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The Non-holonomic Double Pendulum: an Example of Non-linear Non-holonomic System

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Abstract—An example of physically realizable non-linear non-holonomic mechanical system is proposed. The dynamical equations are written following a general method proposed in an earlier paper. In order to make this paper self-contained, an improved and shortened approach to the dynamics of non-holonomic systems is illustrated in preliminary sections.

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1. INTRODUCTION

The aim of this paper is to propose a simple example of non-holonomic mechanical system with non-linear constraints, called **non-holonomic double pendulum**, and to write the corresponding dynamical equations by applying the general methods exposed in earlier papers [1, 2].

In order to make this paper self-contained, an improved and shortened approach to these methods is illustrated in the preliminary Sections 2 and 3.

Our non-holonomic double pendulum consists of two mass-points (P_1, m_1) and (P_2, m_2) moving on the Cartesian plane (x, y). Let ℓ_1 and ℓ_2 be the straight lines passing through P_1 and P_2 and orthogonal to the respective instantaneous velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. If not parallel, these two lines have an intersection point P_0. This point is constrained to move on the y-axis, as shown in Fig. 1.

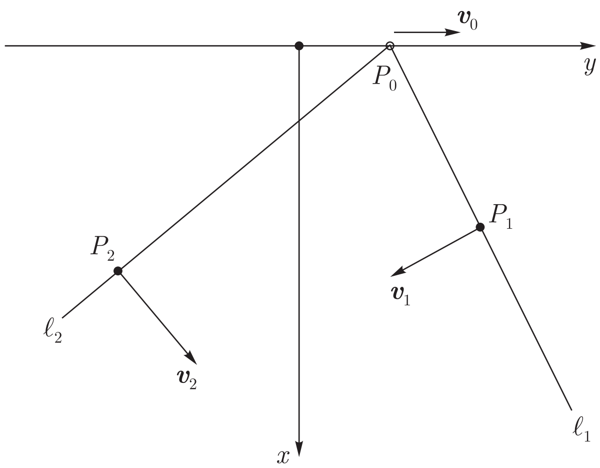

Fig. 1

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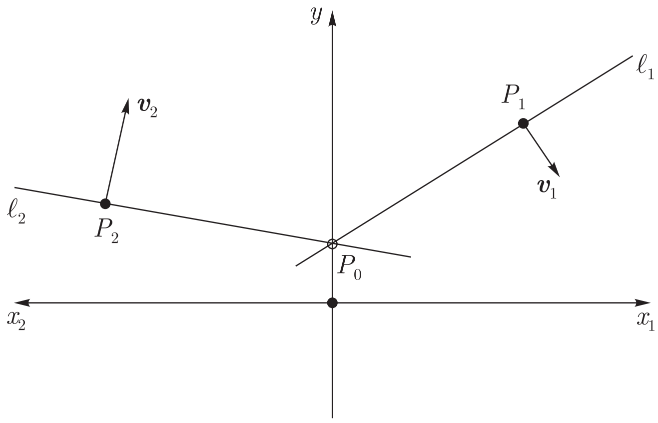

Fig. 2

An equivalent description of this system is given in Fig. 2, where we have splitted the (x, y)-plane into two half-planes (x_1, y) and (x_2, y).

This second description shows how it is possible to create a device that achieves this non-holonomic constraint, as illustrated in Fig. 3, by using mass points placed at the center of sharpened wheels which run along two thin rods ℓ_1 and ℓ_2 pivoting in the point P_0 running along the y-axis.

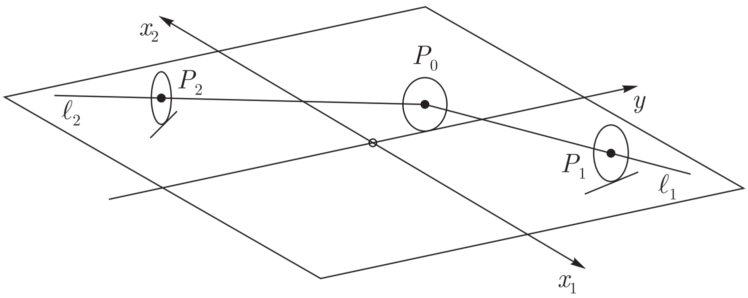

Fig. 3

Figure 4 illustrates a configuration of the double pendulum, given by the location of the two points (P_1, P_2). Note that if the masses of the lines ℓ_1 and ℓ_2 and of the pivoting point P_0 are negligible compared to the masses m_1 and m_2, then in any configuration the pivoting point P_0 can assume any position on the y-axis and its coordinate y_0 plays the role of a **hidden variable**. Thus, the configuration manifold of the double pendulum is $Q = \mathbb{R}^4 = (x_1, y_1, x_2, y_2)$. A set made of the two pairs $(P_1, \boldsymbol{v}_1)$ and $(P_2, \boldsymbol{v}_2)$, where the instantaneous velocities are thought of as vectors attached to the corresponding points, is a **kinematic state** (briefly, a **state**) of the double pendulum. The **state manifold** is the tangent bundle TQ covered by the coordinates $(x_1, y_1, x_2, y_2; \dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2)$

Figure 4 also illustrates two states compatible with the non-holonomic constraints corresponding to the same configuration.

Remark 1. We have to pay attention to those states in which the pivoting point P_0 is undetermined. They occur in the following cases

$$\begin{cases} (a): \quad x_1 = x_2 = 0, \\ (b): \quad \dot{y}_1 = \dot{y}_2 = 0, \\ (c): \quad \boldsymbol{v}_1 = \boldsymbol{v}_2 = 0. \end{cases} \tag{1.1}$$

These are called **singular states** (see Subsection 3.1). $\diamondsuit$

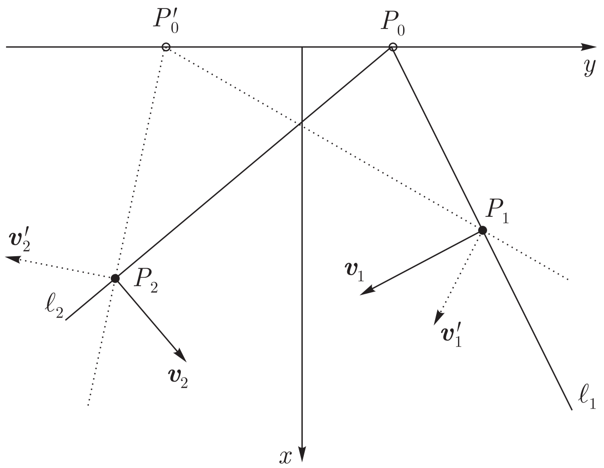

Fig. 4

Figure 5 illustrates two distinct constrained states $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (solid vectors) and $(\boldsymbol{v}_1', \boldsymbol{v}_2')$ (dotted vectors) corresponding to the same configuration (P_1, P_2), but with two distinct hidden locations of the pivoting point, P_0 and P_0'.

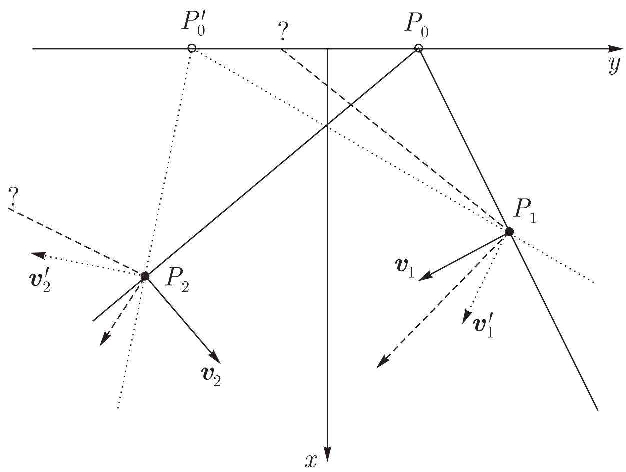

Fig. 5

It is clear that the two sums $\boldsymbol{v}_1 + \boldsymbol{v}_1'$ and $\boldsymbol{v}_2 + \boldsymbol{v}_2'$, represented by the dashed vectors, do not provide a state compatible with the constraint, because the dashed lines orthogonal to them no longer meet in a point of the y-axis. This shows that the non-holonomic constraint is non-linear. The non-linearity is confirmed by the following analytical description.

Theorem 1. *The non-holonomic constraint of the double pendulum is represented by the quadratic-homogeneous equation*

$$\boxed{x_1\,\dot{x}_1\,\dot{y}_2 - x_2\,\dot{x}_2\,\dot{y}_1 + (y_1 - y_2)\,\dot{y}_1\,\dot{y}_2 = 0} \tag{1.2}$$

in the Lagrangian velocities.

Proof. The equations of the lines ℓ_1 and ℓ_2 are:

$$\begin{cases} \ell_1: \quad \dot{x}_1\,(x - x_1) + \dot{y}_1\,(y - y_1) = 0, \\ \ell_2: \quad \dot{x}_2\,(x - x_2) + \dot{y}_2\,(y - y_2) = 0. \end{cases}$$

The point $P_0 = (0, y_0)$ must satisfy both equations,

$$\dot y_1\,(y_0 - y_1) = \dot x_1\, x_1, \qquad \dot y_2\,(y_0 - y_2) = \dot x_2\, x_2.$$

By multiplying the first equation by $\dot y_2$, the second equation by $\dot y_1$, and taking the difference of the two equations so obtained, we get equation (1.2). □

Remark 2. From this proof we get two equations which provide the coordinate y_0 of the point P_0:

$$y_0 = \frac{\dot x_1}{\dot y_1}\, x_1 + y_1, \qquad y_0 = \frac{\dot x_2}{\dot y_2}\, x_2 + y_2. \qquad \diamondsuit \tag{1.3}$$

Remark 3. Non-holonomic constraints are geometrically represented by a subset $C \subset TQ$ of the state-space TQ. By excluding the singular states this subset becomes a submanifold (see Subsection 3.1). In the present case, we have to consider the function

$$C(x, y; \dot x, \dot y) = x_1\, \dot x_1\, \dot y_2 - x_2\, \dot x_2\, \dot y_1 + (y_1 - y_2)\, \dot y_1\, \dot y_2 \tag{1.4}$$

and compute the matrix of the partial derivatives with respect to the Cartesian velocities,

$$\begin{bmatrix} \dfrac{\partial C}{\partial \dot x_1} \\ \dfrac{\partial C}{\partial \dot y_1} \\ \dfrac{\partial C}{\partial \dot x_2} \\ \dfrac{\partial C}{\partial \dot y_2} \end{bmatrix} = \begin{bmatrix} x_1\, \dot y_2 \\ (y_1 - y_2)\, \dot y_2 - x_2\, \dot x_2 \\ -x_2\, \dot y_1 \\ (y_1 - y_2)\, \dot y_1 + x_1\, \dot x_1 \end{bmatrix}.$$

It can be seen that all the elements of this matrix computed for $C = 0$ vanish in one of the three cases (1.1). ♢

Remark 4. It is worthwhile to notice that if we want to consider also the point P_0 endowed with a mass m_0, even very small in comparison with m_1 and m_2, then *the non-holonomic constraint becomes linear*. Indeed, in this case the configuration manifold is $Q \subset \mathbb{R}^5 = (x_1, y_1, x_2, y_2, y_0)$, and Fig. 6 illustrates two distinct configurations.

