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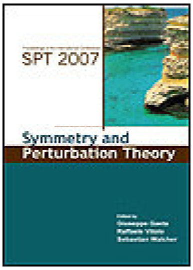

SYMMETRY AND PERTURBATION THEORY

Proceedings of the International Conference on SPT 2007

Otranto, Italy 2 - 9 June 2007

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This proceedings volume is devoted to the interplay of symmetry and perturbation theory, as well as to cognate fields such as integrable systems, normal forms, n-body dynamics and choreographies, geometry and symmetry of differential equations, and finite and infinite dimensional dynamical systems. The papers collected here provide an up-to-date overview of the research in the field, and have many leading scientists in the field among their authors, including: D Alekseevsky, S Benenti, H Broer, A Degasperis, M E Fels, T Gramchev, H Hanssmann, J Krashil'shchik, B Kruglikov, D Krupka, O Krupkova, S Lombardo, P Morando, O Morozov, N N Nekhoroshev, F Oliveri, P J Olver, J A Sanders, M A Teixeira, S Terracini, F Verhulst, P Winternitz, B Zhilinskii.

Contents:

COMPUTING CURVATURE WITHOUT CHRISTOFFEL SYMBOLS

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The Riemann tensor of a given metric, of any dimension and signature, can be computed 'by hand calculation', avoiding the explicit calculation of the $(1/2)\,n^2(n+1)$ Christoffel symbols. The algorithm presented here works with n quadratic form Q^i in the velocity-variables coming from the Lagrange geodesic equations, and with $2n$ cubic forms R^i_0 and R^i_1 generated by them. An example of this method is illustrated: it concerns the application of Geodesic Equivalence theory to General Relativity.

Keywords: Riemannian geometry; Geodesic equivalence.

1. The six steps of the algorithm

We start by describing the algorithm; comments will be given later on.

Step 1.

• Take the covariant components g_{ij} of the given metric tensor, and write the "Kinetic energy" $K = \frac{1}{2}\, g_{ij}\, v^i\, v^j$. • Compute the Lagrange binomials

$$L_i := \frac{d}{dt}\frac{\partial K}{\partial v^i} - \frac{\partial K}{\partial q^i}.$$

Step 2. • Compute the inverse matrix $[g^{ij}]$ of $[g_{ij}]$. • Rise the index of L_i:

$$L^i := g^{ij} L_j = \frac{dv^i}{dt} + \Gamma^i_{hk}\, v^h\, v^k.$$

• Take the quadratic form in L^i: $Q^i := \Gamma^i_{hk}\, v^h\, v^k$. • Do not extract the $\frac{1}{2}\, n^2(n+1)$ Christoffel symbols! We work with the n quadratic forms Q^i only.

Step 3. • Compute the total formal derivative of Q^i w.r.to a 'new time' $\bar{t}$, $R^i \doteq \dfrac{dQ^i}{d\bar{t}}$, by setting $\dfrac{dq^i}{d\bar{t}} = \bar{v}^i$ and $\dfrac{dv^i}{d\bar{t}} = \bar{Q}^i$.

Step 4. • Split R^i into the sum $R^i = R^i_0 + R^i_1$: R^i_0 is the part of R^i which does not contain the terms $\bar{Q}^i$. • In R^i_1 replace all $\bar{Q}^i$ by $\bar{Q}^i = -Q^i(\bar{v})$ i.e., put $v^i \mapsto \bar{v}^i$ in Q^i and change the sign. • Result: R^i_0 are 3^{rd}-degree homogeneous polynomials in $v^h v^i \bar{v}^j$, and R^i_1 are 3^{rd}-degree homogeneous polynomials in $v^h \bar{v}^i \bar{v}^j$.

