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# A General Method for Writing the Dynamical Equations of Nonholonomic Systems with Ideal Constraints

**S. Benenti***

*Department of Mathematics, University of Turin*
*Via Carlo Alberto 10, 10123 Torino, Italy*

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**Abstract**—The basic notions of the dynamics of nonholonomic systems are revisited in order to give a general and simple method for writing the dynamical equations for linear as well as non-linear kinematical constraints. The method is based on the representation of the constraints by parametric equations, which are interpreted as dynamical equations, and leads to first-order differential equations in normal form, involving the Lagrangian coordinates and auxiliary variables (the use of Lagrangian multipliers is avoided). Various examples are illustrated.

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## 1. INTRODUCTION

In this article I propose a simplified and improved version of a part of my paper “A ‘user-friendly’ approach to the dynamics of nonholonomic systems”, *SIGMA*, 2007, vol. 3.[1)] The aim is to provide a simple and effective algorithm for building up the dynamical equations of whatever nonholonomic mechanical system, with linear as well non-linear kinematical ideal constraints, in a form that does not involve Lagrangian multipliers or any other quantities related to the reactive forces. Examples of application of this method have been illustrated in the cited paper, together with historical comments and an essential list of references. Here, some further examples are proposed with the purpose first, of showing how the distinction between linear and non-linear constraints may be very subtle and second, of illustrating how some non-linear nonholonomic systems can be physically realized.

Let us consider a holonomic mechanical system with configuration manifold $Q$ of dimension $n$ and with generic Lagrangian coordinates $q = (q^i)$. Let us denote by $(q, \dot{q}) = (q^i, \dot{q}^i)$ the associated natural coordinates on the tangent bundle $TQ$.

As we know, by definition of *holonomic system*, we are actually dealing with a set of *material points* $(P_\nu, m_\nu)$, where $\nu$ is an index taking integer or continuous[2)] values. This is our **microscopic view-point**. These points cannot occupy any arbitrary position in the three-dimensional Euclidean space. They are one another linked by **positional constraints**[3)] in such a way that all possible configurations they can assume in the space form a set $Q$ endowed with a differentiable-manifold structure. This is our **macroscopic view-point**.

The relation between our two view-points is realized by the position-vectors $\mathbf{r}_\nu$ of each point $P_\nu$, expressed as functions of the Lagrangian coordinates

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^i). \tag{1}$$

*E-mail: sergio.benenti@unito.it

1) Available at the web address http://www.emis.de/journals/SIGMA/.

2) For simplicity, but without loss of generality, we shall consider below only the case of a finite number of material points.

3) Here we assume that these constraints are time-independent. However, it is not difficult to extend what we are going to do to time-dependent constraints.

A **motion** of the system is then represented by parametrized curves $q^i = q^i(t)$ on $Q$, where the parameter $t$ represents the time, and Eqs. (1) provide the motion of each single particle: $\mathbf{r}_\nu(t) = \mathbf{r}_\nu\left(q^i(t)\right)$. Along with the motion the instantaneous velocities are

$$\mathbf{v}_\nu(t) = \frac{\partial \mathbf{r}_\nu}{\partial q^i}\,\frac{dq^i}{dt}.$$

As a result, any possible **kinematical state** of our mechanical system is represented, from the microscopic view-point, by the set of vectors

$$\mathbf{r}_\nu = \mathbf{r}_\nu(q^i), \qquad \mathbf{v}_\nu(q^i, \dot{q}^i) = \frac{\partial \mathbf{r}_\nu}{\partial q^i}\,\dot{q}^i, \tag{2}$$

where $(\dot{q}^i) \in \mathbb{R}$. This explains that, from the macroscopic view-point, the tangent bundle $TQ$ represents the **space of the kinematical states**.

## 2. KINEMATICAL CONSTRAINTS

When the space of the kinematical states is well defined, then we can think of additional constraints involving the velocities. From the macroscopic view-point they are represented by a subset $C \subset TQ$. In most cases this subset, with the exception of **singular points** (or **singular states**), is a submanifold represented by local implicit equations

$$C^a(q, \dot{q}) = 0, \quad a = 1, \dots r, \tag{3}$$

satisfying the **regularity condition**

$$rank\left[\frac{\partial C^a}{\partial \dot{q}^i}\right] = r,$$

at all points of $C$ itself. Then these equations can be transformed into parametric equations

$$\dot{q}^i = \psi^i(q, z) \tag{4}$$

with parameters $z = (z_\alpha)$, $\alpha = 1, \dots, m < n$, and with the regularity condition

$$rank\left[\psi^i_\alpha\right] = m, \qquad \psi^i_\alpha(q, z) \doteq \frac{\partial \psi^i}{\partial z_\alpha}. \tag{5}$$

It is very important, for the following discussion, to observe that $(q^i, z_\alpha)$ can be interpreted as coordinates on $C$, which is then a submanifold of dimension $n + m$.

A **nonholonomic constraint** is a kinematical constraint which is not a differential consequence of constraints on the configurations.

A nonholonomic constraint is **linear** if it admits a representation by implicit equations (3) linear in the Lagrangian velocities,

$$C^a_i(q)\,\dot{q}^i = 0, \quad a = 1, \dots r, \tag{6}$$

or, equivalently, by parametric equations (4) linear in the parameters,

$$\dot{q}^i = \psi^i_\alpha(q)\,z_\alpha. \tag{7}$$

**Remark 1.** If $n$ is the dimension of the configuration manifold and $r$ is the number of the independent implicit equations describing the nonholonomic constraints, then codim $(C) = r$ and $\dim(C) = 2n - r$. On the other hand, $\dim(C) = n + m$, where $m$ is the number of parameters necessary for a parametric representation of $C$. It follows that this number is given by

$$m = n - r. \tag{8}$$

## 3. THE DYNAMICAL EQUATIONS

The leading idea of our approach is to consider the parametric constraint equations (4) as first-order dynamical equations:

$$\frac{dq^i}{dt} = \psi^i(q, z). \tag{9}$$

Since the right hand sides depend not only on the Lagrangian coordinates, but also on the parameters $z = (z_\alpha)$, our aim is to complete the dynamical equations (9) by first-order equations of the kind

$$\frac{dz_\alpha}{dt} = Z^\alpha(q, z). \tag{10}$$

So, the problem is to find the explicit expressions of the functions $Z^\alpha$. Of course, for reaching this result we have to start from basic postulates.

First, for the dynamics of each material point $(P_\nu, m_\nu)$ we believe in the Newton equation

$$m_\nu\, \mathbf{a}_\nu = \mathbf{A}_\nu + \mathbf{R}_\nu, \tag{11}$$

where: $\mathbf{a}_\nu$ is the acceleration, $\mathbf{A}_\nu$ is the **active force** (due to external fields and internal interactions) and $\mathbf{R}_\nu$ is the **reactive force** (it is *a priori* unknown but it has the role of making the constraints satisfied).

**Remark 2.** The idea of 'reactive force' arises from the Newtonian philosophy, according which any action deviating a point from the uniform rectilinear motion (in an inertial reference frame) is a 'force', mathematically represented by a vector. Thus, the presence of a kinematical constraint must be represented by a vector, called 'reactive force', to be summed to the 'active force', which in turn represents the action of fields present in the space and independent from the constraints (gravitational, electromagnetical, centrifugal, Coriolis, etc.).

Second, we assume that the nonholonomic constraints are *ideal* (or *perfect*). What this means will be explained in Section 4. This notion concerns with the *physical behavior* of the constraints, which is translated into precise mathematical assumptions.

**Remark 3.** For a holonomic mechanical systems without kinematical constraints, by a well-known process, we pass from the microscopic level i.e., form the Newton equations, to the macroscopic one i.e., to the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}^i}\right) - \frac{\partial K}{\partial q^i} = A_i + R_i, \tag{12}$$

where

$$K = \tfrac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j \tag{13}$$

is the **kinetic energy**, by setting

$$g_{ij}(q) \doteq \textstyle\sum_\nu m_\nu\, \partial_i \mathbf{r}_\nu \cdot \partial_j \mathbf{r}_\nu, \tag{14}$$

$$A_i(q, \dot{q}) \doteq \sum_\nu \mathbf{A}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i}, \qquad R_i \doteq \sum_\nu \mathbf{R}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i}. \tag{15}$$

$A_i$ and $R_i$ are the **active Lagrangian forces** and the **reactive Lagrangian forces**, respectively. Of course, the Lagrange equations are useless without some suitable *constitutive conditions* on the reactive forces $R_i$.

For composing the dynamical equations we have to carry out the following items:

1. Choose Lagrangian coordinates $(q^i)$ and write the $r$ constraint equations $C^a(q, \dot{q}) = 0$. Transform them into parametric equations $\dot{q}^i = \psi^i(q, z)$ by choosing $m$ parameters $z = (z_\alpha)$, and compute the matrix $[\psi^i_\alpha]$.

2. Compute the kinetic energy $K$ and write the Lagrange equations with active forces $A_i$ only. They assume the form

$$g_{ij}\,\ddot{q}^j + \Gamma_{hki}(q)\,\dot{q}^h\,\dot{q}^k = A_i(q,\dot{q}).$$

3. Put at the left hand side the terms containing the $\ddot{q}^i$ only,

$$g_{ij}\,\ddot{q}^j = A_i(q,\dot{q}) - \Gamma_{hki}(q)\,\dot{q}^h\,\dot{q}^k. \tag{16}$$

4. Take the parametric constraint equations (4) and replace all $\dot{q}^i = \psi^i(q,z)$ in the right hand side of Eq. (16). We get functions depending on $(q,z)$ only, say

$$\bar{Z}_i(q,z) \doteq A_i(q,\psi) - \Gamma_{hki}(q)\,\psi^h\psi^k. \tag{17}$$

5. Compute the quantities

$$Z_\alpha(q,z) \doteq \bar{Z}_i\;\psi^i_\alpha, \qquad \psi^i_\alpha \doteq \frac{\partial\psi^i}{\partial z_\alpha}. \tag{18}$$

6. Compute the $r \times r$-matrix $[G_{\alpha\beta}]$ whose elements are the functions

$$G_{\alpha\beta}(q,z) \doteq g_{ij}\,\psi^i_\alpha\,\psi^i_\beta. \tag{19}$$

7. Compute the inverse matrix

$$[G^{\alpha\beta}] = [G_{\alpha\beta}]^{-1}. \tag{20}$$

8. Rise the index of $Z_\alpha$ by means of $G^{\alpha\beta}$:

$$Z^\alpha(q,z) = G^{\alpha\beta}\,Z_\beta. \tag{21}$$

9. Find the Lagrangian components $A_i$ of the active forces by computing their power $W = A_i\,\dot{q}^i$ or by computing their potential energy $V$:

$$A_i = -\frac{\partial V}{\partial q^i}. \tag{22}$$

Now, we are ready to write the dynamical equations:

**Theorem 1.** *Let $Q$ be the configuration manifold of a mechanical system with $n$ degrees of freedom and Lagrangian coordinates $q^i$. Let us impose nonholonomic constraints represented by parametric equations*

$$\dot{q}^i = \psi^i(q,z),$$

*with $r$ parameters $z = (z_\alpha)$. Assume that the constraints are ideal. Then the motions are given by the integral curves of the dynamical system*

$$\begin{cases} \dfrac{dq^i}{dt} = \psi^i(q,z), \\[2ex] \dfrac{dz_\alpha}{dt} = Z^\alpha(q,z), \end{cases} \tag{23}$$

*where the functions $Z^\alpha(q,z)$ are computed following the nine steps listed above.*

Any integral curve of this system is a set of functions $q^i(t)$ and $z_\alpha(t)$ depending on initial conditions, but we are interested on $q^i(t)$ only — these functions gives the actual motions of the system, compatible with the constraints. The $z_\alpha(t)$ play the role of *auxiliary functions*.

## 4. IDEAL CONSTRAINTS

At any fixed kinematical state, the acceleration $\mathbf{a}_\nu$ of the point $P_\nu$ can be obtained by a formal derivative with respect to $t$ of the expression of $\mathbf{v}_\nu$ in (2):

$$\mathbf{a}_\nu = \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j}\, \dot{q}^i \dot{q}^j + \frac{\partial \mathbf{r}_\nu}{\partial q^i}\, \frac{d\dot{q}^i}{dt}.$$

Consequently, the velocities and the accelerations compatible with the constraints can be obtained through the parametric equations (4),

$$\begin{aligned} \mathbf{v}_\nu(q,z) &= \frac{\partial \mathbf{r}_\nu}{\partial q^i}\, \psi^i, \\ \mathbf{a}_\nu(q,z,\dot{z}) &= \frac{\partial^2 \mathbf{r}_\nu}{\partial q^i \partial q^j}\, \psi^i \psi^j + \tfrac{\partial \mathbf{r}_\nu}{\partial q^i} \left( \tfrac{\partial \psi^i}{\partial q^j}\, \psi^j + \tfrac{\partial \psi^i}{\partial z_\alpha}\, \dot{z}_\alpha \right), \end{aligned}$$

by introducing new parameters $\dot{z}_\alpha$.

For a simpler writing it is convenient to introduce the following symbols:

$$\partial_i = \frac{\partial}{\partial q^i}, \qquad \psi^i_j = \frac{\partial \psi^i}{\partial q^j}, \qquad \psi^i_\alpha = \frac{\partial \psi^i}{\partial z_\alpha}.$$

Thus, the acceleration $\mathbf{a}_\nu$ can be written

$$\mathbf{a}_\nu(q,z,\dot{z}) = \partial_{ij}\mathbf{r}_\nu\, \psi^i \psi^j + \partial_i \mathbf{r}_\nu\, (\psi^i_j\, \psi^j + \psi^i_\alpha\, \dot{z}_\alpha). \tag{24}$$

Let us decompose this vector into the sum

$$\mathbf{a}_\nu = \mathbf{a}_{0\nu} + \mathbf{a}_{\alpha\nu}\, \dot{z}_\alpha \tag{25}$$

by introducing the vectors

$$\mathbf{a}_{0\nu}(q,z) = \partial_{ij}\mathbf{r}_\nu\, \psi^i \psi^j + \partial_i \mathbf{r}_\nu\, \psi^i_j\, \psi^j, \qquad \mathbf{a}_{\alpha\nu}(q,z) = \partial_i \mathbf{r}_\nu\, \psi^i_\alpha. \tag{26}$$

**Definition 1.** *The vectors*

$$\mathbf{w}_\nu(q,z,\dot{z}) \doteq \mathbf{a}_{\alpha\nu}\, \dot{z}_\alpha = \partial_i \mathbf{r}_\nu\, \psi^i_\alpha\, \dot{z}_\alpha, \tag{27}$$

*which are linear in the parameters $\dot{z}_\alpha$, assuming all values in $\mathbb{R}$, are the* **virtual displacements** *at the state determined by the values of $(q,z)$.*

The introduction of these vectors is motivated by the following items:

- It is customary to associate the intuitive idea of "virtual displacement" with that of "virtual velocity", as a limit of a "small" displacement between two configurations of the system. Instead, within the present context, a "virtual displacement" is a "small" displacement between kinematical states (configurations plus velocities), so it is associated with the intuitive idea of "virtual acceleration". This viewpoint turns out to be coherent with the philosophy of the Gauss principle, which deals with accelerations (see Section 5).
- With this definition the Gauss principle becomes a consequence of the Newton dynamical equations (Theorem 4).
- The reactive forces of ideal constraints are not dissipative (Theorem 3).

