1}\,L^{-1}\,T^{2}$	[c]
G_N	$M\,L^{3}\,T^{-2}$	[d]

[a] According to our conventions (Section 1.4) the coordinates q^α are length-dimensional, thus the metric tensor components $g_{\alpha\beta}$, $g^{\alpha\beta}$ are dimensionless and $\mathrm{Dim}\,(R^{\alpha\beta}) = \mathrm{Dim}\,(R) = L^{-2}$. Then from the Einstein equations (4.3) it follows that: $\mathrm{Dim}\,(\Lambda) = \mathrm{Dim}\,(\chi\,T^{\alpha\beta}) = L^{-2}$.

[b] Equations (4.12) shows that $\mathrm{Dim}\,(T^{oo}) = \mathrm{Dim}\,(e)$ and $\mathrm{Dim}\,(T^{ab}) = \mathrm{Dim}\,(p)$. From the entries of Table 1.4 (page 10) $\mathrm{Dim}\,(e) = \mathrm{Dim}\,(p) = M\,L^{-1}\,T^{-2}$.

[c] $\mathrm{Dim}\,(\chi) = \mathrm{Dim}\,(\chi\,T^{\alpha\beta})/\mathrm{Dim}\,(T^{\alpha\beta}) = L^{-2}/(M\,L^{-1}\,T^{-2})$.

[d] $\mathrm{Dim}\,(\chi) = L^{-4}\,T^{4}\cdot\mathrm{Dim}\,(G_N) \Longrightarrow \mathrm{Dim}\,(G_N) = \mathrm{Dim}\,(\chi)\,L^{4}\,T^{-4}$
$= M^{-1}\,L^{-1}\,T^{2}\,L^{4}\,T^{-4} = M^{-1}\,L^{3}\,T^{-2}$. •

4.2 The energy-momentum tensor

According the above postulates, the core of a cosmological model is the choice of an energy-momentum tensor $T^{\alpha\beta}$ for the galactic fluid. As for any isotopic symmetric tensor, it must have the form (4.5):

$$\boxed{\boldsymbol{T} = \phi(t)\,\partial_{0} \otimes \partial_{0} + \psi(t)\,\widetilde{g}^{ab}\,\partial_a \otimes \partial_b} \qquad \boxed{T^{\alpha\beta} : \begin{cases} T^{00} = \phi(t) \\ T^{a0} = 0 \\ T^{ab} = \psi(t)\,\widetilde{g}^{ab} \end{cases}} \tag{4.5}$$

The two characteristic functions $\phi(t)$ and $\psi(t)$ must satisfy the conservation equation

$$\boxed{a\,\dot{\phi} + 3\,\dot{a}\,\left(\phi + a^2\,\psi\right) = 0} \tag{4.6}$$

which is equivalent to the four conservation equations $\nabla_\alpha T^{\alpha\beta} = 0$ (Theorem 3.4). An equivalent form of this equation is

$$\boxed{(\phi\,a^3)^{\dot{}} = -3\,a^4\,\dot{a}\,\psi} \tag{4.7}$$

> *Nothing can be changed in the definition* (4.5) *and in equations* (4.6) *and* (4.7) *without violating our postulates. The only degrees of freedom we will have in the following will be the choice of the energy-momentum tensor (Section 4.2.1) and of the equation of state (Section 5.1).*

Theorem 4.1 – *Let $T^{\alpha\beta}$ be an isotropic energy-momentum tensor with characteristic functions $\phi(t)$ and $\psi(t)$. Then:*

(i) *The ten Einstein equations are equivalent to the two differential equations*

$$\boxed{\begin{cases} \dfrac{\dot{a}^2}{c^2} = \frac{1}{3}\,a^2\,(\Lambda + \chi\,\phi) - \widetilde{K} \\[2ex] 2\,\dfrac{\ddot{a}}{c^2} = a\left[\frac{2}{3}\,\Lambda - \chi\,(\psi\,a^2 + \frac{1}{3}\phi)\right] \end{cases}} \tag{4.8}$$

(ii) *Due to the conservation law* (4.6) *the second equation* (4.8) *is a differential consequence of the first one.*

Proof – (i) The contravariant components of the metric tensor and of the Einstein tensor are given in (3.12). Then the Einstein field equations

$$\begin{cases} G^{00} = \chi\,T^{00} - \Lambda\,g^{00} \\ G^{ab} = \chi\,T^{ab} - \Lambda\,g^{ab} \end{cases}$$

are equivalent to

$$\iff \begin{cases} \dfrac{3}{c^2\,a^2}\left(\dot a^2 + c^2\,\widetilde K\right) = \chi\,\phi + \Lambda \\ -\dfrac{1}{c^2\,a^4}\left(2\,a\,\ddot a + \dot a^2 + c^2\widetilde K\right)\,\widetilde g^{ab} = \chi\,\psi\,\widetilde g^{ab} - \Lambda\,a^{-2}\,\widetilde g^{ab} \end{cases}$$

$$\iff \begin{cases} \dot a^2 = \frac13\,c^2\,a^2\left(\chi\,\phi + \Lambda\right) - c^2\,\widetilde K \\ \left(2\,a\,\ddot a + \dot a^2 + c^2\widetilde K\right)\,\widetilde g^{ab} = c^2\,a^4\left(\Lambda\,a^{-2} - \chi\,\psi\right)\widetilde g^{ab} \end{cases}$$

$$\iff \begin{cases} \dot a^2 = c^2\left[\frac13\,a^2\,\left(\Lambda + \chi\,\phi\right) - \widetilde K\right] \\ 2\,a\,\ddot a + \dot a^2 + c^2\widetilde K = c^2\,a^2\left(\Lambda - \chi\,\psi\,a^2\right) \end{cases}$$

Substitute the first equation into the second one:

$$\iff \begin{cases} \dot a^2 = c^2\left[\frac13\,a^2\,(\Lambda + \chi\,\phi) - \widetilde K\right] \\ 2\,a\,\ddot a + c^2\left[\frac13\,a^2\,(\Lambda + \chi\,\phi) - \widetilde K\right] + c^2\widetilde K = c^2\,a^2\,(\Lambda - \chi\,\psi\,a^2) \end{cases}$$

$$\iff \begin{cases} \dot a^2 = c^2\left[\frac13\,a^2\,(\Lambda + \chi\,\phi) - \widetilde K\right] \\ 2\,a\,\ddot a + \frac13\,c^2\,a^2\,(\Lambda + \chi\,\phi) = c^2\,a^2\,(\Lambda - \chi\,\psi\,a^2) \end{cases}$$

$$\iff \begin{cases} \dot a^2 = c^2\left[\frac13\,a^2\,(\Lambda + \chi\,\phi) - \widetilde K\right] \\ 2\,a\,\ddot a = c^2\,a^2\left[\Lambda - \chi\,\psi\,a^2 - \frac13\,(\Lambda + \chi\,\phi)\right] \end{cases}$$

$$\iff \begin{cases} \dot a^2 = c^2\left[\frac13\,a^2\,(\Lambda + \chi\,\phi) - \widetilde K\right] \\ \ddot a = \frac12\,c^2\,a\left[\frac23\,\Lambda - \chi\left(\psi\,a^2 + \frac13\phi\right)\right]. \qquad \blacksquare \end{cases}$$

(ii) Differentiate the first equation (4.8):

$$\frac{2\,\dot a\,\ddot a}{c^2} = \tfrac23\,(\chi\,\phi + \Lambda)\,a\,\dot a + \tfrac13\,\chi\,a^2\,\dot\phi. \tag{4.9}$$

Substitute into this equation the conservation law (4.6) in the form $a\,\dot\phi = -3\,\dot a\,\left(\phi + a^2\,\psi\right)$. Then

$$(4.9) \implies \frac{2\,\dot a\,\ddot a}{c^2} = \tfrac23\,(\chi\,\phi + \Lambda)\,a\,\dot a - \chi\,\left(\phi + a^2\,\psi\right)\,a\,\dot a$$

$$\implies \frac{2\,\ddot a}{c^2\,a} = \tfrac23\,(\chi\,\phi + \Lambda) - \chi\,\left(\phi + a^2\,\psi\right)$$

$$\implies \frac{2\,\ddot a}{c^2\,a} = \tfrac23\,\Lambda - \chi\,(\tfrac13\,\phi + a^2\,\psi). \iff \text{second equation (4.8).} \quad \blacksquare$$

4.2.1 The energy-momentum tensor of the galactic fluid

Theorem 4.2 – *Let $\mathcal{V}(t)$ be the volume of any arbitrary co-moving portion of the galactic fluid, $\epsilon(t)$ and $p(t)$ the* **energy density** *and the* **pressure** *in that portion. Then the energy conservation law*

$$\boxed{\frac{d}{dt}\,(\epsilon\,\mathcal{V}) = -p\,\frac{d\mathcal{V}}{dt}} \tag{4.10}$$

holds if and only if the energy-momentum tensor of the galactic fluid is that of a **perfect fluid**[1]

$$\boxed{T^{\alpha\beta} = (e+p)\;U^\alpha\,U^\beta + p\,g^{\alpha\beta}} \tag{4.11}$$

where U^α is the unitary four-velocity of the galactic fluid

$$U^\alpha \stackrel{\text{def}}{=} c^{-1}\,\frac{d\gamma^\alpha}{dt} : \begin{cases} U^0 = 1, \\ U^a = 0, \end{cases} \qquad g_{\alpha\beta}\,U^\alpha\,U^\beta = -1.$$

Note that

$$\begin{cases} T^{00} = \epsilon + p - p = \epsilon, \\ T^{a0} = 0, \\ T^{ab} = p\,g^{ab}. \end{cases} \tag{4.12}$$

Lemma 4.1 – *The conservation law* (4.10) *is equivalent to the equation*

$$\boxed{a\,\dot\epsilon + 3\,(\epsilon+p)\,\dot a = 0} \tag{4.13}$$

Proof – Due to (1.45), $\dfrac{\mathcal{V}(t)}{a^3(t)} = \text{const.} = \dfrac{\mathcal{V}(t_*)}{a^3(t_*)}$.

$$\mathcal{V} = \frac{\mathcal{V}(t_*)}{a^3(t_*)}\,a^3(t), \quad \dot{\mathcal{V}} = 3\,\frac{\mathcal{V}(t_*)}{a^3(t_*)}\,a^2(t)\,\dot a(t) = 3\,\frac{\dot a}{a}\,\mathcal{V}.$$

$$(4.13) \iff \dot\epsilon\,\mathcal{V} + \epsilon\,\dot{\mathcal{V}} = -p\,\dot{\mathcal{V}} \iff \dot\epsilon\,\mathcal{V} + (\epsilon+p)\,\dot{\mathcal{V}} = 0$$
$$\iff \dot\epsilon + 3\,(\epsilon+p)\,h = 0. \quad \blacksquare$$

Proof of Theorem 4.2. Equation (4.13) is in perfect agreement with equation (4.6) $a\,\dot\phi + 3\,\dot a\,\left(\phi + a^2\,\psi\right) = 0$ with

$$\boxed{\phi = \epsilon(t), \quad \psi = a^{-2}\,p} \tag{4.14}$$

Then from (4.5) we get (4.12). $\blacksquare$

[1] See [25] p. 127, [15] p. 132, [11] p. 14, [12] p. 23. Due to the different conventions, there are changes of sign.

Remark 4.2 – Since $U^\beta U_\beta = -1$, the four-velocity U^α is an eigenvector of $T^{\alpha\beta}$ with eigenvalue $-\epsilon$:

$$T^{\alpha\beta} U_\beta = -\epsilon\, U^\alpha.$$

Moreover, any vector X^α orthogonal to U^α is an eigenvector of $T^{\alpha\beta}$ with eigenvalue p:

$$T^{\alpha\beta} X_\beta = p\, X^\alpha. \quad \bullet$$

Remark 4.3 – With the substitution (4.14) $\phi = \epsilon(t), \quad \psi = a^{-2}\, p$ the dynamical equations (4.8) read respectively

$$\boxed{\frac{\dot a^2}{c^2} = \tfrac{1}{3}\, a^2\, (\Lambda + \chi\, \epsilon) - \widetilde{K}} \tag{4.15}$$

$$\boxed{\frac{\ddot a}{c^2} = \tfrac{1}{2}\, a \left[\tfrac{2}{3}\, \Lambda - \chi\, (p + \tfrac{1}{3}\, \epsilon)\right]} \tag{4.16}$$

Equation (4.16) is a differential consequence of (4.15) and of the conservation law (4.13) (Theorem 4.1, item (iii)). •

Remark 4.4 – The **acceleration equation** (4.16) highlights a salient feature of the relativistic cosmic dynamics: it can be interpreted as a Newtonian dynamical equation of a point subjected to three forces competing with each other, namely

$$\begin{cases} F_\Lambda(a) \overset{\text{def}}{=} \tfrac{1}{3}\, c^2\, \Lambda\, a, \\ F_p(a) \overset{\text{def}}{=} -\tfrac{1}{2}\, c^2\, \chi\, p\, a, \\ F_\epsilon(a) \overset{\text{def}}{=} -\tfrac{1}{6}\, c^2\, \chi\, \epsilon\, a. \end{cases}$$

Since Λ and χ are positive constants, F_Λ acts as a centrifugal force (with center $a = 0$), as well as F_p and F_ϵ if $p(t)$ and $\epsilon(t)$ are negative. On the contrary, F_p and F_e are attractive towards $a = 0$ when $p(t)$ and $\epsilon(t)$ are positive. •

4.3 Comments on the Friedman equations

The equations of Friedman are definitely the most cited equations in the texts of cosmology, where they appear written in various different forms. Actually, by **Friedman equations** one should understand the dynamical equations appearing in the original work *Über die Krümmung des Raumes* by A. Friedman (1922) which are written exactly as follows:

$$\boxed{\begin{cases} (4) \quad \dfrac{R'^2}{R^2} + \dfrac{2\, R\, R''}{R^2} + \dfrac{c^2}{R^2} - \lambda = 0 \\ (5) \quad \dfrac{3\, R'^2}{R^2} + \dfrac{3\, c^2}{R^2} - \lambda = \varkappa\, c^2\, \varrho \end{cases}} \qquad \begin{cases} R' = \dfrac{dR}{dx_4}, \\ R'' = \dfrac{d^2 R}{dx_4^2}. \end{cases} \tag{4.17}$$

where ϱ is declared to be the density of mass and $\varkappa$ *eine Konstante*. The coordinate x_4 is time-dimensional and the signature of the metric is $(---+)$. These equations come from the Einstein field equations[2]

$$\text{(A)} \quad R_{ik} - \tfrac{1}{2}\, g_{ik}\, \bar{R} + \lambda\, g_{ik} = -\varkappa\, T_{ik}, \qquad \begin{cases} \bar{R} = g^{ik}\, R_{ik} \\ T_{ik} = 0, \quad i,k \neq 4 \\ T_{44} = c^2\, \varrho\, g_{44} \end{cases}$$

for $i = k = 1,2,3$ and for $i = k = 4$, respectively. Looking at the energy tensor components we observe that (i) *Friedman takes into account the cosmological constant*, (ii) the kinetic pressure p is not present, so *Friedman deals with a dust galactic fluid*. Furthermore, as proved below, (iii) *Friedman deals with a positive spatial curvature*.