Step 5. • Compute the A-**symbols**: $A^i{}_{\ell mn} := \frac{1}{2}\,\dfrac{\partial^3 R^i_0}{\partial \bar{v}^m\,\partial v^\ell\,\partial v^n}$

• Compute the B-**symbols**: $B^i{}_{\ell mn} := -\,\frac{1}{4}\,\dfrac{\partial^3 R^i_1}{\partial \bar{v}^n\,\partial \bar{v}^\ell\,\partial v^m}$

Step 6. • Compute the Riemann tensor:

$$R^i{}_{\ell mn} = A^i{}_{\ell mn} - A^i{}_{\ell nm} + B^i{}_{\ell mn} - B^i{}_{\ell nm}.$$

2. Explanation

(1) L^i have the form $L^i = \frac{dv^i}{dt} + \Gamma^i_{hk}\, v^h\, v^k$

(2) R^i have the form

$$R^i = \partial_m \Gamma^i_{\ell n}\,\bar{v}^m\,v^\ell\,v^n + 2\,\Gamma^i_{km}\,\bar{Q}^k\,v^m = \partial_m \Gamma^i_{\ell n}\,\bar{v}^m\,v^\ell\,v^n - 2\,\Gamma^i_{km}\,\Gamma^k_{\ell n}\,\bar{v}^\ell\,\bar{v}^n\,v^m$$

(3) The Riemann tensor is defined as (see Eisenhart,[1] p.19)

$$R^i{}_{\ell mn} := \partial_m \Gamma^i_{\ell n} - \partial_n \Gamma^i_{\ell m} + \Gamma^i_{km}\,\Gamma^k_{\ell n} - \Gamma^i_{kn}\Gamma^k_{\ell m}.$$

If we introduce the symbols $A^i{}_{\ell mn} := \partial_m \Gamma^i_{\ell n}$ and $B^i{}_{\ell mn} := \Gamma^i_{km}\,\Gamma^k_{\ell n}$, then the Riemann tensor can be written

$$R^i{}_{\ell mn} = A^i{}_{\ell mn} - A^i{}_{\ell nm} + B^i{}_{\ell mn} - B^i{}_{\ell nm}. \tag{1}$$

(4) The expression $R^i = \partial_m \Gamma^i_{\ell n}\,\bar{v}^m\,v^\ell\,v^n - 2\,\Gamma^i_{km}\,\Gamma^k_{\ell n}\,\bar{v}^\ell\,\bar{v}^n\,v^m$ in item 2 shows that these symbols can be obtained by three partial derivatives, as in the definitions of Step 5.

Proof After the splitting $R^i = \partial_m \Gamma^i_{\ell n}\,\bar{v}^m\,v^\ell\,v^n + 2\,\Gamma^i_{k\ell}\,\bar{Q}^k\,v^\ell = R^i_0 + R^i_1$, it follows that

$$\frac{\partial R^i_0}{\partial v^\ell} = 2\,\partial_m \Gamma^i_{\ell n}\,\bar{v}^m\,v^n, \quad \frac{\partial^2 R^i_0}{\partial v^\ell\,\partial v^n} = 2\,\partial_m \Gamma^i_{\ell n}\,\bar{v}^m, \quad \frac{\partial^3 R^i_0}{\partial \bar{v}^m\,\partial v^\ell\,\partial v^n} = 2\,\partial_m \Gamma^i_{\ell n},$$

$$\frac{\partial R^i_1}{\partial v^m} = 2\,\Gamma^i_{km}\,\bar{Q}^k, \qquad \frac{\partial^2 R^i_1}{\partial \bar{v}^\ell \partial v^m} = 2\,\Gamma^i_{km}\,\frac{\partial \bar{Q}^k}{\partial \bar{v}^\ell} = -\,4\,\Gamma^i_{km}\,\Gamma^k_{\ell n}\,\bar{v}^n.$$

$$\frac{\partial^3 R^i_1}{\partial \bar{v}^n \partial \bar{v}^\ell \partial v^m} = 2\,\Gamma^i_{km}\,\frac{\partial \bar{Q}^k}{\partial \bar{v}^\ell} = -\,4\,\Gamma^i_{km}\,\Gamma^k_{\ell n}. \qquad \square$$

Remark 2.1. The splitting $R^i = R^i_0 + R^i_1$ is useful for shortening the "by hand" calculations. It is not necessary when using a software. Indeed,

$$\frac{\partial^3 R^i_0}{\partial \bar{v}^m\,\partial v^\ell\,\partial v^n} = \frac{\partial^3 R^i}{\partial \bar{v}^m\,\partial v^\ell\,\partial v^n}, \qquad \frac{\partial^2 R^i_1}{\partial \bar{v}^n\,\partial \bar{v}^\ell \partial v^m} = \frac{\partial^2 R^i}{\partial \bar{v}^n\,\partial \bar{v}^\ell \partial v^m}.$$