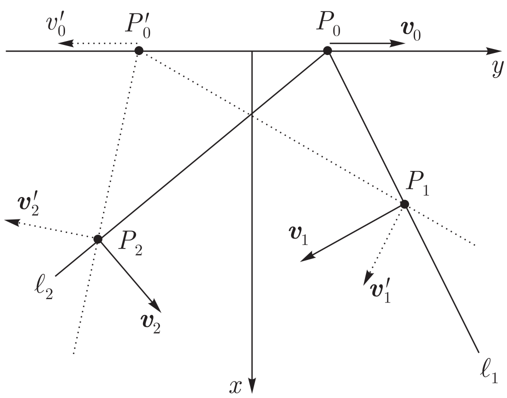

Fig. 6

Note that, apparently, Fig. 6 does not differ from Fig. 4, except for the mere graphic fact that the pivoting point P_0 in Fig. 6 is marked with a bullet, instead of a circle, just for emphasizing that now it has a mass. Figure 7 illustrates two distinct constrained kinematic states $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_0)$ (solid vectors) and $(\boldsymbol{v}_1', \boldsymbol{v}_2', \boldsymbol{v}_0')$ (dotted vectors) corresponding to the same configuration (P_1, P_2, P_0).

We observe that the sums $\boldsymbol{v}_1 + \boldsymbol{v}_1'$, $\boldsymbol{v}_2 + \boldsymbol{v}_2'$, and $\boldsymbol{v}_0 + \boldsymbol{v}_0'$ provide a third state, at the same configuration, which is compatible with the constraints. This shows that the non-holonomic

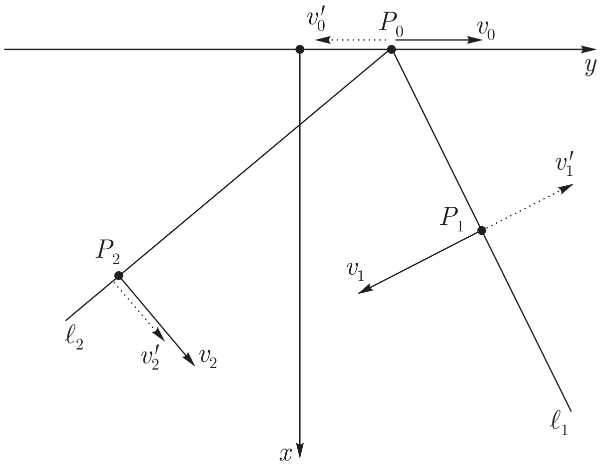

Fig. 7

constraint is linear. This is confirmed by the fact that now the velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must be orthogonal to the vectors P_0P_1 and P_0P_2, respectively: $P_0P_1 \cdot \boldsymbol{v}_1 = 0$, $P_0P_2 \cdot \boldsymbol{v}_2 = 0$. Thus, the non-holonomic constraint is represented by the two linear equations

$$x_1\,\dot{x}_1 + (y_1 - y_0)\,\dot{y}_1 = 0, \quad x_2\,\dot{x}_2 + (y_2 - y_0)\,\dot{y}_2 = 0. \quad \diamondsuit$$

2. HOLONOMIC SYSTEMS

2.1. Geometry

A **holonomic mechanical system** is a set of material points $\{P_\nu,\ \nu \in \mathcal{B}\}$ free to occupy any position in the Euclidean three-space or subject to constraints, called *internal constraints*, such that their possible configurations with respect to a reference frame form an n-dimensional manifold Q called the **configuration manifold**.

Here, ν is an index labeling the points of the system and belonging to a certain set $\mathcal{B}$,[1)] which may be *continuous* or *discrete*. For simplicity, but without loss of generality for our purposes, we assume that the system is made of a finite number N of points P_ν. Then $\nu = 1, 2, \ldots, N$.

Local coordinates (q^i) on Q are called **Lagrangian coordinates**.

For each point P_ν, there exists a map from Q to the Euclidean vector space $\mathbb{E}_3$,

$$\boldsymbol{r}_\nu \colon Q \to \mathbb{E}_3, \tag{2.1}$$

which gives the position vector $\boldsymbol{r}_\nu(q) = OP_\nu$ (in the chosen reference frame with origin at the point O) at the configuration $q \in Q$. These maps are represented by vector-valued functions of the Lagrangian coordinates,

$$\boldsymbol{r}_\nu = \boldsymbol{r}_\nu(q^i). \tag{2.2}$$

These functions play a fundamental role in the holonomic system theory, because they define a link between the configurations as points of Q and the configurations as configurations of points in the Euclidean space.

2.2. Kinematics

A **motion** of a holonomic system is represented by a curve $I \to Q \colon t \mapsto q(t)$ on the configuration manifold Q, where $I \subset \mathbb{R}$ is an interval and the parameter $t \in I$ is interpreted as *time*. If $q^i = q^i(t)$ are local parametric equations of this curve, then the positions, the velocities and the accelerations of the points P_ν along with the motion are determined by the functions

$$\begin{cases} \boldsymbol{r}_\nu(t) = \boldsymbol{r}_\nu(q^i(t)), \\ \boldsymbol{v}_\nu(t) = \dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\dfrac{dq^i}{dt}, \\ \boldsymbol{a}_\nu(t) = \dfrac{\partial^2 \boldsymbol{r}_\nu}{\partial q^j \partial q^i}\,\dfrac{dq^i}{dt}\,\dfrac{dq^j}{dt} + \dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\dfrac{d^2 q^i}{dt^2}. \end{cases} \tag{2.3}$$

1) $\mathcal{B}$ stands for "body".

A **kinematic state** (briefly, a **state**) of a holonomic system is a collection of pairs position-velocity $(\boldsymbol{r}_\nu, \boldsymbol{v}_\nu)$ compatible with the internal constraints. The set of all possible kinematic states is represented by the tangent bundle TQ of Q. If (q^i) are Lagrangian coordinates on Q and $(q^i, \dot{q}^i)$ are the corresponding **natural coordinates** on TQ, then the coordinates $(\dot{q}^i)$ are called **Lagrangian velocities**. Due to the first two equations (2.3), the kinematic states in the domain of the coordinates $(q, \dot{q}) = (q^i, \dot{q}^i)$ are given by the vector-valued functions

$$\begin{cases} \boldsymbol{r}_\nu = \boldsymbol{r}_\nu(q^i), \\ \boldsymbol{v}_\nu = \dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\, \dot{q}^i, \end{cases} \tag{2.4}$$

where $(\dot{q}^i)$ can assume any value in $\mathbb{R}^n$.

2.3. Dynamics

The dynamics of each point P_ν of a holonomic system is assumed to be governed by the **Newton equation**

$$m_\nu\, \boldsymbol{a}_\nu = \boldsymbol{A}_\nu + \boldsymbol{R}_\nu, \tag{2.5}$$

where $\boldsymbol{A}_\nu$ is the **active force** acting on P_ν and $\boldsymbol{R}_\nu$ is the **reactive force**, due to the internal constraints imposed to all points.

The active forces $\boldsymbol{A}_\nu$ are in general known functions depending on the kinematic states of the system. The reactive forces $\boldsymbol{R}_\nu$, which are *a priori* unknown, have the role of maintaining the internal constraints satisfied along with any motion of the system. Then we have to specify the type of reactive forces the constraints are able to supply. This means to assume a **constitutive condition** of the constraints.

In the fundamental approach to the holonomic dynamics it is assumed that the constraints are *ideal*.

Definition 1. *A holonomic system is called* **ideal** *(or* **perfect***) if the reactive forces obey to equation*

$$\sum_\nu \boldsymbol{R}_\nu \cdot \boldsymbol{w}_\nu = 0, \tag{2.6}$$

for all virtual displacements $\boldsymbol{w}_\nu$.

Definition 2 (*Intuitive*). *A* **virtual displacement** *is a small displacement from a state* $s_0 = (q_0, \dot{q}_0) \in TQ$ *to a neighboring state* $s = (q_0, \dot{q})$ *at the same configuration* $q_0 \in Q$.

Definition 3 (*Rigorous*). *Let* $c\colon I \to TQ\colon t \mapsto (q^i(t), \dot{q}^i(t))$ *be a state-valued curve starting from* s_0 *and preserving the initial configuration* (q_0),

$$c(0) = (q_0, \dot{q}_0), \qquad c(t) = (q_0, \dot{q}(t)).$$

then the tangent vector to this curve at $t = 0$ *is a* **virtual displacement from the state** $s_0 = (q_0, \dot{q}_0)$.

The curve $c(t)$ generates a one-parameter family of states compatible with the internal constraints,

$$\begin{cases} \boldsymbol{r}_\nu = \boldsymbol{r}_\nu(q_0^i) = \text{constant}, \\ \boldsymbol{v}_\nu(t) = \left.\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\right|_{q^i = q_0^i} \dot{q}^i(t). \end{cases} \tag{2.7}$$

It follows that

$$\begin{cases} \dfrac{d\boldsymbol{r}_\nu}{dt} = 0, \\ \dfrac{d\boldsymbol{v}_\nu}{dt} = \left.\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\right|_{q^i = q_0^i} \dfrac{d\dot{q}^i}{dt}. \end{cases} \tag{2.8}$$

Hence, if we set

$$\left.\frac{d\dot{q}^i}{dt}\right|_{t=0} = w^i, \quad \left.\frac{d\boldsymbol{v}_\nu}{dt}\right|_{t=0} = \boldsymbol{w}_\nu, \tag{2.9}$$

then we can write

$$\boxed{\boldsymbol{w}_\nu(q_0, w) = \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\bigg|_{q^i=q^i_0} w^i} \tag{2.10}$$

with $w = (w^i) \in \mathbb{R}^n$. This formula gives the virtual displacements of the points P_ν at a state $s_0 = (q_0, \dot{q}_0)$.

Remark 5. Due to the Newton equations (2.5), the definition (2.6) of ideal constraint is equivalent to equation

$$\sum_\nu (m_\nu\, \boldsymbol{a}_\nu - \boldsymbol{A}_\nu) \cdot \boldsymbol{w}_\nu = 0. \tag{2.11}$$

This equation expresses the so-caled **virtual work principle** or **D'Alembert–Lagrange principle**. $\diamondsuit$

Due to (2.10), we can write

$$\boldsymbol{w}_\nu = \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\, w^i, \tag{2.12}$$

so that equation (2.11) becomes equivalent to

$$\sum_\nu m_\nu\, \boldsymbol{a}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} = \sum_\nu \boldsymbol{A}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}, \tag{2.13}$$

since the w^i are arbitrary. Furthermore, due to the expression (2.3) of the accelerations, we can write

$$\sum_\nu m_\nu\, \boldsymbol{a}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} = \sum_\nu m_\nu \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^j \partial q^k} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \frac{dq^j}{dt} \frac{dq^k}{dt} + \sum_\nu m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^j} \frac{d^2 q^j}{dt^2}.$$

Then, if we put

$$\left.\begin{aligned} &\textstyle\sum_\nu m_\nu \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^j} = g_{ij}, \\ &\textstyle\sum_\nu m_\nu \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^j \partial q^k} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} = \Gamma_{jk,i}, \\ &\textstyle\sum_\nu \boldsymbol{A}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} = A_i, \end{aligned}\right\} \tag{2.14}$$

equation (2.13) reads

$$g_{ij} \frac{d^2 q^j}{dt^2} + \Gamma_{jk,i} \frac{dq^j}{dt} \frac{dq^k}{dt} = A_i. \tag{2.15}$$

This is just the form assumed by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}^i}\right) - \frac{\partial K}{\partial q^i} = A_i \tag{2.16}$$

where K is the **kinetic energy**:

$$K = \tfrac{1}{2} \sum_\nu m_\nu \boldsymbol{v}_\nu \cdot \boldsymbol{v}_\nu = \tfrac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j, \tag{2.17}$$

and A_i are the so-called **Lagrangian active forces**. The remarkable fact is that these coefficients can be interpreted as the covariant components of a metric tensor on the configuration manifold Q. It follows that $\Gamma_{jk,i}$ are the Christoffel symbols of the metric g_{ij}. By using the inverse matrix $[g^{ij}]$ of $[g_{ij}]$, equations (2.15) become equivalent to

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{hk} \frac{dq^h}{dt} \frac{dq^k}{dt} = A^i. \tag{2.18}$$

where $\Gamma^i_{hk} = g^{ij}\, \Gamma_{hk,j}$ are the coefficients of the Levi-Civita connection associated with this metric, and $A^i = g^{ij}$ are the **contravariant Lagrangian forces**.