In our theory we have seen that the Einstein equations determined by the energy-momentum tensor (4.11) reduce to the differential equations (4.15) and (4.16), namely

$$\begin{cases} \dfrac{\dot{a}^2}{c^2} = \tfrac{1}{3}\, a^2\, (\Lambda + \chi\, e) - \widetilde{K}, \\ \dfrac{\ddot{a}}{c^2} = \tfrac{1}{2}\, a \left[\tfrac{2}{3}\, \Lambda - \chi\, (p + \tfrac{1}{3}\, e)\right]. \end{cases} \tag{4.18}$$

Let us compare these equations with the Friedman equations (4.17). To do this we rewrite them as

$$\begin{cases} [4] \quad 2\, R\, R'' + {R'}^2 + c^2 - \lambda\, R^2 = 0, \\ [5] \quad {R'}^2 + c^2 - \tfrac{1}{3}\, \left(\lambda + \varkappa\, c^2 \varrho\right) R^2 = 0, \end{cases} \tag{4.19}$$

Subtract side by side [4] – [5]:

$$2\, R\, R'' - \lambda\, R^2 + \tfrac{1}{3}\, \left(\lambda + \varkappa\, c^2 \varrho\right) R^2 = 0.$$

Since $R \neq 0$, we get $2\, R'' - \frac{2}{3}\, \lambda\, R + \frac{1}{3}\, \varkappa\, c^2 \varrho\, R = 0$, i.e.

$$R'' = \tfrac{1}{6}\, R\, \left(2\, \lambda - \varkappa\, c^2 \varrho\right). \tag{4.20}$$

If in (4.20) we put

$$\begin{cases} dx_4 = dt \\ R = a \end{cases} \qquad \begin{cases} R' = \dot{a} \\ R'' = \ddot{a} \end{cases} \tag{4.21}$$

then we get the equation $\ddot{a} = \frac{1}{6}\, a\, \left(2\, \lambda - \varkappa\, c^2 \varrho\right)$ which coincides with the second equation (4.18) with $p = 0$,

$$\frac{\ddot{a}}{c^2} = \tfrac{1}{6}\, a\, (2\, \Lambda - \chi\, \epsilon)$$

[2] The comparison with our Einstein equations (4.3) $R^{\alpha\beta} + \left(\Lambda - \frac{1}{2}\, R\right) g^{\alpha\beta} = \chi\, T^{\alpha\beta}$ shows a difference of sign in the right side. This is due to the different signature of the metric.

provided that $2\,\lambda - \varkappa\, c^2 \varrho = c^2\,(2\,\Lambda - \chi\,\epsilon)$ i.e.

$$\lambda = c^2\,\Lambda, \quad \varkappa\,\varrho = \chi\,\epsilon. \tag{4.22}$$

In turn, due to the substitutions (4.21), the second equation [5] in (4.19) reads

$$\dot{a}^2 = \tfrac{1}{3}\,\big(\lambda + \varkappa\, c^2 \varrho\big)\, a^2 - c^2.$$

Due to (4.22),

$$\dot{a}^2 = \tfrac{1}{3}\, c^2\, a^2\,(\Lambda + \chi\,\epsilon) - c^2.$$

This equation coincides with our first equation (4.18)

$$\frac{\dot{a}^2}{c^2} = \tfrac{1}{3}\, a^2\,(\Lambda + \chi\,\epsilon) - \widetilde{K}$$

provided that $\widetilde{K} = 1$. This proves item (iii) above.

Chapter 5

Barotropic dynamics of a single-component universe

5.1 Preamble

The dynamics of the scale factor $a(t)$ is so far governed by the two first-order differential equations (4.15) and (4.13). These two equations involve three unknown functions $a(t)$, $\epsilon(t)$ and $p(t)$. We need a further equation. This equation should express the physical characteristics of the galactic fluid and should be dictated by physical arguments. The galactic fluid may have different components (typically mass, radiation, etc.). Assuming that the energy densities ϵ_i of these components are additive, we can write the total energy density ϵ as the sum

$$\epsilon = \sum_i \epsilon_i.$$

Beside the densities we have consider the pressures p_i of each component. Then we must assume that all of these variables are bound each other by means of certain **equations of state**. In the case in which there is no interaction between the components, these equations are separated between them, that is to say of the type

$$p_i = f_i(\epsilon_i).$$

In the simplest case, we can consider linear equations

$$p_i = w_i\, \epsilon_i$$

where w_i are dimensionless constants called **barotropic parameters**. In this chapter we confine our analysis to a single-component universe with the equation of state

$$\boxed{p = w\, \epsilon} \tag{5.1}$$

As a consequence, the **barotropic dynamics** will involve two functions in the cosmic time t, $a(t) > 0$ and $\epsilon(t)$, together with a constant parameter w. This dynamics is governed by the differential equations

$$\boxed{a\,\dot{\epsilon} + 3\,(w+1)\,\epsilon\,\dot{a} = 0} \tag{5.2}$$

$$\boxed{\frac{\dot{a}^2}{c^2} = \tfrac{1}{3}\,a^2\,(\Lambda + \chi\,\epsilon) - K_\sharp} \tag{5.3}$$

$$\boxed{\frac{\ddot{a}}{c^2} = \tfrac{1}{2}\,a\left[\tfrac{2}{3}\,\Lambda - \chi\,(w + \tfrac{1}{3})\,\epsilon\right]} \tag{5.4}$$

which are respectively called **fluid equation**, **velocity equation** and **acceleration equation**.

Equations (5.2) and (5.4) come from equations (4.13) and (4.16) with the substitution $p = w\,\epsilon$. Equation (5.3) is nothing but equation (4.15). The acceleration equation (5.4) is a differential consequence of the fluid and the velocity equations (Remark 4.3).

Our purpose is to find and classify all possible solutions $a(t)$ of these dynamical equations. We call them **profiles of the universe**. Such a classification should depend on the value of the parameter w and on the value of the spatial curvature $K_\sharp$, specially on its sign. Moreover, two profiles differing by a translation along the t-axis have to be considered equivalent.

5.2 Basic theorems

In the analysis of a barotropic dynamics it turns out to be convenient to replace the parameter w with the new parameter[1]

$$\boxed{u \overset{\text{def}}{=} w + 1} \tag{5.5}$$

and introduce the new constants

$$\boxed{\begin{aligned} \lambda &\overset{\text{def}}{=} \tfrac{1}{3}\,c^2\Lambda > 0 \\ \mu_\sharp &\overset{\text{def}}{=} \tfrac{1}{3}\,\chi\,c^2\,\epsilon_\sharp > 0 \end{aligned}} \qquad \boxed{\mathrm{Dim}\,(\lambda) = \mathrm{Dim}\,(\mu_\sharp) = T^{-2}} \tag{5.6}$$

$\epsilon_\sharp$ being the value of $\epsilon(t)$ at the normalization time $t_\sharp$.

[1] Table of conversion:

$u = w + 1$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$
w	$-\frac{4}{3}$	-1	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$

The parameter u is used by other authors. See e.g. [5], p. 33, where it is denoted by Γ.

Theorem 5.1 – *The evolution $a(t, t_\sharp)$ of the scale factor is governed by the single first-order differential equation*

$$\boxed{\dot a^2 = \lambda\, a^2 + \mu_\sharp\, a^{2-3u} - c^2\, K_\sharp} \tag{5.7}$$

which can also be written as

$$\boxed{h^2 = \frac{\dot a^2}{a^2} = \lambda + \frac{\mu_\sharp}{a^{3u}} - c^2\, \frac{K_\sharp}{a^2}} \tag{5.8}$$

Proof – (5.2) $\iff \dfrac{\dot\epsilon}{\epsilon} + 3u\,\dfrac{\dot a}{a} = 0 \iff d\log\epsilon + 3u\, d\log a = 0 \iff$

$$\boxed{\epsilon(t)\, a^{3u}(t, t_\sharp) = \text{constant in } t} \tag{5.9}$$

As $a(t_\sharp, t_\sharp) = 1$, equation (5.9) is equivalent to

$$\boxed{\epsilon(t)\, a^{3u}(t, t_\sharp) = \epsilon_\sharp} \tag{5.10}$$

Substituting the expression $\epsilon(t) = \epsilon_\sharp\, a^{-3u}(t, t_\sharp)$ coming from this last equation into the velocity equation (5.3) we get

$$\frac{\dot a^2}{c^2} = \tfrac{1}{3}\, a^2 \left[\Lambda + \chi\, \epsilon_\sharp\, a^{-3u}\right] - K_\sharp = \tfrac{1}{3}\left[a^2\, \Lambda + \chi\, \epsilon_\sharp\, a^{2-3u}\right] - K_\sharp \Longrightarrow (5.7), (5.6). \quad \blacksquare$$

Remark 5.1 – The dynamical equation (5.8) must be invariant under any change $t_\sharp \mapsto t_\flat$ of the normalization time of the scale parameter $a(t, t_\sharp)$, as explained in Remark 1.9, page 12. We know that the definition of the Hubble parameter h is invariant. Since also the constant λ is invariant, the invariant condition of (5.8) reduces to equation

$$\tfrac{1}{3}\,\chi\, \frac{\epsilon_\sharp}{a^{3u}(t, t_\sharp)} - \frac{K_\sharp}{a^2(t, t_\sharp)} = \tfrac{1}{3}\,\chi\, \frac{\epsilon_*}{a^{3u}(t, t_\flat)} - \frac{K_*}{a^2(t, t_\flat)}, \tag{5.11}$$

to be satisfied for all $t, t_\sharp, t_\flat$. For $t = t_\flat$ we get

$$\tfrac{1}{3}\,\chi\, \frac{\epsilon_\sharp}{a^{3u}(t_\flat, t_\sharp)} - \frac{K_\sharp}{a^2(t_\flat, t_\sharp)} = \tfrac{1}{3}\,\chi\, \epsilon_* - K_*.$$

Due to (1.41), $K(t_\flat) = \dfrac{K_\sharp}{a^2(t_\flat, t_\sharp)}$, this equation reduces to $\epsilon_\sharp = \epsilon_*\, a^{3u}(t_\flat, t_\sharp)$. This last equation still holds for all values of $(t_\flat, t_\sharp)$, and by putting $t_\flat = t$ we get equation (5.10) that, as we have seen above, is a consequence of the fluid equation (5.2). This proves that *the dynamical equation* (5.8) *satisfies the required invariance condition.* •

Remark 5.2 – By virtue of Theorem (1.11) (page 21) the following equations are equivalent:

$$\epsilon(t)\,\mathcal{V}^u(U,t) = \text{const.}\ \ \forall\, U = \text{co-moving domain}), \tag{5.12}$$

$$\epsilon(t)\,a^{3u}(t,t_\sharp) = \text{constant in } t \tag{5.13}$$

$$a\,\dot\epsilon + 3\,u\,\epsilon\,\dot a = 0 \quad \text{(fluid equation)}, \tag{5.14}$$

$$h = -\tfrac{1}{3u}\,\frac{\dot\epsilon}{\epsilon}. \tag{5.15}$$

This shows that the energy density $\epsilon(t)$ is a conserved density of order equal to the barotropic parameter u. •

Theorem 5.2 – *If the spatial curvature is negative then any physical length has a permanent superluminal expansion or contraction.*

Note that this theorem holds whatever u.

