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ALGEBRAIC CONSTRUCTION OF THE QUADRATIC FIRST INTEGRALS FOR A SPECIAL CLASS OF ORTHOGONAL SEPARABLE SYSTEMS.

SERGIO BENENTI*

Abstract. With the notion of L-pencil, based on the notion of L-tensor, we construct a new class of Stäckel systems such that the quadratic first integrals associated with the orthogonal separation are computed by a coordinate-independent algebraic process.

Key words. Completely integrable systems, separation of variables, algebraic computation of first integrals.

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1. Introduction.

1.1. Stäckel systems. A **Stäckel system** (S-system) on a Riemannian manifold (Q_n, g) is an orthogonal coordinate system (q^i) which allows the integration by (additive) separation of variables of the Hamilton-Jacobi equation of the geodesic flow. More precisely, a S-system is an equivalence class of such coordinates, being equivalent two coordinate systems related by a rescaling (i.e., by a coordinate transformation with a diagonal Jacobian matrix).

A celebrated theorem of Eisenhart, revised in [3, 4], shows that the existence of a S-system is equivalent to the (local) existence of a n-dimensional linear space of Killing two-tensors with common normal eigenvectors; we call such a space a **Killing-Stäckel space** (KS-space). This is equivalent to the existence of a complete system of quadratic first integrals in involution of the geodesic flow. Any separable coordinate system (q^i) associated with a KS-space is such that the differentials dq^i are common eigenforms.

It is also known that a KS-space is completely determined by one of its elements, called **characteristic Killing tensor** (although not unique) with normal eigenvectors and simple eigenvalues. However, if a characteristic tensor is given, the full KS-space can be determined by integrating a system of PDE's.

1.2. L-systems. There exists a special class of S-systems for which the whole KS-space can be generated by a single symmetric two-tensor $\mathbf{L}$ through a pure algebraic process. Such a tensor $\mathbf{L}$ is a *torsionless conformal Killing tensor with simple eigenvalues* and it has been called **L-tensor** [2, 3]. This process is based on the following theorem (see [1] for a proof and further details):

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Theorem 1.1. *Let* $\mathbf{L}$ *be a symmetric 2-tensor. The tensors*

$$(\mathbf{K}_a) = (\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_{n-1})$$

defined by the sequence

$$\begin{aligned} &\mathbf{K}_0 = \mathbf{I}, \\ &\mathbf{K}_a = \tfrac{1}{a}\,\mathrm{tr}\,(\mathbf{K}_{a-1}\mathbf{L})\,\mathbf{I} - \mathbf{K}_{a-1}\,\mathbf{L}, \quad 1 < a < n. \end{aligned} \tag{1.1}$$

form a basis of a KS-space if and only if $\mathbf{L}$ *is a L-tensor.*

Since we are on a Riemannian manifold, any symmetric 2-tensor (covariant or contravariant) can be interpreted as a $(1,1)$-tensor i.e., as a linear endomorphism on the space of vector fields or on the space of one-forms. In the recursive formula (1.1) all tensors must be interpreted in this sense. In particular, we observe that the identity operator $\mathbf{I}$ is the $(1,1)$-tensor associated with the metric tensor.

We call **L-sequence** (or **L-chain**) a sequence of the kind (1.1) and **L-system** a S-system having this property.

The geodesic flow of an asymmetric ellipsoid (Jacobi) as well as of any asymmetric hyperquadrics of a Euclidean space, are examples of L-systems. Within this framework, we can also deal with cofactor and bi-cofactor systems (see [5] and the bibliography therein).

1.3. L-pencils. Although L-systems form a very special class of S-systems, they can be used for defining other classes. This idea has been recently developed by Błaszak [6]. In the present note, I point out the existence of a further class of S-systems for which the KS-spaces can be constructed by an L-sequences. We call them **LP-systems**, since they are based on the notion of **L-pencil**:

Definition 1.1. *A* **L-pencil** *is a linear combination*

$$\mathbf{L}_m = \mathbf{M} + m\,\mathbf{N} \tag{1.2}$$

which is a L-tensor for all values of $m \in \mathbb{R}$*.*

If we compute the L-sequence (1.1) of a L-pencil (1.2), then each $\mathbf{K}_a(m)$ is a polynomial at most of degree a in the parameter m. It is clear that $\mathbf{K}_a(0)$ form a L-system and that all the coefficients of these polynomials are Killing tensors. On the other hand, since the Killing tensors generated by a L-sequence commute as linear operators and are in involution as first integrals, we get