This is a well-known matter. But, in the following, we need an explicit reference to the above written formulae.

At last, it is important to observe that the second-order dynamical equations (2.18) can be transformed into a first-order system

$$\boxed{\boldsymbol{D}: \begin{cases} \dfrac{dq^i}{dt} = \dot{q}^i, \\ \dfrac{d\dot{q}^i}{dt} = -\Gamma^i_{hk}\,\dot{q}^h\,\dot{q}^k + A^i, \end{cases}} \tag{2.19}$$

associated with a vector field $\boldsymbol{D}$ on TQ. This passage should also clarify the different meaning of the symbols $\dot{q}^i$ and d^i/dt: $\dot{q}^i$ denotes a coordinate on TQ (i.e., a Lagrangian velocity), dq^i/dt denotes the derivative of the function $q^i(t)$.

3. NON-HOLONOMIC SYSTEMS

A **non-holonomic system** is a holonomic system with *additional constraints* on the velocities.[2)]

3.1. Geometry

Non-holonomic constraints are geometrically represented by a subset C of the state manifold TQ. The constraint sets C we are going to consider, apart from a set of **singular states**, satisfy the following **regularity conditions**:

- C is a submanifold of dimension $n + m$, with $m < n = \dim Q$.
- For all $q \in Q$, the set $C_q = C \cap T_qQ$ is a submanifold of dimension m.
- The restriction to C of the tangent fibration $\tau_Q : TQ \to Q$ is a *surjective submersion*.[3)]

Then we call C a **constraint submanifold**. A constraint submanifold can be represented in two ways:

- By m equations,
$$\dot{q}^i = \psi^i(q, z), \tag{3.1}$$
where $z = (z^\alpha)$, $\alpha = 1, \ldots, m < n$ are called *parameters*. This is a **parametric representation**. Note that (q, z) can be interpreted as local coordinates on C.
- By $r = n - m$ *independent* equations,
$$C^a(q, \dot{q}) = 0, \quad a = 1, \ldots r. \tag{3.2}$$
'Independent' means that the differentials dC^a are linearly independent at each point of C. This is an **implicit representation**.

In these two representations, the regularity conditions are equivalent to

$$\left.\begin{aligned} &\operatorname{rank}\left[\frac{\partial\psi^i}{\partial z^\alpha}\right]_{n\times m} = m \quad \text{(maximal rank)} \\ &\operatorname{rank}\left[\frac{\partial C^a}{\partial \dot{q}^i}\right]_{n\times r} = r \quad \text{(maximal rank)} \end{aligned}\right\} \tag{3.3}$$

respectively.

By differentiating the identity $C^a(q, \psi(q, z)) = 0$,

$$dC^a = \left(\frac{\partial C^a}{\partial q^i} + \frac{\partial C^a}{\partial \dot{q}^j}\frac{\partial \psi^j}{\partial q^i}\right) dq^i + \frac{\partial C^a}{\partial \dot{q}^j}\frac{\partial \psi^j}{\partial z^\alpha}\, dz^\alpha = 0,$$

we get the following relations between the two representations,

$$\left.\begin{aligned} &\frac{\partial C^a}{\partial q^i} + \frac{\partial C^a}{\partial \dot{q}^j}\frac{\partial \psi^j}{\partial q^i} = 0 \\ &\frac{\partial C^a}{\partial \dot{q}^j}\frac{\partial \psi^j}{\partial z^\alpha} = 0 \end{aligned}\right\} \tag{3.4}$$

[2)]Such constraints are usually called **non-holonomic constraints** and consequently the mechanical system is called "non-holonomic".

[3)]This definition of regularity is taken from [3, 4].

3.2. Kinematics

In any motion of a holonomic system the velocities and the accelerations of the points P_ν are given by (2.3). If the non-holonomic constraints are described by parametric equations (3.1), then the derivatives dq^i/dt must be replaced by $\psi^i(q(t), z(t))$. We find that the velocities and the accelerations of the points P_ν along with any motion compatible with the constraints, are given by

$$\left.\begin{aligned} \boldsymbol{v}_\nu(t) &= \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\psi^i, \\ \boldsymbol{a}_\nu(t) &= \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^i \partial q^j}\,\psi^i\psi^j + \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\left(\frac{\partial \psi^i}{\partial q^j}\,\psi^j + \frac{\partial \psi^i}{\partial z^\alpha}\,\frac{dz^\alpha}{dt}\right) \end{aligned}\right\} \tag{3.5}$$

where in $\psi^i(q, z)$ the variables q^i and z^α are replaced by their functions in t, $q^i = q^i(t)$ and $z^\alpha = z^\alpha(t)$.

By these expressions we can write these vectors as functions of the states compatible with the constraints, namely:

$$\left.\begin{aligned} \boldsymbol{v}_\nu(q, z) &= \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\psi^i, \\ \boldsymbol{a}_\nu(q, z, \dot{z}) &= \frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^i \partial q^j}\,\psi^i\psi^j + \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\left(\frac{\partial \psi^i}{\partial q^j}\,\psi^j + \frac{\partial \psi^i}{\partial z^\alpha}\,\dot{z}^\alpha\right) \end{aligned}\right\} \tag{3.6}$$

Further parameters $\dot{z} = (\dot{z}^\alpha)$ are introduced for representing the accelerations.

3.3. Dynamics I

For the study of the foundations of the non-holonomic dynamics it is convenient to rely on the parametric representation of the constraints.

This dynamics is based on the same principles of the holonomic dynamics. We just have to adapt the notion of virtual displacement to the case in which non-holonomic constraints are present. To do this we have to impose that the curves considered in Definition 3 be compatible with these constraints. This simply amounts to replace the derivatives dq^i/dt in equations (2.7) by $\psi^i(q, z)$. Then equations (2.8) are replaced by

$$\begin{cases} \dfrac{d\boldsymbol{r}_\nu}{dt} = 0, \\ \dfrac{d\boldsymbol{v}_\nu}{dt} = \left(\dfrac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\dfrac{\partial \psi^i}{\partial z^\alpha}\right)_{q^i = q^i_0} \dfrac{dz^\alpha}{dt}. \end{cases} \tag{3.7}$$

The final result is that equation (2.10) is replaced by

$$\boxed{\boldsymbol{w}_\nu(q_0, z_0, \dot{z}) = \left[\frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\frac{\partial \psi^i}{\partial z^\alpha}\right]_0 \dot{z}^\alpha} \tag{3.8}$$

with $\dot{z} = (\dot{z}^\alpha) \in \mathbb{R}^m$, where $[\ldots]_0$ means the computation at $q = q_0$ and $z = z_0$. This formula gives the virtual displacements of the points P_ν at a state $s_0 = (q_0, z_0)$.

Remark 6. The acceleration (3.5) can be decomposed into a sum

$$\boldsymbol{a}_\nu = \boldsymbol{a}_{0\nu} + \boldsymbol{a}_{\alpha\nu}\,\dot{z}^\alpha \tag{3.9}$$

where

$$\left.\begin{aligned} \boldsymbol{a}_{0\nu} &= \left(\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^i \partial q^j}\,\psi^i + \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\frac{\partial \psi^i}{\partial q^j}\right)\psi^j \\ \boldsymbol{a}_{\alpha\nu} &= \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}\,\frac{\partial \psi^i}{\partial z^\alpha}. \end{aligned}\right\} \tag{3.10}$$

We observe that the first vector $\boldsymbol{a}_{0\nu}$ in the decomposition (3.9) depends only on the state of the system, whereas the second vector $\boldsymbol{a}_{\alpha\nu}\,\dot{z}^\alpha$ is a virtual displacement,

$$\boldsymbol{a}_{\alpha\nu}\,\dot{z}^\alpha = \boldsymbol{w}_\nu(q, z, \dot{z}). \qquad \diamondsuit \tag{3.11}$$

Theorem 2. *The dynamics of a non-holonomic system with ideal constraints represented by parametric equations* $\dot{q}^i = \psi^i(q, z)$ *is governed by the differential equations*

$$\boxed{\boldsymbol{Z}: \begin{cases} \dfrac{dq^i}{dt} = \psi^i(q, z) \\ \dfrac{dz^\alpha}{dt} = G^{\alpha\beta} \dfrac{\partial \psi^i}{\partial z^\beta} \left(A_i - \Gamma_{jk,i}\, \psi^j\, \psi^k - g_{ij} \dfrac{\partial \psi^j}{\partial q^h}\, \psi^h \right) \end{cases}} \tag{3.12}$$

where $[G^{\alpha\beta}]$ *is the inverse matrix of* $[G_{\alpha\beta}]$ *defined by*

$$\boxed{G_{\alpha\beta} = g_{ij} \frac{\partial \psi^i}{\partial z^\alpha} \frac{\partial \psi^j}{\partial z^\beta}} \tag{3.13}$$

Remark 7. The dynamical equations (3.12) are the first-order equations associated with a vector field $\boldsymbol{Z}$ on the constraint manifold C. $\diamondsuit$

Remark 8. If we introduce the vectors

$$\boldsymbol{\psi}^\alpha = \left[G^{\alpha\beta} \frac{\partial \psi^i}{\partial z^\beta} \right]$$

and the co-vector

$$\boldsymbol{\phi} = \left[A_i - g_{ij} \frac{\partial \psi^j}{\partial q^h}\, \psi^h \right],$$

then the dynamical equations (3.12) can be put in the synthetic form

$$\boxed{\begin{cases} \dfrac{dq^i}{dt} = \psi^i(q, z) \\ \dfrac{dz^\alpha}{dt} = \langle \boldsymbol{\psi}^\alpha, \boldsymbol{\phi} \rangle \end{cases}} \quad \diamondsuit \tag{3.14}$$

Proof. The second equation (3.5) and the definition (3.11) of virtual displacement imply

$$\begin{aligned} \boldsymbol{a}_\nu \cdot \boldsymbol{w}_\nu &= \left[\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^j \partial q^k} \psi^j \psi^k + \frac{\partial \boldsymbol{r}_\nu}{\partial q^j} \left(\frac{\partial \psi^j}{\partial q^k} \psi^k + \frac{\partial \psi^j}{\partial z^\alpha} \frac{dz^\alpha}{dt} \right) \right] \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \frac{\partial \psi^i}{\partial z^\beta} \dot{z}^\beta \\ &= \left[\frac{\partial^2 \boldsymbol{r}_\nu}{\partial q^j \partial q^k} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \psi^j \psi^k + \frac{\partial \boldsymbol{r}_\nu}{\partial q^j} \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i} \left(\frac{\partial \psi^j}{\partial q^k} \psi^k + \frac{\partial \psi^j}{\partial z^\alpha} \frac{dz^\alpha}{dt} \right) \right] \frac{\partial \psi^i}{\partial z^\beta} \dot{z}^\beta. \end{aligned}$$

Hence,

$$\sum_\nu m_\nu\, \boldsymbol{a}_\nu \cdot \boldsymbol{w}_\nu = \left[\Gamma_{jk,i}\, \psi^j\, \psi^k + g_{ij} \left(\frac{\partial \psi^j}{\partial q^k}\, \psi^k + \frac{\partial \psi^j}{\partial z^\alpha} \frac{dz^\alpha}{dt} \right) \right] \frac{\partial \psi^i}{\partial z^\beta}\, \dot{z}^\beta.$$

Since $(\dot{z}^\beta) \in \mathbb{R}^m$ in the definition of virtual displacement can take arbitrary values, the virtual work principle (2.11) becomes equivalent to the differential equations

$$\left[g_{ij} \left(\frac{\partial \psi^j}{\partial q^h} \frac{dq^h}{dt} + \frac{\partial \psi^j}{\partial z^\alpha} \frac{dz^\alpha}{dt} \right) + \Gamma_{jk,i} \frac{dq^j}{dt} \frac{dq^k}{dt} - A_i \right] \frac{\partial \psi^i}{\partial z^\beta} = 0. \tag{3.15}$$

where g_{ij}, $\Gamma_{jk,i}$ and A_i are defined as in (2.14). By setting $\psi^i = dq^i/dt$ (first equations (3.12)) and by introducing the symbols $G_{\alpha\beta}$ (3.13), equations (3.15) are transformed into equations

$$G_{\alpha\beta} \frac{dz^\alpha}{dt} + \left(g_{ij} \frac{\partial \psi^j}{\partial q^h}\, \psi^h + \Gamma_{jk,i}\, \psi^j\, \psi^k - A_i \right) \frac{\partial \psi^i}{\partial z^\beta} = 0. \tag{3.16}$$

In the non-singular states the matrix

$$\left[\frac{\partial \psi^i}{\partial z^\alpha} \right]$$

has maximal rank. Thus, the symmetric $m \times m$-matrix $[G_{\alpha\beta}]$ is regular and positive-definite. By using the inverse matrix $[G^{\alpha\beta}]$, equations (3.16) can be solved with respect to dz^α/dt. $\square$

3.4. Dynamics II

So far, our analysis was based on the parametric representation of non-holonomic constraints. Now we analyze the dynamics based on the implicit representation, that leads us to a re-visitation of the *Lagrange multipliers method*.