Proof – Let us go back to Section 1.6, equation (1.29), $\dot\ell(t) = \dot a(t,t_\sharp)\,\ell_\sharp$, where $\ell_\sharp$ is the length of a curve (not necessarily a geodesic) on the quotient manifold and $\ell(t)$ is the corresponding length on the spatial section S_t. Due to equation (5.7) we have

$$\dot\ell^2(t) = \dot a^2(t,t_\sharp)\,\ell_\sharp^2 = \left[\lambda\,a^2 + \mu_\sharp\,a^{-(3w+1)} - c^2\,K_\sharp\right]\ell_\sharp^2. \tag{5.16}$$

If $K_\sharp < 0$ then $\dot\ell^2 = \left[\lambda\,a^2 + \mu_\sharp\,a^{-(3\,w+1)} + c^2\,|K_\sharp|\right]\,\ell_\sharp^2$. This shows that $\dot\ell^2 > c^2$. ∎

Remark 5.3 – The superluminal condition $\ell > c^2$ of any physical length during all the life of the universe has no physical sense. Thus, our theory leads to consider <u>inadmissible</u> the barotropic models with <u>negative</u> spatial curvature. Similarly, the <u>positive</u> curvature models are not covered here because they are deemed to be incompatible as a result of astrophysical observations. •

5.3 The profiles of the barotropic flat models

In the vastness of the cosmological mathematical models, the flat barotropic models have the rare property that the whole evolution in time of the scale factor admits an analytical expression in terms of elementary functions (exponentials, or hyperbolic functions), whatever the value of the parameter u.

We will denote by $a(u;t,t_\sharp)$ the scale factor of a barotropic universe with barotropic parameter $u = w+1$ and reference time $t_\sharp$.

Theorem 5.3 – *The profiles $a(u;t,t_\sharp)$ of a barotropic flat model admit the following two equivalent representations*

$$\boxed{a(u;t,t_\sharp) = \left[\tfrac{1}{4}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\frac{(e^{u\beta t}-1)^2}{e^{u\beta t}}\right]^{\frac{1}{3u}} \quad \left\{\begin{array}{l}\text{exponential}\\ \text{form}\end{array}\right.} \tag{5.17}$$

$$\boxed{a(u;t,t_\sharp) = \left[\tfrac{1}{2}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\big(\cosh(u\beta t)-1\big)\right]^{\frac{1}{3u}} \quad \left\{\begin{array}{l}\text{hyperbolic}\\ \text{form}\end{array}\right.} \tag{5.18}$$

where

$$\boxed{\beta \overset{\text{def}}{=} \sqrt{3\,\Lambda}\,c} \quad \boxed{\mathrm{Dim}\,(\beta) = T^{-1}} \tag{5.19}$$

and $\epsilon_\sharp$ is the value of the energy density $\epsilon(t)$ at the refrence time $t_\sharp$.

Proof – With $K_\sharp = 0$ equation (5.8) reads

$$\frac{\dot a^2}{a^2} = \lambda + \frac{\mu_\sharp}{a^{3u}}$$

and is equivalent to

$$\frac{da}{a\,\sqrt{(1+b\,a^{-3u})}} = \sqrt{\lambda}\,dt, \quad b \overset{\text{def}}{=} \frac{\mu_\sharp}{\lambda}. \tag{5.20}$$

The left-hand side is integrable in terms of elementary functions:

$$\int \frac{da}{a\,\sqrt{(1+b\,a^{-3u})}} = \tfrac{1}{3u}\,\log\frac{\sqrt{a^{3u}+b}+\sqrt{a^{3u}}}{\sqrt{a^{3u}+b}-\sqrt{a^{3u}}} + \text{constant}.$$

Thus from (5.20) we get

$$\log\frac{\sqrt{a^{3u}+b}+\sqrt{a^{3u}}}{\sqrt{a^{3u}+b}-\sqrt{a^{3u}}} = 3u\,\sqrt{\lambda}\,(t-t_*)$$

with an arbitrary t_*. However, there is no loss of generality in assuming $t_* = 0$. In this case $a(0) = 0$.[2] By setting $\beta = 3\,\sqrt{\lambda} = \sqrt{3\,\Lambda}\,c$ we can write

$$\frac{\sqrt{a^{3u}+b}+\sqrt{a^{3u}}}{\sqrt{a^{3u}+b}-\sqrt{a^{3u}}} = e^{u\,\beta\,t}.$$

[2] The physical meaning of a scale factor is invariant under translations along the t-axis.

In order to solve this equation with respect to a^{3u} we put

$$\boxed{X \overset{\text{def}}{=} e^{u\beta t}} \tag{5.21}$$

and

$$A \overset{\text{def}}{=} \sqrt{a^{3u}+b}, \quad B \overset{\text{def}}{=} \sqrt{a^{3u}}.$$

Note that $A^2 - B^2 = b$. Then we have the following sequence of implications:
$\dfrac{A+B}{A-B} = X \Longrightarrow A+B = X\,(A-B) \Longrightarrow b = X\,(A-B)^2$ [†]
$\Longrightarrow b = X\,(A^2+B^2-2\,A\,B) \Longrightarrow b = X\,\big(2\,A^2+b-2\,A\,B\big)$
$\Longrightarrow \frac{1}{2}\,\dfrac{X-1}{X}\,b = B\,(A-B)$. Due to [†], $A-B = \sqrt{\dfrac{b}{X}}$. Then
$\Longrightarrow \frac{1}{2}\,\dfrac{X-1}{X}\,b = B\,\sqrt{\dfrac{b}{X}} \Longrightarrow \frac{1}{2}\,\dfrac{X-1}{\sqrt{X}}\,\sqrt{b} = B \Longrightarrow B^2 = \frac{1}{4}\,\dfrac{(X-1)^2}{X}\,b$
As $B^2 = a^{3u}$, $X = e^{u\,\beta\,t}$ and $b = \dfrac{\mu_\sharp}{\lambda}$, we finally get

$$a^{3u} = \tfrac{1}{4}\,\frac{\mu_\sharp}{\lambda}\,\frac{(e^{u\beta t}-1)^2}{e^{u\beta t}}.$$

Due to the definitions (5.6) we have

$$\frac{\mu_\sharp}{\lambda} = \frac{\chi}{\Lambda}\,\epsilon_\sharp, \tag{5.22}$$

$\epsilon_\sharp$ being the value of $\epsilon(t)$ at the normalization time $t_\sharp$, and (5.17) is proved. The profile (5.18) follows from (5.17) by observing that

$$2\,(\cosh(z)-1) = e^z + e^{-z} - 2 = e^{-z}\left[e^{2z}+1-2\,e^z\right] = \frac{(e^z-1)^2}{e^z}. \quad \blacksquare$$

Remark 5.4 – The profiles (5.17) and (5.17) show the convenience of introducing the **dimensionless time**

$$\boxed{x \overset{\text{def}}{=} \beta\,t} \tag{5.23}$$

so that they assume the form

$$\boxed{a(u;x,x_\sharp) = \left[\tfrac{1}{4}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\frac{(e^{ux}-1)^2}{e^{ux}}\right]^{\frac{1}{3u}} \quad \begin{cases}\text{exponential}\\ \text{form}\end{cases}} \tag{5.24}$$

$$\boxed{a(u;x,x_\sharp) = \left[\tfrac{1}{2}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\big(\cosh(ux)-1\big)\right]^{\frac{1}{3u}} \quad \begin{cases}\text{hyperbolic}\\ \text{form}\end{cases}} \tag{5.25}$$

These profiles will be plotted later on (Section 5.7) since we need more information about the magnitude of the constants that are involved. •

5.4 The profiles of the Hubble parameter

Theorem 5.4 – *The profiles $h(u;x)$ of the Hubble parameter of a barotropic flat model admit the following two equivalent representations*

$$\boxed{h(u;x) \stackrel{\text{def}}{=} \frac{1}{a}\,\frac{da}{dx} = \tfrac{1}{3}\,\frac{e^{ux}+1}{e^{ux}-1}} \tag{5.26}$$

$$\boxed{h(u;t) \stackrel{\text{def}}{=} \frac{1}{a}\,\frac{da}{dt} = \tfrac{1}{3}\,\beta\,\frac{e^{u\beta t}+1}{e^{u\beta t}-1}} \tag{5.27}$$

Proof – By setting, as above, $X \stackrel{\text{def}}{=} e^{ux}$ and observing that $X' = u\,x$ from (5.24) we get

$$3\,u\,\frac{d\log a}{dX} = 2\,\frac{d\log(X-1)}{dX} - \frac{d\log X}{dX} = \frac{2}{X-1} - \frac{1}{X} = \frac{X+1}{X\,(X-1)} \Longrightarrow$$

$$\boxed{\frac{da}{dX} = \frac{a}{3u}\,\frac{X+1}{X\,(X-1)}} \tag{5.28}$$

$$\Longrightarrow \frac{da}{dx} = \frac{da}{dX}\,u\,X = \frac{a}{3u}\,\frac{X+1}{X\,(X-1)}\,u\,X \Longrightarrow$$

$$\boxed{\frac{da}{dx} = \tfrac{1}{3}\,a\,\frac{X+1}{X-1} = \tfrac{1}{3}\,a\,\frac{e^{ux}+1}{e^{ux}-1}} \tag{5.29}$$

$\Longrightarrow$ (5.26). As $da/dt = \beta\,da/dx$ we get (5.27). ■

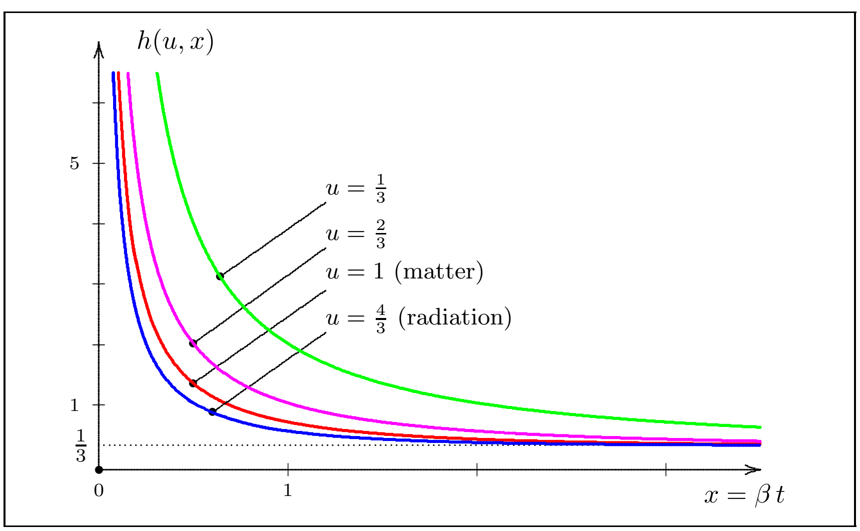


Figure 5.1: Graphs of $h(u;x)$.

Remark 5.5 – The evolution of the Hubble parameter does not depend on the normalization time $x_\sharp$, in accordance with Theorem 1.9, page 15. •

5.5 Cosmological data

The dimensionless time $x = \beta t$ has no practical significance as long as we do not know the value of the constant β. The estimates of β and other constants introduced in this theory are listed in Table 5.2 and are inferred from Table 5.1 containing basic cosmological data taken from [4].

name	symbol	estimate	note
tropical year (2011)		$31.5569522 \cdot 10^6\, s$	
speed of light	c	$299.792458 \cdot 10^3\, km\, s^{-1}$	
age of the universe	t_0	$\simeq 13.81 \pm 0.05\, Gyr$	
Hubble parameter today	H_0	$\simeq 0.6882972691 \cdot 10^{-10}\, yr^{-1}$	
gravitational constant	G_N	$\simeq 6.67408 \cdot 10^{-11}\, m^3\, kg^{-1} s^{-2}$	
dark energy density	Ω_Λ	$0.685^{+0.017}_{-0.016}$	

Table 5.1: Basic cosmological data.

constant	dimension	estimate	note
Λ	L^{-2}	$\simeq 1.087769524444 \cdot 10^{-52}\, m^{-2}$	[1]
β	T^{-1}	$\simeq \begin{cases} 5.41563983302 \cdot 10^{-18}\, s^{-1} \\ 0.170901087343\, Gyr^{-1} \end{cases}$	[2]
$\dfrac{1}{\beta}$	T	$\simeq \begin{cases} 0.18465038865819 \cdot 10^{18}\, s \\ 5.8513378442864\, Gyr \end{cases}$	
χ	$L^{-1} M^{-1} T^2$	$\simeq 2.07657899185574 \cdot 10^{-43}\, m^{-1}\, kg^{-1}\, s^2$	[3]
$\dfrac{\Lambda}{\chi}$	$L^{-1} M\, T^{-2}$	$\simeq 5.238276649771 \cdot 10^{-10}\, m^{-1}\, kg\, s^{-2}$	

Table 5.2: Supplementary data.

Notes.