Theorem 1.2. *Let* $\mathbf{H}_a$ *denote the coefficient of maximal degree in the parameter* m *of* $\mathbf{K}_a(m)$*. Then:* (i) *all tensors* $\mathbf{H}_a$ *are Killing tensors;* (ii) *they commute as linear operators,* $\mathbf{H}_a\mathbf{H}_b = \mathbf{H}_b\mathbf{H}_a$*;* (iii) *they commute in the Lie-Schouten brackets,* $[\mathbf{H}_a, \mathbf{H}_b] = 0$*.*

Item (iii) means that the quadratic functions on T^*Q associated with these tensors, $P_1 = H_1^{ij} p_i p_j$ and $P_2 = H_2^{ij} p_i p_j$, are in involution i.e., $\{P_1, P_2\} = 0$.

Definition 1.2. *We call* **effective** *a L-pencil for which the tensors* $\mathbf{H}_a$ *are linearly independent.*

In this case they form a KS-space. Starting from this theorem and this definition we can get the following two main results.

Theorem 1.3. *A L-pencil* $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{N}$ *is effective if and only if* $\mathbf{M}$ *is a L-tensor and* $\mathbf{N}$ *has the form* $\mathbf{N} = \mathbf{X} \otimes \mathbf{X}^\flat$, *where* $\mathbf{X}$ *is a conformal Killing vector field whose associated one-form* $\mathbf{X}^\flat$ *is closed,* $d\mathbf{X}^\flat = 0$.

Theorem 1.4. *Let* $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{X} \otimes \mathbf{X}^\flat$ *be an effective L-pencil. Then:* (i) *the CKV* $\mathbf{X}$ *is a translation or a dilatation;* (ii) *all tensors* $\mathbf{K}_a$ *of the L-sequence are linear in* m *i.e., of the form*

$$\mathbf{K}_a = \mathbf{K}_{a0} + m\,\mathbf{H}_a;$$

(iii) $\mathbf{X}$ *is an eigenvector with zero eigenvalues of all* $\mathbf{H}_a$, $\mathbf{H}_a\mathbf{X} = 0$; (iv) *the restrictions of the tensors* $\mathbf{H}_a$ *to any leaf of the foliation orthogonal to* $\mathbf{X}$ *form a L-system on that leaf.*

We call **LP-system** a Stäckel system generated by a L-pencil.

Remark 1.1. A fundamental example of LP-system, that inspired the definition of L-pencil and guided this research, is the asymmetric spherical-conical web in $\mathbb{R}^n$, where $\mathbf{M}$ is a symmetric constant matrix (note that in this case $\mathbf{M}$ is a Killing tensor) with distinct eigenvalues, and $\mathbf{X} = \mathbf{r}$ is the radius vector, whose components at a point $(x_1, \ldots, x_n)$ are just $(x_1, \ldots, x_n)$ [2, 3]. $\diamondsuit$

Remark 1.2. As shown by Theorem 1.4, LP-systems are of two types, according to the type of the vector $\mathbf{X}$: **dilatational** or **translational**. $\diamondsuit$

Remark 1.3. These two theorems have an intrinsic character: any L-sequence can be computed by using any suitable coordinate systems. However, they have a local meaning, not only because the global existence of special objects, like conformal Killing vectors (in particular, dilatations, etc.), is known to be impossible on certain kinds of Riemannian manifolds, but also because of the structure itself of a KS-space. For instance, the independence of the tensors $\mathbf{H}_a$ may occur in an open subset of Q, with the exception of a closed **singular set**. $\diamondsuit$

2. Differential conditions. The torsion $\mathbf{H}(\mathbf{A})$ of a $(1,1)$-tensor $\mathbf{A}$ is defined by

$$H^k_{ij}(\mathbf{A}) \doteq 2\big(A^h_{[i}\partial_{|h|}A^k_{j]} - A^k_m\partial_{[i}A^m_{j]}\big). \tag{2.1}$$

For the torsion the following **additive rule** holds

$$\mathbf{H}(\mathbf{A}+\mathbf{B}) = \mathbf{H}(\mathbf{A}) + \mathbf{H}(\mathbf{B}) + 2\,\mathbf{T}(\mathbf{A},\mathbf{B}). \tag{2.2}$$

where[1]

$$T^k_{ij}(\mathbf{A},\mathbf{B}) \doteq A^h_{[i}\partial_{|h|}B^k_{j]} - A^k_m\partial_{[i}B^m_{j]} + B^h_{[i}\partial_{|h|}A^k_{j]} - B^k_m\partial_{[i}A^m_{j]}. \tag{2.3}$$

[1] The tensor $\mathbf{T}(\mathbf{A},\mathbf{B})$ hab been introduced by Okubo, [12], formula 3.9.