**Definition 2.** *Nonholonomic constraints are said to be* **ideal** *(or* **perfect***) if*

$$\textstyle\sum_\nu \mathbf{R}_\nu \cdot \mathbf{w}_\nu = 0 \tag{28}$$

*for all virtual displacements $\mathbf{w}_\nu$.*

**Remark 4.** We consider Eq. (28) as a **constitutive condition** for the constraints: it says which kind of reactive forces the constraints are able to supply in order to be satisfied along any motion. It is straightforward to check that for linear nonholonomic constraints, as well as for holonomic constraints (which do not involve velocities) Eq. (28) reduces to the classical **virtual work principle**.

**Theorem 2.** *Let*

$$R_i \doteq \sum_{\nu} \mathbf{R}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q^i}, \tag{29}$$

*be the Lagrangian reactive forces. Then the definition* (28) *of ideal constraint is equivalent to the following equations,*

$$R_i\, \psi^i_\alpha = 0. \tag{30}$$

*Proof.* Write Eq. (28) taking into account Eqs. (27) and (29)

$$0 = \textstyle\sum_\nu \mathbf{R}_\nu \cdot \mathbf{a}_{\alpha\nu}\, \dot{z}_\alpha = \sum_\nu \mathbf{R}_\nu \cdot \partial_i \mathbf{r}_\nu\, \psi^i_\alpha\, \dot{z}_\alpha = R_i\, \psi^i_\alpha\, \dot{z}_\alpha.$$

This must hold for any choice of the numbers $\dot{z}_\alpha$. □

Note: Eq. (30) means that $R_i\, \dot{q}^i = 0$, for all $\dot{q}^i$ compatible with the constraints. This shows that

**Theorem 3.** *The ideal constraints do not dissipate energy.*

## 5. THE GAUSS PRINCIPLE

At the microscopic level we introduce the quantity

$$G \doteq \tfrac{1}{2} \textstyle\sum_\nu m_\nu \left(\mathbf{a}_\nu - \dfrac{\mathbf{A}_\nu}{m_\nu}\right)^2. \tag{31}$$

The active forces $\mathbf{A}_\nu$ are known functions of the state $(q, \dot{q})$. Thus, due to the parametric equations of the constraints, $G$ becomes a function of $(q, z)$. Moreover, even for active forces depending on the velocities, $\mathbf{A}_\nu$ does not depend on $\dot{z}$,

$$\frac{\partial \mathbf{A}_\nu}{\partial \dot{z}_\alpha} = 0.$$

Thus, due to Eqs. (25) and (26), along any motion satisfying the constraints we have

$$\frac{\partial G}{\partial \dot{z}_\alpha} = \textstyle\sum_\nu m_\nu \left(\mathbf{a}_\nu - \dfrac{\mathbf{A}_\nu}{m_\nu}\right) \cdot \dfrac{\partial \mathbf{a}_\nu}{\partial \dot{z}_\alpha} = \sum_\nu m_\nu \left(\mathbf{a}_\nu - \dfrac{\mathbf{A}_\nu}{m_\nu}\right) \cdot \mathbf{a}_{\alpha\nu}. \tag{32}$$

Furthermore,

$$\begin{aligned} \frac{\partial^2 G}{\partial \dot{z}_\alpha \partial \dot{z}_\beta} &= \textstyle\sum_\nu m_\nu \dfrac{\partial \mathbf{a}_\nu}{\partial \dot{z}_\beta} \cdot \mathbf{a}_{\alpha\nu} = \sum_\nu m_\nu\, \mathbf{a}_{\beta\nu} \cdot \mathbf{a}_{\alpha\nu} \\ &= \textstyle\sum_\nu m_\nu\, \psi^i_\alpha\, \psi^j_\beta\, \partial_i \mathbf{r}_\nu \cdot \partial_j \mathbf{r}_\nu = g_{ij}\, \psi^i_\alpha\, \psi^j_\beta. \end{aligned}$$

Then, if we introduce the functions

$$G_{\alpha\beta} \doteq g_{ij}\, \psi^i_\alpha\, \psi^j_\beta, \tag{33}$$

we get

$$\frac{\partial^2 G}{\partial \dot{z}_\alpha \partial \dot{z}_\beta} = G_{\alpha\beta}. \tag{34}$$

Since the matrix $[\psi^i_\alpha]$ has maximal rank, the symmetric matrix $[G_{\alpha\beta}]$ is regular and positive-definite as well as $[g_{ij}]$.

**Theorem 4.** *Assume the Newton equations $m_\nu \mathbf{a}_\nu = \mathbf{A}_\nu + \mathbf{R}_\nu$ for each point $P_\nu$. Then, at any state of any actual motion the quantity $G$ takes a minimal value* (Gauss principle) *if and only if the constraints are ideal.*

*Proof.* Write the Newton equations in the form

$$m_\nu \left( \mathbf{a}_\nu - \frac{\mathbf{A}_\nu}{m_\nu} \right) = \mathbf{R}_\nu.$$

Then, due to Eqs. (25), (26) and (29),

$$\frac{\partial G}{\partial \dot{z}_\alpha} = \sum_\nu \mathbf{R}_\nu \cdot \frac{\partial \mathbf{a}_\nu}{\partial \dot{z}_\alpha} = \sum_\nu \mathbf{R}_\nu \cdot \mathbf{a}_{\alpha\nu} = R_i\, \psi^i_\alpha. \tag{35}$$

This shows that for ideal constraints, see Eq. (30), the Newton equations imply

$$\frac{\partial G}{\partial \dot{z}_\alpha} = 0, \tag{36}$$

at any state along any actual motion. Due to Eq. (34), being $[G_{\alpha\beta}]$ positive, at the stationary states for which Eq. (36) holds, the function $G$ has a strong minimum. (ii) Conversely, assume that the Gauss principle holds true. Then Eq. (36) is satisfied, so that from (35) we get $R_i\, \psi^i_\alpha = 0$. This means that the constraints are ideal (Theorem 2). $\square$

**Remark 5.** The vector $\mathbf{A}_\nu / m_\nu$ is the acceleration of the point $P_\nu$ in a **free motion**, free from the constraints. Let us denote it by $\mathbf{a}^f_\nu$. As a consequence, the function $G$ can be also defined as

$$G \doteq \tfrac{1}{2} \sum_\nu m_\nu \big( \mathbf{a}_\nu - \mathbf{a}^f_\nu \big)^2, \tag{37}$$

and Theorem 4 can be reformulated as follows:

**Theorem 5.** *Let $\mathbf{r}_\nu(t)$ and $\mathbf{r}^f_\nu(t)$ <u>two</u> motions of the system $P_\nu$ such that for $t = t_0$ the corresponding states coincide i.e.,*

$$\mathbf{r}_\nu(t_0) = \mathbf{r}^f_\nu(t_0), \qquad \mathbf{v}_\nu(t_0) = \mathbf{v}^f_\nu(t_0).$$

*Assume that $\mathbf{r}^f_\nu(t)$ is a free motion. Then, at this state, and for any motion compatible with ideal constraints, the actual accelerations $\mathbf{a}_\nu(t_0)$, are such that $G$ takes a minimal value.*

## 6. THE GIBBS–APPELL EQUATIONS

Let us go back to the definition (31) of the function $G$. If we introduce the functions

$$S \doteq \tfrac{1}{2} \sum_\nu m_\nu\, \mathbf{a}^2_\nu, \qquad S_1 \doteq \tfrac{1}{2} \sum_\nu \tfrac{1}{m_\nu}\, \mathbf{A}^2_\nu, \qquad S_2 \doteq \sum_\nu \mathbf{A}_\nu \cdot \mathbf{a}_\nu, \tag{38}$$

then we have the decomposition

$$G = S + S_1 - S_2.$$

The function $S$ is called the **energy of the accelerations**. We observe that

$$\frac{\partial S_1}{\partial \dot{z}_\alpha} = 0$$

and that, due to Eq. (24) and the definition (15) of active Lagrangian force,

$$\frac{\partial S_2}{\partial \dot{z}_\alpha} = \sum_\nu \mathbf{A}_\nu \cdot \partial_i \mathbf{r}_\nu\, \psi^i_\alpha = A_i\, \psi^i_\alpha.$$

Thus,

$$\frac{\partial G}{\partial \dot{z}_\alpha} = \frac{\partial S}{\partial \dot{z}_\alpha} - A_i\, \psi^i_\alpha.$$

Due to the Gauss principle (Theorem 4), this proves

**Theorem 6.** *The Gauss principle is equivalent to equations*

$$\frac{\partial S}{\partial \dot{z}_\alpha} = A_\alpha, \tag{39}$$

*where the function*

$$S(q, z, \dot{z}) \doteq \tfrac{1}{2} \sum_\nu m_\nu \, \mathbf{a}_\nu^2 \tag{40}$$

*is determined by the expression* (24) *of the accelerations, and*

$$A_\alpha \doteq A_i \psi_\alpha^i. \tag{41}$$

Eqs. (39) are the celebrated **Gibbs–Appell equations**.

**Remark 6.** The quantities $A_\alpha$ can be computed by writing the **virtual power** of the active forces:

$$W \doteq \textstyle\sum_\nu \mathbf{A}_\nu \cdot \mathbf{w}_\nu = \sum_\nu \mathbf{A}_\nu \cdot \partial_i \mathbf{r}_\nu \, \psi_\alpha^i \, \dot{z}_\alpha = A_i \, \psi_\alpha^i \, \dot{z}_\alpha = A_\alpha \, \dot{z}_\alpha. \tag{42}$$

## 7. THE NORMAL FORM OF THE GIBBS–APPELL EQUATIONS

Both sides of the Gibbs–Appell equations (39) are functions of $(q, z, \dot{z})$. Let us solve them w.r.to the variables $\dot{z}_\alpha$. To this end, it is crucial to observe that by using Eqs. (24) we get for the function $S$ (40) the expression

$$S = \tfrac{1}{2} \, g_{ij} \, \psi_\alpha^i \psi_\beta^j \, \dot{z}_\alpha \dot{z}_\beta + \textstyle\sum_\nu m_\nu \, \partial_{ij} \mathbf{r}_\nu \cdot \partial_k \mathbf{r}_\nu \, \psi^i \psi^j \psi_\alpha^k \, \dot{z}_\alpha + S_0,$$

where $S_0$ is a function dependent on $(q, z)$ only. Then, this function is not involved by the Gibbs–Appell equations and $S$ can be replaced by

$$S_* = \tfrac{1}{2} \, g_{ij} \, \psi_\alpha^i \psi_\beta^j \, \dot{z}_\alpha \dot{z}_\beta + \textstyle\sum_\nu m_\nu \, \partial_{ij} \mathbf{r}_\nu \cdot \partial_k \mathbf{r}_\nu \, \psi^i \psi^j \psi_\alpha^k \, \dot{z}_\alpha. \tag{43}$$

This new function $S_*$ assumes a very interesting expression. Let us introduce the functions

$$\xi_{ijk}(q) \doteq \textstyle\sum_\nu m_\nu \, \partial_{ij} \mathbf{r}_\nu \cdot \partial_k \mathbf{r}_\nu.$$

Since

$$\xi_{ijk} = \textstyle\sum_\nu m_\nu \, \partial_i (\partial_j \mathbf{r}_\nu \cdot \partial_k \mathbf{r}_\nu) - \sum_\nu m_\nu \, (\partial_j \mathbf{r}_\nu \cdot \partial_{ik} \mathbf{r}_\nu) = \partial_i g_{jk} - \xi_{ikj},$$

by a cyclic permutation of the indices we get

$$\xi_{ijk} + \xi_{ikj} = \partial_i g_{jk}, \qquad \xi_{jki} + \xi_{jik} = \partial_j g_{ki}, \qquad \xi_{kij} + \xi_{kji} = \partial_k g_{ij}.$$

By summing the first two equations and subtracting the third one, since $\xi_{ijk}$ is symmetric in the first two indices, we get $\xi_{ijk} = \Gamma_{ijk}$, where

$$\Gamma_{ijk} \doteq \tfrac{1}{2} \, (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$$

are the Christoffel symbols of the metric tensor $g_{ij}$ (the coefficients of the Levi-Civita connection). As a consequence, if we recall the definition (33) of $G_{\alpha\beta}$, the function $S_*$ (43) can be written as

$$S_* = \tfrac{1}{2} \, G_{\alpha\beta} \, \dot{z}_\alpha \dot{z}_\beta + \Gamma_{ijk} \, \psi^i \psi^j \psi_\alpha^k \, \dot{z}_\alpha,$$

and the Gibbs–Appell equations (39) assume the form

$$G_{\alpha\beta} \, \dot{z}_\beta + \Gamma_{ijk} \, \psi^i \psi^j \psi_\alpha^k = A_\alpha. \tag{44}$$

Then we can prove

**Theorem 7.** *The Gibbs–Appell equations* (39) *are equivalent to equations*

$$\dot{z}_\alpha = G^{\alpha\beta} \, (A_\beta - \Gamma_{ijk} \, \psi^i \psi^j \psi_\beta^k), \tag{45}$$

*where* $[G^{\alpha\beta}]$ *the inverse matrix of* $[G_{\alpha\beta}]$.

*Proof.* Indeed, as we remarked in Section 5, the matrix $[G_{\alpha\beta}]$ is regular. If we apply the inverse matrix $[G^{\alpha\beta}]$ to Eqs. (44), then we get Eqs. (45). □

We can call equations (45) the **normal form** of the Gibbs–Appell equations (39). At this point we have proved our main theorem, Theorem 1.

## 8. EXAMPLES

This section is devoted to the application of the general method proposed by Theorem 1 for writing the dynamical equations of a nonholonomic system.

We are going to analyze four **paradigmatic** examples, all dealing with two mass-points $(P_1, m_1)$ and $(P_2, m_2)$ in the Euclidean plane $\mathbb{R}^2 = (x, y)$ endowed with the natural metric.[4)] Their Cartesian coordinates and their velocities will be denoted by $(x_1, y_1)$, $(x_2, y_2)$, and by $\mathbf{v}_1$, $\mathbf{v}_2$, respectively.

**Example 8.1**: two free points $P_1$ and $P_2$ with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ constrained to be orthogonal to the line $P_1P_2$.

**Example 8.2**: two points $P_1$ and $P_2$ connected by a rigid and massless segment; the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are constrained to be orthogonal to the segment $P_1P_2$.