If we introduce the **Lagrangian reactive forces**

$$R_i \doteq \sum_\nu \boldsymbol{R}_\nu \cdot \frac{\partial \boldsymbol{r}_\nu}{\partial q^i}, \tag{3.17}$$

then, because of (3.11), the definition (2.6) of ideal constraint becomes equivalent to the equations

$$\boxed{R_i \, \frac{\partial \psi^i}{\partial z^\alpha} = 0} \tag{3.18}$$

On the other hand, because of the regularity conditions (3.3), the r vectors $\boldsymbol{C}^a = [C^{ai}]$ defined by

$$C^{ai} \doteq g^{ij} \, \frac{\partial C^a}{\partial \dot{q}^j}, \tag{3.19}$$

are pointwise independent, as well as the m vectors

$$\boldsymbol{\psi}_\alpha = \left[\frac{\partial \psi^i}{\partial z^\alpha} \right].$$

The second set of equations (3.4) is equivalent to

$$\boldsymbol{C}^a \cdot \boldsymbol{\psi}_\alpha = 0.$$

As a consequence, these vectors form an orthogonal splitting of the tangent space T_qQ at each $q \in Q$, and, if a vector $\boldsymbol{v}$ is such that $\boldsymbol{v} \cdot \boldsymbol{\psi}_\alpha = 0$, that is

$$v_i \, \frac{\partial \psi^i}{\partial z^\alpha} = 0$$

for all α, then $\boldsymbol{v}$ is a linear combination of the vectors $\boldsymbol{C}^a$:

$$v_i = \lambda_a \, \frac{\partial C^a}{\partial \dot{q}^i}.$$

Hence, by applying this result to equation (3.18), we find that

Theorem 3. *Non-holonomic constraints defined by r equations $C^a(q, \dot{q}) = 0$ are ideal if and only if there exist r functions $\lambda_a(q, \dot{q})$ such that*

$$\boxed{R_i = \lambda_a \, \frac{\partial C^a}{\partial \dot{q}^i}} \tag{3.20}$$

Theorem 4. *Let the non-holonomic constraints be defined by r equations $C^a(q, \dot{q}) = 0$. If the constraints are ideal then the motions of the non-holonomic system are represented by the integral curves, whose initial conditions satisfy the constraints, of the dynamical equations*

$$\boxed{\boldsymbol{D} : \begin{cases} \dfrac{dq^i}{dt} = \dot{q}^i, \\ \dfrac{d\dot{q}^i}{dt} = F^\ell \left(\delta^i_\ell - G_{ab} \, \dfrac{\partial C^b}{\partial \dot{q}^\ell} \, \dfrac{\partial C^a}{\partial \dot{q}^j} \, g^{ij} \right) - G_{ab} \, \dfrac{\partial C^b}{\partial q^\ell} \, \dfrac{\partial C^a}{\partial \dot{q}^j} \, g^{ij} \, \dot{q}^\ell \end{cases}} \tag{3.21}$$

where

$$\boxed{F^i(q, \dot{q}) = A^i - \Gamma^i_{hk} \, \dot{q}^h \, \dot{q}^k} \tag{3.22}$$

and $[G_{ab}]$ is the inverse of the $r \times r$ symmetric matrix $[G^{ab}]$ defined by

$$\boxed{G^{ab} = g^{ij} \, \frac{\partial C^a}{\partial \dot{q}^i} \, \frac{\partial C^b}{\partial \dot{q}^j}} \tag{3.23}$$

Equations (3.21) are the first-order equations associated with a vector field $\boldsymbol{D}$ on TQ, tangent to the constraint submanifold C.

Proof. 1. Following the method applied to the holonomic system one can prove that *if the non-holonomic constraints are ideal then the Newton equations* $m_\nu \boldsymbol{a}_\nu = \boldsymbol{A}_\nu + \boldsymbol{R}_\nu$ *are equivalent to Lagrange equations of the form*

$$g_{ij} \frac{d^2 q^j}{dt^2} + \Gamma_{jk,i} \frac{dq^j}{dt} \frac{dq^k}{dt} = A_i + \lambda_a \frac{\partial C^a}{\partial \dot{q}^i}. \tag{3.24}$$

2. In turn, these second-order differential equations are equivalent to the first-order system

$$\boldsymbol{D}_\lambda : \begin{cases} \dfrac{dq^i}{dt} = \dot{q}^i, \\ \dfrac{d\dot{q}^i}{dt} = g^{ij} \left(A_i - \Gamma_{hk,i}\, \dot{q}^h \dot{q}^k + \lambda_a \dfrac{\partial C^a}{\partial \dot{q}^j} \right) \end{cases} \tag{3.25}$$

associated with a vector field $\boldsymbol{D}_\lambda$ on TQ. Actually, this is a *family* of vector fields depending on the Lagrangian multipliers $\lambda = (\lambda_a)$. Thus, we have to see if in this family there exist values of λ for which the constraint equations are invariant with respect to D_λ. This is equivalent to look for values of λ such that D_λ is tangent to the constraint submanifold C.

3. As a derivation, the vector field $\boldsymbol{D}_\lambda$ is given by

$$\boldsymbol{D}_\lambda = \dot{q}^i \frac{\partial}{\partial q^i} + \left(\lambda_a \frac{\partial C^a}{\partial \dot{q}^j} g^{ji} - \Gamma^i_{hk}\, \dot{q}^h \dot{q}^k + A^i \right) \frac{\partial}{\partial \dot{q}^i}. \tag{3.26}$$

Then $\boldsymbol{D}_\lambda$ is tangent to C if and only if

$$\boldsymbol{D}_\lambda C^a(q, \dot{q}) = f^a(C^1, \ldots, C^r)$$

where f^a are smooth functions of r variables, $f^a(x^1, \ldots, x^r)$, such that $f^a(0, \ldots, 0) = 0$. Since the restriction to C of D_λ does not depend on these functions (they vanish on C) and we are interested only on the integral curves of D_λ lying on C, we can assume $f^a = 0$ without loss of generality. Then equations $\boldsymbol{D}_\lambda C^a(q, \dot{q}) = 0$ assume the form

$$\dot{q}^i \frac{\partial C^a}{\partial q^i} + G^{ab} \lambda_b + F^i \frac{\partial C^a}{\partial \dot{q}^i} = 0,$$

where F^i and $[G^{ab}]$ are defined as in (3.22) and (3.13).

4. Because of the regularity conditions, $\det[G^{ab}] \neq 0$. By using the inverse matrix $[G_{ab}]$ we can solve these equations with respect to the Lagrangian multipliers. This proves that $\boldsymbol{D}_\lambda$ *is tangent to* C *if and only if*

$$\boxed{\lambda_a(q, \dot{q}) = -G_{ab} \left(F^i \frac{\partial C^b}{\partial \dot{q}^i} + \dot{q}^i \frac{\partial C^b}{\partial q^i} \right)} \tag{3.27}$$

By substituting these values of the Langrangian multipliers into the second group of the dynamical equations (3.25) we get the differential system (3.21). □

Remark 9. Formula (3.27) allows the computation of the reactive Lagrangian forces,

$$\boxed{R_i(q, \dot{q}) = -G_{ab} \left(F^j \frac{\partial C^b}{\partial \dot{q}^j} + \dot{q}^j \frac{\partial C^b}{\partial q^j} \right) \frac{\partial C^a}{\partial \dot{q}^i}} \quad \diamondsuit \tag{3.28}$$

Remark 10. The right-hand sides D^i of the second group of the dynamical equations (3.21) can be constructed by computing, in the order, the following objects.

1. The kinetic energy $K = \frac{1}{2} g_{ij}\, \dot{q}^i \dot{q}^j$ and the inverse matrix $[g^{ij}]$ of $[g_{ij}]$.
2. F^i are obtained by the Lagrange equations written in the form
$$\frac{d\dot{q}^i}{dt} = A^i - \Gamma^i_{hk}\, \dot{q}^h \dot{q}^k = F^i.$$
3. $C^a_i = \dfrac{\partial C^a}{\partial \dot{q}^i}$.

4. $C^{ai} = g^{ij}\, C_i^a$.
5. $G^{ab} = g^{ij}\, C_i^a\, C_j^b = C^{ai}\, C_i^b$.
6. $[G_{ab}] = [G^{ab}]^{-1}$.
7. $X_a^i = G_{ab}\, C^{bi}$.
8. $\pi_i^j = C_i^a\, X_a^j$.
9. $(\delta_j^i - \pi_j^i)\, F^j$.
10. $D^i = (\delta_j^i - \pi_j^i)\, F^j - X_a^i\, \dot{q}^\ell\, \dfrac{\partial C^b}{\partial q^\ell}$. ◊

We have to pay attention to the special case of non-holonomic constraints defined by a single equations.

Theorem 5. *If the non-holonomic constraints are defined by a single equation* $C(q, \dot{q}) = 0$*, then the components* D^i *are*

$$\boxed{D^i = (\delta_\ell^i - \pi_\ell^i)\, F^\ell - \dot{q}^\ell\, \frac{\partial C}{\partial q^\ell}\, X^i} \tag{3.29}$$

These components can be constructed by computing, in the order, the following objects.

1. *As above.*
2. *As above.*
3. $C_i = \dfrac{\partial C}{\partial \dot{q}^i}$.
4. $C^i = g^{ij}\, C_j$.
5. $G = C^i\, C_i$.
6. $X^i = G^{-1}\, C^i$.
7. $\pi_i^j = C_i\, X^j$.
8. $(\delta_j^i - \pi_j^i)\, F^j$.

4. DYNAMICS THE DOUBLE PENDULUM

Let us apply Theorem 5 to the double pendulum. In order to show how this theorem can be put into practice, the calculations are performed in all the details, as may appear in a *worksheet* written by a software.

4.1. Worksheet

For the double pendulum, a natural choice of the Lagrangian coordinates is

$$(q^1, q^2, q^3, q^4) = (x_1, y_1, x_2, y_2).$$

Note that the constraint equation (1.2) can be written in the two equivalent forms

$$\begin{cases} (y_1 - y_2)\, \dot{y}_2 - x_2\, \dot{x}_2 = -x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}, \\ (y_1 - y_2)\, \dot{y}_1 + x_1\, \dot{x}_1 = x_2\, \dfrac{\dot{x}_2\, \dot{y}_1}{\dot{y}_2}. \end{cases} \tag{4.1}$$

These equations are used throughout the following calculation.