[1] **Estimate of** $\Lambda \stackrel{\text{def}}{=} \dfrac{3H_0^2}{c^2}\,\Omega_\Lambda$:

$H_0 \simeq 0.6882972691 \cdot 10^{-10}\, yr^{-1}$

$H_0^2 \simeq 0.47375313065051 \cdot 10^{-20}\, yr^{-2}$

$3\,H_0^2 \simeq 1.42125939195 \cdot 10^{-20}\, yr^{-2}$

$3\,H_0^2/c \simeq 0.00474081103117677496 7434 \cdot 10^{-20}\, yr^{-2} \cdot 10^{-6}\, m^{-1}\, s$

$3\,H_0^2/c^2 \simeq 1.581364342119899149509 \cdot 10^{-5} \cdot 10^{-20}\, yr^{-2} \cdot 10^{-12}\, m^{-2}\, s^2$

$\Lambda = 3\,H_0^2/c^2 \cdot \Omega_\Lambda \simeq 1.581364342119899149509 \cdot 10^{-37}\, yr^{-2}\, m^{-2}\, s^2 \cdot 0.685$

$\Lambda \simeq 1.08324574352130917414 \cdot 10^{-37}\, yr^{-2}\, m^{-2}\, s^2$

$yr/s = 3.15569522 \cdot 10^7, \quad yr^2/s^2 = 9.9584123215308484 \cdot 10^{14}$

$\boxed{\Lambda \simeq 0.108776952444422833994 \cdot 10^{-51}\, m^{-2}}$ ■

[2] **Estimate of** $\beta \stackrel{\text{def}}{=} \sqrt{3\,\Lambda}\, c$:

$3\,\Lambda \simeq 3.26330857333326850198 2 \cdot 10^{-52}\, m^{-2}$

$\sqrt{3\,\Lambda} \simeq 1.806463000820462002629 \cdot 10^{-26}\, m^{-1}$

$c = 299.792458 \cdot 10^6\, m\, s^{-1}$

$\beta = \sqrt{3\,\Lambda}\, c \simeq 541.5639833020223204638 \cdot 10^{-26}\, m^{-1} \cdot 10^6\, m\, s^{-1}$

$\simeq 5.415639833020223204638 \cdot 10^{-18}\, s^{-1}.$

$1\, yr = 31.5569522 \cdot 10^6\, s.$

$\beta \simeq 1.709010873430351653021 \cdot 10^{-10}\, yr^{-1}.$ ■

[3] **Estimate of** $\chi \stackrel{\text{def}}{=} \dfrac{8\,\pi\, G_N}{c^4}$:

$c^{-4} \simeq 1.237990147236120239125 \cdot 10^{-34}\, m^{-4}\, s^4$

$\pi \simeq 3.14159265358979323846 26433$

$8\,\pi\, c^{-4} \simeq 31.11408601418833454066 \cdot 10^{-34}\, m^{-4}\, s^4$

$G_N = 6.67408(31) \cdot 10^{-11}\, m^3\, kg^{-1} s^{-2}$

$\chi \simeq 207.657899185574 \cdot 10^{-34}\, m^{-4}\, s^4 \cdot 10^{-11}\, m^3\, kg^{-1} s^{-2}$

$\boxed{\chi \simeq 2.07657899185574 \cdot 10^{-43}\, m^{-1}\, s^2\, kg^{-1}}$ ■

5.6 The age of the universe

Theorem 5.5 – *If H_0 is the present-day value of the Hubble parameter, then the age of the universe is*

$$x_0 = \beta\, t_0 = \frac{1}{u}\,\log\frac{3\,H_0+\beta}{3\,H_0-\beta} \tag{5.30}$$

Proof – Equation (5.26) is solvable with respect to $X = e^{ux}$:

$h(u,x) = \frac{1}{3}\,\dfrac{X+1}{X-1} \Longrightarrow 3\,(X-1)\,h(u,x) = X+1$

$\Longrightarrow [3\,h(u,x)-1]\,X = 3\,h(u,x)+1 \Longrightarrow X = \dfrac{3\,h(u,x)+1}{3\,h(u,x)-1}.$

Because of (5.27), $h(u,x) = \beta^{-1}\,h(u,t) \Longrightarrow X = \dfrac{3\,h(u,t)+\beta}{3\,h(u,t)-\beta}.$

As $X = e^{ux} \Longrightarrow$

$$x = \beta\, t = \frac{1}{u}\,\log\frac{3\,h(u,t)+\beta}{3\,h(u,t)-\beta} \tag{5.31}$$

The profile (5.17) satisfies the initial condition $a(0) = 0$ with $t_\alpha = 0$. Then the beginning of the universe corresponds to $t = x = 0$, so that equation (5.31) applied to the present epoch provides the age of the universe. ■

According to the formula (5.30), for computing the age of the universe we only need the values of H_0 and $\beta = \sqrt{3, \Lambda}\, c$:

$$\left.\begin{array}{l} H_0 \simeq 0.6882972691 \cdot 10^{-10}\, yr^{-1} \\ \beta \simeq 1.70901087343 \cdot 10^{-10}\, yr^{-1} \end{array}\right\} \Longrightarrow \begin{array}{l} u\,x_0 \simeq \log 10.6043969244822 \\ \simeq 2.36126871930698527084 9 \end{array} \Longrightarrow$$

$$u\,x_0 \simeq 2.3612687193 \tag{5.32}$$

$$u\,t_0 = \frac{u\,x_0}{\beta} \simeq 13.81658101781 \cdot 10^9\, yr \tag{5.33}$$

Remark 5.6 – For $u = 1$ this estimate is very close to that supplied by the astronomers [4] (2015) $t_0 \simeq 13.81 \pm 0.05\; Gyr$. This means that the primordial phase of radiation dominance has an irrelevant influence on the evaluation of the present-day age of the universe. •

5.7 The *‘exact’* profiles of the flat barotropic universes

If we consider sufficiently reliable the estimate of the age of the universe found above, then we should consider equally reliable the choice of the present-day time t_0 as reference time for the scale factor. In doing so we get a ‘sufficiently reliable’ (or ‘*exact*’) numerical evaluation of the universe profile, for any value of the parameter u.

Theorem 5.6 – *With the present-day reference time x_0 the profiles of the universe are*

$$\boxed{a(u;x,x_0) = \Big[\mathsf{c}_0\,\big(\cosh(ux)-1\big)\Big]^{\frac{1}{3u}} = \left[\tfrac{1}{2}\,\mathsf{c}_0\,\frac{(e^{ux}-1)^2}{e^{ux}}\right]^{\frac{1}{3u}}} \tag{5.34}$$

$$\boxed{\mathsf{c}_0 \overset{\text{def}}{=} \frac{1}{\cosh(ux_0)-1} \simeq 0.2299194811} \tag{5.35}$$

Proof – With $x_\sharp = x_0$ the profiles (5.25) read

$$a(u;x,x_0) = \Big[\tfrac{1}{2}\,\frac{\chi}{\Lambda}\,\epsilon_0\,\big(\cosh(ux)-1\big)\Big]^{\frac{1}{3u}}.$$

By imposing the normalization condition $a(u;x_0,x_0)=1$ we get

$$\tfrac{1}{2}\,\frac{\chi}{\Lambda}\,\epsilon_0\,\big(\cosh(ux_0)-1\big) = 1$$

i.e.

$$\boxed{\tfrac{1}{2}\,\frac{\chi}{\Lambda}\,\epsilon_0 = \mathsf{c}_0} \tag{5.36}$$

with c_0 defined by (5.35). ■

The graphs of $a(u;x,x_0)$ are plotted in Fig. 5.2 with respect to the variable ux for some relevant values of u. Whatever u, they all pass through the point $(x_0,1)$, as expected. In Fig. 5.3 the profiles are plotted with respect to the variable x. In both representation we observe (i) a different way of approaching the origin $x=0$ and (ii) the presence of inflection points x_{ip} for certain values of u.

Theorem 5.7 – (i) *The profiles approach the beginning of the universe $x=0$*

in different ways:

$$(5.37)\qquad \begin{cases} u > \frac{2}{3} \quad \Longrightarrow \quad \lim_{x\to 0} \dfrac{da}{dx} = +\infty. \\ u = \frac{2}{3} \quad \Longrightarrow \quad \lim_{x\to 0} \dfrac{da}{dx} = \frac{1}{3}\sqrt{2\,\mathsf{c}_0}. \\ u < \frac{2}{3} \quad \Longrightarrow \quad \lim_{x\to 0} \dfrac{da}{dx} = 0. \end{cases}$$

(ii) *For $u > \frac{2}{3}$ there is an inflection point at the time*

$$(5.38)\qquad \boxed{x_{ip}(u) = \frac{1}{u}\,\log\left[3\,u - 1 + \sqrt{(3\,u-1)^2 - 1}\right] = \frac{1}{u}\,\mathrm{arccosh}(3\,u - 1)}$$

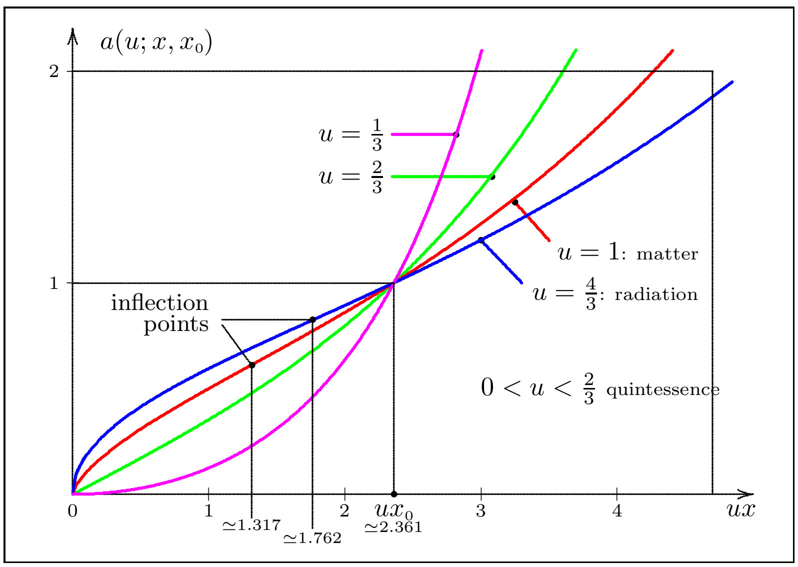


Figure 5.2: Graphs of $a(u;x,x_0)$ in the variable ux.

Remark 5.7 – This theorem indicates the value $u = \frac{2}{3}$ (corresponding to $w = -\frac{1}{3}$) as a **threshold parameter**: for any small variation of this value the profile $a(u,x)$ changes radically. Due to this sort of 'instability' we should consider **inadmissible** the case $u = \frac{2}{3}$. •

Proof – (i) (5.26) and (5.34) $\Longrightarrow$

$$\frac{da}{dx} = \frac{1}{3}\,a\,\frac{e^{ux}+1}{e^{ux}-1} = \frac{1}{3}\,\left(\tfrac{1}{2}\,\mathsf{c}_0\right)^{\frac{1}{3u}}\left[\frac{(e^{ux}-1)^2}{e^{ux}}\right]^{\frac{1}{3u}}\frac{e^{ux}+1}{e^{ux}-1}$$
$$= \frac{1}{3}\,\left(\tfrac{1}{2}\,\mathsf{c}_0\right)^{\frac{1}{3u}}\,e^{-\frac{x}{3}}\,\left(e^{ux}+1\right)\left(e^{ux}-1\right)^{\frac{2}{3u}-1}.$$

$$\lim_{x\to 0}\frac{da}{dx} = \tfrac{2}{3}\left(\tfrac{1}{2}\,\mathsf{c}_0\right)^{\frac{1}{3u}}\left(e^{ux}-1\right)^{\frac{2}{3u}-1} =$$

$$= \begin{cases} 0 & \iff \frac{2}{3u}-1>0 \iff \frac{2}{3}>u, \\ \frac{1}{3}\sqrt{2\,\mathsf{c}_0} & \iff \frac{2}{3u}-1=0 \iff \frac{2}{3}=u, \\ +\infty & \iff \frac{2}{3u}-1<0 \iff \frac{2}{3}<u. \end{cases}$$

Note that $\frac{1}{3}\sqrt{2\,\mathsf{c}_0} \simeq 0.2260378$.

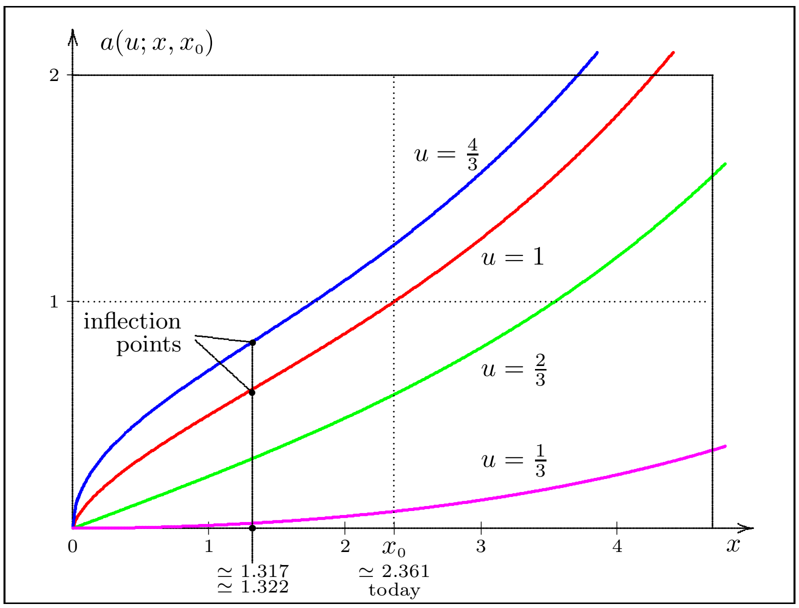


Figure 5.3: Graphs of $a(u;x,x_0)$ in the variable x.

(ii) (5.29) $\Longrightarrow$ $\dfrac{d^2a}{dx^2} = \frac{1}{3}\left[\dfrac{da}{dx}\,\dfrac{X+1}{X-1} + a\,\dfrac{d}{dx}\,\dfrac{X+1}{X-1}\right]$

$$= \tfrac{1}{3}\,a\left[\tfrac{1}{3}\,\frac{(X+1)^2}{(X-1)^2} + \frac{u\,X\,(X-1)-(X+1)\,u\,X}{(X-1)^2}\right] = \tfrac{1}{3}\,a\,\frac{\frac{1}{3}\,(X+1)^2-2\,u\,X}{(X-1)^2}.$$

$$\frac{d^2a}{dx^2}=0 \iff \tfrac{1}{3}\,(X+1)^2-2\,u\,X=0 \iff X^2+2\,(1-3u)\,X+1=0$$

$\iff$ $X = 3u-1 \pm \sqrt{(3u-1)^2-1}$. With the $-$ sign $X<1$, rejected. $X=e^{ux} \Longrightarrow$ (5.38). ■

Remark 5.8 – The inflection point marks the transition from decelerated to accelerated expansion.