**Example 8.3**: two free points $P_1$ and $P_2$ with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ constrained to be parallel.

**Example 8.4**: two points $P_1$ and $P_2$ connected by a rigid and massless segment; the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are constrained to be parallel.

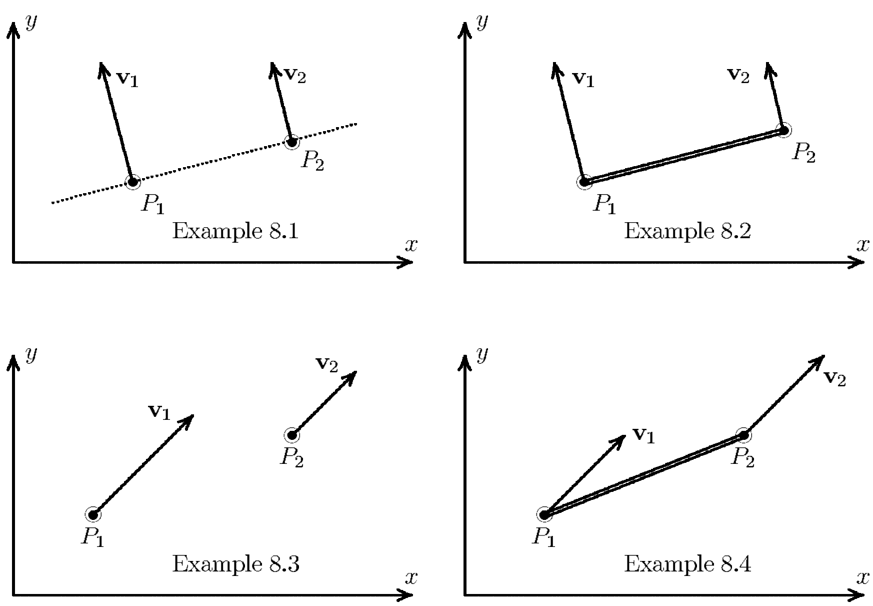


**Fig. 1.** Examples.

Since a great interest is due to distinction between *linear* and *non-linear* constraints, by means of these examples we will observe that a non-linear constraint may be:

1. True non-linear,
2. Apparently non-linear, but in fact linear,
3. Alternatively, linear nonholonomic and integrable (i.e., holonomic).

[4)] Such a class of examples has been proposed by D. Zekovich in the papers "Examples of Non-linear Nonholonomic Constraints in Classical Mechanics", *Moscow Univ. Math. Bull.*, 1991, Vol. 46, No. 1, pp. 44–47, and "On the motion of an integrable system with non-linear nonholonomic coupling. The resonant case", *Moscow Univ. Math. Bull.*, 1993, Vol. 48, No. 1, pp. 37–41.

*8.1. Two Mass-points with Velocities Orthogonal to the Joining Straight Line*

This mechanical system has been proposed by Zekovich (1991) as an example of non-linear nonholonomic system. In fact, the constraints are linear. The misleading argument is that the condition *'$\mathbf{v}_1$ is parallel to $\mathbf{v}_2$'* is of course expressed by equation $\mathbf{v}_1 \times \mathbf{v}_2 = 0$, i.e., by the quadratic equation $\dot{x}_1\,\dot{y}_2 - \dot{x}_2\,\dot{y}_1 = 0$, but in this case it may be considered as a consequence of two independent linear conditions namely, *'$\mathbf{v}_1$ is orthogonal to $P_1P_2$'* and *'$\mathbf{v}_2$ is orthogonal to $P_1P_2$'*, which are expressed by two linear equations:

$$\begin{cases} \dot{x}_1\,(x_2 - x_1) + \dot{y}_1\,(y_2 - y_1) = 0, \\ \dot{x}_2\,(x_2 - x_1) + \dot{y}_2\,(y_2 - y_1) = 0. \end{cases}$$

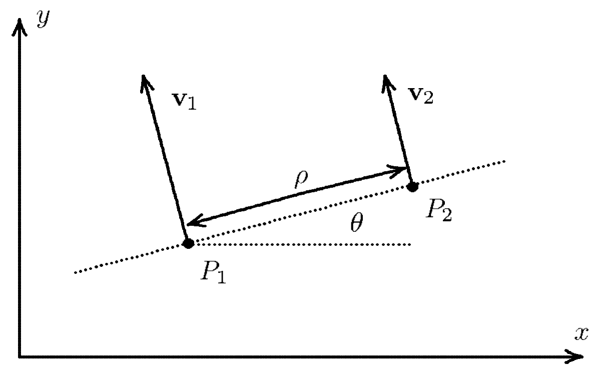


**Fig. 2.** Example 8.1 in polar coordinates.

For the study of these constraints it seems to be convenient to use the Lagrangian coordinates

$$(q^1, q^2, q^3, q^4) = (x, y, \theta, \rho),$$

where $(x, y) = (x_1, y_1)$ and $(\rho, \theta)$ are the polar coordinates of $P_2$ w.r.to the center $P_1$ (see Fig. 2). It follows that

$$\begin{cases} x_1 = x, \\ y_1 = y, \\ x_2 = x + \rho\,\cos\theta, \\ y_2 = y + \rho\,\sin\theta, \end{cases} \qquad \begin{cases} \dot{x}_1 = \dot{x}, \\ \dot{y}_1 = \dot{y}, \\ \dot{x}_2 = \dot{x} + \dot{\rho}\,\cos\theta - \rho\,\sin\theta\,\dot{\theta}, \\ \dot{y}_2 = \dot{y} + \dot{\rho}\,\sin\theta + \rho\,\cos\theta\,\dot{\theta}, \end{cases}$$

$$P_1P_2 = \begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \end{bmatrix},$$

$$\begin{aligned} P_1P_2 \cdot \mathbf{v}_1 &= \rho\,(\dot{x}\,\cos\theta + \dot{y}\,\sin\theta), \\ P_1P_2 \cdot \mathbf{v}_2 &= \rho\,(\dot{x}\,\cos\theta + \dot{y}\,\sin\theta) \\ &\quad + \rho\,(\dot{\rho}\,\cos\theta - \rho\,\sin\theta\,\dot{\theta})\,\cos\theta \\ &\quad + \rho\,(\dot{\rho}\,\sin\theta + \rho\,\cos\theta\,\dot{\theta})\,\sin\theta \\ &= \rho\,(\dot{x}\,\cos\theta + \dot{y}\,\sin\theta + \dot{\rho}). \end{aligned}$$

Hence, the constraint equations are

$$\rho\,(\dot x\,\cos\theta + \dot y\,\sin\theta) = 0, \qquad \rho\,(\dot x\,\cos\theta + \dot y\,\sin\theta + \dot\rho) = 0,$$

and they become equivalent to the system of equations

$$\begin{cases} \rho\,(\dot x\,\cos\theta + \dot y\,\sin\theta) = 0, \\ \rho\,\dot\rho = 0. \end{cases}$$

If we consider the second equation as a dynamical equation,

$$\rho\,\frac{d\rho}{dt} = 0,$$

then we find that $\rho =$ *constant* during any motion. This means that, *once the initial positions of the two points are fixed*, then the system looses a degree of freedom and its dynamics becomes equivalent to that of two points joined by a massless rigid segment of length $\rho$, with Lagrangian coordinates

$$(q^1, q^2, q^3) = (x, y, \theta),$$

and submitted to the linear constraint

$$\dot x\,\cos\theta + \dot y\,\sin\theta = 0,$$

representing the fact that the velocity $\mathbf{v}_1$ of the point $P_1$ is orthogonal to the segment.[5)] Actually, this is one of the most common pedagogical example. Anyway, let us try to apply our method to it. Going back to Remark 1 and formula (8), we see that in this case we need $m = n - r = 3 - 1 = 2$ parameters $(z_1, z_2)$ for a parametric representation of the constraints. We can consider for instance the equations

$$\begin{cases} \dot x = v\,\sin\theta, \\ \dot y = -\,v\,\cos\theta, \\ \dot\theta = \omega, \end{cases}$$

with parameters $z_1 = v$, the scalar velocity of $P_1$, and $z_2 = \omega$, the scalar angular velocity. Then, according to the notation introduced in Sections 2 and 3, we have

$$\begin{cases} \psi^1 = \dot x = v\,\sin\theta, \\ \psi^2 = \dot y = -\,v\,\cos\theta, \\ \psi^3 = \dot\theta = \omega, \end{cases} \tag{46}$$

and

$$[\psi^i_\alpha] = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\alpha \text{ index of line}). \tag{47}$$

Since

$$\begin{cases} \dot x_1 = \dot x, \\ \dot y_1 = \dot y, \end{cases} \qquad \begin{cases} \dot x_2 = \dot x - \rho\,\sin\theta\,\dot\theta, \\ \dot y_2 = \dot y + \rho\,\cos\theta\,\dot\theta, \end{cases}$$

the kinetic energy is

$$\begin{aligned} K &= \tfrac{1}{2}\,m_1\,(\dot x^2 + \dot y^2) + \tfrac{1}{2}\,m_2\,(\dot x^2 + \dot y^2 - 2\rho\,\sin\theta\,\dot x\,\dot\theta + 2\rho\,\cos\theta\,\dot y\,\dot\theta + \rho^2\,\dot\theta^2) \\ &= \tfrac{1}{2}\,m\,(\dot x^2 + \dot y^2) + \tfrac{1}{2}\,m_2\,\rho\,(\rho\,\dot\theta^2 - 2\sin\theta\,\dot x\,\dot\theta + 2\cos\theta\,\dot y\,\dot\theta), \end{aligned} \tag{48}$$

[5)] Notice that from the rigid body kinemtaics we know that also the velocity $\mathbf{v}_2$ has, consequently, the same property.

where $m = m_1 + m_2$, and the Lagrange equations with active forces $(A_i) = (A_x, A_y, A_\theta)$ are

$$\begin{cases} \dfrac{d}{dt}(m\,\dot x - m_2\,\rho\,\sin\theta\,\dot\theta) = A_x, \\ \dfrac{d}{dt}(m\,\dot y + m_2\,\rho\,\cos\theta\,\dot\theta) = A_y, \\ m_2\rho\,\left(\dfrac{d}{dt}(\rho\,\dot\theta - \sin\theta\,\dot x + \cos\theta\,\dot y) + \cos\theta\,\dot x\,\dot\theta + \sin\theta\,\dot y\,\dot\theta\right) = A_\theta, \end{cases}$$

$$\begin{cases} m\,\ddot x - m_2\rho\,(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta) = A_x, \\ m\,\ddot y + m_2\rho\,(-\sin\theta\,\dot\theta^2 + \cos\theta\,\ddot\theta) = A_y, \\ m_2\,\rho\,\left(\rho\,\ddot\theta - \cos\theta\,\dot x\,\dot\theta - \sin\theta\,\ddot x - \sin\theta\,\dot y\,\dot\theta + \cos\theta\,\ddot y + \cos\theta\,\dot x\,\dot\theta + \sin\theta\,\dot y\,\dot\theta\right) = A_\theta, \end{cases}$$

$$\begin{cases} m\,\ddot x - m_2\,\rho\,(\cos\theta\,\dot\theta^2 + \sin\theta\,\ddot\theta) = A_x, \\ m\,\ddot y + m_2\,\rho\,(-\sin\theta\,\dot\theta^2 + \cos\theta\,\ddot\theta) = A_y, \\ m_2\,\rho\,\left(\rho\,\ddot\theta - \sin\theta\,\ddot x + \cos\theta\,\ddot y\right) = A_\theta, \end{cases}$$

The Lagrange equations in the form (16) are

$$\begin{cases} m\,\ddot x - m_2\,\rho\,\sin\theta\,\ddot\theta = m_2\,\rho\,\cos\theta\,\dot\theta^2 + A_x, \\ m\,\ddot y + m_2\,\rho\,\cos\theta\,\ddot\theta = m_2\,\rho\,\sin\theta\,\dot\theta^2 + A_y, \\ m_2\,\rho\,\left(\rho\,\ddot\theta - \sin\theta\,\ddot x + \cos\theta\,\ddot y\right) = A_\theta. \end{cases}$$

According to item 4 of our method i.e., formula (17) of Section 3, we substitute Eqs. (46) into the right hand sides of these equations in order to compute the components $\bar Z_i(q,z) \doteq A_i(q,\psi) - \Gamma_{hki}(q)\,\psi^h\psi^k$. We obtain

$$\begin{cases} \bar Z_1 = m_2\,\rho\,\cos\theta\,\omega^2 + A_x, \\ \bar Z_2 = m_2\,\rho\,\sin\theta\,\omega^2 + A_y, \\ \bar Z_3 = A_\theta. \end{cases}$$

Recall that $\omega = z_2$ is the second parameter. Item 5: computation of the components $Z_\alpha = \bar Z_i\,\psi^i_\alpha$. In the present case, due to (47), we get

$$\begin{cases} Z_1 = \sin\theta\,(m_2\rho\,\cos\theta\,\omega^2 + A_x) - \cos\theta\,(m_2\rho\,\sin\theta\,\omega^2 + A_y) \\ \quad\;\; = \sin\theta\,A_x - \cos\theta\,A_y, \\ Z_2 = A_\theta, \end{cases}$$

Item 6: compute the matrix $[G_{\alpha\beta} = g_{ij}\,\psi^i_\alpha\,\psi^i_\beta]$. From the expression (48) of the kinetic energy we get

$$[g_{ij}] = \begin{bmatrix} m & 0 & -m_2\,\rho\,\sin\theta \\ 0 & m & m_2\,\rho\,\cos\theta \\ -m_2\,\rho\,\sin\theta & m_2\,\rho\,\cos\theta & m_2\,\rho^2 \end{bmatrix}$$

$$= m_2\,\rho \begin{bmatrix} \mu & 0 & -\sin\theta \\ 0 & \mu & \cos\theta \\ -\sin\theta & \cos\theta & \rho \end{bmatrix},$$

where

$$\mu \doteq \frac{m}{m_2\,\rho} = \frac{m_1 + m_2}{m_2\,\rho}.$$

Hence,

$$G_{\alpha\beta} = \rho\,m_2\,\Big(\psi^1_\alpha\psi^1_\beta\,g_{11} + \psi^2_\alpha\psi^2_\beta\,g_{22} + \psi^3_\alpha\psi^3_\beta\,g_{33} + 2\,\psi^1_\alpha\psi^2_\beta\,g_{12} + 2\,\psi^1_\alpha\psi^3_\beta\,g_{13} + 2\,\psi^2_\alpha\psi^3_\beta\,g_{23}\Big)$$

$$= \rho\,m_2\,\Big(\psi^1_\alpha\psi^1_\beta\,\mu + \psi^2_\alpha\psi^2_\beta\,\mu + \psi^3_\alpha\psi^3_\beta\,\rho - 2\,\psi^1_\alpha\psi^3_\beta\,\sin\theta + 2\,\psi^2_\alpha\psi^3_\beta\,\cos\theta\Big).$$

$$G_{11} = \rho\,m_2\,\mu = m.$$

$$G_{12} = -\rho\,m_2\,\big(2\,\sin^2\theta + 2\,\cos^2\theta\big) = -2\,\rho\,m_2.$$

$$G_{22} = \rho^2\,m_2.$$

$$[G_{\alpha\beta}] = \begin{bmatrix} m & -2\,\rho\,m_2 \\ -2\,\rho\,m_2 & \rho^2\,m_2 \end{bmatrix} = \rho\,m_2 \begin{bmatrix} \mu & -2 \\ -2 & \rho \end{bmatrix}.$$