$\boxed{1}$ Kinetic energy.

$$K = K_1 + K_2 = \tfrac{1}{2}\, m_1\, (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}\, m_2\, (\dot{x}_2^2 + \dot{y}_2^2).$$

$$\boxed{[g_{ij}] = \text{diag}\,[m_1, m_1, m_2, m_2], \quad [g^{ij}] = \text{diag}\left[\frac{1}{m_1}, \frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_2}\right]}$$

$\boxed{2}$ The components of the metric are constant; the Christoffel symbols vanish. Thus,

$$\boxed{F^i = A^i}$$

Note that the Lagrangian active forces A_i can be determined by computing the virtual work

$$\delta W = A_i\,\delta q^i = A_{x_1}\delta x_1 + A_{y_1}\delta y_1 + A_{x_2}\delta x_2 + A_{y_2}\delta y_2.$$

Then the contravariant components $A^i = g^{ij}\,A_j$ are given by

$$A^1 = \frac{1}{m_1}\,A_1, \quad A^2 = \frac{1}{m_1}\,A_2,$$
$$A^3 = \frac{1}{m_2}\,A_3, \quad A^4 = \frac{1}{m_2}\,A_4,$$

so that

$$\left.\begin{aligned} m_1\,A^1 = A_{x_1}, \quad m_1\,A^2 = A_{y_1}, \\ m_2\,A^3 = A_{x_2}, \quad m_2\,A^4 = A_{y_2}. \end{aligned}\right\} \tag{4.2}$$

$\boxed{3}$ Compute $C_i = \dfrac{\partial C}{\partial \dot{q}^i}$, where $C(x, y)$ is the left-hand side of the constraint equation (1.2). Use (4.1).

$$[C_i] = \begin{bmatrix} x_1\,\dot{y}_2 \\ -x_2\,\dot{x}_2 + (y_1 - y_2)\,\dot{y}_2 \\ -x_2\,\dot{y}_1 \\ x_1\,\dot{x}_1 + (y_1 - y_2)\,\dot{y}_1 \end{bmatrix} = \begin{bmatrix} x_1\,\dot{y}_2 \\ -\,x_1\,\dfrac{\dot{x}_1\,\dot{y}_2}{\dot{y}_1} \\ -\,x_2\,\dot{y}_1 \\ x_2\,\dfrac{\dot{y}_1\,\dot{x}_2}{\dot{y}_2} \end{bmatrix}.$$

$\boxed{4}$ Compute $C^i = g^{ij}\,C_i$.

$$[C^i] = \begin{bmatrix} \dfrac{x_1}{m_1}\,\dot{y}_2 \\ -\,\dfrac{x_1}{m_1}\,\dfrac{\dot{x}_1\,\dot{y}_2}{\dot{y}_1} \\ -\,\dfrac{x_2}{m_2}\,\dot{y}_1 \\ \dfrac{x_2}{m_2}\,\dfrac{\dot{y}_1\,\dot{x}_2}{\dot{y}_2} \end{bmatrix}.$$

$\boxed{5}$ Compute $G = C^i\,C_i$.

$$\begin{aligned} G &= \frac{1}{m_1}\,x_1^2\,\dot{y}_2^2 + \frac{1}{m_1}\left(x_1\,\frac{\dot{x}_1\,\dot{y}_2}{\dot{y}_1}\right)^2 + \frac{1}{m_2}\,x_2^2\,\dot{y}_1^2 + \frac{1}{m_2}\left(x_2\,\frac{\dot{x}_2\,\dot{y}_1}{\dot{y}_2}\right)^2 \\ &= \frac{x_1^2\,\dot{y}_2^2}{m_1}\left(1 + \frac{\dot{x}_1^2}{\dot{y}_1^2}\right) + \frac{x_2^2\,\dot{y}_1^2}{m_2}\left(1 + \frac{\dot{x}_2^2}{\dot{y}_2^2}\right) \\ &= \frac{x_1^2\,\dot{y}_2^2}{m_1}\,\frac{\dot{x}_1^2 + \dot{y}_1^2}{\dot{y}_1^2} + \frac{x_2^2\,\dot{y}_1^2}{m_2}\,\frac{\dot{x}_2^2 + \dot{y}_2^2}{\dot{y}_2^2} = \frac{x_1^2\,\dot{y}_2^2}{m_1}\,\frac{v_1^2}{\dot{y}_1^2} + \frac{x_2^2\,\dot{y}_1^2}{m_2}\,\frac{v_2^2}{\dot{y}_2^2}, \end{aligned}$$

where

$$\boxed{v_1^2 = \dot{x}_1^2 + \dot{y}_1^2, \qquad v_2^2 = \dot{x}_2^2 + \dot{y}_2^2}$$

Then,

$$\boxed{G = \frac{\Delta}{m_1\, m_2\, \dot{y}_1^2\, \dot{y}_2^2}, \qquad \Delta \doteq m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2}$$

$\boxed{6}$ Compute $X^i = G^{-1}\, C^i$.

$$[X^i] = \frac{m_1\, m_2\, \dot{y}_1^2\, \dot{y}_2^2}{\Delta} \begin{bmatrix} \dfrac{x_1}{m_1}\, \dot{y}_2 \\ -\dfrac{x_1}{m_1}\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1} \\ -\dfrac{x_2}{m_2}\, \dot{y}_1 \\ \dfrac{x_2}{m_2}\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} m_2\, x_1\, \dot{y}_1^2\, \dot{y}_2^3 \\ -\, m_2\, x_1\, \dot{x}_1\, \dot{y}_1\, \dot{y}_2^3 \\ -\, m_1\, x_2\, \dot{y}_2^2\, \dot{y}_1^3 \\ m_1\, x_2\, \dot{x}_2\, \dot{y}_2\, \dot{y}_1^3 \end{bmatrix}.$$

$\boxed{7}$ Compute $\pi_i^j = C_i\, X^j$.

$$[\pi_i^j] = \begin{bmatrix} C_1\, X^j \\ C_2\, X^j \\ C_3\, X^j \\ C_4\, X^j \end{bmatrix} = \begin{bmatrix} x_1\, \dot{y}_2\, X^j \\ -\, x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}\, X^j \\ -\, x_2\, \dot{y}_1\, X^j \\ x_2\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2}\, X^j \end{bmatrix} =$$

$$= \begin{bmatrix} x_1\, \dot{y}_2\, X^1 & x_1\, \dot{y}_2\, X^2 & x_1\, \dot{y}_2\, X^3 & x_1\, \dot{y}_2\, X^4 \\ -\, x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}\, X^1 & -\, x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}\, X^2 & -\, x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}\, X^3 & -\, x_1\, \dfrac{\dot{x}_1\, \dot{y}_2}{\dot{y}_1}\, X^4 \\ -\, x_2\, \dot{y}_1\, X^1 & -\, x_2\, \dot{y}_1\, X^2 & -\, x_2\, \dot{y}_1\, X^3 & -\, x_2\, \dot{y}_1\, X^4 \\ x_2\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2}\, X^1 & x_2\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2}\, X^2 & x_2\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2}\, X^3 & x_2\, \dfrac{\dot{y}_1\, \dot{x}_2}{\dot{y}_2}\, X^4 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} m_2 x_1^2 \dot{y}_1^2 \dot{y}_2^4 & -\, m_2 x_1^2 \dot{x}_1 \dot{y}_1 \dot{y}_2^4 & -\, m_1 x_1 x_2 \dot{y}_1^3 \dot{y}_2^3 & m_1 x_1 x_2 \dot{x}_2 \dot{y}_2^2 \dot{y}_1^3 \\ -\, m_2 x_1^2 \dot{x}_1 \dot{y}_1 \dot{y}_2^4 & m_2 x_1^2 \dot{x}_1^2 \dot{y}_2^4 & m_1 x_1 x_2 \dot{x}_1 \dot{y}_1^2 \dot{y}_2^3 & -\, m_1 x_1 x_2 \dot{x}_1 \dot{x}_2 \dot{y}_1^2 \dot{y}_2^2 \\ -\, m_2 x_1 x_2 \dot{y}_1^3 \dot{y}_2^3 & m_2 x_1 x_2 \dot{x}_1 \dot{y}_1^2 \dot{y}_2^3 & m_1 x_2^2 \dot{y}_1^4 \dot{y}_2^2 & -\, m_1 x_2^2 \dot{x}_2 \dot{y}_1^4 \dot{y}_2 \\ m_2 x_1 x_2 \dot{x}_2 \dot{y}_1^3 \dot{y}_2^2 & -\, m_2 x_1 x_2 \dot{x}_1 \dot{x}_2 \dot{y}_1^2 \dot{y}_2^2 & -\, m_1 x_2^2 \dot{x}_2 \dot{y}_1^4 \dot{y}_2 & m_1 x_2^2 \dot{x}_2^2 \dot{y}_1^4 \end{bmatrix}$$

[8] Compute $[\delta_i^j - \pi_i^j]$.

$$[\delta_i^j - \pi_i^j] = \frac{1}{\Delta}\begin{bmatrix} \Delta - m_2 x_1^2 \dot y_1^2 \dot y_2^4 & m_2 x_1^2 \dot x_1 \dot y_1 \dot y_2^4 & m_1 x_1 x_2 \dot y_1^3 \dot y_2^3 & - m_1 x_1 x_2 \dot x_2 \dot y_2^2 \dot y_1^3 \\ m_2 x_1^2 \dot x_1 \dot y_1 \dot y_2^4 & \Delta - m_2 x_1^2 \dot x_1^2 \dot y_2^4 & - m_1 x_1 x_2 \dot x_1 \dot y_1^2 \dot y_2^3 & m_1 x_1 x_2 \dot x_1 \dot x_2 \dot y_1^2 \dot y_2^2 \\ m_2 x_1 x_2 \dot y_1^3 \dot y_2^3 & - m_2 x_1 x_2 \dot x_1 \dot y_1^2 \dot y_2^3 & \Delta - m_1 x_2^2 \dot y_1^4 \dot y_2^2 & m_1 x_2^2 \dot x_2 \dot y_1^4 \dot y_2 \\ - m_2 x_1 x_2 \dot x_2 \dot y_1^3 \dot y_2^2 & m_2 x_1 x_2 \dot x_1 \dot x_2 \dot y_1^2 \dot y_2^2 & m_1 x_2^2 \dot x_2 \dot y_1^4 \dot y_2 & \Delta - m_1 x_2^2 \dot x_2^2 \dot y_1^4 \end{bmatrix}$$

where

$$\Delta = m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2).$$

Diagonal terms:

$$\begin{aligned} \Delta - m_2\, x_1^2\, \dot y_1^2\, \dot y_2^4 &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2) - m_2\, x_1^2\, \dot y_1^2\, \dot y_2^4 \\ &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot x_1^2\, \dot y_2^4. \end{aligned}$$

$$\begin{aligned} \Delta - m_2\, x_1^2\, \dot x_1^2\, \dot y_2^4 &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2) - m_2\, x_1^2\, \dot x_1^2\, \dot y_2^4 \\ &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_1^2\, \dot y_2^4. \end{aligned}$$

$$\begin{aligned} \Delta - m_1\, x_2^2\, \dot y_1^4\, \dot y_2^2 &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2) - m_1\, x_2^2\, \dot y_1^4\, \dot y_2^2 \\ &= m_1\, x_2^2\, \dot y_1^4\, \dot x_2^2 + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2). \end{aligned}$$

$$\begin{aligned} \Delta - m_1\, x_2^2\, \dot x_2^2\, \dot y_1^4 &= m_1\, x_2^2\, \dot y_1^4\, (\dot x_2^2 + \dot y_2^2) + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2) - m_1\, x_2^2\, \dot x_2^2\, \dot y_1^4 \\ &= m_1\, x_2^2\, \dot y_1^4\, \dot y_2^{)} + m_2\, x_1^2\, \dot y_2^4\, (\dot x_1^2 + \dot y_1^2). \end{aligned}$$