3. An example

Let us apply the algorithm to a kinetic energy of the kind

$$K = \tfrac{1}{2}\, g_{aa}\,(v^a)^2 + \tfrac{1}{2}\, f\, h_{\alpha\beta}\, v^\alpha\, v^\beta. \tag{2}$$

The coordinates are divided into two subsets: $(q^i) = (q^a, q^\alpha)$. The **Roman indices** $a, b, c, \dots$ assume values from 1 to n_R. The **Greek indices** $\alpha, \beta, \gamma, \dots$ assume values from $n_R + 1$ to $n_R + n_G = n$. Moreover, the components g_{aa} and the **conformal factor** f depend on the Roman coordinates only, while the components of the metric tensor $h_{\alpha\beta}$ depend on the Greek coordinates only. The interest of such a metric will be explained below.

3.1. *The computation of the quadratic forms Q^i*

From (2) we get:

$$\begin{aligned}
&\frac{\partial K}{\partial v^a} = g_{aa}\, v^a. \qquad \frac{\partial K}{\partial v^\alpha} = f\, h_{\alpha\beta}\, v^\beta.\\
&\frac{d}{dt}\frac{\partial K}{\partial v^a} = g_{aa}\,\frac{dv^a}{dt} + \partial_b g_{aa}\, v^b\, v^a.\\
&\frac{d}{dt}\frac{\partial K}{\partial v^\alpha} = f\, h_{\alpha\beta}\,\frac{dv^\beta}{dt} + f\,\partial_\gamma h_{\alpha\beta}\, v^\beta\, v^\gamma + \partial_a f\, h_{\alpha\beta}\, v^\beta\, v^a.\\
&\frac{\partial K}{\partial q^a} = \tfrac{1}{2}\,\partial_a g_{bb}\,(v^b)^2 + \tfrac{1}{2}\,\partial_a f\, h_{\alpha\beta}\, v^\alpha\, v^\beta. \qquad \frac{\partial K}{\partial q^\alpha} = \tfrac{1}{2}\, f\,\partial_\alpha h_{\beta\gamma}\, v^\beta\, v^\gamma.
\end{aligned} \tag{3}$$

The Lagrangian binomials are

$$L_a = g_{aa}\,\frac{dv^a}{dt} + \partial_b g_{aa}\, v^b\, v^a - \tfrac{1}{2}\,\partial_a g_{bb}\,(v^b)^2 - \tfrac{1}{2}\,\partial_a f\, h_{\alpha\beta}\, v^\alpha\, v^\beta,$$

$$L_\alpha = f\, h_{\alpha\beta}\,\frac{dv^\beta}{dt} + f\,\partial_\gamma h_{\alpha\beta}\, v^\beta\, v^\gamma + \partial_a f\, h_{\alpha\beta}\, v^\beta\, v^a - \tfrac{1}{2}\, f\,\partial_\alpha h_{\beta\gamma}\, v^\beta\, v^\gamma.$$

Then, the quadratic forms Q^i are given by

$$\begin{aligned}
Q^a &= g^{aa}\,\left(\partial_b g_{aa}\, v^b\, v^a - \tfrac{1}{2}\,\partial_a g_{bb}\,(v^b)^2 - \tfrac{1}{2}\,\partial_a f\, h_{\alpha\beta}\, v^\alpha\, v^\beta\right),\\
Q^\alpha &= h^{\alpha\rho}\,\left(\partial_\gamma h_{\rho\beta} - \tfrac{1}{2}\,\partial_\rho h_{\beta\gamma}\right)\, v^\beta\, v^\gamma + v^\alpha\,\partial_a \log|f|\, v^a.
\end{aligned} \tag{4}$$

Since, for the moment, we do not specify the expressions of g_{aa}, f and $h_{\alpha\beta}$, we can continue our calculation by observing that the Q^i assume the form

$$Q^a = \widehat{\Gamma}^a_{bc}\, v^b\, v^c - \tfrac{1}{2}\, f^a\, h_{\alpha\beta}\, v^\alpha\, v^\beta \qquad Q^\alpha = \widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, v^\gamma + v^\alpha\, v^a\, F_a, \tag{5}$$

where $\widehat{\Gamma}^a_{bc}$ and $\widetilde{\Gamma}^\alpha_{\beta\gamma}$ are the Christoffel symbols of the metric g_{ab} and $h_{\alpha\beta}$, respectively, and $f^a := g^{ab}\,\partial_b f = g^{aa}\,\partial_a f, \quad F_a := \partial_a \log|f|$.