Item 7: the inverse matrix $[G^{\alpha\beta}] = [G_{\alpha\beta}]^{-1}$ is

$$[G^{\alpha\beta}] = \frac{1}{\rho\,m_2\,(\mu\,\rho - 4)} \begin{bmatrix} \rho & 2 \\ 2 & \mu \end{bmatrix}.$$

Item 8: compute $Z^\alpha = G^{\alpha\beta} Z_\beta$,

$$Z^\alpha = G^{\alpha 1} Z_1 + G^{\alpha 2} Z_2 = G^{\alpha 1}(\sin\theta\ A_x - \cos\theta\ A_y) + G^{\alpha 2} A_\theta,$$

$$Z^1 = \frac{\rho\,(\sin\theta\ A_x - \cos\theta\ A_y) + 2\,A_\theta}{\rho\,m_2\,(\mu\,\rho - 4)},$$

$$Z^2 = \frac{2\,(\sin\theta\ A_x - \cos\theta\ A_y) + \mu\,A_\theta}{\rho\,m_2\,(\mu\,\rho - 4)}.$$

In conclusion, taking into account the constraint equations (46), the resulting dynamical equations (23) are the following:

$$\begin{cases} \dfrac{dx}{dt} = v\,\sin\theta, \\ \dfrac{dy}{dt} = -\,v\,\cos\theta, \\ \dfrac{d\theta}{dt} = \omega, \\ \dfrac{dv}{dt} = \frac{\rho\,(\sin\theta\,A_x - \cos\theta\,A_y) + 2\,A_\theta}{\rho\,m_2\,(\mu\,\rho - 4)}, \\ \dfrac{d\omega}{dt} = \frac{2\,(\sin\theta\,A_x - \cos\theta\,A_y) + \mu\,A_\theta}{\rho\,m_2\,(\mu\,\rho - 4)}. \end{cases} \tag{49}$$

### *8.2. Two Mass-points with Velocities Orthogonal to the Joining Massless Rigid Segment*

This example of nonholonomic system does dot differ from the preceding one. Indeed, as we have seen, even if the two points are free to run along their joining line, they preserve their distance along any motion. So, the holonomic constraint of having a constant distance does not play any work.

**Remark 7.** The dynamical behavior of **two coaxial rolling discs** (Fig. 3) is quite similar to those of the preceding two examples: even if the centers $P_1$ and $P_2$ are free to run along the axis, when the initial configuration is fixed, their distance remains constant. If the discs are thin blades and two masses are concentrated in the centers $P_1$ and $P_2$, then we obtain a physical realization of the above examples.

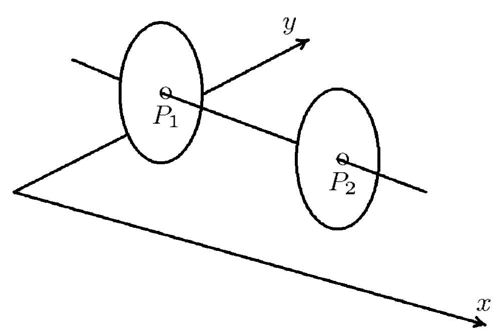


**Fig. 3.** Coaxial rolling discs.

### *8.3. Two Free Points with Parallel Velocities*

Two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are constrained to have parallel velocities $\mathbf{v}_1$ and $\mathbf{v}_2$. This constraint is expressed by the vectorial equation $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ which is equivalent to the single scalar equation

$$\dot{x}_1\,\dot{y}_2 - \dot{x}_2\,\dot{y}_1 = 0. \tag{50}$$

This is clearly a non-linear constraint. The configuration manifold of this system is $Q = \mathbb{R}^4$, with ordered global Lagrangian coordinates

$$(q^1, q^2, q^3, q^4) = (x_1, y_1, x_2, y_2).$$

In order to apply our method we have to represent the implicit constraint equations (50) in a parametric form. We need three parameters $(z_1, z_2, z_3)$. Let us choose the following equations:

$$\begin{cases} \dot{x}_1 = v_1 \cos\theta, \\ \dot{y}_1 = v_1 \sin\theta, \\ \dot{x}_2 = v_2 \cos\theta, \\ \dot{y}_2 = v_2 \sin\theta, \end{cases} \tag{51}$$

with parameters

$$(z_\alpha) = (z_1, z_2, z_3) = (v_1, v_2, \theta).$$

The meaning of the parameters $(v_1, v_2)$ is

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2, \qquad v_2^2 = \dot{x}_2^2 + \dot{y}_2^2,$$

while $\theta$ is the angle of the two vector velocities w.r.to the $x$-axis (oriented anticlockwise). However, even if $v_1$ and $v_2$ represents the intensity of the two velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, we accept the fact that they may assume negative values. Notice that the kinematical state $\mathbf{v}_1 = \mathbf{v}_2 = 0$ i.e., $v_1 = v_2 = 0$ is a singular state.

According to our notation, Eqs. (51) show that

$$\begin{cases} \psi^1 = v_1 \cos\theta, \\ \psi^2 = v_1 \sin\theta, \\ \psi^3 = v_2 \cos\theta, \\ \psi^4 = v_2 \sin\theta. \end{cases}$$

Then we find:

$$[\psi_\alpha^i] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ -v_1 \sin\theta & v_1 \cos\theta & -v_2 \sin\theta & v_2 \cos\theta \end{bmatrix} \quad (\alpha \text{ index of line}).$$

Let $m_1$ and $m_2$ be the masses of the points $P_1$ and $P_2$. Then the kinetic energy is

$$K = \tfrac{1}{2}\, m_1\, (\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}\, m_2\, (\dot{x}_2^2 + \dot{y}_2^2),$$

and

$$[g_{ij}] = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix},$$

$$[G_{\alpha\beta} = g_{ij}\, \psi_\alpha^i \psi_\beta^j] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_1 v_1^2 + m_2 v_2^2 \end{bmatrix},$$

$$[G^{\alpha\beta}] = \begin{bmatrix} \dfrac{1}{m_1} & 0 & 0 \\ 0 & \dfrac{1}{m_2} & 0 \\ 0 & 0 & \dfrac{1}{m_1 v_1^2 + m_2 v_2^2} \end{bmatrix}.$$

The Lagrange equations in the form (16) are

$$m_1\,\ddot{x}_1 = A_1, \quad m_1\,\ddot{y}_1 = A_2, \quad m_2\,\ddot{x}_2 = A_3, \quad m_2\,\ddot{y}_2 = A_4.$$

They shows that $\bar{Z}_i = A_i$. Hence,

$$[Z_\alpha = \bar{Z}_i\,\psi^i_\alpha] = \begin{bmatrix} A_1\,\cos\theta + A_2\,\sin\theta \\ A_3\,\cos\theta + A_4\,\sin\theta \\ v_1\,(A_2\,\cos\theta - A_1\,\sin\theta) + v_2\,(A_4\,\cos\theta - A_3\,\sin\theta) \end{bmatrix},$$

$$[Z^\alpha = G^{\alpha\beta}\,Z_\beta] = \begin{bmatrix} \dfrac{A_1\,\cos\theta + A_2\,\sin\theta}{m_1} \\ \dfrac{A_3\,\cos\theta + A_4\,\sin\theta}{m_2} \\ \dfrac{v_1\,(A_2\,\cos\theta - A_1\,\sin\theta) + v_2\,(A_4\,\cos\theta - A_3\,\sin\theta)}{m_1 v_1^2 + m_2 v_2^2} \end{bmatrix},$$

and the resulting dynamical equations are

$$\begin{cases} \dfrac{dx_1}{dt} = v_1\,\cos\theta, \\ \dfrac{dy_1}{dt} = v_1\,\sin\theta, \\ \dfrac{dx_2}{dt} = v_2\,\cos\theta, \\ \dfrac{dy_2}{dt} = v_2\,\sin\theta, \\ \dfrac{dv_1}{dt} = \dfrac{A_1\,\cos\theta + A_2\,\sin\theta}{m_1}, \\ \dfrac{dv_2}{dt} = \dfrac{A_3\,\cos\theta + A_4\,\sin\theta}{m_2}, \\ \dfrac{d\theta}{dt} = \dfrac{v_1\,(A_2\,\cos\theta - A_1\,\sin\theta) + v_2\,(A_4\,\cos\theta - A_3\,\sin\theta)}{m_1 v_1^2 + m_2 v_2^2}, \end{cases} \tag{52}$$

where the Lagrangian active forces $(A_1, A_2, A_3, A_4)$ are in general known functions of $(x_1, y_1, x_2, y_2)$ and $(v_1, v_2, \theta)$. Let us recall, for a better understanding of the above equations, that $A_1 = A_{x_1}$ i.e., that $A_1$ is the component w.r.to the $x$-axis of the active force $\mathbf{A}_1$ acting on the point $P_1$, and so on:

$$A_1 = A_{x_1}, \quad A_2 = A_{y_1}, \quad A_3 = A_{x_2}, \quad A_4 = A_{y_2}.$$

Notice that, as we remarked above, $v_1 = v_2 = 0$ is a singular state for the constraint; this fact is also revealed by the last of the dynamical equations (52).

Now we consider Eqs. (52) for two special cases of active forces.

*8.3.1. The Inclined Plane*

The points are on an inclined plane and submitted to the gravity. We consider the $x$-axis oriented downward and the $y$-axis horizontal. In this case $A_1 = m_1\, g$, $A_3 = m_2\, g$, $A_2 = A_4 = 0$, where $g$ is the reduced gravitational constant. Then Eqs. (52) become

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= v_1\, \cos\theta, \\ \frac{dy_1}{dt} &= v_1\, \sin\theta, \\ \frac{dx_2}{dt} &= v_2\, \cos\theta, \\ \frac{dy_2}{dt} &= v_2\, \sin\theta, \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{dv_1}{dt} &= g\, \cos\theta, \\ \frac{dv_2}{dt} &= g\, \cos\theta, \\ \frac{d\theta}{dt} &= -\, g\, \sin\theta\, \frac{m_1\, v_1 + m_2\, v_2}{m_1 v_1^2 + m_2 v_2^2}. \end{aligned}\right. \tag{53}$$

For equal masses $m_1 = m_2$, we have a further simplification:

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= v_1\, \cos\theta, \\ \frac{dy_1}{dt} &= v_1\, \sin\theta, \\ \frac{dx_2}{dt} &= v_2\, \cos\theta, \\ \frac{dy_2}{dt} &= v_2\, \sin\theta, \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{dv_1}{dt} &= g\, \cos\theta, \\ \frac{dv_2}{dt} &= g\, \cos\theta, \\ \frac{d\theta}{dt} &= -\, g\, \sin\theta\, \frac{v_1 + v_2}{v_1^2 + v_2^2}. \end{aligned}\right. \tag{54}$$

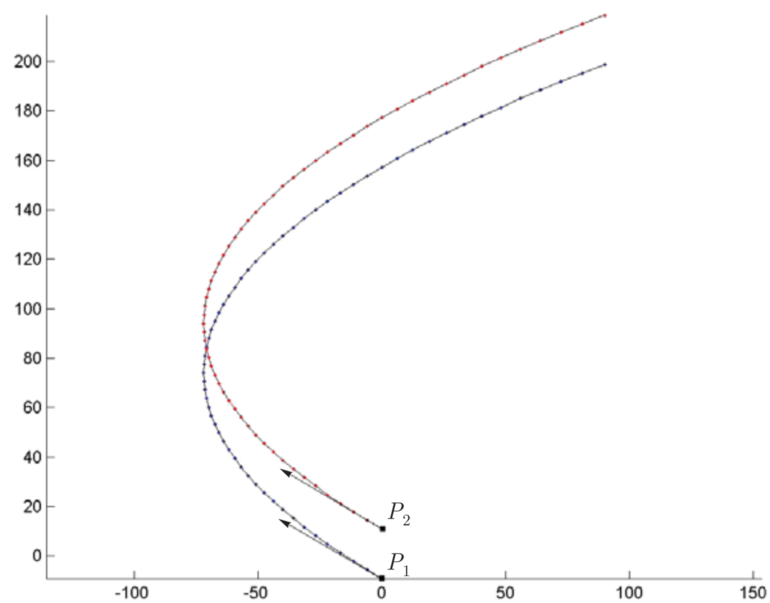


**Fig. 4.** $\mathbf{v}_1 = \mathbf{v}_2$, $m_2 = m_1$.

*8.3.2. Horizontal Plane, the two Points are Linked by an Ideal Spring*

The potential energy of an elastic active force is

$$V = \tfrac{1}{2}\, k\, |P_1P_2|^2 = \tfrac{1}{2}\, k\, \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right),$$