Conclusion (i index of line).

$$\Delta\, [\delta_i^j - \pi_i^j] = \left[\begin{matrix} m_1\, x_2^2\, \dot y_1^4\, v_2^2 + m_2\, x_1^2\, \dot x_1^2\, \dot y_2^4 & m_2\, x_1^2\, \dot x_1\, \dot y_1\, \dot y_2^4 \\ m_2\, x_1^2\, \dot x_1\, \dot y_1\, \dot y_2^4 & m_1\, x_2^2\, \dot y_1^4\, v_2^2 + m_2\, x_1^2\, \dot y_1^2\, \dot y_2^4 \\ m_2\, x_1\, x_2\, \dot y_1^3\, \dot y_2^3 & - m_2\, x_1\, x_2\, \dot x_1\, \dot y_1^2\, \dot y_2^3 \\ - m_2\, x_1\, x_2\, \dot x_2\, \dot y_1^3\, \dot y_2^2 & m_2\, x_1\, x_2\, \dot x_1\, \dot x_2\, \dot y_1^2\, \dot y_2^2 \end{matrix}\right.$$

$$\left.\begin{matrix} m_1\, x_1\, x_2\, \dot y_1^3\, \dot y_2^3 & - m_1\, x_1\, x_2\, \dot x_2\, \dot y_2^2\, \dot y_1^3 \\ - m_1\, x_1\, x_2\, \dot x_1\, \dot y_1^2\, \dot y_2^3 & m_1\, x_1\, x_2\, \dot x_1\, \dot x_2\, \dot y_1^2\, \dot y_2^2 \\ m_1\, x_2^2\, \dot y_1^4\, \dot x_2^2 + m_2\, x_1^2\, \dot y_2^4\, v_1^2 & m_1\, x_2^2\, \dot x_2\, \dot y_1^4\, \dot y_2 \\ m_1\, x_2^2\, \dot x_2\, \dot y_1^4\, \dot y_2 & m_1\, x_2^2\, \dot y_1^4\, \dot y_2 + m_2\, x_1^2\, \dot y_2^4\, v_1^2 \end{matrix}\right]$$

9 Compute $\dot q^\ell \dfrac{\partial C}{\partial q^\ell} = \dot x_1^2\,\dot y_2 + \dot y_1^2\,\dot y_2 - \dot x_2^2\,\dot y_1 - \dot y_1\,\dot y_2^2 = \dot y_2\,v_1^2 - \dot y_1\,v_2^2$.

10 Compute $D^i = (\delta^i_\ell - \pi^i_\ell)\,F^\ell - \dot q^\ell \dfrac{\partial C}{\partial q^\ell} X^i$. Compute ΔD^i with $\Delta = m_1\,x_2^2\,\dot y_1^4\,v_2^2 + m_2\,x_1^2\,\dot y_2^4\,v_1^2$ and $F^i = A^i$:

$$\Delta D^i = \Delta\,(\delta^i_\ell - \pi^i_\ell)\,A^\ell - (\dot y_2\,v_1^2 - \dot y_1\,v_2^2)\begin{bmatrix} m_2\,x_1\,\dot y_1^2\,\dot y_2^3 \\ -\,m_2\,x_1\,\dot x_1\,\dot y_1\,\dot y_2^3 \\ -\,m_1\,x_2\,\dot y_2^2\,\dot y_1^3 \\ m_1\,x_2\,\dot x_2\,\dot y_2\,\dot y_1^3 \end{bmatrix}.$$

10.1 Compute ΔD^1.

$$\begin{aligned}\Delta D^1 &= (m_1\,x_2^2\,\dot y_1^4\,v_2^2 + m_2\,x_1^2\,\dot x_1^2\,\dot y_2^4)\,A^1 + m_2\,x_1^2\,\dot x_1\,\dot y_1\,\dot y_2^4\,A^2 \\ &\quad + m_2\,x_1\,x_2\,\dot y_1^3\,\dot y_2^3\,A^3 - m_2\,x_1\,x_2\,\dot x_2\,\dot y_1^3\,\dot y_2^2\,A^4 \\ &\quad - m_2\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2)\,x_1\,\dot y_1^2\,\dot y_2^3 \\ &= m_1\,x_2^2\,\dot y_1^4\,v_2^2\,A^1 + m_2\,x_1^2\,\dot x_1\,\dot y_2^4\,(\dot x_1\,A^1 + \dot y_1\,A^2) \\ &\quad + m_2\,x_1\,x_2\,\dot y_1\,(\dot y_2\,A^3 - \dot x_2\,A^4) - m_2\,x_1\,\dot y_1^2\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2).\end{aligned}$$

Recall (4.2).

$$\begin{aligned}\Delta D^1 &= x_2^2\,\dot y_1^4\,v_2^2\,A_{x_1} + \frac{m_2}{m_1}\,x_1^2\,\dot x_1\,\dot y_2^4\,(\dot x_1\,A^1 + \dot y_1\,A^2) \\ &\quad + x_1\,x_2\,\dot y_1\,(\dot y_2\,A_{x_2} - \dot x_2\,A_{y_2}) - m_2\,x_1\,\dot y_1^2\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2).\end{aligned} \tag{4.3}$$

10.2 Compute ΔD^2.

$$\begin{aligned}\Delta D^2 &= m_2\,x_1^2\,\dot x_1\,\dot y_1\,\dot y_2^4\,A^1 + (m_1\,x_2^2\,\dot y_1^4\,v_2^2 + m_2\,x_1^2\,\dot y_1^2\,\dot y_2^4)\,A^2 \\ &\quad - m_2\,x_1\,x_2\,\dot x_1\,\dot y_1^2\,\dot y_2^3\,A^3 + m_2\,x_1\,x_2\,\dot x_1\,\dot x_2\,\dot y_1^2\,\dot y_2^2\,A^4 \\ &\quad + m_2\,x_1\,\dot x_1\,\dot y_1\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2) \\ &= m_1\,x_2^2\,\dot y_1^4\,v_2^2\,A^2 + m_2\,x_1^2\,\dot y_1\,\dot y_2^4\,(\dot x_1\,A^1 + \dot y_1\,A^2) \\ &\quad - m_2\,x_1\,x_2\,\dot x_1\,\dot y_1^2\,\dot y_2^2\,(\dot y_2\,A^3 - \dot x_2\,A^4) \\ &\quad + m_2\,x_1\,\dot x_1\,\dot y_1\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2).\end{aligned}$$

$$\begin{aligned}\Delta D^2 &= x_2^2\,\dot y_1^4\,v_2^2\,A_{y_1} + \frac{m_2}{m_1}\,x_1^2\,\dot y_1\,\dot y_2^4\,(\dot x_1\,A_{x_1} + \dot y_1\,A_{y_1})\\ &- x_1\,x_2\,\dot x_1\,\dot y_1^2\,\dot y_2^2\,(\dot y_2\,A_{x_2} - \dot x_2\,A_{y_2})\\ &+ m_2\,x_1\,\dot x_1\,\dot y_1\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2).\end{aligned} \tag{4.4}$$

The expressions of D^3 and D^4 are obtained from (4.3) and (4.4) by symmetry, that is, by interchanging the lower indices 1 and 2.

11 Dynamical equations (3.21) for the double pendulum.

First group:

$$\begin{cases}\dfrac{dx_1}{dt} = \dot x_1,\\ \dfrac{dy_1}{dt} = \dot y_1,\end{cases}\qquad \begin{cases}\dfrac{dx_2}{dt} = \dot x_2,\\ \dfrac{dy_2}{dt} = \dot y_2.\end{cases} \tag{4.5}$$

Second group:

$$\boxed{\begin{aligned}&\frac{d\dot x_1}{dt} = \frac{1}{\Delta}\Big[x_2^2\,\dot y_1^4\,v_2^2\,A_{x_1} + \frac{m_2}{m_1}\,x_1^2\,\dot x_1\,\dot y_2^4\,(\dot x_1\,A_{x_1} + \dot y_1\,A_{y_1}) +\\ &+ x_1\,x_2\,\dot y_1^3\,\dot y_2^2\,(\dot y_2\,A_{x_2} - \dot x_2\,A_{y_2}) + m_2\,x_1\,(\dot y_1\,v_2^2 - \dot y_2\,v_1^2)\,\dot y_1^2\,\dot y_2^3\Big]\\ &\frac{d\dot y_1}{dt} = \frac{\dot y_1}{\Delta}\Big[x_2^2\,\dot y_1^3\,v_2^2\,A_{y_1} + \frac{m_2}{m_1}\,x_1^2\,\dot y_2^4\,(\dot x_1\,A_{x_1} + \dot y_1\,A_{y_1}) +\\ &+ x_1\,x_2\,\dot x_1\,\dot y_1\,\dot y_2^2\,(\dot x_2\,A_{y_2} - \dot y_2\,A_{x_2}) + m_2\,x_1\,\dot x_1\,\dot y_2^3\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2)\Big]\end{aligned}} \tag{4.6}$$

$$\boxed{\begin{aligned}&\frac{d\dot x_2}{dt} = \frac{1}{\Delta}\Big[x_1^2\,\dot y_2^4\,v_1^2\,A_{x_2} + \frac{m_1}{m_2}\,x_2^2\,\dot x_2\,\dot y_1^4\,(\dot x_2\,A_{x_2} + \dot y_2\,A_{y_2}) +\\ &+ x_1\,x_2\,\dot y_2^3\,\dot y_1^2\,(\dot y_1\,A_{x_1} - \dot x_1\,A_{y_1}) + m_1\,x_2\,(\dot y_2\,v_1^2 - \dot y_1\,v_2^2)\,\dot y_2^2\,\dot y_1^3\Big]\\ &\frac{d\dot y_2}{dt} = \frac{\dot y_2}{\Delta}\Big[\,x_1^2\,\dot y_2^3\,v_1^2\,A_{y_2} + \frac{m_1}{m_2}\,x_2^2\,\dot y_1^4\,(\dot x_2\,A_{x_2} + \dot y_2\,A_{y_2}) +\\ &+ x_1\,x_2\,\dot x_2\,\dot y_2\,\dot y_1^2\,(\dot x_1\,A_{y_1} - \dot y_1\,A_{x_1}) + m_1\,x_2\,\dot x_2\,\dot y_1^3\,(\dot y_1\,v_2^2 - \dot y_2\,v_1^2)\Big]\end{aligned}} \tag{4.7}$$

where

$$\begin{cases}\Delta = m_1\,x_2^2\,\dot y_1^4\,v_2^2 + m_2\,x_1^2\,\dot y_2^4\,v_1^2,\\ v_1^2 = \dot x_1^2 + \dot y_1^2, \quad v_2^2 = \dot x_2^2 + \dot y_2^2.\end{cases} \tag{4.8}$$

These dynamical equations hold for whatever active forces: $\boldsymbol{A}_1 = [A_1, A_2] = [A_{x_1}, A_{y_1}]$ acting on P_1, and $\boldsymbol{A}_2 = [A_3, A_4] = [A_{x_2}, A_{y_2}]$ acting on P_2 (Fig. 8).

Note that, in the dynamical equations,

$$\begin{cases}\dot x_1\,A_{x_1} + \dot y_1\,A_{y_1} = \boldsymbol{v}_1 \cdot \boldsymbol{A}_1,\\ \dot x_1\,A_{y_1} - \dot y_1\,A_{x_1} = \boldsymbol{v}_1 \times \boldsymbol{A}_1 \cdot \boldsymbol{k},\end{cases} \tag{4.9}$$

(the same for 1 replaced by 2) where $\boldsymbol{k}$ is the unit vector orthogonal to the Cartesian plane.