In the definition (2.1), as well as in (2.3), the partial derivatives $\partial_i = \partial/\partial q^i$ can be replaced by the covariant derivatives ∇_i associated with *any symmetric linear connection* (in particular, the Levi-Civita connection):

$$
\begin{aligned}
&H^k_{ij}(\mathbf{A}) \doteq 2\big(A^h_{[i}\nabla_{|h|}A^k_{j]} - A^k_m\nabla_{[i}A^m_{j]}\big),\\
&T^k_{ij}(\mathbf{A},\mathbf{B}) = A^h_{[i}\nabla_{|h|}B^k_{j]} - A^k_m\nabla_{[i}B^m_{j]} + B^h_{[i}\nabla_{|h|}A^k_{j]} - B^k_m\nabla_{[i}A^m_{j]}.
\end{aligned}
$$

By applying the additive rule (2.2) to equation $\mathbf{H}(\mathbf{M} + m\,\mathbf{N}) = 0$ we get

THEOREM 2.1. *The tensor* $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{N}$ *is a L-pencil if and only if* (i) $\mathbf{M}$ *is a L-tensor,* (ii) $\mathbf{N}$ *is a torsionless CKT and* (iii)

$$
\mathbf{T}(\mathbf{M},\mathbf{N}) = 0. \tag{2.4}
$$

3. Algebraic conditions. Let us compute the first elements of the L-chain (1.1) for a L-pencil $\mathbf{L} = \mathbf{M} + m\,\mathbf{N}$. The first step of the L-chain (1.1) gives

$$
\begin{aligned}
\mathbf{K}_1 &= \operatorname{tr}\mathbf{L}_m\,\mathbf{I} - \mathbf{L}_m = (\operatorname{tr}\mathbf{M} + m\operatorname{tr}\mathbf{N})\,\mathbf{I} - \mathbf{M} - m\,\mathbf{N}\\
&= (\mu + m\,\nu)\,\mathbf{I} - \mathbf{M} - m\,\mathbf{N},
\end{aligned}
$$

where

$$
\mu \doteq \operatorname{tr}\mathbf{M}, \qquad \nu \doteq \operatorname{tr}\mathbf{N}.
$$

Hence,

$$
\mathbf{K}_1 = \mathbf{K}_{10} + m\,\mathbf{K}_{11} \quad \begin{cases} \mathbf{K}_{10} \doteq \mu\,\mathbf{I} - \mathbf{M},\\ \mathbf{K}_{11} \doteq \nu\,\mathbf{I} - \mathbf{N}, \end{cases} \tag{3.1}
$$

and

$$
\mathbf{H}_1 = \mathbf{K}_{11} = \nu\,\mathbf{I} - \mathbf{N} \tag{3.2}
$$

The second step gives

$$
\mathbf{K}_2 = \mathbf{K}_{20} + m\,\mathbf{K}_{21} + m^2\,\mathbf{K}_{22}, \tag{3.3}
$$

with

$$
\begin{aligned}
\mathbf{K}_{20} &\doteq \tfrac{1}{2}\,(\mu^2 - \operatorname{tr}\mathbf{M}^2)\,\mathbf{I} - \mu\,\mathbf{M} + \mathbf{M}^2,\\
\mathbf{K}_{21} &\doteq (\mu\nu - \operatorname{tr}\mathbf{MN})\,\mathbf{I} - \nu\mathbf{M} - \mu\,\mathbf{N} + \mathbf{MN} + \mathbf{NM},\\
\mathbf{K}_{22} &\doteq \tfrac{1}{2}\,(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2.
\end{aligned}
$$

REMARK 3.1. These equations shows that we have to deal with two types of L-systems,

$$\text{(3.4)} \qquad \textbf{Type-1}: \quad \mathbf{K}_{22} \doteq \tfrac{1}{2}(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2 = 0,$$

$$\text{(3.5)} \qquad \textbf{Type-2}: \quad \mathbf{K}_{22} \doteq \tfrac{1}{2}(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2 \neq 0.$$