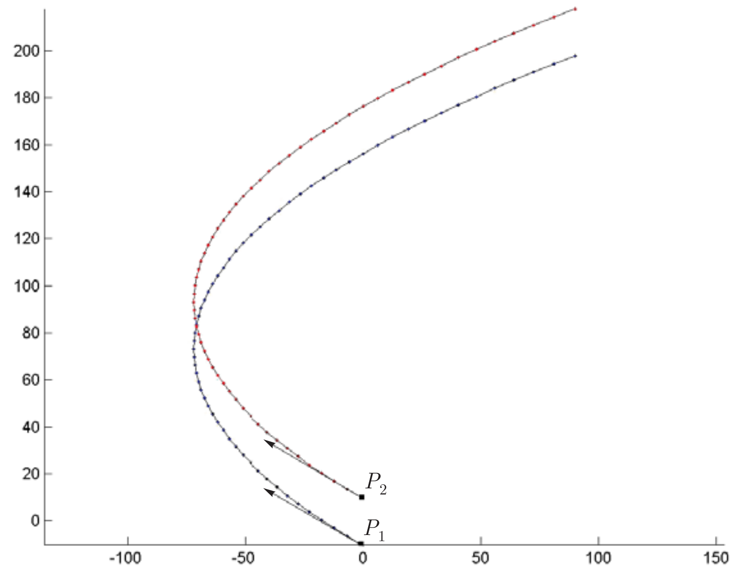


**Fig. 5.** $\mathbf{v}_1 = \mathbf{v}_2$, $m_2 = 10\,m_1$.

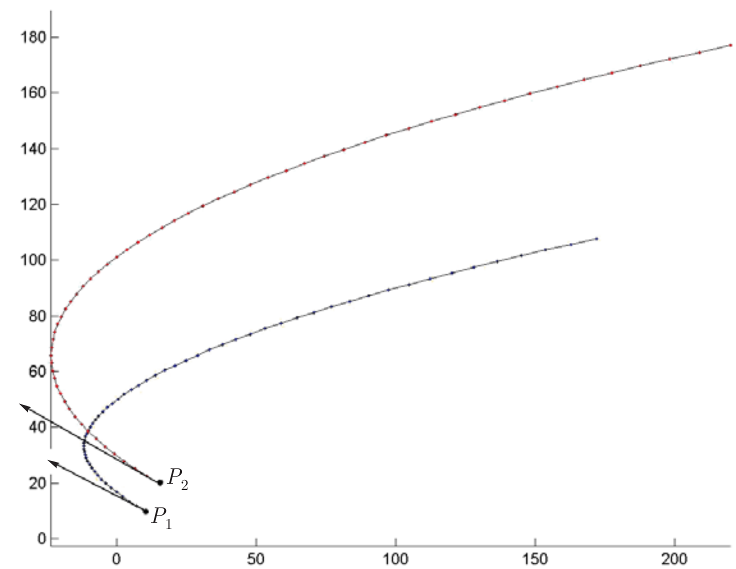


**Fig. 6.** $\mathbf{v}_1 = \frac{2}{3}\,\mathbf{v}_2$, $m_2 = 2m_1$.

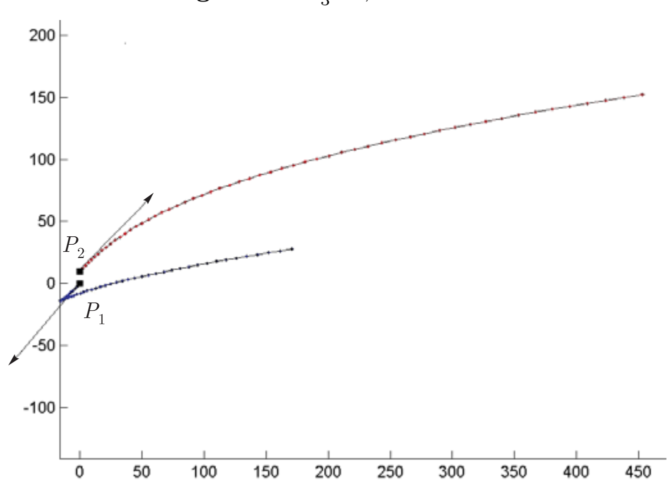


**Fig. 7.** $\mathbf{v}_2 = -\mathbf{v}_1$, $m_1 = m_2$.

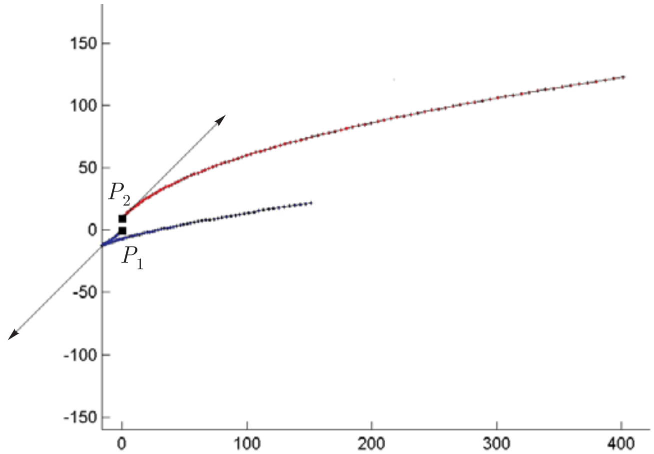


**Fig. 8.** $\mathbf{v}_2 = -\mathbf{v}_1$, $m_2 = 3\,m_2$.

where $k > 0$ is a constant. It follows that

$$A_1 = -\frac{\partial V}{\partial x_1} = k\,(x_2 - x_1), \qquad A_3 = -\frac{\partial V}{\partial x_2} = k\,(x_1 - x_2),$$
$$A_2 = -\frac{\partial V}{\partial y_1} = k\,(y_2 - y_1), \qquad A_4 = -\frac{\partial V}{\partial y_2} = k\,(y_1 - y_2).$$

Then the dynamical equations (52) become

$$\left\{\begin{aligned} \frac{dx_1}{dt} &= v_1\,\cos\theta, \\ \frac{dy_1}{dt} &= v_1\,\sin\theta, \\ \frac{dx_2}{dt} &= v_2\,\cos\theta, \\ \frac{dy_2}{dt} &= v_2\,\sin\theta, \end{aligned}\right. \quad \left\{\begin{aligned} \frac{dv_1}{dt} &= k\,\frac{(x_2 - x_1)\,\cos\theta + (y_2 - y_1)\,\sin\theta}{m_1}, \\ \frac{dv_2}{dt} &= k\,\frac{(x_1 - x_2)\,\cos\theta + (y_1 - y_2)\,\sin\theta}{m_2}, \\ \frac{d\theta}{dt} &= k\,\frac{v_1\,[(y_2 - y_1)\,\cos\theta - (x_2 - x_1)\,\sin\theta]}{m_1 v_1^2 + m_2 v_2^2} \\ &\quad + k\,\frac{v_2\,[(y_1 - y_2)\,\cos\theta - (x_1 - x_2)\,\sin\theta]}{m_1 v_1^2 + m_2 v_2^2}. \end{aligned}\right. \tag{55}$$

Let us look at the numerical solutions of these equations for various initial conditions. We have of course a very large number of cases. A first classification of them is the following (see Fig. 9):

- Case A — Opposite initial velocities perpendicular to the segment $P_1P_2$:
- Case B — Equioriented initial velocities perpendicular to the segment $P_1P_2$:
- Case C — Opposite initial velocities inclined w.r.to the segment $P_1P_2$:
- Case D — Equioriented initial velocities inclined w.r.to the segment $P_1P_2$:

Each one of these cases has actually three sub-cases determined by the values of the scalar velocities: (1) $v_1 = v_2$ (as in the above pictures), (2) $v_1 \neq v_2$ both different from zero, and (3) $v_1 = 0$. Furthermore, each sub-case should be considered with equal masses as well as different masses. So, they are too many. Here, only a few significant cases are illustrated with graphics and comments.

• Case A. For opposite velocities with equal intensity, we observe the difference between the two sub-cases of equal masses (Fig. 10) and different masses (Fig. 11).

• Case A. For opposite velocities with different intensities we observe a strange phenomenon. For equal masses, nothing of special (Fig. 12). But for different masses (say, $m_2 = 2\,m_2$) we observe a very strong qualitative dependence on the initial distance between the two points: see Fig. 13,

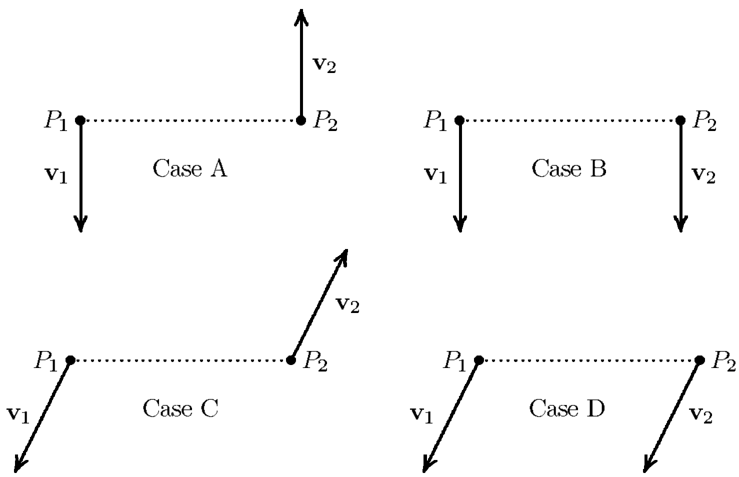


**Fig. 9.** Cases of in initial conditions.

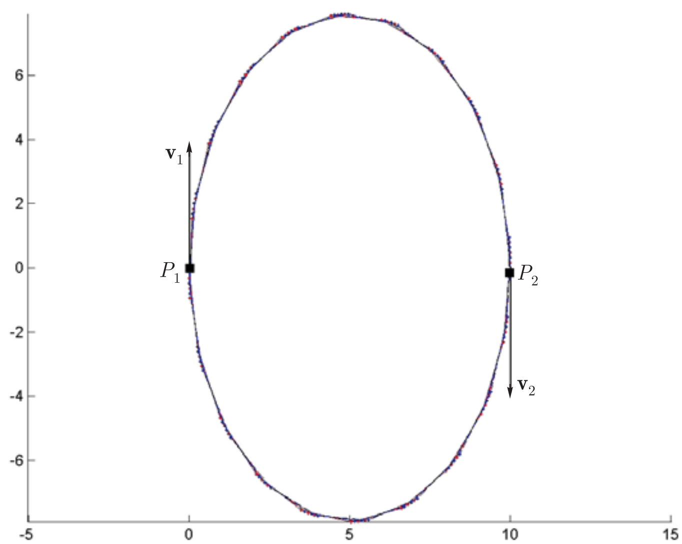


**Fig. 10.** Case A. $\mathbf{v}_1 = -\mathbf{v}_2, \ \perp P_1P_2, \ m_1 = m_2$.

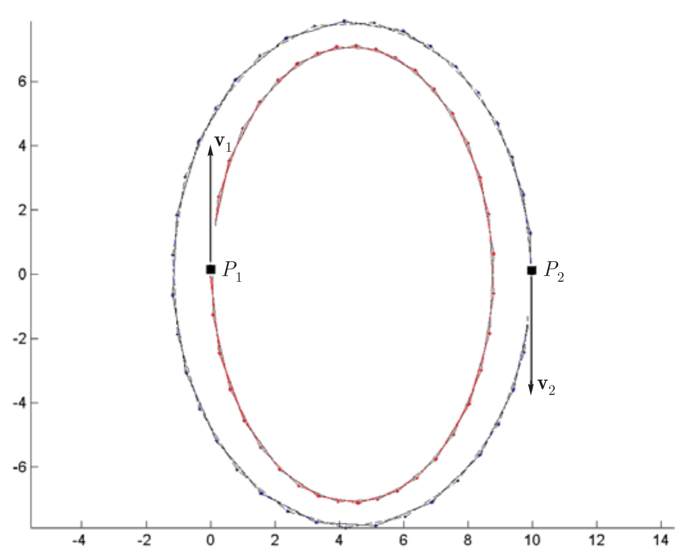


**Fig. 11.** Case A. $\mathbf{v}_1 = -\mathbf{v}_2, \ \perp P_1P_2, \ m_2 = 2\,m_1$.

Fig. 14 and Fig. 16. On the other hand, we observe a significant change in the orbits if we modify the ratio between the masses. For instance, if we maintain the same initial conditions of Fig. 14 but we put $m_2 = 3\,m_1$ we get the orbits of Fig. 15, which are of a different kind.

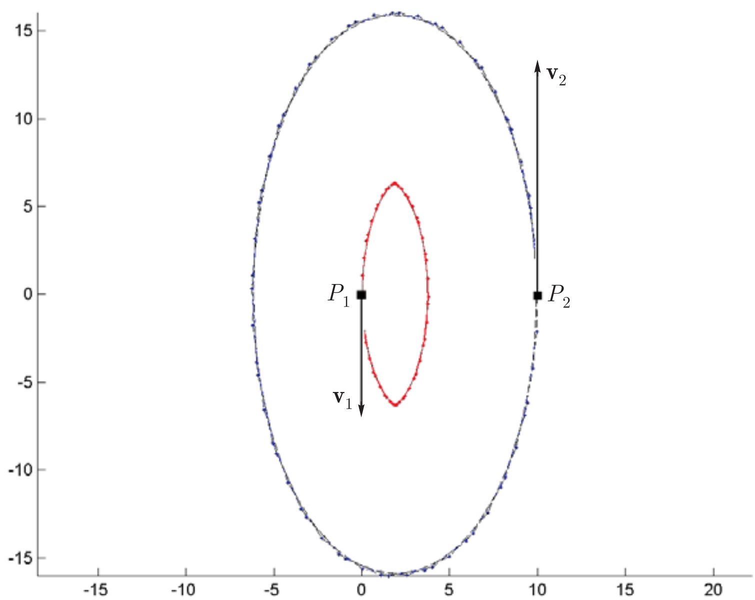


**Fig. 12.** Case A. $\mathbf{v}_2 = -2\,\mathbf{v}_1,\ \perp P_1P_2,\ m_2 = m_1.$

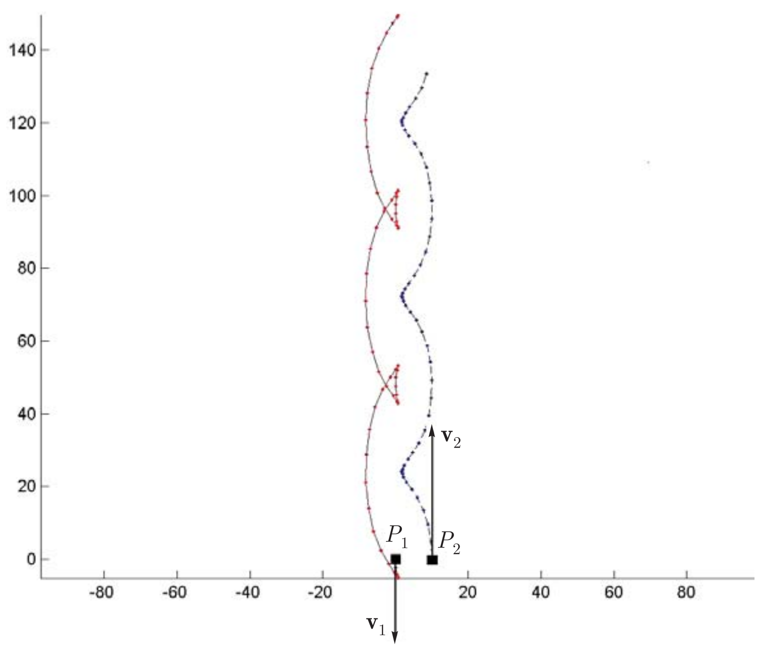


**Fig. 13.** Case A. $\mathbf{v}_2 = -2\,\mathbf{v}_1,\ \perp P_1P_2,\ m_2 = 2\,m_1.$

• Case B. For equal initial velocities $\mathbf{v}_1 = \mathbf{v}_2$, perpendicular to the segment $P_1P_2$, we observe a strange fact: the trajectories are parallel straight lines, Fig. 17. But do not forget the constraints: even if the two points are attracted by a spring, they cannot converge since they must have parallel velocities. Further experiments show that this behavior does not depend on the values of the masses.