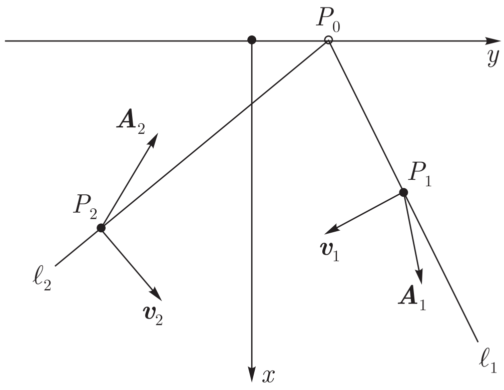

Fig. 8

4.2. The Energy Constants

If K_ν and W_ν are the kinetic energy of the point P_ν and the power of the active force acting on the same point,

$$K_\nu = \tfrac{1}{2}\, m_\nu\, \boldsymbol{v}_\nu^2, \quad W_\nu = \boldsymbol{v}_\nu \cdot \boldsymbol{A}_\nu,$$

then

$$\frac{d}{dt}\sum_\nu K_\nu = \sum_\nu W_\nu.$$

For the double pendulum we observe that this equation splits into two equations,

$$\frac{dK_1}{dt} = W_1, \quad \frac{dK_2}{dt} = W_2.$$

Theorem 6. *Let W_1 be the power of the active forces acting on P_1 and K_1 its kinetic energy. Then*

$$\boxed{\frac{dK_1}{dt} = W_1}$$

Proof.

$$\frac{dK_1}{dt} = \dot{x}_1 \frac{d\dot{x}_1}{dt} + \dot{y}_1 \frac{d\dot{y}_1}{dt} = \dot{x}_1\, D^1 + \dot{y}_1\, D^2,$$

and

$$W_1 = \boldsymbol{v}_1 \cdot \boldsymbol{A}_1 = \dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}.$$

Compute $\Delta\,(\dot{x}_1\, D^1 + \dot{y}_1\, D^2)$:

$$\left[\begin{aligned} &= \dot{x}_1\, x_2^2\, \dot{y}_1^4\, v_2^2\, A_{x_1} \\ &+ \frac{m_2}{m_1}\, x_1^2\, \dot{y}_2^4\, \dot{x}_1^2\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \\ &+ \dot{x}_1\, x_1\, x_2\, \dot{y}_1^3\, \dot{y}_2^2\, (\dot{y}_2\, A_{x_2} - \dot{x}_2\, A_{y_2}) \\ &+ m_2\, x_1\, \dot{x}_1\, (v_2^2\, \dot{y}_1 - v_1^2\, \dot{y}_2)\, \dot{y}_1^2\, \dot{y}_2^3 \end{aligned}\right. \qquad \left[\begin{aligned} &+ x_2^2\, \dot{y}_1^5\, v_2^2\, A_{y_1} \\ &+ \frac{m_2}{m_1}\, x_1^2\, \dot{y}_1^2\, \dot{y}_2^4\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \\ &+ x_1\, x_2\, \dot{x}_1\, \dot{y}_1^3\, \dot{y}_2^2\, (\dot{x}_2\, A_{y_2} - \dot{y}_2\, A_{x_2}) \\ &+ m_2\, x_1\, \dot{x}_1\, \dot{y}_1^2\, \dot{y}_2^3\, \left(v_1^2\, \dot{y}_2 - v_2^2\, \dot{y}_1\right). \end{aligned}\right.$$

The last terms of the columns cancel each other.

$$\left[\begin{array}{l} = \dot{x}_1\, x_2^2\, \dot{y}_1^4\, v_2^2\, A_{x_1} \\ \\ +\dfrac{m_2}{m_1}\, x_1^2\, \dot{y}_2^4\, \dot{x}_1^2\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \\ \\ +\dot{x}_1\, x_1\, x_2\, \dot{y}_1^3\, \dot{y}_2^2\, (\dot{y}_2\, A_{x_2} - \dot{x}_2\, A_{y_2}) \end{array}\right. \quad \left[\begin{array}{l} +x_2^2\, \dot{y}_1^5\, v_2^2\, A_{y_1} \\ \\ +\dfrac{m_2}{m_1}\, x_1^2\, \dot{y}_1^2\, \dot{y}_2^4\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \\ \\ +x_1\, x_2\, \dot{x}_1\, \dot{y}_1^3\, \dot{y}_2^2\, (\dot{x}_2\, A_{y_2} - \dot{y}_2\, A_{x_2}) \end{array}\right.$$

The last terms of the columns cancel each other.

$$\left[\begin{array}{l} = x_2^2\, \dot{y}_1^4\, v_2^2\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) + \dfrac{m_2}{m_1}\, x_1^2\, \dot{y}_2^4\, v_1^2\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \\ = \dfrac{\Delta}{m_1}\, (\dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}) \end{array}\right.$$

since $\Delta \doteq m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2$. Consequently,

$$m_1\, (\dot{x}_1\, D^1 + \dot{y}_1\, D^2) = \dot{x}_1\, A_{x_1} + \dot{y}_1\, A_{y_1}.$$

□

As a consequence, in the case of conservative forces with potentials $V_1(x_1, y_1)$ and $V_2(x_2, y_2)$, we have two separated constants of energy,

$$K_1 + V_1 = E_1, \quad K_2 + V_2 = E_2. \tag{4.10}$$

4.3. Special Cases and Examples

The dynamic equations (4.6) and (4.7) are very cumbersome. When the active forces are assigned, to study the behavior of the double pendulum is almost always necessary to resort to numerical integration. Nevertheless, let us see how we can analyze them in some simple examples.

4.3.1. Spontaneous Motions

These are the motions in absence of active forces. In this case, the dynamical equations (4.6) and (4.7) reduce to

$$\begin{aligned} \frac{d\dot{x}_1}{dt} &= \frac{m_2\, x_1\, (\dot{y}_1\, v_2^2 - \dot{y}_2\, v_1^2)\, \dot{y}_1^2\, \dot{y}_2^3}{m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2}, \\ \frac{d\dot{y}_1}{dt} &= \frac{m_2\, x_1\, \dot{x}_1\, \dot{y}_1\, \dot{y}_2^3\, (\dot{y}_2\, v_1^2 - \dot{y}_1\, v_2^2)}{m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \frac{d\dot{x}_2}{dt} &= \frac{m_1\, x_2\, (\dot{y}_2\, v_1^2 - \dot{y}_1\, v_2^2)\, \dot{y}_2^2\, \dot{y}_1^3}{m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2}, \\ \frac{d\dot{y}_2}{dt} &= \frac{m_1\, x_2\, \dot{x}_2\, \dot{y}_2\, \dot{y}_1^3\, (\dot{y}_1\, v_2^2 - \dot{y}_2\, v_1^2)}{m_1\, x_2^2\, \dot{y}_1^4\, v_2^2 + m_2\, x_1^2\, \dot{y}_2^4\, v_1^2}. \end{aligned} \tag{4.12}$$

For these motions the kinetic energies of the two points are constant,

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = \text{const.}, \quad v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = \text{const.}$$

Example 1. Motions with $\boldsymbol{v}_2(t) = 0$. Equations (4.12) are identically satisfied. Equations (4.11) give

$$\mapsto \begin{cases} \dfrac{d\dot{x}_1}{dt} = -\dfrac{\dot{y}_1^2}{x_1} \\ \dfrac{d\dot{y}_1}{dt} = \dfrac{\dot{x}_1\, \dot{y}_1}{x_1}. \end{cases}$$

The solution is

$$\begin{cases} x_1 = A\cos(\omega\, t + \phi) \\ y_1 = A\sin(\omega\, t + \phi) + B \end{cases} \qquad \begin{cases} x_1(0) = A\cos\phi \\ y_1(0) = A\sin\phi + B, \end{cases}$$

Figure 9 illustrates the initial conditions. The fixed point P_2 lies on the x-axis.

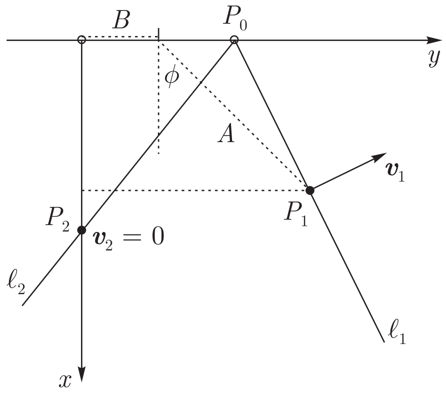

Fig. 9

The coordinate of the point P_0 is given by the first equation (1.3),

$$y_0 = \frac{\dot{x}_1}{\dot{y}_1}\, x_1 + y_1.$$

Since $\dot{y}_1 = \omega\, x_1$,

$$\begin{aligned} y_0 &= \frac{\dot{x}_1}{\dot{y}_1}\, x_1 + y_1 = \frac{\dot{x}_1}{\omega} + y_1 \\ &= \frac{-\omega\, A\sin(\omega\, t + \phi)}{\omega} + A\sin(\omega\, t + \phi) + B = B. \end{aligned}$$

Thus, also P_0 remains at rest. Fig. 10 illustrates the orbit of P_1.

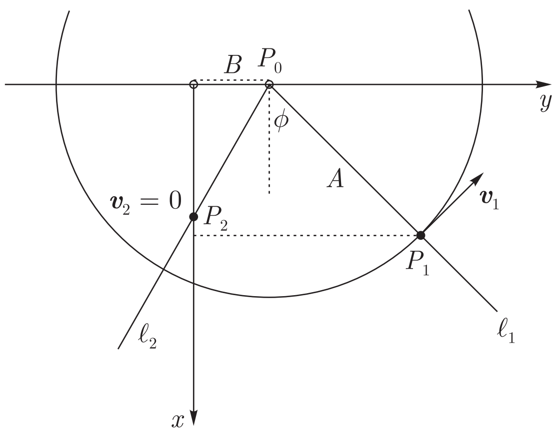

Fig. 10

Example 2. Motions with $\boldsymbol{v}_2$ of constant direction. Fig. 11 illustrates the initial conditions; the point P_2 lies on the x-axis; its velocity $\boldsymbol{v}_2(0)$ is assigned as well as the position of P_1.

Consequently, the position of P_0 and the direction of $\boldsymbol{v}_1$ are uniquely determined. In order to find a motion such that $\boldsymbol{v}_2$ as a constant direction, we have to find a suitable value of the scalar velocity v_1.

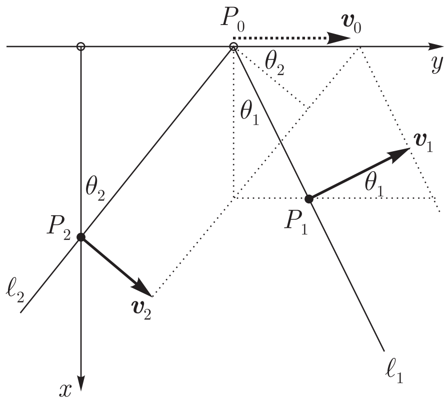

Fig. 11

Figure 11 shows a geometrical construction of the length of $\boldsymbol{v}_1$ (dotted lines) through the construction of the velocity $\boldsymbol{v}_0$ of P_0. Two angles θ_1 and θ_2 are put in evidence, such that

$$v_0 = v_1 \tan\theta_1, \quad y_1 - y_0 = x_1 \tan\theta_1,$$

and

$$v_0 = v_2 \tan\theta_2, \quad x_2 \tan\theta_2 = y_0, \quad \tan\theta_2 = \frac{\dot{x}_2}{\dot{y}_2}.$$

It follows that

$$v_1 = v_2 \frac{\tan\theta_2}{\tan\theta_1} = v_2\, x_1 \frac{\tan\theta_2}{y_1 - y_0} = \frac{v_2\, x_1}{y_1 - y_0} \frac{\dot{x}_2}{\dot{y}_2},$$

where

$$y_0 = \frac{x_2\, \dot{x}_2}{\dot{y}_2}.$$

Thus,

$$v_1 = \frac{x_1\, \dot{x}_2\, v_2}{y_1\, \dot{y}_2 - x_2\, \dot{x}_2}. \tag{4.13}$$

This formula provides the scalar velocity v_1 in terms of $\boldsymbol{v}_2 = (\dot{x}_2, \dot{y}_2)$, x_2 and (x_1, y_1).