Component	w	u	$x_{ip}(u)$	$t_{ip}(u) = \beta^{-1}\, x_{ip}(u)$
Matter	0	1	1.316958	$7.70601\, Gyr$
Radiation	$\frac{1}{3}$	$\frac{4}{3}$	1.322067	$7.73591\, Gyr$

Table 5.3: Estimate of the inflection time.

The two times $x_{ip}(1)$ and $x_{ip}(\frac{4}{3})$ are very close. The phase of accelerated expansion starts $\simeq 6.08 - 6.11$ billion years ago. Note that x_{ip} does not depend on the choice of the normalization time. •

5.8 The profiles of the energy density

Theorem 5.8 – *In the barotropic flat models the profiles $e(u;x)$ of the energy density do not depend on the reference time $t_\sharp$ and admit the following two equivalent representations*

$$\boxed{\epsilon(u;x) = \frac{\Lambda}{\chi}\,\frac{4\,e^{ux}}{(1-e^{ux})^2}} \qquad \boxed{\epsilon(u;x) = \frac{\Lambda}{\chi}\,\frac{2}{\cosh(ux)-1}} \tag{5.39}$$

Proof – $$\begin{cases} (5.24) \implies a^{3u}(u;x,x_\sharp) = \frac{1}{4}\,\epsilon_\sharp\,\dfrac{\chi}{\Lambda}\,\dfrac{(e^{ux}-1)^2}{e^{ux}}. \\ (5.25) \implies a^{3u}(u;x,x_\sharp) = \frac{1}{2}\,\epsilon_\sharp\,\dfrac{\chi}{\Lambda}\,\big(\cosh(ux)-1\big). \end{cases}$$

$$(5.10) \implies \epsilon(u;x,x_\sharp) = \frac{\epsilon_\sharp}{a^{3u}(u;x,x_\sharp)} \implies (5.39). \ \blacksquare$$

Remark 5.9 – The evolution of the energy density does not depend on the choice of the normalization time but only on the parameter u and on the ratio Λ/χ. As a consequence, since we know a 'reliable' numerical value of Λ/χ (see Table 5.2)

$$\frac{\Lambda}{\chi} \simeq 5.238276649771 \cdot 10^{-10}\, m^{-1}\, kg\, s^{-2} \tag{5.40}$$

then we can get a 'reliable' numerical estimate of the evolution of the energy density for any value of the parameter u (Fig. 5.4). The formula to be used for plotting $\epsilon(u;x)$ is

$$\epsilon(u;x) = 5.238276649771 * 2\,\frac{1}{\cosh(u*x)-1}. \quad \bullet$$

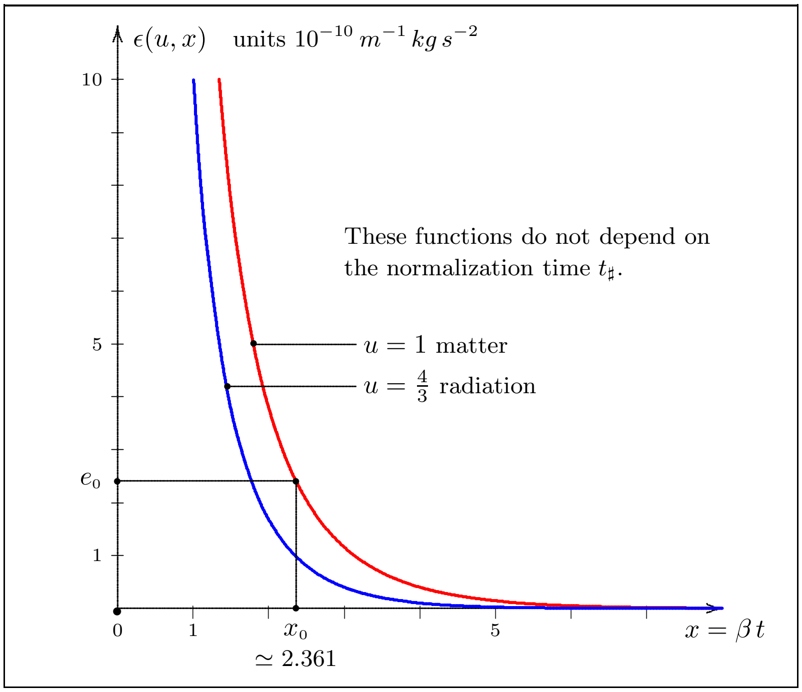


Figure 5.4: Graphs of $\epsilon(u, x)$.

Remark 5.10 – The present-day value ϵ_0 of the energy density does not depend on the barotropic parameter u. Due to (5.36), (5.35) and (5.40) we have

$$\epsilon_0 = 2\,\mathsf{c}_0\,\frac{\Lambda}{\chi} \simeq 2 * 0.2299194811 * 5.238276649771 \cdot 10^{-10}\,m^{-1}\,kg\,s^{-2},$$

$$\epsilon_0 \simeq 2.408763697 \cdot 10^{-10}\,m^{-1}\,kg\,s^{-2}. \quad \bullet \tag{5.41}$$

5.9 Superluminal recession speed and the Hubble radius

Let $\ell_\sharp$ be the distance of two galaxies at the reference time $t_\sharp$. This distance evolves with time according to the law (1.28)

$$\ell(u;t,t_\sharp) = a(u;t,t_\sharp)\,\ell_\sharp, \tag{5.42}$$

with an expansion speed $\dot\ell(u;t,t_\sharp) = \dot a(u;t,t_\sharp)\,\ell_\sharp$ that may becomes greater than the light speed c.

Theorem 5.9 – *The recession speed* $\dot\ell(u;t,t_\sharp)$ *is superluminal*

$$\dot\ell(u;t,t_\sharp) \ge c$$

in the time interval defined by the inequality

$$\boxed{(X+1)^{3u}\,(X-1)^{2-3u} \ge \mathsf{C}\,X} \quad \boxed{X \overset{\text{def}}{=} e^{ux}} \quad \boxed{x \overset{\text{def}}{=} \beta t} \tag{5.43}$$

where the dimensionless constant C *is defined by*

$$\mathsf{C} \overset{\text{def}}{=} 4 \cdot 3^{\frac{3u}{2}}\,\frac{\Lambda^{1-\frac{3u}{2}}}{\chi\,\epsilon_\sharp\,\ell_\sharp^{3u}}. \tag{5.44}$$

This constant does not depend on the choice of the reference time $t_\sharp$.

Proof – (i) $\dot\ell(u;t,t_\sharp) = \beta\,\ell'(u;x,x_\sharp)$ and $\ell' = a'\,\ell_\sharp \Longrightarrow$

$$\dot\ell \ge c \iff \ell' \ge \frac{c}{\beta} = \frac{1}{\sqrt{3\Lambda}} \iff a'\,\ell_\sharp \ge \frac{1}{\sqrt{3\Lambda}} \iff$$

$$a' \ge \frac{1}{\sqrt{3\Lambda}\,\ell_\sharp}. \tag{5.45}$$

(ii) (5.29): $a' = \dfrac{da}{dx} = \frac{1}{3}\,a\,\dfrac{X+1}{X-1}$. (5.24): $a(u;x,x_\sharp) = \left[\frac{1}{4}\,\dfrac{\chi}{\Lambda}\,\epsilon_\sharp\,\dfrac{(X-1)^2}{X}\right]^{\frac{1}{3u}}$

$$\Longrightarrow a' = \tfrac{1}{3}\left[\tfrac{1}{4}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\frac{(X-1)^2}{X}\right]^{\frac{1}{3u}}\frac{X+1}{X-1}$$

$$\Longrightarrow [a']^{3u} = [\tfrac{1}{3}]^{3u}\,\tfrac{1}{4}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\frac{(X-1)^2}{X}\,\frac{(X+1)^{3u}}{(X-1)^{3u}}.$$

(iii) (5.45) $\iff [a']^{3u} \ge \dfrac{1}{[\sqrt{3\Lambda}]^{3u}\,\ell_\sharp^{3u}}$

$$\iff \quad [\tfrac{1}{3}]^{3u}\,\tfrac{1}{4}\,\frac{\chi}{\Lambda}\,\epsilon_\sharp\,\frac{(X-1)^2}{X}\,\frac{(X+1)^{3u}}{(X-1)^{3u}} \ge \frac{1}{[\sqrt{3\Lambda}]^{3u}\,\ell_\sharp^{3u}}$$

$$\iff \quad \frac{(X-1)^{2-3u}\,(X+1)^{3u}}{X} \ge \frac{4\cdot 3^{3u}\,\Lambda}{\chi\,[\sqrt{3\Lambda}]^{3u}\,\epsilon_\sharp\,\ell_\sharp^{3u}} = \frac{4\cdot 3^{\frac{3u}{2}}}{\chi\,\Lambda^{\frac{3u-2}{2}}\,\epsilon_\sharp\,\ell_\sharp^{3u}} \Longrightarrow$$

$$\dot\ell \ge c \iff (X-1)^{2-3u}\,(X+1)^{3u} \ge \mathsf{C}\,X,$$

where C is defined as in (5.44). (iv) By virtue of (5.10) $\epsilon(t)\,a^{3u}(t,t_\sharp)\,\ell_\sharp^{3u} = \epsilon_\sharp\,\ell_\sharp^{3u} =$ constant in t, i.e.

$$\epsilon(t)\,\ell^{3u}(t) = \epsilon_\sharp\,\ell_\sharp^{3u}, \quad \forall\, t.$$

This shows that the product $\epsilon_\sharp\,\ell_\sharp^{3u}$ does not depend on the choice of the reference time $t_\sharp$. Hence, also C is independent. ∎

Theorem 5.10 – *The constant* C *has the form*

$$\boxed{\mathsf{C} = \left(\frac{\mathsf{L}_\sharp}{\ell_\sharp}\right)^{3u}} \tag{5.46}$$

where

$$\boxed{\mathsf{L}_\sharp \stackrel{\text{def}}{=} \sqrt{\frac{3}{\Lambda}} \left(\frac{(1-e^{ux_\sharp})^2}{e^{ux_\sharp}}\right)^{\frac{1}{3u}} = \sqrt{\frac{3}{\Lambda}} \left[2\left(\cosh(ux_\sharp)-1\right)\right]^{\frac{1}{3u}}} \tag{5.47}$$

Proof – (5.44): $\mathsf{C} \stackrel{\text{def}}{=} 4 \cdot 3^{\frac{3u}{2}} \dfrac{\Lambda^{1-\frac{3u}{2}}}{\chi\, \epsilon_\sharp\, \ell_\sharp^{3u}}$.

(5.39): $\epsilon(u;x) = \dfrac{\Lambda}{\chi} \dfrac{4\, e^{ux}}{(1-e^{ux})^2} = \dfrac{\Lambda}{\chi} \dfrac{2}{\cosh(ux)-1}$

$$\mathsf{C} = 4 \cdot 3^{\frac{3u}{2}} \frac{\Lambda^{1-\frac{3u}{2}}}{\chi\, \ell_\sharp^{3u}} \cdot \frac{\chi}{\Lambda} \frac{(1-e^{ux_\sharp})^2}{4\, e^{ux_\sharp}} = 3^{\frac{3u}{2}} \frac{\Lambda^{-\frac{3u}{2}}}{\ell_\sharp^{3u}} \cdot \frac{(1-e^{ux_\sharp})^2}{e^{ux_\sharp}}$$

$$\mathsf{C} = \left(\frac{3}{\Lambda}\right)^{\frac{3u}{2}} \frac{(1-e^{ux_\sharp})^2}{e^{ux_\sharp}} \cdot \frac{1}{\ell_\sharp^{3u}} = \left[\left(\frac{3}{\Lambda}\right)^{\frac{1}{2}} \left(\frac{(1-e^{ux_\sharp})^2}{e^{ux_\sharp}}\right)^{\frac{1}{3u}} \cdot \frac{1}{\ell_\sharp^{3u}}\right]^{3u} \implies (5.47).$$

The alternative expression of $\mathsf{L}_\sharp$ follows from $2\left(\cosh(z)-1\right) = \dfrac{(e^z-1)^2}{e^z}$. ∎

We can continue the analysis of the superluminal recession speed only with the specification of the barotropic parameter u. We will consider the case $u=1$: dust-matter universe.

Theorem 5.11 – *For* $u=1$ *the superluminal expansion condition* (5.43) *is equivalent to*

$$\boxed{\begin{aligned} &\dot{\ell}(1;t,t_\sharp) \geq c \\ &\iff f(\mathsf{C};X) \stackrel{\text{def}}{=} X^3 + (3-\mathsf{C})\,X^2 + (3+\mathsf{C})\,X + 1 \geq 0 \end{aligned}} \tag{5.48}$$

with the constant C *given by*

$$\boxed{\mathsf{C} = \left(\frac{\mathsf{L}_\sharp}{\ell_\sharp}\right)^3} \quad \boxed{\mathsf{L}_\sharp = \sqrt{\frac{3}{\Lambda}} \sqrt[3]{\frac{(X_\sharp-1)^2}{X_\sharp}}} \quad \boxed{\begin{aligned} X &\stackrel{\text{def}}{=} e^x \\ X_\sharp &\stackrel{\text{def}}{=} e^{x_\sharp} \end{aligned}} \tag{5.49}$$

Proof – (5.43) $\implies (X+1)^3\,(X-1)^{-1} \geq \mathsf{C}\,X \iff (X+1)^3 \geq \mathsf{C}\,X\,(X-1)$
$\iff X^3 + 3\,X^2 + 3\,X + 1 \geq \mathsf{C}\,X^2 - \mathsf{C}\,X$

$\iff X^3 + 3\,X^2 - \mathsf{C}\,X^2 + \mathsf{C}\,X + 3\,X + 1 \geq 0 \iff$ (5.48). ∎

According to this theorem the analysis of the occurrence of the superluminal phenomenon is reduced to the analysis of the roots of the cubic polynomial $f(\mathsf{C}; X)$, whose coefficients depends on the constant C. The graphs of $f(\mathsf{C}, X)$ are plotted in Fig. 5.5 for various values of C. Some relevant facts must be highlighted:

1. As shown by (5.49) C is the cube of the ratio of two lengths, $\mathsf{L}_\sharp$ and $\ell_\sharp$. $\mathsf{L}_\sharp$ has the property of being computable regardless of the given value of galactic distance $\ell_\sharp$ (see item 7 below).
2. The analysis makes sense only in the interval $X = e^x \geq 1$, corresponding to $t \geq 0$, since $t = 0$ is the date of birth of the universe.
3. Whatever C, $f(\mathsf{C}, 0) = 1$ and $f(\mathsf{C}, 1) = 8$: all graphs pass through the points (0,1) and (1,8). Furthermore, $f(\mathsf{C}, X)$ has a real negative root close to $X = 0$ and the other real roots (if any) are located in the unbounded interval $X > 1$.