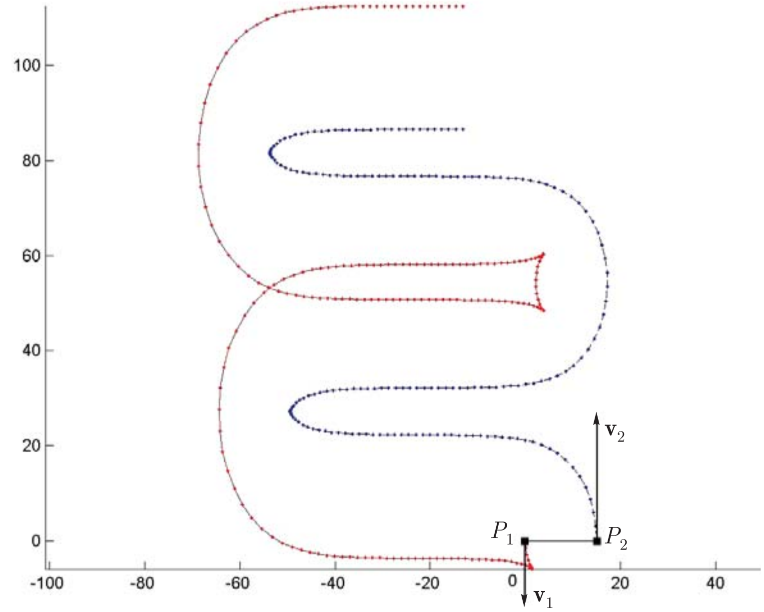


**Fig. 14.** Case A. $\mathbf{v}_2 = -2\,\mathbf{v}_1$, $\perp P_1P_2$, $m_2 = 2\,m_1$. A larger initial distance $|P_1P_2|$.

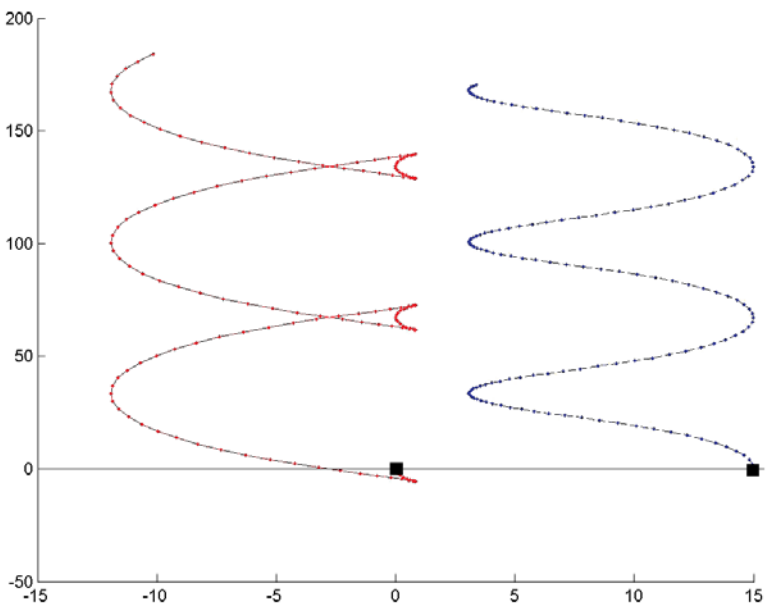


**Fig. 15.** Case A. The same initial conditions as in Fig. 14 but with $m_2 = 3\,m_1$. The $x$-axis is expanded.

• Case B. For parallel and equioriented initial velocities, but with different intensities $v_1 \neq v_2$, we observe a rather different behavior w.r.to the previous case: see Fig. 18. Further experiments show that this qualitative behavior does not depend on the values of the masses.

• Case C. Recall that this case concerns with opposite initial velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ which are inclined w.r.to the segment $P_1P_2$. For velocities of equal intensity, Fig. 19 shows the qualitative behavior for equal masses. For different masses we have a different result as shown by Fig. 20.

• Case C. For opposite initial velocities of different intensity, say $\mathbf{v}_2 = -2\,\mathbf{v}_1$, we observe very different kinds of orbits, depending on the initial distance $|P_1P_2|$ between the two points. For equal masses $m_1 = m_2$ these orbits are illustrated in Fig. 21, 22 and 23. For different masses $m_1 \neq m_2$ we have a similar phenomenon: see Fig. 24 and Fig. 25.

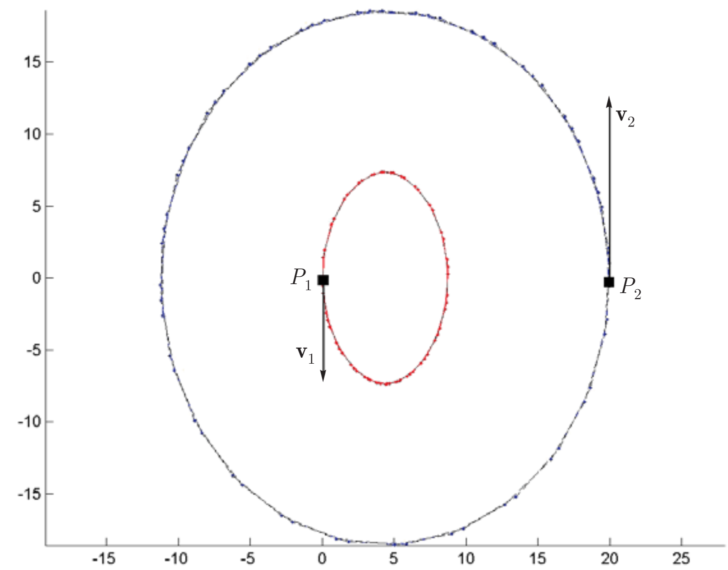


**Fig. 16.** Case A. $\mathbf{v}_2 = -2\,\mathbf{v}_1,\ \perp P_1P_2,\ m_2 = 2\,m_1$. A much larger initial distance $|P_1P_2|$.

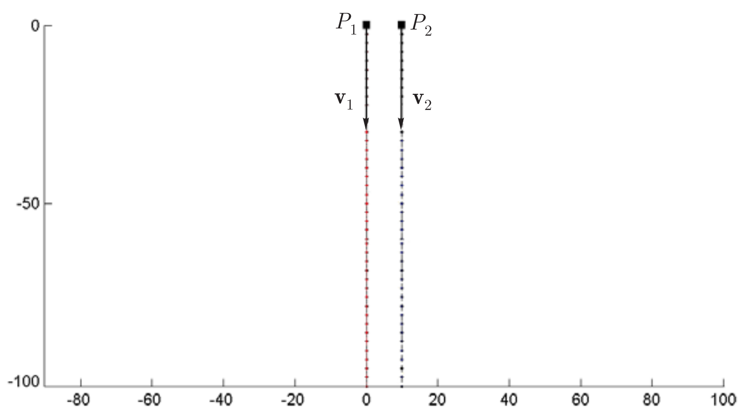


**Fig. 17.** Case B. $\mathbf{v}_1 = \mathbf{v}_2,\ \perp P_1P_2,\ m_2 = m_1$ or $m_2 = \alpha\,m_1$.

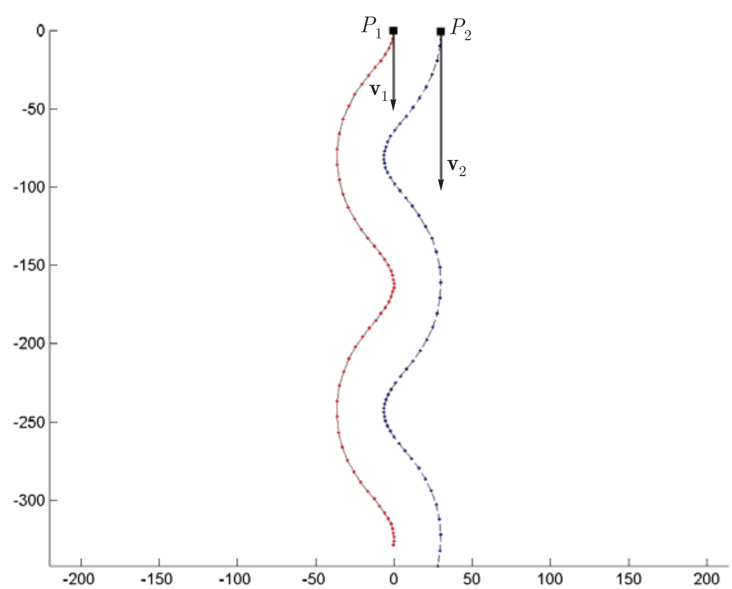


**Fig. 18.** Case B. $\mathbf{v}_2 = 2\,\mathbf{v}_1,\ \perp P_1P_2,\ m_2 = m_1$ or $m_2 = \alpha\,m_1$.

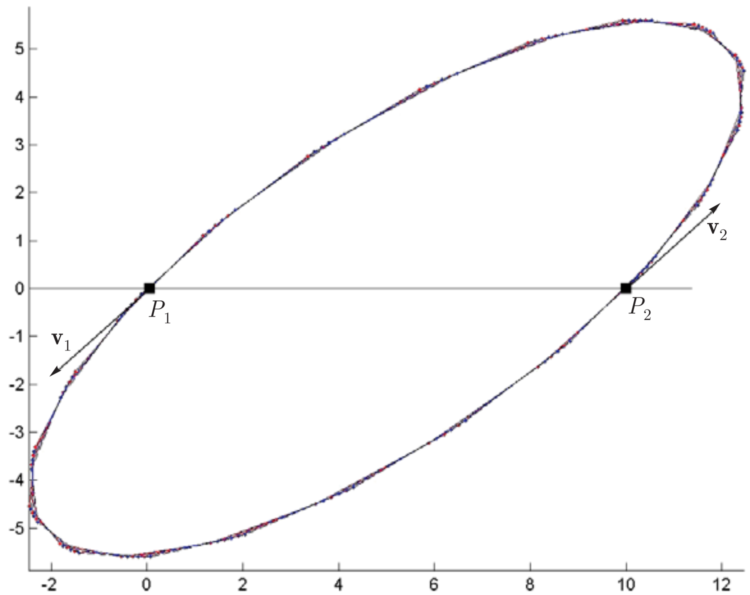


**Fig. 19.** Case C. $\mathbf{v}_1 = -\mathbf{v}_2,\ m_2 = m_1$.

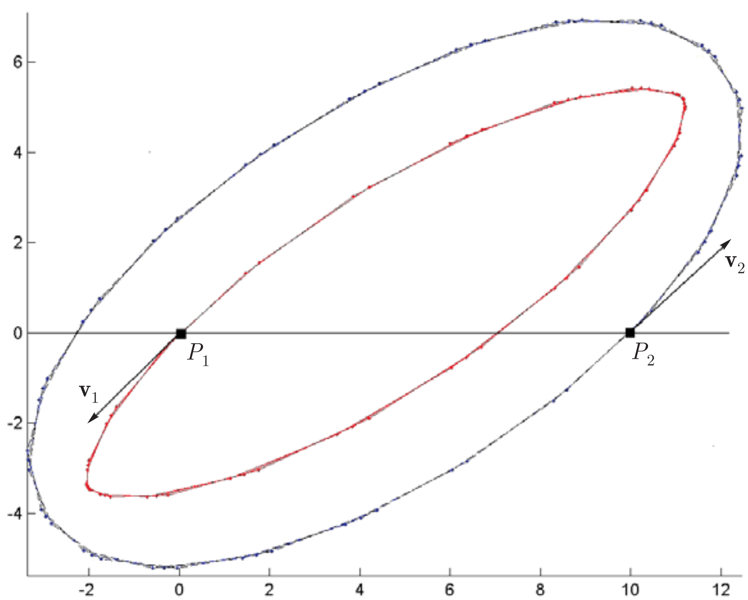


**Fig. 20.** Case C. $\mathbf{v}_1 = -\mathbf{v}_2,\ m_2 = 2\,m_1$.

• Case D. Recall that this case concerns with equioriented initial velocities inclined w.r.to the segment $P_1P_2$. In Fig. 26 and 27 we have $\mathbf{v}_1 = \mathbf{v}_2$, for equal and different masses, respectively. In Fig. 28 and 29 we have $\mathbf{v}_2 = 2\,\mathbf{v}_2$, for equal and different masses, respectively, and with the same locations of the initial positions $P_1$ and $P_2$.

In Figs. 29 and 31 we observe a strange behavior of the initial motion of the point $P_1$: it seems to move in an opposite direction of the initial velocity $\mathbf{v}_1$. But if we use a zoom over its neighborhood, then we understand what happens (Fig. 32).

### *8.4. Two Mass-points with Constant Distance and Parallel Velocities*

Two mass-points $(P_1, m_1)$ e $(P_2, m_2)$ running on a plane are linked by a rigid segment of negligible mass and of length $\ell$. The configuration manifold of this mechanical system is $Q_3 = \mathbb{R}^2 \times \mathbb{S}_1$. Let

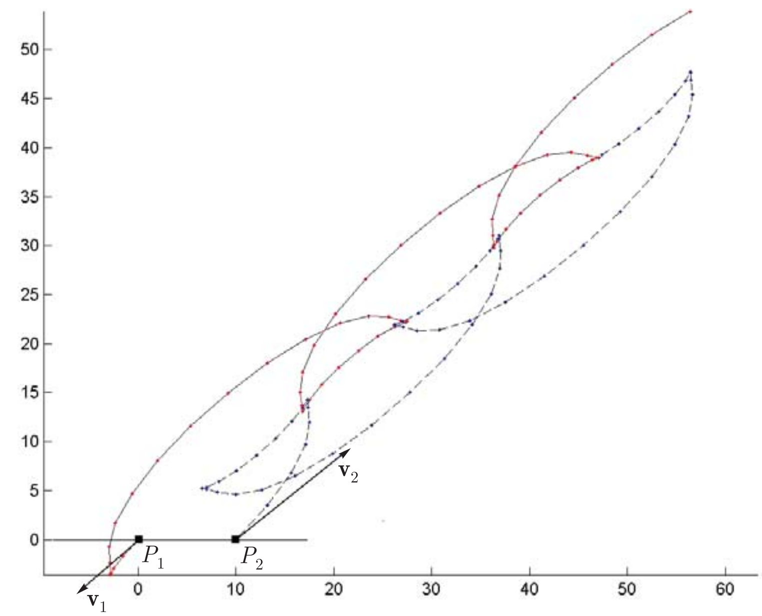


**Fig. 21.** Case C. $\mathbf{v}_2 = -2\,\mathbf{v}_1,\ m_2 = m_1$.

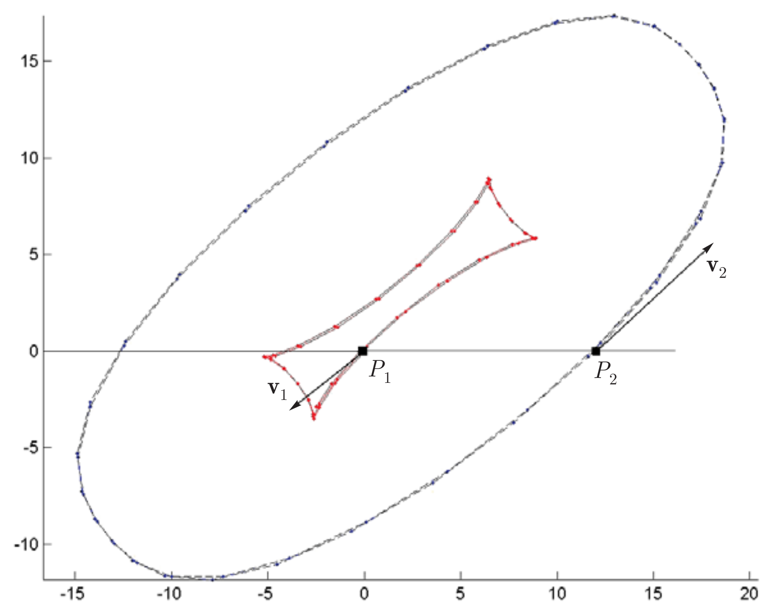


**Fig. 22.** Case C. The same as in Fig. 21, but with a larger distance $|P_1P_2|$.

us use the Lagrangian coordinates

$$(q^1, q^2, q^3) = (x, y, \phi),$$

where $(x, y)$ are the Cartesian coordinates of $P_1$ and $\phi$ is the angle of the segment $P_1P_2$ w.r.to the $x$-axis (oriented anticlockwise). Hence, we can write

$$OP_1 = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad OP_2 = \begin{bmatrix} x + \ell\ \cos\phi \\ y + \ell\ \sin\phi \end{bmatrix},$$