3.2. *The computation of the cubic forms R^i*

According to Step 3, from (5) we get

$$
\begin{aligned}
R^a = \frac{dQ^a}{d\bar{t}} &= \partial_e \widehat{\Gamma}^a_{bc}\, v^b\, v^c\, \bar{v}^e - \tfrac{1}{2}\,\partial_e f^a\, h_{\alpha\beta}\, v^\alpha\, v^\beta\, \bar{v}^e - f^a\,\partial_\gamma h_{\alpha\beta}\, v^\alpha\, v^\beta\, \bar{v}^\gamma \\
&\quad + 2\,\widehat{\Gamma}^a_{bc}\, v^b\, \bar{Q}^c - f^a\, h_{\alpha\beta}\, v^\alpha\, \bar{Q}^\beta. \\
R^\alpha = \frac{dQ^\alpha}{d\bar{t}} &= \partial_\rho \widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, v^\gamma\, \bar{v}^\rho + \partial_e F_a\, v^\alpha\, v^a\, \bar{v}^e + 2\,\widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, \bar{Q}^\gamma \\
&\quad + F_a\,(\bar{Q}^\alpha\, v^a + v^\alpha\, \bar{Q}^a).
\end{aligned}
$$

According to Step 4 we have

$$
\begin{aligned}
R^a_0 &= \partial_e \widehat{\Gamma}^a_{bc}\, v^b\, v^c\, \bar{v}^e - \tfrac{1}{2}\,\partial_e f^a\, h_{\alpha\beta}\, v^\alpha\, v^\beta\, \bar{v}^e - f^a\,\partial_\gamma h_{\alpha\beta}\, v^\alpha\, v^\beta\, \bar{v}^\gamma \\
R^\alpha_0 &= \partial_\rho \widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, v^\gamma\, \bar{v}^\rho + \partial_e F_a\, v^\alpha\, v^a\, \bar{v}^e,
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
R^a_1 &= 2\,\widehat{\Gamma}^a_{bc}\, v^b\, \bar{Q}^c - f^a\, h_{\alpha\beta}\, v^\alpha\, \bar{Q}^\beta, \\
R^\alpha_1 &= 2\,\widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, \bar{Q}^\gamma + F_a\,(\bar{Q}^\alpha\, v^a + v^\alpha\, \bar{Q}^a),
\end{aligned}
$$

where, according to (5), we have to substitute

$$
\bar{Q}^a = -\,\widehat{\Gamma}^a_{de}\,\bar{v}^d\,\bar{v}^e + \tfrac{1}{2}\, f^a\, h_{\lambda\mu}\,\bar{v}^\lambda\,\bar{v}^\mu, \qquad \bar{Q}^\alpha = -\,\widetilde{\Gamma}^\alpha_{\lambda\mu}\,\bar{v}^\lambda\,\bar{v}^\mu - F_e\,\bar{v}^e\,\bar{v}^\alpha.
$$

Then,

$$
\begin{aligned}
R^a_1 &= \widehat{\Gamma}^a_{bc} v^b \left(f^c h_{\lambda\mu} \bar{v}^\lambda \bar{v}^\mu - 2\,\widehat{\Gamma}^c_{de} \bar{v}^d \bar{v}^e \right) + f^a h_{\alpha\beta} v^\alpha \left(\widetilde{\Gamma}^\beta_{\lambda\mu} \bar{v}^\lambda \bar{v}^\mu + F_e \bar{v}^e \bar{v}^\beta \right) \\
R^\alpha_1 &= -\,2\widetilde{\Gamma}^\alpha_{\beta\gamma} v^\beta \left(\widetilde{\Gamma}^\gamma_{\lambda\mu} \bar{v}^\lambda \bar{v}^\mu + F_e \bar{v}^e \bar{v}^\gamma \right) \\
&- F_a\,\left[\left(\widetilde{\Gamma}^\alpha_{\lambda\mu}\,\bar{v}^\lambda\,\bar{v}^\mu + F_e\,\bar{v}^e\,\bar{v}^\alpha \right) v^a + v^\alpha\,\left(\widehat{\Gamma}^a_{de}\,\bar{v}^d\,\bar{v}^e - \tfrac{1}{2}\, f^a\, h_{\lambda\mu}\,\bar{v}^\lambda\,\bar{v}^\mu \right) \right].
\end{aligned} \tag{7}
$$