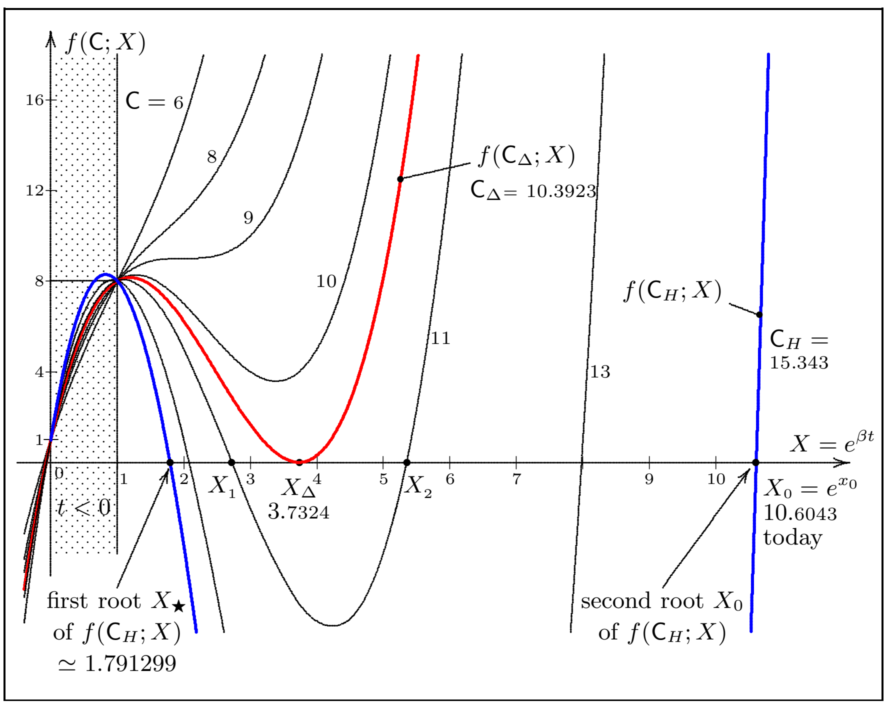


Figure 5.5: Graphs of $f(\mathsf{C}; X)$.

4. There exists a discriminant value

$$\boxed{\mathsf{C}_\Delta \simeq 10.3923}$$

for which $f(\mathsf{C}_\Delta, X)$ is tangent to the X-axis at a point $X_\Delta \simeq 3.7324$. The point X_Δ corresponds to the value[3]

$$\boxed{t_\Delta \simeq 7.706513\, Gyr}$$

of the cosmic time.

5. For $\mathsf{C} < \mathsf{C}_\Delta$, there are no real roots $X > 1$ of $f(\mathsf{C}; X)$ and we have a permanent superluminal recession speed. For $\mathsf{C} > \mathsf{C}_\Delta$ the polynomial $f(\mathsf{C}; X)$ has two (positive) simple roots $X_1 < X_2$. The recession speed is subluminal in the interval (X_1, X_2) delimited by these roots and containing X_Δ.

6. There exists a special value $\mathsf{C}_H > \mathsf{C}_\Delta$ given by[4]

$$\boxed{\mathsf{C}_H = \frac{(e^{x_0}+1)^3}{e^{x_0}\,(e^{x_0}-1)} \simeq 15.34304824} \tag{5.50}$$

such that the polynomial $f(\mathsf{C}_H, X)$ has a root

$$X_0 \simeq 10.604396924 > X_\Delta \tag{5.51}$$

corresponding to the present-day time

$$t_{\scriptscriptstyle 0} \simeq 13.816581\, Gyr \quad (x_{\scriptscriptstyle 0} \simeq 2.3612687193) \tag{5.52}$$

and a root

$$X_\star \simeq 1.791299 < X_\Delta \tag{5.53}$$

corresponding to the cosmic time

$$t_\star \simeq 3.41098505\, Gyr \quad (x_\star \simeq 0.582941054). \tag{5.54}$$

---

[3] $X_\Delta = e^{\beta t_\Delta} \Longrightarrow \beta\, t_\Delta = \log X_\Delta \simeq 1.317051 \Longrightarrow t_\Delta = \beta^{-1}\, \log X_\Delta \simeq 7.706513.$

[4] Proof – $f(\mathsf{C}, X_0) \overset{\text{def}}{=} X_0^3 + (3-\mathsf{C})\, X_0^2 + (3+\mathsf{C})\, X_0 + 1$ with $X \overset{\text{def}}{=} e^{x_0}$, $x_0 \overset{\text{def}}{=} \beta t_0$.
$f(\mathsf{C}, X_0) = 0 \iff X_0^3 + (3-\mathsf{C})\, X_0^2 + (3+\mathsf{C})\, X_0 + 1 = 0$

$\iff X_0^3 + 3\, X_0^2 - \mathsf{C}\, X_0^2 + 3\, X_0 + \mathsf{C}\, X_0 + 1 = 0$

$\iff X_0^3 + 3\, X_0^2 + 3\, X_0 + 1 = \mathsf{C}\, X_0\, (X_0 - 1) \iff (X_0+1)^3 = \mathsf{C}\, X_0\, (X_0 - 1)$

$\iff \mathsf{C}_H = \dfrac{(X_0+1)^3}{X_0\,(X_0-1)}.\quad \mathsf{C}_H \simeq \dfrac{10.60439692 + 1)^3}{(10.60439692 \cdot (10.60439692 - 1)}$
$\simeq 15.34304824.$

The estimate (5.53) of the first root $X_1 < X_\Delta$ is a matter of numerical analysis. It follows that

$$t_\star = \beta^{-1} \log X_\star \simeq \frac{0.582941054}{0.170901087343}\, Gyr \simeq 3.41098505\, Gyr. \quad \blacksquare$$

7. Since the definition of C does not depend on the choice of the reference time $t_\sharp$ (Theorem 5.9) we can write the definition (5.49) for $\mathsf{C} = \mathsf{C}_H$ by taking the reference times $t_\star$ and t_0 of item 6 above:

$$\text{(5.55)} \qquad \begin{cases} \mathsf{C}_H = \left(\dfrac{\mathsf{L}(t_\star)}{\ell(t_\star)}\right)^3, \quad \mathsf{L}(t_\star) = \sqrt{\dfrac{3}{\Lambda}}\,\sqrt[3]{\dfrac{(X_\star - 1)^2}{X_\star}}. \\ \mathsf{C}_H = \left(\dfrac{\mathsf{L}(t_0)}{\ell(t_0)}\right)^3, \quad \mathsf{L}(t_0) = \sqrt{\dfrac{3}{\Lambda}}\,\sqrt[3]{\dfrac{(X_0 - 1)^2}{X_0}}. \end{cases}$$

Since $\mathsf{L}(t_\star)$ and $\mathsf{L}(t_0)$ are computable (see below), and C_H is known (item 4), from (5.55) we can derive the lengths $\ell(t_\star)$ and $\ell(t_0)$:

$$\text{(5.56)} \qquad \begin{cases} \ell(t_\star) = \dfrac{\mathsf{L}(t_\star)}{\sqrt[3]{\mathsf{C}_H}}, \\ \ell(t_0) = \dfrac{\mathsf{L}(t_0)}{\sqrt[3]{\mathsf{C}_H}}, \end{cases} \qquad \mathsf{C}_H = \frac{(e^{x_0}+1)^3}{e^{x_0}\,(e^{x_0}-1)} \simeq 15.34304824.$$

For the way in which we arrived to its definition, $\ell(t_0)$ is the present-day distance of two galaxies crossing the boundary beyond which the recession velocity exceeds the speed of light. This boundary is called **Hubble radius** (of the **Hubble sphere**). In turn, $\ell(t_\star)$ is the distance at the time t_* at the early universe when their recession speed crossed this boundary in the opposite sense: from superluminal to subluminal.

Computation of $\ell(t_\star)$ and $\ell(t_0)$.

1. $\Lambda \simeq 1.0877695244444422834 \cdot 10^{-52}\, m^{-2}$ (table 5.2) $\Longrightarrow$

$\Longrightarrow \dfrac{3}{\Lambda} \simeq 2.7579371664528356 \cdot 10^{52}\, m^2$

$\Longrightarrow \sqrt{\dfrac{3}{\Lambda}} \simeq 1.66070381659489 \cdot 10^{26}\, m = 1.66070381659489 \cdot 10^{23}\, km.$

Conversion to light-years: $10^{23}\, km \simeq 10.570234105227\, Gly \Longrightarrow$

$$\boxed{\sqrt{\frac{3}{\Lambda}} \simeq 17.554028120851\, Gly}$$

2. $\mathsf{C}_H \simeq 15.34304824 \Longrightarrow \boxed{\sqrt[3]{\mathsf{C}_H} \simeq 2.4848712052}$

3. $X_\star \simeq 1.791299 \Longrightarrow \dfrac{(X_\star - 1)^2}{X_\star} \simeq 0.349553 \Longrightarrow \sqrt[3]{\dfrac{(X_\star - 1)^2}{X_\star}} \simeq 0.7044297$

$\Longrightarrow \mathsf{L}(t_\star) \simeq 17.55402812 \cdot 0.7044297\, Gly \Longrightarrow \boxed{\mathsf{L}(t_\star) \simeq 12.36557\, Gly}$

4. $\ell(t_\star) = \dfrac{\mathsf{L}(t_\star)}{\sqrt[3]{\mathsf{C}_H}} \simeq \dfrac{12.36557}{2.4848712052}\,Gly \Longrightarrow \boxed{\ell(t_\star) \simeq 4.976342\,Gly}$

5. $X_0 \simeq 10.604396924 \Longrightarrow \dfrac{(X_0-1)^2}{X_0} \simeq 8.69869743 \Longrightarrow \sqrt[3]{\dfrac{(X_0-1)^2}{X_0}} \simeq 2.056607463 \Longrightarrow \mathsf{L}(t_0) \simeq 17.55402812 \cdot 2.056607463\,Gly \Longrightarrow$

$\boxed{\mathsf{L}(t_0) \simeq 36.10174523\,Gly}$

6. $\ell(t_0) = \dfrac{\mathsf{L}(t_0)}{\sqrt[3]{\mathsf{C}_H}} \simeq \dfrac{36.10174523}{2.4848712052}\,Gly \Longrightarrow \boxed{\ell(t_0) \simeq 14.52861828\,Gly}$

We can obtain the evolution of the Hubble radius in the cosmic time by suppressing the index 0 in (5.55) and (5.56):

$$\left.\begin{array}{l} \mathsf{L}(t) = \sqrt{\dfrac{3}{\Lambda}}\sqrt[3]{\dfrac{(e^x-1)^2}{e^x}} \\ \mathsf{C}_H = \dfrac{(e^x+1)^3}{e^x\,(e^x-1)} \end{array}\right\} \Longrightarrow \ell_H(t) = \frac{\mathsf{L}(t)}{\sqrt[3]{\mathsf{C}_H}} = \sqrt{\frac{3}{\Lambda}}\,\frac{\sqrt[3]{(e^x-1)^3}}{\sqrt[3]{(e^x+1)^3}} \Longrightarrow$$

$$\boxed{\ell_H(t) = \sqrt{\frac{3}{\Lambda}}\,\frac{e^{\beta t}-1}{e^{\beta t}+1}} \tag{5.57}$$

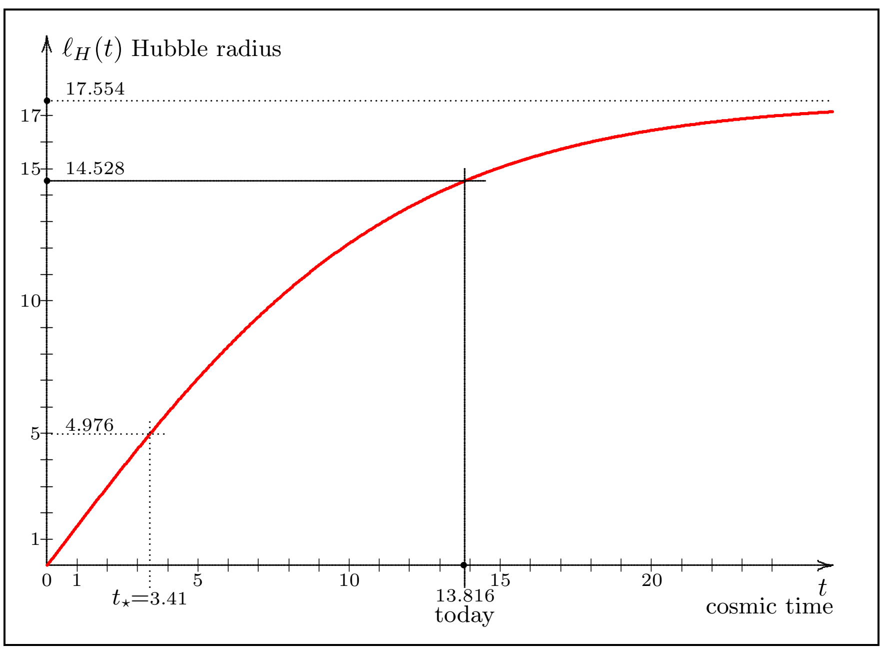


Figure 5.6: Graph of the Hubble radius $\ell_H(t)$.