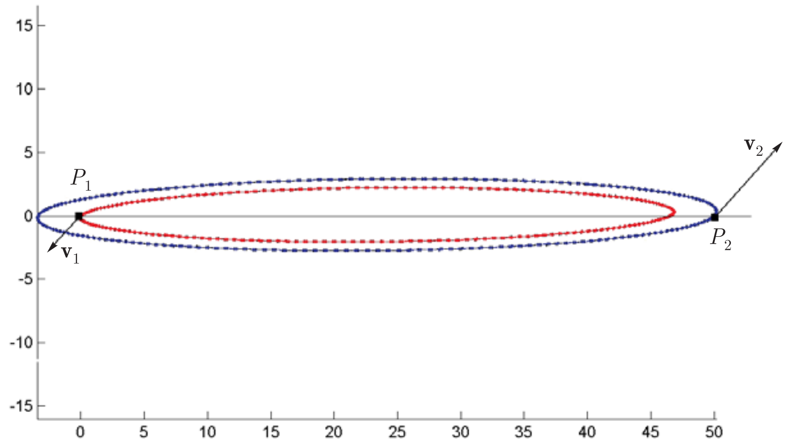


**Fig. 23.** Case C. The same as in Fig. 22, but with a larger distance $|P_1P_2|$.

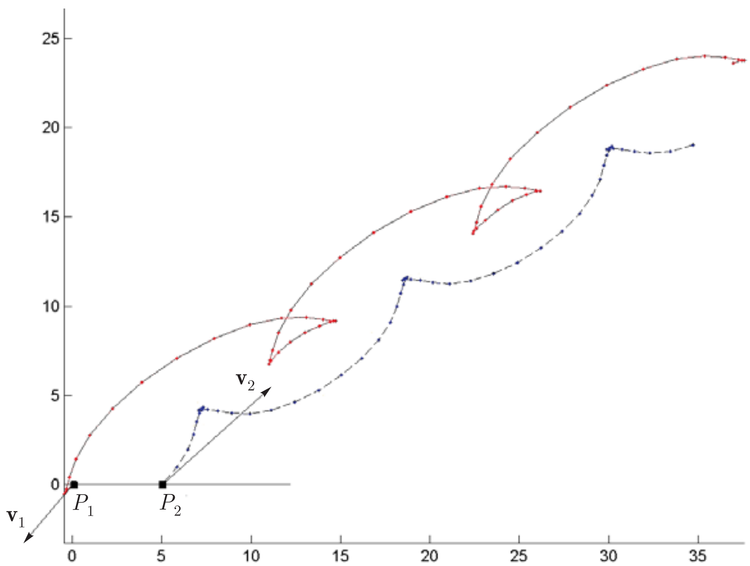


**Fig. 24.** Case C. Different masses: $m_2 = 2\,m_1$.

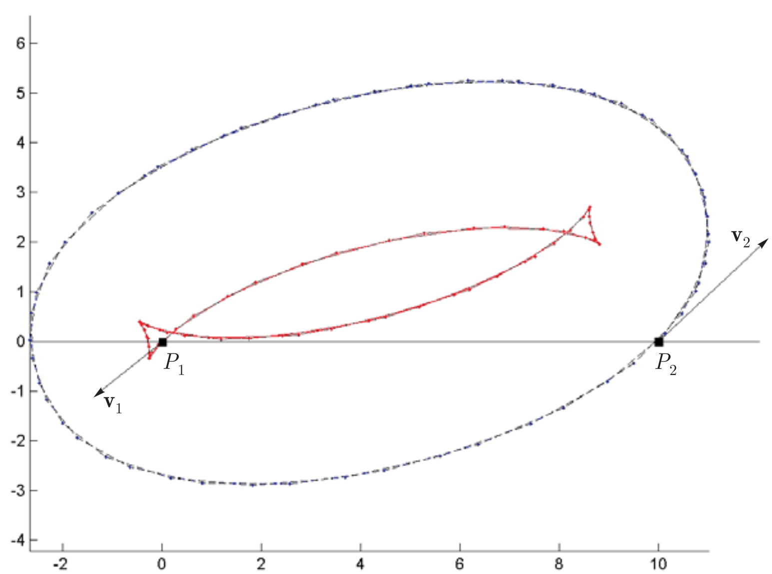


**Fig. 25.** Case C. Different masses $m_2 = 2\,m_1$. A larger initial distance $|P_1P_2|$.

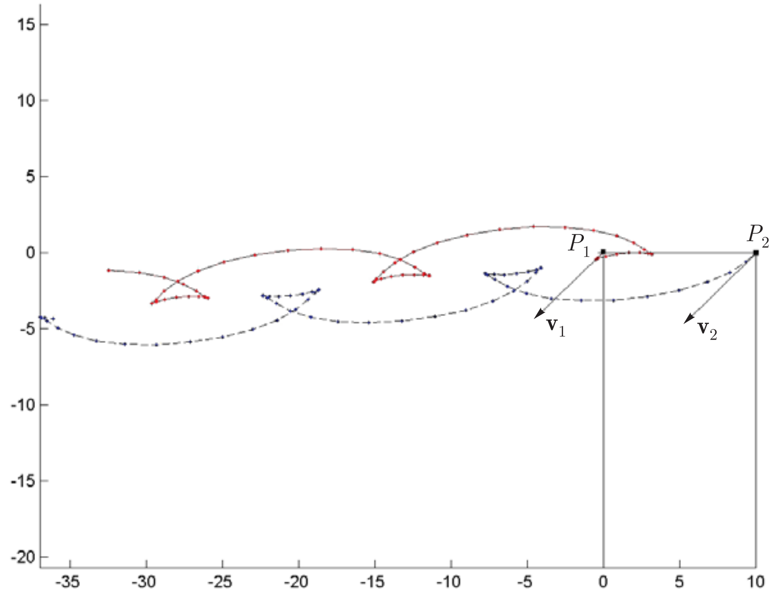


**Fig. 26.** Case D. $\mathbf{v}_1 = \mathbf{v}_2$, $m_1 = m_2$, $P_1 = (0, 0)$, $P_2 = (10, 0)$.

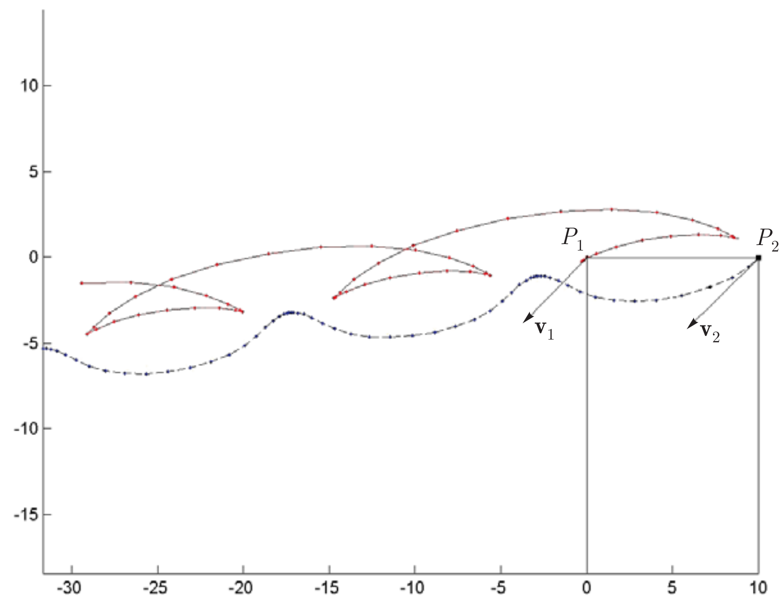


**Fig. 27.** Case D. The same as in Fig. 26 but with $m_2 = 4\,m_1$.

and in a generic kinematical state the velocities of the two points are

$$\mathbf{v}_1 = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} \dot{x} - \ell\ \sin\phi\ \dot{\phi} \\ \dot{y} + \ell\ \cos\phi\ \dot{\phi} \end{bmatrix}. \tag{56}$$

The segment $P_1P_2$ behaves, from a kinematical viewpoint, as any flat rigid body on the plane. A first consequence is that, at any kinematical state, the velocity $\mathbf{v}_P$ of all points $P$ of the line containing the segment, including the endpoints, have the same projection onto the line itself. This projection is given by