These are algebraic conditions involving the tensor $\mathbf{N}$ only. For a type-1 L-pencil we have

$$\text{(3.6)} \qquad \mathbf{H}_2 = \mathbf{K}_{21} = (\mu\nu - \operatorname{tr}\mathbf{MN})\,\mathbf{I} - \nu\mathbf{M} - \mu\,\mathbf{N} + \mathbf{MN} + \mathbf{NM},$$

and for a type-2 L-pencil,

$$\begin{aligned} \mathbf{H}_2 = \mathbf{K}_{22} &= \tfrac{1}{2}(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2 \\ &= \tfrac{1}{2}\operatorname{tr}(\mathbf{H}_1\,\mathbf{N})\,\mathbf{I} - \mathbf{H}_1\,\mathbf{N}. \end{aligned}$$

This last expression compared with (3.2) shows that $\mathbf{H}_1$ and $\mathbf{H}_2$ are the first two elements of the L-chain generated by the torsionless tensor $\mathbf{N}$. Going back to Eq. (3.3) we observe that the coefficient of m^3 of

$$\mathbf{K}_3 = \tfrac{1}{3}\operatorname{tr}(\mathbf{K}_2\mathbf{L}_m)\,\mathbf{I} - \mathbf{K}_2\mathbf{L}_m$$

is the tensor

$$\mathbf{K}_{33} = \tfrac{1}{3}\operatorname{tr}(\mathbf{K}_{22}\,\mathbf{N})\,\mathbf{I} - \mathbf{K}_{22}\,\mathbf{N} = \tfrac{1}{3}\operatorname{tr}(\mathbf{H}_2\,\mathbf{N})\,\mathbf{I} - \mathbf{H}_2\,\mathbf{N}.$$

This is sufficient to show that *a type-2 L-pencil* $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{N}$ *generates the L-chain of* $\mathbf{N}$. This means that if $\mathbf{N}$ has distinct eigenvalues (i.e., it is a L-tensor) the L-pencil $\mathbf{L}_m$ generates nothing new. In the case of non-distinct eigenvalues the $\mathbf{N}$-chain generates a space of dimension $< n$ of Killing tensors (see Appendix 12.2 for an example). Hence, in both cases, *the type-2 L-pencils have no interest*, and hereafter we consider type-1 L-pencils only. $\diamondsuit$

4. The eigenvalues of N. Eq. (3.4) written in the form

$$\text{(4.1)} \qquad \mathbf{N}^2 - \nu\,\mathbf{N} = \tfrac{1}{2}(\operatorname{tr}\mathbf{N}^2 - \nu^2)\,\mathbf{I},$$

shows that

$$\operatorname{tr}\mathbf{N}^2 - \nu^2 = \tfrac{n}{2}(\operatorname{tr}\mathbf{N}^2 - \nu^2),$$

i.e., for $n > 2$, $\operatorname{tr}\mathbf{N}^2 = \nu^2 = (\operatorname{tr}\mathbf{N})^2$. Thus, from (4.1), $\mathbf{N}^2 = \nu\,\mathbf{N}$. Let ν_i be the eigenvalues of $\mathbf{N}$. Then the diagonalization of this equation[2] yields equation $\nu_i^2 = \nu\,\nu_i$, which is equivalent to

$$\nu_i \sum_{k\neq i} \nu_k = 0, \quad \forall\, i.$$

[2] We are in a pure algebraic context. If N_i^j are the components of $\mathbf{N}$ with respect to any basis, the tensor $N_{ik} = g_{jk}\,N_i^j$ is symmetric. Hence, N_{ij} and g_{ij} can be simultaneously diagonalized in a canonical basis.

Let us consider the simplest case $n = 3$. Then we have

$$\nu_1(\nu_2 + \nu_3) = 0, \quad \nu_2(\nu_3 + \nu_1) = 0, \quad \nu_3(\nu_1 + \nu_2) = 0.$$

Assume $\nu_2 \neq 0$ and $\nu_3 \neq 0$. Then,

$$\nu_1(\nu_2 + \nu_3) = 0, \quad \nu_3 = -\nu_1, \quad \nu_2 = -\nu_1.$$

If we replace the last two equations into the first one, we get $\nu_1 = 0$. Hence, $\nu_2 = \nu_3 = 0$: absurd.