Chapter 6

Transmission of photons

A human lives in a celestial body O and observes in its sky another celestial body B by means of optical or electro-magnetic devices. This requires that B is capable of emitting electro-magnetic signals in form of 'photons' in a broad sense i.e., in form of particles whose world-lines in space-time are light-like geodesics. We also assume that the world-lines of B and O are time-like geodesics belonging to the galactic fluid.[1] The observer asks a number of questions: how far is B? On what date the photons that I am receiving now have been issued by B? When B has appeared in my sky? As long as it will be visible? ...

The focus of this chapter is to give an answer to these (and related) questions.

6.1 Emission-reception of photons

A photon is emitted by B at the time t_e and received by O at the time t_r. Its world-line is depicted in Fig. 6.1. At any intermediate time t (in the figure two of them are marked: $t_1 < t_2$) a length $\ell(t_e, t)$ is defined: it is the distance traveled by the photon till the time t measured on the spatial section S_t.[2]

If we look at the photon progression through the cosmic monitor, i.e. on the quotient manifold (Fig. 6.2) then we observe that it moves along a geodesic joining B to O with a speed given by equation (2.32):

$$v(t) = \frac{ds}{dt} = \frac{c}{a(t, t_\sharp)}. \tag{6.1}$$

[1] In other words, we assume that B and O have to be considered as particles of the galactic fluid.

[2] It is a synchronous distance at the time t, as defined in Section 1.6.

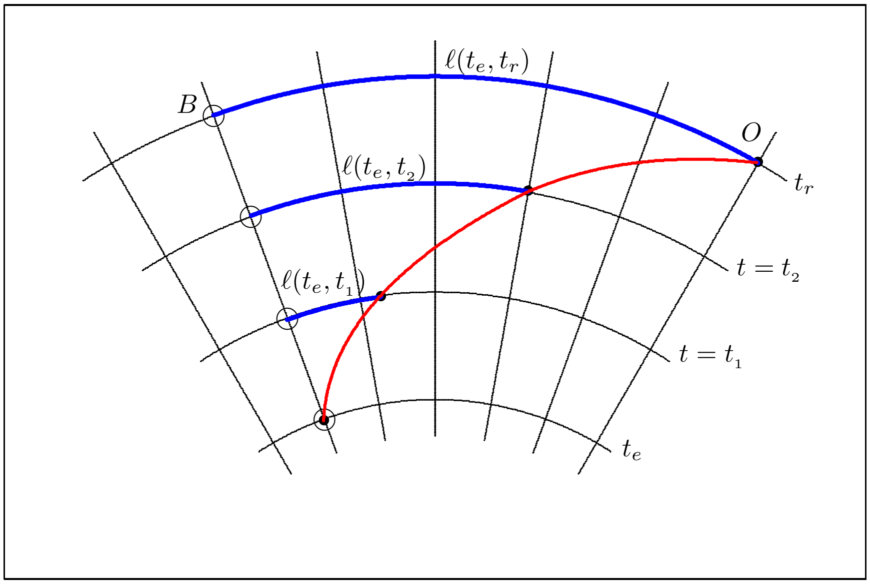


Figure 6.1: Photon world-line from B to O.

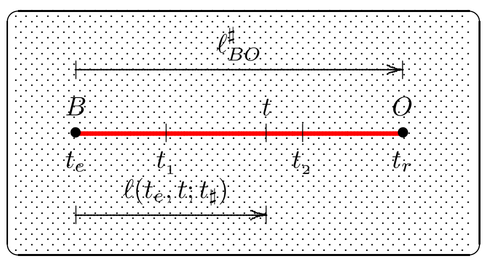


Figure 6.2: Photon trajectory observed on the cosmic monitor.

On the other hand, the synchronous traveled distance in space-time (depicted in Fig. 6.1) is given by

$$\ell(t_e, t) = a(t, t_\sharp)\, \ell(t_e, t; t_\sharp), \tag{6.2}$$

where $\ell(t_e, t; t_\sharp)$ is the traveled distance on the cosmic monitor, whose metric is that of the spatial section at the reference time $t_\sharp$.[3] Then at the time $t > t_e$ the traveled distance on the monitor is given by

$$\ell(t_e, t; t_\sharp) = c \int_{t_e}^{t} \frac{dt'}{a(t', t_\sharp)}. \tag{6.3}$$

[3] The symbol $t_\sharp$ has been inserted in this notation to emphasize the dependence on the reference time of the scale factor.

Consequently, the photon will reach O at the time t_r if and only if

$$\boxed{\ell(t_e, t_r; t_\sharp) \stackrel{\text{def}}{=} c \int_{t_e}^{t_r} \frac{dt'}{a(t', t_\sharp)}} = \boxed{\ell^\sharp_{BO} \stackrel{\text{def}}{=} \text{distance from } B \text{ to } O} \tag{6.4}$$

provided that the integral exists as a finite number. In this case we say that B is **visible to O at the time t_r**. This property is independent from the choice of $t_\sharp$ and has a significant geometrical interpretation illustrated in Fig. 6.3: the shaded area delimited by the graph of $c/a(t, t_\sharp)$ upon the emission-reception interval $[t_e, t_r]$ represents the integral (6.4) hence the co-moving distance $\ell^\sharp_{BO}$. Note that, from the dimensional view-point, this 'area' is in fact a time times a velocity quantity, i.e. a length-dimensional quantity.

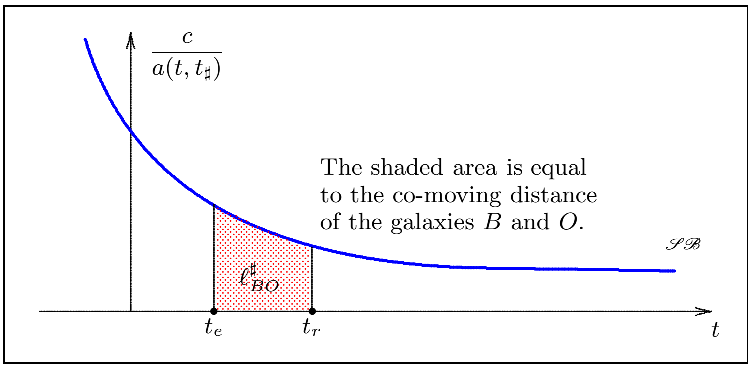


Figure 6.3: Geometrical interpretation of equation (6.4).

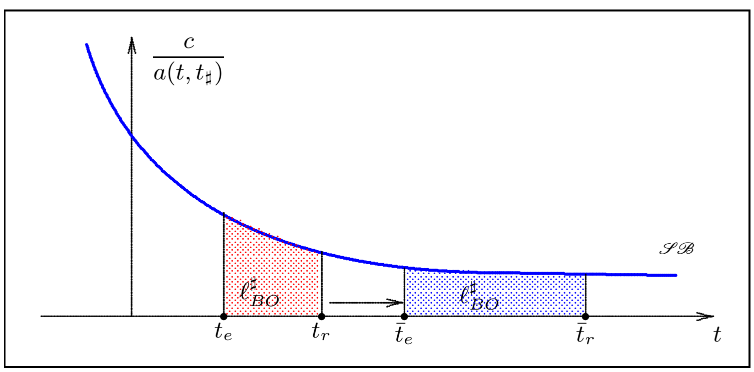


Figure 6.4: Shift of the emission-reception intervals preserving the shaded area.

If a second photon is emitted at $\bar{t}_e > t_e$ then we have a different reception time $\bar{t}_r > t_r$. The two emission-reception intervals have (in general) different magnitudes: $\bar{t}_r - \bar{t}_e \neq t_r - t_e$. However, the two shaded areas over these intervals remain unchanged since both of them are equal to $\ell^\sharp_{BO}$, according to equation (6.3) (Fig. 6.4). In other words, the shaded area behaves as a *planar incompressible fluid* constrained to stay under the graph of $c/a(t, t_\sharp)$ and upon the emission-reception interval $[t_e, t_r]$.

6.2 Event horizon

The topic of the previous section is based on the existence of a finite reception time t_r. However, one could object that the photon might not have enough time to reach O before the end of the universe (at the time t_ω). In fact, this happens when

$$\boxed{\ell(t_e, t_\omega; t_\sharp) \stackrel{\text{def}}{=} c \int_{t_e}^{t_\omega} \frac{dt'}{a(t', t_\sharp)} < \ell_{BO}^\sharp} \tag{6.5}$$

as illustrated in Fig. 6.5:

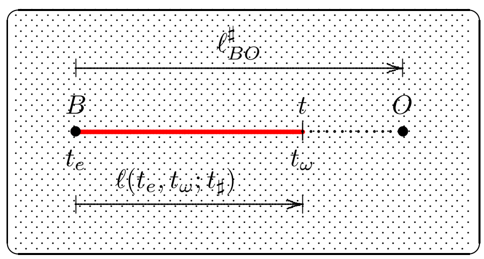


Figure 6.5: Graphic representation of (6.5).

Alternatively, one can consider the boundary case where the reception time coincides with the finish time of the universe. In this case, denoting by $t_\star$ the emission time, we have

$$\boxed{\ell(t_\star, t_\omega; t_\sharp) \stackrel{\text{def}}{=} c \int_{t_\star}^{t_\omega} \frac{dt'}{a(t', t_\sharp)} = \ell_{BO}^\sharp} \tag{6.6}$$

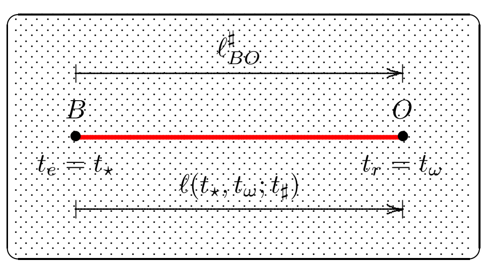


Figure 6.6: Graphic representation of (6.6).

If $\bar{t}_e > t_\star$ is a second emission time, then

$$\ell(\bar{t}_e, t_\omega; t_\sharp) < \ell(t_\star, t_\omega; t_\sharp)$$

i.e.

$$\ell(\bar{t}_e, t_\omega; t_\sharp) < \ell^\sharp_{BO}$$

and we fall in the previous case: *any photon emitted after $t_\star$ never reaches O.* So, *$t_\star$ is the boundary after which no event occurring on B will be observed from O*: it is the **event boundary** of B.

This general argument finds a concrete and fruitful application in the case of a dust-matter flat universe. Indeed, in the case we have an estimate of the age of the universe t_0, so that we can take t_0 as reference time. Hence, by virtue of (5.34) and (5.35), the boundary $t_\star$ for which (6.6) holds,

$$\ell(t_\star, +\infty; t_0) \stackrel{\text{def}}{=} c \int_{t_\star}^{+\infty} \frac{dt'}{a(t', t_0)} = \ell^0_{BO} \tag{6.7}$$

is implicitly defined by equation

$$\frac{c}{\sqrt[3]{\mathsf{c}_0}} \int_{t_\star}^{+\infty} \frac{dt'}{\sqrt[3]{\cosh(\beta\, t') - 1}} = \ell^0_{BO}, \tag{6.8}$$

where $\mathsf{c}_0 \simeq 0.2299194811$ (dimensionless) and $\beta \simeq 0.170901087343\, Gyr^{-1}$.

For $t_\star = t_0$ (today) we find what is called the (present-day) **radius of the event horizon**:

$$\boxed{R_{eh}(t_0) \stackrel{\text{def}}{=} \ell(t_0, +\infty; t_0) \simeq 16.702920561\, Gly} \tag{6.9}$$

The **event horizon** *encloses the set of bodies B from which the photons emitted by t_0 onwards will never be received by an observer O in the future.*[4]

6.3 Particle horizon

According to the physics of the early universe there is a date $t_\star > 0$ in which the photons began to spread freely in the universe, as a consequence of a

[4] If the time t is expressed in Gyr units and we want $\ell_{eh}(t_0)$ expressed in light-years units, then we have to put $c = 1$ in (6.12) (the speed of light is equal to one light-year per year). Since

$$\frac{1}{\sqrt[3]{\mathsf{c}_0}} \simeq \frac{1}{\sqrt[3]{0.2299194811}}\, Gly \simeq 1.6323304282\, Gly,$$

$\beta \simeq 0.170901087343\, Gyr^{-1}$, and $t_0 \simeq 13.81658101781\, Gyr$, the formula to be used for the estimation of the radius is

$$1.6323304282 * \int_{13.81658101781}^{\text{very large}} \frac{dx}{\sqrt[3]{\cosh(0.170901087343 * x) - 1}}.$$

Our estimate is in agreement with that provided in [14]: $\simeq 16,4\, Glyr$.

phenomenon called *recombination*, whose current estimate is $t_\star \simeq 378,000\,yr$. Thus, according to (6.3), the distance traveled by a photon from $t_\star$ to a time t is given by

$$\ell(t_\star, t; t_\sharp) = c\int_{t_\star}^{t} \frac{dz}{a(z, t_\sharp)}. \tag{6.10}$$

If at the time t the co-moving distance $\ell^\sharp_{BO}$ is greater than $\ell(t_\star, t; t_\sharp)$,

$$\ell(t_\star, t; t_\sharp) < \ell^\sharp_{BO}, \tag{6.11}$$

then the body B is *not yet visible* to the observer O. In this regard it must be noted that, due to (1.19),

$$\ell(t_\star, t; t_\sharp) = c\int_{t_\star}^{t} \frac{dz}{a(z, t_\sharp)} = \frac{c}{a(t_*, t_\sharp)}\int_{t_\star}^{t} \frac{dz}{a(z, t_\flat)} = a(t_\sharp, t_\flat)\,\ell(t_\star, t_0; t_\flat)$$

and that, due to (1.21),

$$\ell^\sharp_{BO} = a(t_\sharp, t_\flat)\,\ell^\flat_{BO}.$$

Hence, in changing the reference time, the inequality (6.11) remains invariant. More precisely: both members are multiplied by the same factor $a(t_\sharp, t_\flat)$.