$$\tau \doteq \dot{x}\ \cos\phi + \dot{y}\ \sin\phi. \tag{57}$$

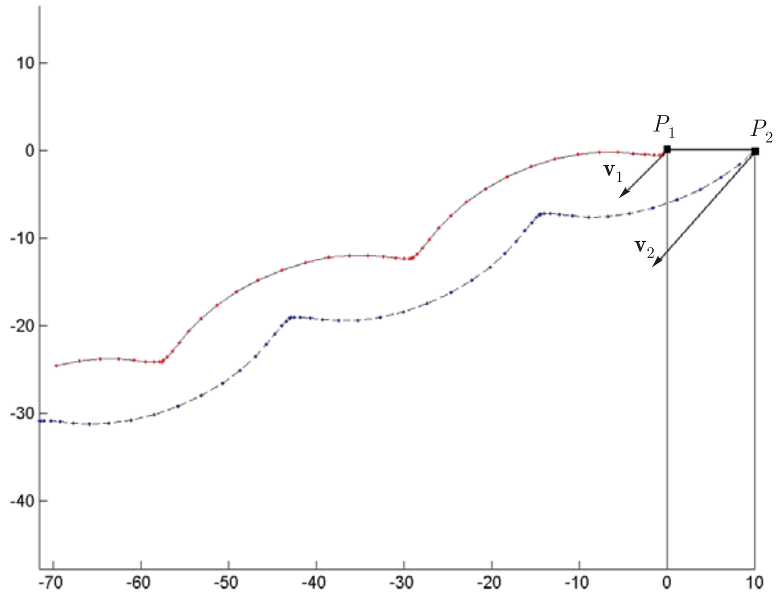


**Fig. 28.** Case D. $\mathbf{v}_2 = 2\,\mathbf{v}_1$, $m_1 = m_2$, $P_1 = (0, 0)$, $P_2 = (10, 0)$.

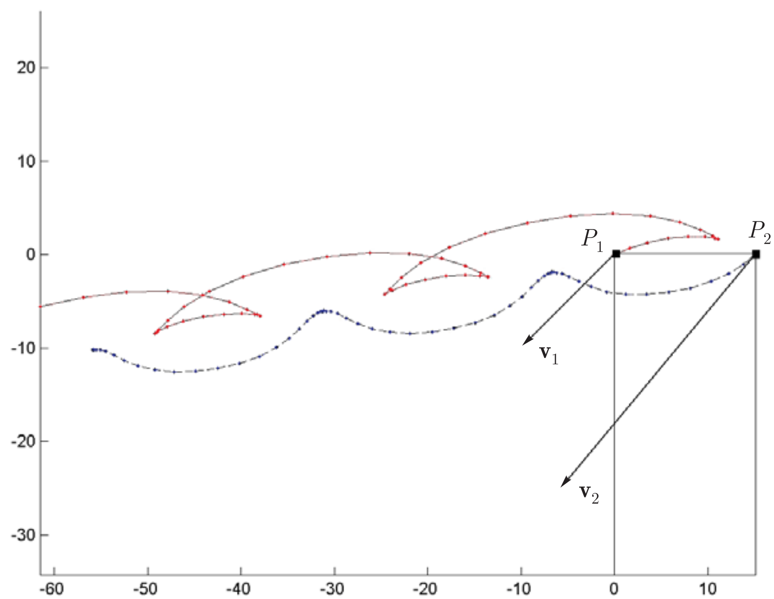


**Fig. 29.** Case D. The same as in Fig. 28 but with $m_2 = 3\,m_1$ and $P_2 = (15, 0)$.

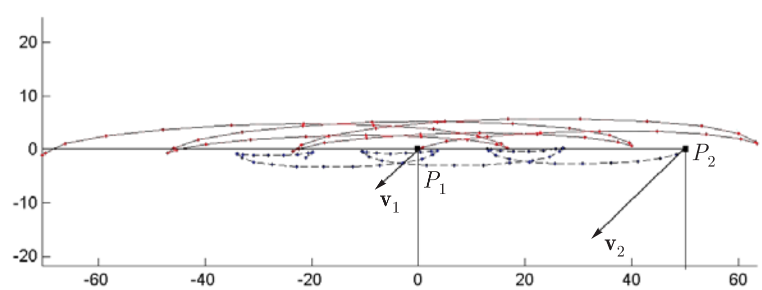


**Fig. 30.** Case D. $\mathbf{v}_2 = 2\,\mathbf{v}_1$, $m_2 = 3\,m_1$, $P_1 = (0, 0)$, $P_2 = (50, 0)$.

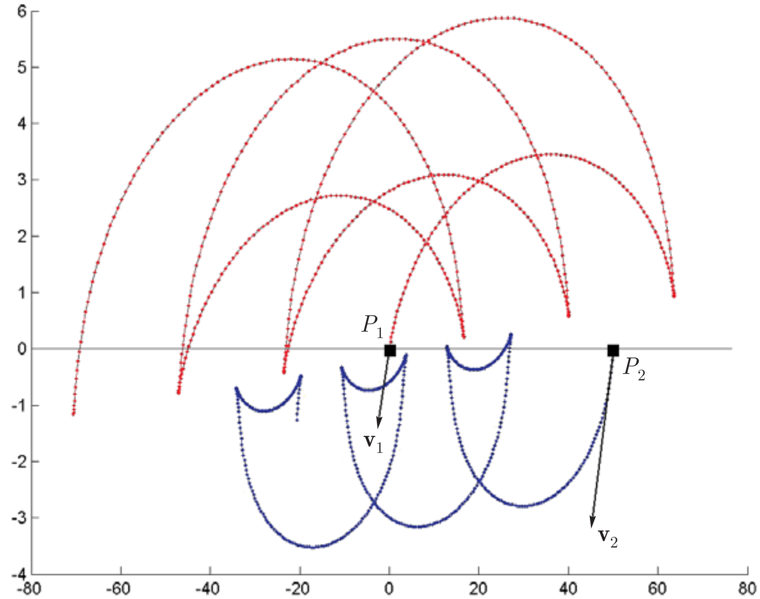


**Fig. 31.** Case D. The same initial conditions of Fig. 30 but with the $y$-axis expanded.

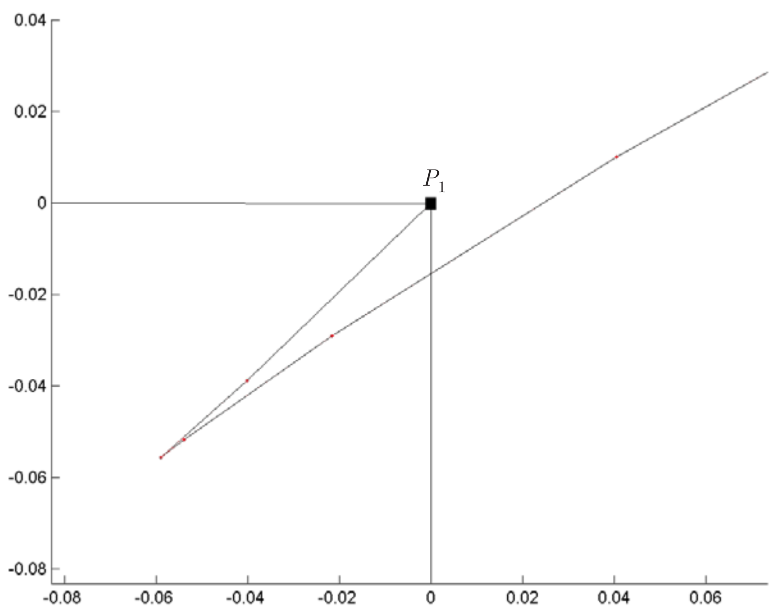


**Fig. 32.** Zoom over $P_1$ in Fig. 29 and Fig. 31.

Indeed, if we introduce the unit vector from $P_1$ to $P_2$,

$$\mathbf{d} = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix},$$

then from(56) we obtain

$$\mathbf{v}_1 \cdot \mathbf{d} = \mathbf{v}_2 \cdot \mathbf{d} = \dot{x}\ \cos\phi + \dot{y}\ \sin\phi. \tag{58}$$

A second consequence is that we have two kinds of **kinematical states** (see Fig. 33):

1. **Translational state**: all points $P$ of the segment (including the endpoints) have the same velocity: $\mathbf{v}_P = \mathbf{v}_1 = \mathbf{v}_2$.

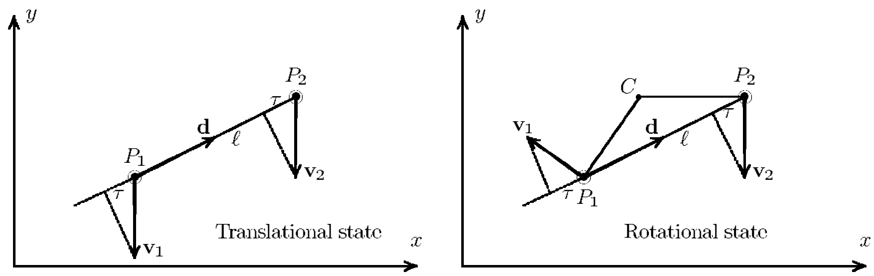


**Fig. 33.** Kinematical states of a rigid segment.

2. **Rotational state**: there exists a point $C$ on the plane such that, for each $P$ of the segment (including $P_1$ and $P_2$) we have

$$\mathbf{v}_P(t) = \dot{\phi}(t)\, \mathbf{k} \times CP, \tag{59}$$

where $\mathbf{k}$ is a unit vector orthogonal to the plane and oriented according to the orientation of $\phi$. The point $C$ is the **instantaneous center of motion** and the vector $\dot{\phi}\,\mathbf{k}$ is the **instantaneous angular velocity**.

In Example 8.2 we have considered the case in which the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are constrained to be orthogonal to the segment. Now, let us consider the case in which *the two points $P_1$ and $P_2$ are constrained to have parallel velocities.* This nonholonomic constraint is expressed by the condition $\mathbf{v}_1 \times \mathbf{v}_2 = 0$ which, due to (56), is equivalent to equation

$$\begin{vmatrix} \dot{x} & \dot{y} \\ \dot{x} - \ell\, \sin\phi\, \dot{\phi} & \dot{y} + \ell\, \cos\phi\, \dot{\phi} \end{vmatrix} = 0,$$

i.e., to the quadratic equation

$$\tau\, \dot{\phi} = (\dot{x}\, \cos\phi + \dot{y}\, \sin\phi)\, \dot{\phi} = 0. \tag{60}$$

This is a constraint of a special kind: *it is a quadratic constraint* **reducible** *to two linear constraints*,

$$\tau = \dot{x}\, \cos\phi + \dot{y}\, \sin\phi = 0, \tag{61}$$

and

$$\dot{\phi} = 0. \tag{62}$$

They must be alternatively or simultaneously satisfied. As a consequence, the dynamics of our system splits into two types:[6)]

• **Dynamics of type I**: the dynamics obeys to the linear nonholonomic constraint $\tau = 0$. In this case, the kinematical states compatible with the constraint are such that $\mathbf{v}_1$ and $\mathbf{v}_2$ are both orthogonal to the segment, as illustrated in Fig. 34, to which we should add the cases where one of the two velocities, or both, are zero.

• **Dynamics of type II**: the dynamics obeys to the linear constraint $\dot{\phi} = 0$. This is an integrable constraint, since it is equivalent to

$$\phi = \phi_0 = \text{constant}.$$

In this case the mechanical system behaves as a holonomic system with 2 degrees of freedom $(x, y)$. All motions are then translational. There are *transitional states* at which a change of the dynamical behavior may occur. A detailed analysis of this nonholonomic system will be developed in another paper.

[6)]This interesting fact has been already remarked by Zekovich (1991).

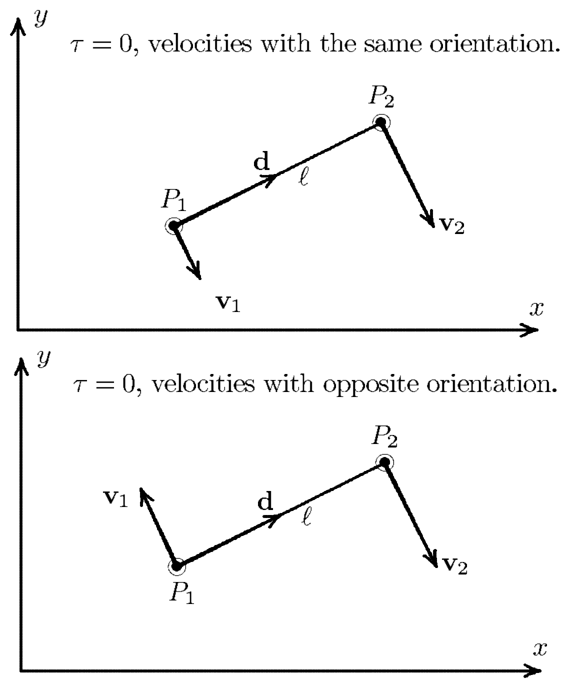


**Fig. 34.** Possible states in the dydamics of type I: $\tau = 0$.

## 9. A TENTATIVE PROJECT OF A DEVICE

I propose a possible concrete realization of a non-linear nonholonomic system considered above: two mass-points moving on the plane with parallel velocities. The problem is twofold: (1) how to realize a mass-point and, (2) how to realize a parallel transport in the plane. We can solve these two problems by two devices, denoted by Device A and Device B. Device A is illustrated in Fig. 35 and 36.

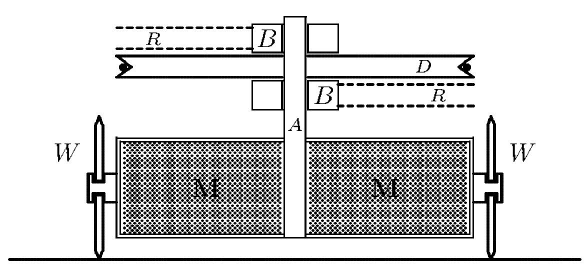


**Fig. 35.** Device A, vertical section SS.

The *"mass-point"* consists of a small cylinder $M$ filled with mercury. This heavy liquid metal has the property of non-adherence to the walls, if the walls are made of a suitable material. In this way we avoid any rotational effects: the particles of mercury will have only instantaneous translational motions i.e., at each instant they velocities are equal. On the other hand, the heaviness of mercury will make negligible all the masses of the components of both the devices, which however should be made of very light, but strong rigid material.[7)]

To the cylinder we attach, in diametrical opposition, two vertical wheels with sharp edges $W$, which lean on the plane of motion. The plane may be iced or a plane of billiard. In both cases, the wheels can roll only in the direction perpendicular to their own axes. The cylinder $M$ is rigidly connected to its vertical axis $A$. Upon the cylinder we attach to $A$, in the order, a ball bearing $B$,

[7)] There are in fact other ways of realizing material points, by using heavy rigid materials instead of mercury. Notice moreover that, instead of a cylinder, is some cases it is better to use a sphere filled with mercury.

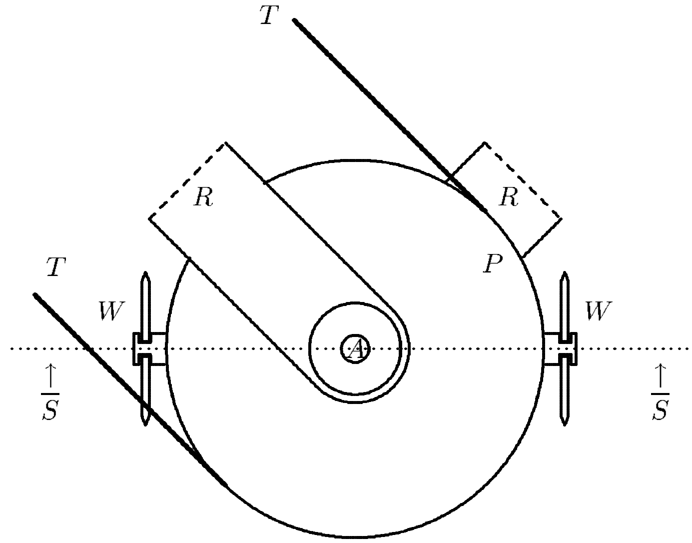


**Fig. 36.** Device A, from above.

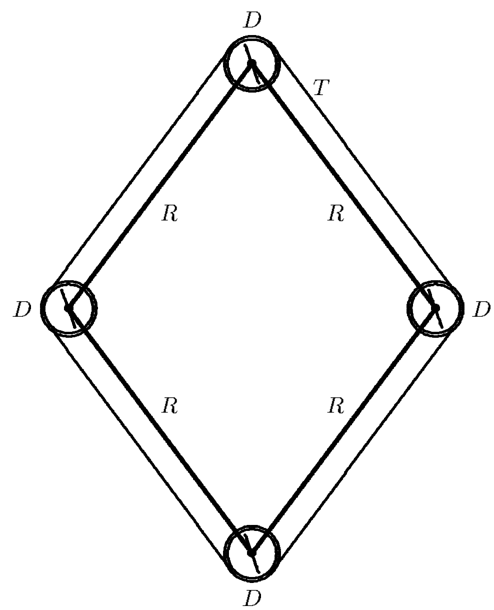


**Fig. 37.** Device B.

a pulley $D$ and another ball bearing $B$. Actually, they belong to the Device B, together with the thread $T$ and the rods $R$.

Device B has the role of realizing the parallel transport in the plane (Fig. 37). It is made of four rods $R$ of equal length which can pivot at their end points, so that they compose a rhombus. The pivoting is without, thanks to the ball bearings $B$. Four discs $D$ are hinged on these end points. Actually, the discs are pulleys connecting a closed tight thread $T$. It is remarkable the fact that, whatever configuration assumes the rhombus, the length of the thread does not change. In this way, in any configuration, the four discs rotate of a same angle. In other words, if we draw four parallel ticks on them, these ticks remain one another parallel at each instant of any motion of the device.

These two devices are combined together as follows. If we fix two devices A under two discs $D$ (it is of course convenient to choose two opposite discs) with the four wheels $W$ in a parallel position, then we get a device with two mass points which can move only with parallel velocities. Of course this device has two limits: the two mass points cannot have (1) the same location and, (2) a distance greater than the double of the length of the rods. However, it possible to invent special initial conditions as well as some modification of the device, for which these circumstances do not occur.

**Notes.** (1) This device does not work very well when both velocities of the mass points are zero; this in agreement with the fact that such a state is a singular state for the theoretical constraint, as remarked above. (2) The devices $A$ and $B$ can be used for realizing a nonholonomic system made of a single point on a plane submitted to a kinematical constraint of the kind

$$f(\mathbf{x}, \dot{\mathbf{x}})\, \dot{x} = g(\mathbf{x}, \dot{\mathbf{x}})\, \dot{y},$$

where $\mathbf{x} = (x, y)$, and $f$, $g$ are (almost) arbitrary smooth functions of $(\mathbf{x}, \dot{\mathbf{x}})$. The axis $A$ of one of the discs is fixed at a point on the plane and submitted to the action of a (fine) stepping motor. The opposite disc is a nonholonomic mass point (Device A). What we need is an electronic device capable of detecting the position and the velocity of this point, of elaborating these data according to the given functions $f$ and $g$, and of transmitting the result to the stepping motor. Note that the stepping motor does not spend energy in controlling the direction of the wheels of the nonholonomic point.

## ACKNOWLEDGMENTS

I am grateful to Alexey Borisov for his encouragement to complete a first version of my article by adding the graphics of numerical experiments on Eqs. 52 and for making me aware of the papers of Zekovich and providing me their English versions.

The graphics shown above are due to the work of a student of mine, Dario Orlando, who is compiling his thesis by carrying on many numerical simulations of the dynamics of several concrete nonholonomic systems, based on the method presented here.