3.3. *The computation of the A-symbols*

According to Step 5, we have to compute the partial derivatives of R^i_0 with respect to the variables v^i and $\bar{v}^i$. In doing this δ-symbols will arise systematically, since $(\partial v^i/\partial v^j) = (\partial \bar{v}^i/\partial \bar{v}^j) = \delta^i_j$. However, the calculation can be shortened if we consider that

$$\left.\begin{array}{l}
\sum_a X_a\,\delta^a_\ell = X_\ell,\ \ \sum_a Y_\alpha\,\delta^\alpha_\ell = Y_\ell.\\
\widehat{\Gamma}^a_{\ell m} = 0,\ \ \text{when one of the two indices } (\ell, m) \text{ is Greek.}\\
\widetilde{\Gamma}^\alpha_{\ell m} = 0,\ \ h_{\ell m} = 0\ \ \text{when one of the two indices } (\ell, m) \text{ is Roman.}\\
f^\alpha = 0,\ \ F_\alpha = 0.\\
\Gamma^a_{bc},\ \ f^a,\ \ F_a\ \ \text{depend on the Roman coordinates only.}\\
\widetilde{\Gamma}^\alpha_{\beta\gamma},\ \ h_{\alpha\beta}\ \ \text{depend on the Greek coordinates only.}
\end{array}\right\} \tag{8}$$

Then, from (6) we derive

$$\begin{aligned}
\frac{\partial R^a_0}{\partial \bar{v}^m} &= \partial_m \widehat{\Gamma}^a_{bc}\, v^b\, v^c - \tfrac{1}{2}\,\partial_m f^a\, h_{\alpha\beta}\, v^\alpha\, v^\beta - f^a\,\partial_m h_{\alpha\beta}\, v^\alpha\, v^\beta,\\
\frac{\partial R^\alpha_0}{\partial \bar{v}^m} &= \partial_m \widetilde{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, v^\gamma + \partial_m F_a\, v^\alpha\, v^a.\\
\frac{\partial^2 R^a_0}{\partial v^\ell\,\partial \bar{v}^m} &= 2\,\partial_m \widehat{\Gamma}^a_{b\ell}\, v^b - \partial_m f^a\, h_{\alpha\ell}\, v^\alpha - 2\, f^a\,\partial_m h_{\alpha\ell}\, v^\alpha,\\
\frac{\partial^2 R^\alpha_0}{\partial v^\ell\,\partial \bar{v}^m} &= 2\,\partial_m \widetilde{\Gamma}^\alpha_{\beta\ell}\, v^\beta + \delta^\alpha_\ell\,\partial_m F_a\, v^a + \partial_m F_\ell\, v^\alpha.\\
\frac{\partial^3 R^a_0}{\partial v^n\,\partial v^\ell\,\partial \bar{v}^m} &= 2\,\partial_m \widehat{\Gamma}^a_{\ell n} - \partial_m f^a\, h_{\ell n} - 2\, f^a\,\partial_m h_{\ell n},\\
\frac{\partial^3 R^\alpha_0}{\partial v^n\,\partial v^\ell\,\partial \bar{v}^m} &= 2\,\partial_m \widetilde{\Gamma}^\alpha_{\ell n} + \delta^\alpha_\ell\,\partial_m F_n + \delta^\alpha_n\,\partial_m F_\ell.
\end{aligned}$$