On the other hand, the computation of $\dfrac{d}{dt}\dfrac{\dot{y}_2}{\dot{x}_2}$ with the intervention of the dynamical equations shows that the condition $\dot{y}_2/\dot{x}_2 = \text{constant}$ implies

$$x_2\, (\dot{y}_1\, v_2^2 - \dot{y}_2\, v_1^2)\, \dot{y}_2\, \dot{y}_1^3\, v_2^2 = 0,$$

that is, $\dot{y}_1\, v_2^2 = \dot{y}_2\, v_1^2$. Then the dynamical equations imply

$$\frac{d\dot{x}_1}{dt} = \frac{d\dot{y}_1}{dt} = \frac{d\dot{x}_2}{dt} = \frac{d\dot{y}_2}{dt} = 0.$$

The velocities $\mathbf{v_1}$ and $\boldsymbol{v}_2$ are constant. Moreover, since

$$y_0 = \frac{\dot{x}_2}{\dot{y}_2}\, x_2 + y_2,$$

also the point P_0 moves with a constant velocity

$$\dot{y}_0 = \frac{\dot{x}_2^2}{\dot{y}_2} + \dot{y}_2 = \frac{v_2^2}{\dot{y}_2}.$$

This is in agreement with the previous considerations. The conclusion is that *uniform rectilinear motions of the two points of the double pendulum are possible, provided that the velocities and the starting positions satisfy equation* (4.13).

4.3.2. Constant gravitational field

The active forces are parallel to the x-axes and proportional to the masses (Fig. 12).

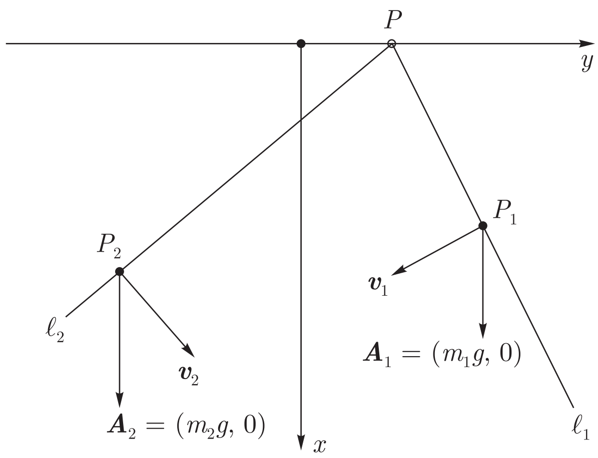

Fig. 12

One can think of a possible realization of such a non-holonomic system by means of two inclined planes on a horizontal desk, as shown in Fig. 13.

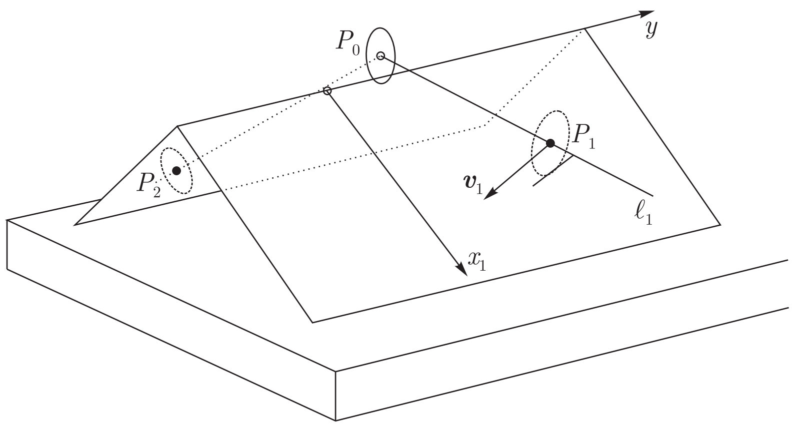

Fig. 13

The active forces are

$$\begin{cases} A_{x_1} = m_1\, g, \\ A_{x_2} = m_2\, g, \end{cases} \qquad \begin{cases} A_{y_1} = 0, \\ A_{y_2} = 0, \end{cases}$$

where g is the gravity acceleration reduced to the inclined planes. As shown by a straightforward calculation, the dynamical equations (4.6) and (4.7) reduce to

$$\boxed{\begin{aligned} \frac{d\dot{x}_1}{dt} &= g + \frac{m_2\,\Theta}{\Delta}\,x_1\,\dot{y}_1^2\,\dot{y}_2^3\,, & \frac{d\dot{x}_2}{dt} &= g - \frac{m_1\,\Theta}{\Delta}\,x_2\,\dot{y}_2^2\,\dot{y}_1^3 \\ \frac{d\dot{y}_1}{dt} &= -\frac{m_2\,\Theta}{\Delta}\,x_1\,\dot{x}_1\,\dot{y}_1\,\dot{y}_2^3\,, & \frac{d\dot{y}_2}{dt} &= \frac{m_1\,\Theta}{\Delta}\,x_2\,\dot{x}_2\,\dot{y}_2\,\dot{y}_1^3 \end{aligned}} \tag{4.14}$$

with

$$\left.\begin{aligned} \Delta &= m_1\,x_2^2\,\dot{y}_1^4\,v_2^2 + m_2\,x_1^2\,\dot{y}_2^4\,v_1^2, \\ \Theta &= (g\,x_2 + v_2^2)\,\dot{y}_1 - (g\,x_1 + v_1^2)\,\dot{y}_2. \end{aligned}\right\} \tag{4.15}$$

Note that Θ is skew-symmetric in the indices 1 and 2.

Example 3. Coinciding motions. Assume that, from the very beginning and at each instant t, the two points have the same positions and velocities. This kind of motion is compatible with the dynamical equations. Indeed, in this case Θ is constantly zero and the dynamical equations give

$$\frac{d\dot{x}_1}{dt} = g, \quad \frac{d\dot{y}_1}{dt} = 0, \quad \frac{d\dot{x}_2}{dt} = g, \quad \frac{d\dot{y}_2}{dt} = 0.$$

These are the equations of a free-falling particle (Fig. 14).

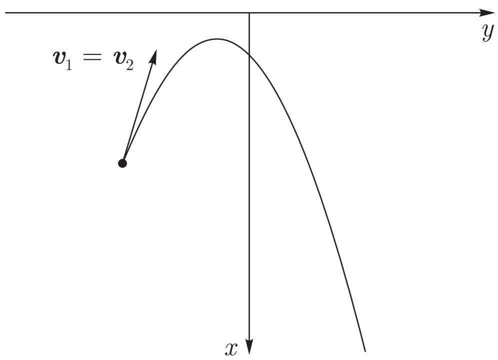

Fig. 14

Example 4. Symmetrical motions. Consider a motion for which

$$\begin{cases} x_2 = x_1, \quad \dot{x}_2 = \dot{x}_1, \\ y_2 = -\,y_1, \quad \dot{y}_2 = -\,\dot{y}_2, \end{cases}$$

for all t (Fig. 15).

Proposition 1. *Such a motion is compatible with the dynamical equations iff* $m_1 = m_2$.

Proof. (i) Assume that the dynamical equations admit such a solution. In this case they give

$$\begin{aligned} \frac{d\dot{x}_1}{dt} &= g - \frac{m_2\,\Theta}{\Delta}\,x_1\,\dot{y}_1^5\,, & \frac{d\dot{x}_1}{dt} &= g - \frac{m_1\,\Theta}{\Delta}\,x_1\,\dot{y}_1^5\,, \\ \frac{d\dot{y}_1}{dt} &= \frac{m_2\,\Theta}{\Delta}\,x_1\,\dot{x}_1\,\dot{y}_1^4\,, & \frac{d\dot{y}_1}{dt} &= \frac{m_1\,\Theta}{\Delta}\,x_1\,\dot{x}_1\,\dot{y}_4\,, \end{aligned}$$

with

$$\Delta = (m_1 + m_2)\,x_1^2\,\dot{y}_1^4\,v_1^2, \quad \Theta = 2\,(g\,x_1 + v_1^2)\,\dot{y}_1.$$

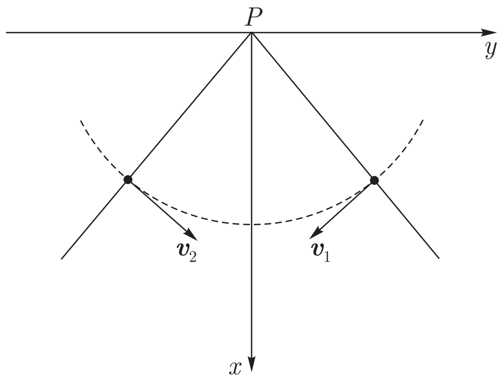

Fig. 15

These equations are simultaneously satisfied iff $m_1/\Delta = m_2/\Delta$ i.e., iff $m_1 = m_2$. Conversely, for $m_1 = m_2$ the problem is totally symmetric and the symmetry of the kinematical states is preserved during the motion. □

Proposition 2. *In this motion* $y_0 = 0$ *and* $r_1 = r_2 = \text{constant}$.

Proof. For $y_1 = -y_2$ and $x_1 = x_2$ the constraint equation (1.2) reduces to

$$x_1\,\dot{x}_1 + y_1\,\dot{y}_1 = 0 \qquad (PP_1 \cdot \boldsymbol{v}_1 = 0). \tag{4.16}$$

Because of (1.3) we find

$$y_0 = \frac{\dot{x}_1}{\dot{y}_1}\,x_1 + y_1 = 0.$$

Furthermore,

$$\frac{dr_1^2}{dt} = \frac{d}{dt}(x_1^2 + y_1^2) = 2x_1\,\dot{x}_1 + 2y_1\,\dot{y}_1 = 0.$$

□

Note that the integration of the dynamical equations, with $m_1 = m_2 = m$, $x = x_1 = x_2$ and $y = y_1 = -y_2$, reduces to the integration of the differential system

$$\begin{cases} \dfrac{d\dot{x}}{dt} = g - \dfrac{g\,x + v^2}{x\,v^2}\,\dot{y}^2, \\ \dfrac{d\dot{y}}{dt} = \dfrac{g\,x + v^2}{x\,v^2}\,\dot{x}\,\dot{y}. \end{cases} \tag{4.17}$$

Since $r_i = r_2 = \ell = \text{const.}$, we can write

$$\begin{cases} x = \ell\,\cos\theta, & \dot{x} = -\ell\,\dot{\theta}\,\sin\theta. \\ y = \ell\,\sin\theta, & \dot{y} = \ell\,\dot{\theta}\,\cos\theta, \end{cases} \qquad v^2 = \ell^2\,\dot{\theta}^2,$$

where θ is the angular deviation of the pendulum PP_1 from the x-axis. Then the second equation (4.17) takes the form

$$\ell\,(\ddot{\theta}\,\cos\theta - \dot{\theta}^2\,\sin\theta) = -(g\,\cos\theta + \ell\,\dot{\theta}^2)\,\sin\theta,$$

which reduces to $\ell\,\ddot{\theta} + g\,\sin\theta = 0$. This is the equation of the *mathematical pendulum.*

I am grateful to Nicolas Petit who has verified the validity of the results presented here through various numerical simulations. These simulations, along with additional arguments on the non-holonomic double pendulum, will be the subject of a future paper.

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