In the case of a dust-matter flat model, for which we have an estimate of the age of the universe t_0, we can take t_0 as reference time. Hence, by virtue of (5.34) and (5.35), from (6.10) we get

$$\ell(t_\star, t; t_0) = c\int_{t_\star}^{t} \frac{dt'}{a(t', t_0)} = \frac{c}{\sqrt[3]{\mathsf{c}_0}}\int_{t_\star}^{t} \frac{dt'}{\sqrt[3]{\cosh(\beta\, t') - 1}} \tag{6.12}$$

where $\mathsf{c}_0 \simeq 0.2299194811$. For $t = t_0$ we find[5]

$$\boxed{R_{ph}(t_0) \overset{\text{def}}{=} \ell(t_\star, t_0; t_0) \simeq 45.627196784\,Gly} \tag{6.13}$$

The meaning of this length is the following: *a radiating body B is currently not visible by an observer O if*

$$R_{ph}(t_0) < \ell^0_{BO}, \tag{6.14}$$

where ℓ^0_{BO} is the present-day proper distance of B and O. $R_{ph}(t_0)$ is called the (present-day) **radius of the visible universe** or **radius of the particle horizon**.[6] The result (6.13) is in good agreement with the current estimate of $\simeq$ 46 billion light years.

[5] For this evaluation we follow the same procedure as for $R_{eh}(t_0)$, footnote of page 86. Since $t_\star \simeq 0.000378\,Gyr$ the formula to be used for this computation is

$$1.63233 * \int_{0.000378}^{13.816581} \frac{dx}{\sqrt[3]{\cosh(0.1709 * x) - 1}}.$$

[6] See for instance [16] (Section 2.2) and [21].

6.4 Red-shift

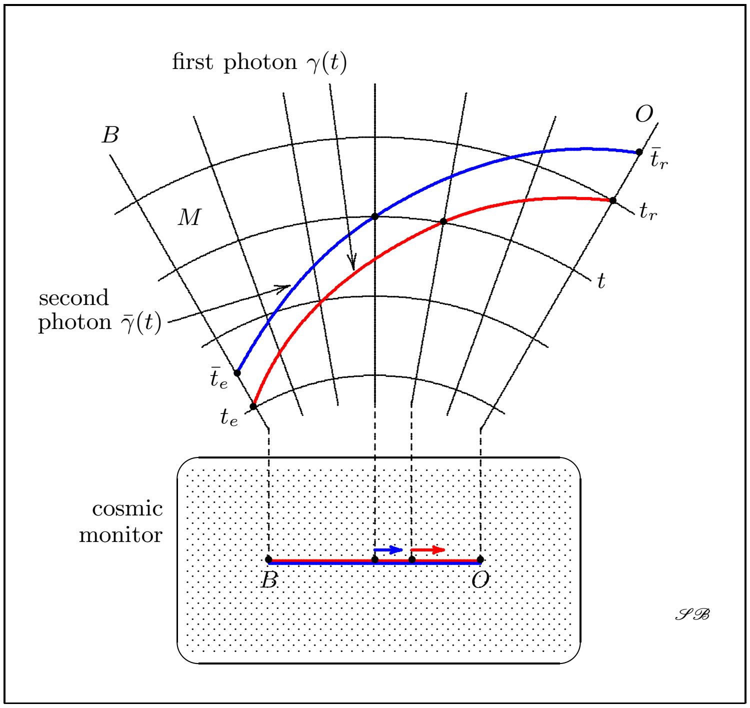


Figure 6.7: Two photons observed on the cosmic monitor.

Let us go back to the end of Section 6.1, p. 84. Assume that the emission time $\bar{t}_e$ of the second photon is very close to t_e (Fig. 6.7). From Fig. 6.8 we infer that the areas over the intervals $I_{er} = [t_e, t_r]$ and $\bar{I}_{er} = [\bar{t}_e, \bar{t}_r]$ are both equal to the co-moving distance $\ell^{\sharp}_{BO}$. Since the central blank area is a common part of these two areas, these two shaded areas are equal. If the magnitude of the intervals $I_e = [t_e, \bar{t}_e]$ and $I_r = [t_r, \bar{t}_r]$ are 'extremely smaller' than the intervals I_{er} and $\bar{I}_{er}$, then the shaded areas can be considered equal to *width* $\times$ *height* of the rectangles where they are contained. So, we can write with 'great precision'[7]

$$\frac{I_e}{a(t_e, t_\sharp)} = \frac{I_r}{a(t_r, t_\sharp)}$$

i.e.,

$$\boxed{\frac{a(t_r, t_\sharp)}{a(t_e, t_\sharp)} = \frac{I_r}{I_e}} \tag{6.15}$$

[7] This argument is taken, with minor modifications, from [13], pp.126-127.

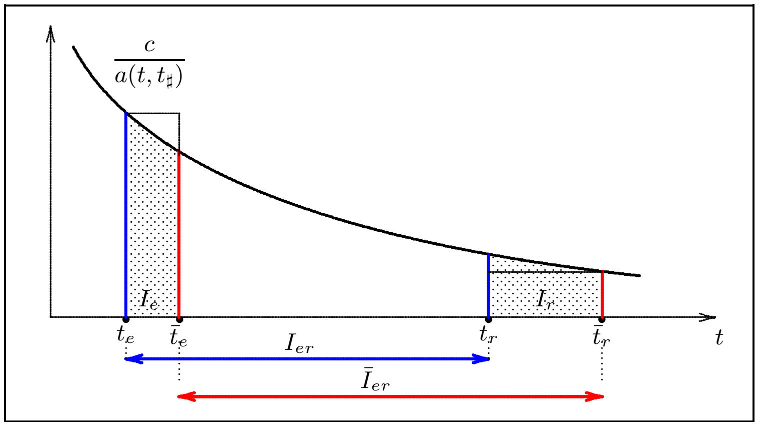


Figure 6.8: The two shaded areas are equal.

This argument can be correctly applied when the two events of emission of the two photons correspond to successive crests of a monochromatic light-wave emitted by B with wavelength

$$\lambda_e = c\, I_e.$$

In this case the two reception-events correspond to successive crests of the same light-wave received by O with wavelength

$$\lambda_r = c\, I_r.$$

Then (6.15) is translated into equation

$$\boxed{\frac{a(t_r, t_\sharp)}{a(t_e, t_\sharp)} = \frac{\lambda_r}{\lambda_e}} \tag{6.16}$$

This formula describes the well-known **spectral shift phenomenon**:

$$\begin{cases} a(t_r, t_\sharp) > a(t_e, t_\sharp) & \Longleftrightarrow \quad \lambda_r > \lambda_e \quad \Longleftrightarrow \\ a(t_r, t_\sharp) < a(t_e, t_\sharp) & \Longleftrightarrow \quad \lambda_r < \lambda_e \quad \Longleftrightarrow \end{cases}$$

$$\begin{cases} \Longleftrightarrow & \text{shift of the original wavelength towards the red} \\ \Longleftrightarrow & \text{shift of the original wavelength towards the blue} \end{cases}$$

Remark 6.1 – The term ‘shift’ sounds like ‘translational displacement’, and this may cause a misunderstanding. In fact, if we write (6.16) in the form

$$\lambda_r = \frac{a(t_r, t_\sharp)}{a(t_e, t_\sharp)}\, \lambda_e$$

then we observe that the spectrum of a galaxy is multiplied by $a(t_r, t_\sharp)/a(t_e, t_\sharp)$ and not translated as a whole. •

6.5 Red-shift versus the emission-time

If we introduce the so-called **red-shift parameter**

$$\boxed{z \stackrel{\text{def}}{=} \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1} \tag{6.17}$$

then equation (6.16) can be written as

$$\boxed{\frac{a(t_r, t_\sharp)}{a(t_e, t_\sharp)} = 1 + z} \tag{6.18}$$

This formula can be used for determining the time of emission t_e of a photon from a galaxy B knowing the red-shift z observed from another galaxy O. Indeed, since the reception time t_r is equal to the today time t_0, then by taking the reference time $t_\sharp$ of the profile $a(t, t_\sharp)$ equal to t_0 equation (6.18) reduces to

$$\boxed{\frac{1}{a(t_e, t_0)} = 1 + z} \tag{6.19}$$

because $a(t_0, t_0) = 1$. As a consequence, if we know the analytic expression of the profile $a(t, t_0)$ then from (6.19) we can extract the emission time t_e as a function of z.

We apply this result to the barotropic flat model with $u = 1$ (pure matter).

Theorem 6.1 – *If we know the present-day red-shift z of a galaxy B observed from a galaxy O, then the dimensionless emission time x_e is given by*

$$\boxed{x_e = \operatorname{arccosh}(y) = \log\left(y + \sqrt{y^2 - 1}\right)} \tag{6.20}$$

$$\boxed{y \stackrel{\text{def}}{=} \frac{1}{\mathsf{c}_0\,(1+z)^3} + 1 = \frac{1}{\mathsf{c}_0}\left(\frac{\lambda_e}{\lambda_r}\right)^3 + 1, \quad \mathsf{c}_0 \simeq 0.2299194811} \tag{6.21}$$

Proof – From (5.34) we derive $a^3(1; x, x_0) = \mathsf{c}_0\,\big(\cosh(x) - 1\big)$

$\Longrightarrow \mathsf{c}_0^{-1}\, a^3(1; x, x_0) = \cosh(x) - 1 \Longrightarrow \cosh(x) = \mathsf{c}_0^{-1}\, a^3(1; x, x_0) + 1$

$\Longrightarrow x = \operatorname{arccosh}[\mathsf{c}_0^{-1}\, a^3(1; x, x_0) + 1] \Longrightarrow x_e = \operatorname{arccosh}[\mathsf{c}_0^{-1}\, a^3(1; x_e, x_0) + 1]$.

By virtue of (6.19) we obtain the emission dimensionless time x_e. ■

The emission time t_e in Gyr is given by $t_e = x_e/\beta$ with $\beta^{-1} \simeq 5.8513378\, Gyr$ (table 5.2, page 68). Thus, the formula to be used for computing the emission cosmic time t_e is

$$\boxed{t_e(z) = 5.8513378 * \operatorname{arccosh}\left(\frac{1}{0.22991948 * (1+z)^3} + 1\right)} \tag{6.22}$$

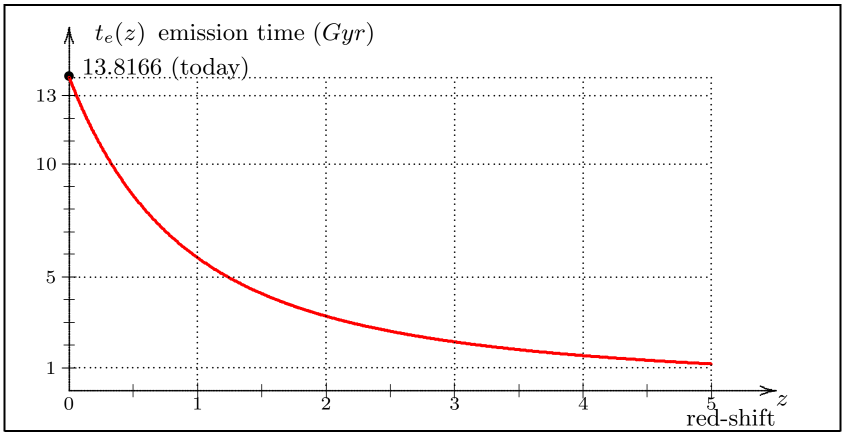


Figure 6.9: Red-shift z versus the emission time t_e.

red-shift z	emission time $t_e\,(Glyr)$	red-shift z	emission time $t_e\,(Glyr)$
0.0	13.8166	1.0	5.8543
0.0005	...	1.1	5.4696
0.001	...	1.2	5.1234
0.002	...	1.3	4.8109
0.003	...	1.4	4.5278
0.004	...	1.5	4.2705
0.005	...	1.6	4.0360
0.006	...	1.7	3.8216
0.1	12.4646	1.8	3.6251
0.2	11.2925	1.9	3.4445
0.3	10.2722	2.0	3.2782
0.4	9.3808	2.1	3.1246
0.5	8.5990	2.2	2.9824
0.6	7.9107	2.3	2.8505
0.7	7.3027	2.4	2.7280
0.8	6.7633	2.5	2.6138
0.9	6.2832	2.6	2.5073

Table 6.1: Red-shift z versus the emission time t_e.