Hence, according to Step 5, the A-symbols are

$$\begin{aligned}
A^a_{\ell m n} &= \partial_m \widehat{\Gamma}^a_{\ell n} - \tfrac{1}{2}\,\partial_m f^a\, h_{\ell n} - f^a\,\partial_m h_{\ell n},\\
A^\alpha_{\ell m n} &= \partial_m \widetilde{\Gamma}^\alpha_{\ell n} + \tfrac{1}{2}\,(\delta^\alpha_\ell\,\partial_m F_n + \delta^\alpha_n\,\partial_m F_\ell).
\end{aligned} \tag{9}$$

3.4. *The computation of the B-symbols*

We have to compute the partial derivatives of R^i_1. By a calculation similar to that done for the A-symbols, and still taking into account the rules (8), we get

$$\begin{aligned}
\frac{\partial^3 R^a_1}{\partial \bar{v}^n\,\partial \bar{v}^\ell\,\partial v^m} &= 2\,\widehat{\Gamma}^a_{mc}\, f^c\, h_{\ell n} - 4\,\widehat{\Gamma}^a_{mc}\,\widehat{\Gamma}^c_{\ell n} + 2\, f^a\, h_{m\beta}\,\widetilde{\Gamma}^\beta_{\ell n}\\
&\quad + F_\ell\, f^a\, h_{mn} + F_n\, f^a\, h_{\ell m},\\
\frac{\partial^3 R^\alpha_1}{\partial \bar{v}^n\,\partial \bar{v}^\ell\,\partial v^m} &= -\,4\,\widetilde{\Gamma}^\alpha_{m\gamma}\,\widetilde{\Gamma}^\gamma_{\ell n} - 2\,\widetilde{\Gamma}^\alpha_{mn}\, F_\ell - 2\,\widetilde{\Gamma}^\alpha_{\ell m}\, F_n\\
&\quad - \left(2\,\widetilde{\Gamma}^\alpha_{\ell n} - F_\ell\,\delta^\alpha_n + F_n\,\delta^\alpha_\ell\right) F_m - \delta^\alpha_m\, F_a\,\left(2\,\widehat{\Gamma}^a_{\ell n} - f^a\, h_{\ell n}\right).
\end{aligned}$$

Hence, according to Step 5, the B-symbols are

$$
\begin{aligned}
B^{a}_{\ \ell mn} &= \widehat{\Gamma}^{a}_{mc}\,\widehat{\Gamma}^{c}_{\ell n} - \tfrac{1}{2}\widehat{\Gamma}^{a}_{mc} f^{c} h_{\ell n} - \tfrac{1}{2} f^{a} h_{m\beta}\widetilde{\Gamma}^{\beta}_{\ell n} - \tfrac{1}{4}\,F_{\ell} f^{a} h_{mn} \\
&\quad - \tfrac{1}{4}F_{n}\, f^{a} h_{\ell m}, \\
B^{\alpha}_{\ \ell mn} &= \widetilde{\Gamma}^{\alpha}_{m\gamma}\,\widetilde{\Gamma}^{\gamma}_{\ell n} + \tfrac{1}{2}\widetilde{\Gamma}^{\alpha}_{mn} F_{\ell} + \tfrac{1}{2}\widetilde{\Gamma}^{\alpha}_{\ell m} F_{n} \\
&\quad + \tfrac{1}{4}\left(2\widetilde{\Gamma}^{\alpha}_{\ell n} - F_{\ell}\delta^{\alpha}_{n} + F_{n}\delta^{\alpha}_{\ell}\right) F_{m} + \tfrac{1}{4}\delta^{\alpha}_{m} F_{a}\,\left(2\widehat{\Gamma}^{a}_{\ell n} - f^{a} h_{\ell n}\right).
\end{aligned}
\tag{10}
$$

3.5. *The Ricci tensor*

Following Eisenhart[1] (see p.21), the Ricci tensor is defined by $R_{\ell m} = R^{i}_{\ \ell mi}$. Hence, according to (1),

$$
R_{\ell m} = A^{i}_{\ \ell mi} - A^{i}_{\ \ell im} + B^{i}_{\ \ell mi} - B^{i}_{\ \ell im}. \tag{11}
$$

In our example the Ricci tensor components are the sum of A-terms and B-terms, $R_{\ell m} = {}_{A}R_{\ell m} + {}_{B}R_{\ell m}$, with

$$
\begin{aligned}
{}_{A}R_{\ell m} &:= \textstyle\sum_{a}(A^{a}_{\ \ell ma} - A^{a}_{\ \ell am}) + \sum_{\alpha}(A^{\alpha}_{\ \ell m\alpha} - A^{\alpha}_{\ \ell\alpha m}), \\
{}_{B}R_{\ell m} &:= \textstyle\sum_{a}(B^{a}_{\ \ell ma} - B^{a}_{\ \ell am}) + \sum_{\alpha}(B^{\alpha}_{\ \ell m\alpha} - B^{\alpha}_{\ \ell\alpha m}).
\end{aligned}
\tag{12}
$$

Starting from (9) we get

$$
\begin{aligned}
{}_{A}R_{\ell m} &= \partial_{m}\widehat{\Gamma}^{a}_{\ell a} - \partial_{a}\widehat{\Gamma}^{a}_{\ell m} + \partial_{m}\widetilde{\Gamma}^{\alpha}_{\ell\alpha} - \partial_{\alpha}\widetilde{\Gamma}^{\alpha}_{\ell m} \\
&- \tfrac{1}{2}\,\partial_{m} f^{a}\, h_{\ell a} + \tfrac{1}{2}\,\partial_{a} f^{a}\, h_{\ell m} - f^{a}\,\partial_{m} h_{\ell a} + f^{a}\,\partial_{a} h_{\ell m} + \tfrac{1}{2}\,n_{G}\,\partial_{m} F_{\ell}. \\
{}_{B}R_{\ell m} &= \widehat{\Gamma}^{a}_{me}\,\widehat{\Gamma}^{e}_{\ell a} - \widehat{\Gamma}^{a}_{ae}\,\widehat{\Gamma}^{e}_{\ell m} + \widetilde{\Gamma}^{\alpha}_{m\gamma}\,\widetilde{\Gamma}^{\gamma}_{\ell\alpha} - \widetilde{\Gamma}^{\alpha}_{\alpha\gamma}\,\widetilde{\Gamma}^{\gamma}_{\ell m} \\
&- \tfrac{1}{4}\,n_{G}\,F_{\ell}\,F_{m} + \tfrac{1}{4}\,(n_{G} - 2)\,F_{a}\,f^{a}\,h_{\ell m} + \tfrac{1}{2}\,\widehat{\Gamma}^{a}_{ae}\,f^{e}\,h_{\ell m} - \tfrac{1}{2}\,n_{G}\,F_{a}\,\widehat{\Gamma}^{a}_{\ell m}.
\end{aligned}
$$

Then we consider the three essential cases (b, c), (λ, μ), (λ, b) of the pair (ℓ, m), and the result is the following

$$
\begin{aligned}
R_{bc} &= \widehat{R}_{bc} - \tfrac{1}{4}\,n_{G}\,\left(F_{b}\,F_{c} + 2\,\widehat{\nabla}_{b}F_{c}\right), \qquad R_{\lambda b} = 0, \\
R_{\lambda\mu} &= \widetilde{R}_{\lambda\mu} + \tfrac{1}{4}\,\left((n_{G} - 2)\,F_{a}\,f^{a} + 2\,\widehat{\nabla}_{a} f^{a}\right)\,h_{\lambda\mu},
\end{aligned}
\tag{13}
$$

being $\widehat{R}_{bc}$ and $\widetilde{R}_{\lambda\mu}$ the Ricci tensors associated with the metrics g_{aa} and $h_{\alpha\beta}$, respectively.

4. How to use this example

Assume $n = 4$, $n_R = n_G = 2$:

$$\begin{bmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & f\,h_{33} & f\,h_{34} \\ 0 & 0 & f\,h_{43} & f\,h_{44} \end{bmatrix} \tag{14}$$

Since $n_G = 2$, the components (13) become

$$\begin{aligned} R_{bc} &= \widehat{R}_{bc} - \tfrac{1}{2}\left(F_b\,F_c + 2\,\widehat{\nabla}_b F_c\right), \\ R_{\lambda\mu} &= \widetilde{R}_{\lambda\mu} + \tfrac{1}{2}\,\widehat{\nabla}_a f^a\,h_{\lambda\mu}, \qquad R_{\lambda b} = 0. \end{aligned} \tag{15}$$

Now we assume that g_{aa} and f have the form

$$g_{aa} = e_a\,(c - u^a)(u^2 - u^1), \qquad f = (c - u^1)(c - u^2), \tag{16}$$

where u^a is a function of q^a only, c is a constant and $e_a = \pm 1$, according to the signature.

As shown in my Tutorial Paper[2] for SPT-2004 (Appendix A, Theorem A.4.1), a metric of this kind admits an equivalent metric. Two metrics g and $\bar{g}$ on a manifold Q_n are said to be **equivalent** if they have the same unparametrized geodesics. A metric g admits an equivalent metric $\bar{g}$ iff it admits a (non-singular) J-tensor $\mathbf{J}$. A J-tensor is a torsionless trace-type conformal Killing tensor. If there exists a J-tensor $\mathbf{J}$ with eigenvalues (u_i), then there exist **standard coordinates** (q^i) such that the components of g and $\mathbf{J}$ assume a certain **standard form** and such that $u_i(q^i)$. This form depends on the multiplicity of the eigenvalues. The metric (14)-(16) is just one of the four possible cases, for $n = 4$, corresponding to the case in which two eigenvalues are simple and one is double. The equivalent metric $\bar{g}_{ij}$ can be computed by means of $\mathbf{J}$, following the rule (described in Ref.2):

$$\bar{g}_{ij} = \frac{1}{\mu^2}\,A_{ij}, \qquad \mu := \det[J^i_h], \qquad [A^i_h] = \mathrm{cof}\,[J^i_h],$$

where cof$[-]$ is the cofactor-matrix operator.

A crucial property of the geodesic equivalence is that the affine parameters t and $\bar{t}$ of two corresponding geodesics are related by the equation $dt/d\bar{t} = \mu$.

Hence, a spontaneous question arises: can Metric Equivalence Theory find any application in General Relativity? In other words, can an empty (i.e., Ricci-flat) space-time admit an equivalent metric?

We can give a first partial answer to this question by considering the metric (14)-(16) and by imposing the condition $R_{ij} = 0$: this metric is Ricci-flat if and only if

$$\widehat{R}_{bc} = \tfrac{1}{2}\left(F_b\,F_c + 2\,\widehat{\nabla}_b F_c\right), \qquad \widetilde{R}_{\lambda\mu} + \tfrac{1}{2}\,\widehat{\nabla}_a f^a\,h_{\lambda\mu} = 0. \tag{17}$$

It can be shown that for an orthogonal metric of the kind (16), $\widehat{R}_{bc} = 0$ for $b \neq c$. Then the first equation (17) is equivalent to

$$\widehat{\nabla}_a F_b = -\tfrac{1}{2}\,F_a\,F_b, \qquad a \neq b. \tag{18}$$

The second equation (17) has the form

$$\widetilde{R}_{\lambda\mu} = \tfrac{1}{2}\,\kappa\,h_{\lambda\mu}, \qquad \kappa := -\,\widehat{\nabla}_a f^a,$$

where $\widetilde{R}_{\lambda\mu}$ and $h_{\lambda\mu}$ are functions of the Greek coordinates, while κ is a function of the Roman coordinates. It follows that κ is a constant and the two-dimensional manifolds q^a = constant are Einstein manifolds. It can be shown that, while Eq. (18) is identically satisfied, the second equation (17) is satisfied iff (i) u^a = constant and (ii) $\kappa = 0$. This means that, (i) g_{aa} and f are constant and, (ii) the Ricci tensor is zero and the submanifolds q^a = constant are flat (since they have dimension 2). Hence: the metric (14)-(16) is Ricci-flat and admits an equivalent metric if and only if it is flat. This shows that geodesic equivalence does not occur in General Relativity, at least in the case considered here – one of the four possible cases. But it can be conjectured that this happens also for the remaining three cases. A complete discussion of this matter, with detailed calculation, will be available on my personal web-site.

Credits. Bibliographical references and more recent results about the Geodesic Equivalence Theory can be found in Ref.2 . I wish to thank L. Fatibene, of my Department, for testing this method of computing the curvature tensors by a software. In spite of several simplification routines of the output sheet, the final formulae were really cumbersome. On the contrary, the "by-hand" calculation has shown the advantage of several step-by-step significant simplifications, associated with a better understanding of the meaning of the written formulae.

References

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