For $n = 4$,

$$\begin{aligned}
&\nu_1(\nu_2 + \nu_3 + \nu_4) = 0, \\
&\nu_2(\nu_3 + \nu_4 + \nu_1) = 0, \\
&\nu_3(\nu_4 + \nu_1 + \nu_2) = 0, \\
&\nu_4(\nu_1 + \nu_2 + \nu_3) = 0.
\end{aligned}$$

Assume $\nu_3, \nu_4 \neq 0$:

$$\begin{aligned}
&\nu_1(\nu_2 + \nu_3 + \nu_4) = 0, \\
&\nu_2(\nu_3 + \nu_4 + \nu_1) = 0, \\
&\nu_4 = -\nu_1 - \nu_2, \\
&\nu_3 = -\nu_1 - \nu_2.
\end{aligned}$$

Replace the last two equations into the first two:

$$\begin{aligned}
&\nu_1(\nu_2 + 2\nu_1) = 0, \\
&\nu_2(2\nu_2 + \nu_1) = 0, \\
&\nu_4 = -\nu_1 - \nu_2, \\
&\nu_3 = -\nu_1 - \nu_2.
\end{aligned}$$

Take the difference of the first two equations:

$$\nu_1(\nu_2 + 2\nu_1) - (\nu_2(2\nu_2 + \nu_1)) = 2(\nu_1^2 + \nu_2^2) = 0.$$

This implies $\nu_1 = \nu_2 = 0$, and consequently $\nu_3 = \nu_4 = 0$: absurd. The above calculations can be extended to any dimension n. The result is

Proposition 4.1. *In a L-pencil the tensor* $\mathbf{N}$ *cannot have two eigenvalues different from zero.*

Since the case $\mathbf{N} = 0$ is excluded, we have proved

Proposition 4.2. *In a L-pencil the tensor* $\mathbf{N}$ *has only one eigenvalue different from zero.*

It follows that if $\mathbf{U}$ is a unit eigenvector corresponding to the non-zero eigenvalue ν_1, then $\mathbf{N} = \nu_1 \mathbf{U} \otimes \mathbf{U}^\flat$, i.e., $N_i^j = \nu_1\, U_i U^j$. It is not restrictive to assume hereafter

$$\nu_1 > 0.$$

This corresponds to replace $\mathbf{N}$ by $-\mathbf{N}$, or m by $-m$, in the L-pencil. Hence, if we introduce the vector field

$$\mathbf{X} \doteq \sqrt{\nu_1}\,\mathbf{U},$$

then it is proved that

Theorem 4.1. *In a L-pencil $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{N}$, the tensor $\mathbf{N}$ has the form*

$$\mathbf{N} = \mathbf{X} \otimes \mathbf{X}^\flat, \quad N_i^j = X_i X^j,$$

and

$$\nu \doteq \operatorname{tr} \mathbf{N} = \nu_1 = \mathbf{X} \cdot \mathbf{X}.$$

5. The differential properties of the vector X. The aim of this section is to prove

Theorem 5.1. *The vector field $\mathbf{X}$ is a dilatation or a translation.*

Let us recall that a **conformal Killing vector** (CKV) is a vector field $\mathbf{X}$ on a Riemannian manifold characterized by equation

$$\{P(\mathbf{X}), P(\mathbf{G})\} = \psi\, P(\mathbf{G}), \tag{5.1}$$

where $P(\mathbf{X}) \doteq X^i p_i$, $P(\mathbf{G}) = g^{ij}\, p_i p_j$, and f a scalar field on the manifold Q. If $\psi = 0$, $\mathbf{X}$ is a **Killing vector**. If $\psi =$ constant $\neq 0$, $\mathbf{X}$ is a **dilatation**. If $\psi = 0$ and $\mathbf{X} \cdot \mathbf{X} =$ constant, $\mathbf{X}$ is a **translation**.[3]

To prove Theorem 5.1 we need preliminary statements.

Theorem 5.2. *The vector field $\mathbf{X}$ is a conformal Killing vector.*

Proof. A conformal Killing tensor (CKT) $\mathbf{N}$ is characterized by equation

$$\{P(\mathbf{N}), P(\mathbf{G})\} = -2\, P(\mathbf{C}) P(\mathbf{G})$$

where $\mathbf{C}$ is a vector field and $P(\mathbf{C}) = C^i p_i$. Being $\mathbf{N} = \mathbf{X} \otimes \mathbf{X}^\flat$ a CKT, and $P(\mathbf{N}) = (X^i p_i)^2 = P^2(\mathbf{X})$, equation

$$\{P^2(\mathbf{X}), P(\mathbf{G})\} = -2\, P(\mathbf{C}) P(\mathbf{G}) \tag{5.2}$$

[3] An *infinitesimal translation* is a Killing vector with constant length or, equivalently, a vector field whose integral curves are geodesics [9], §72.

holds. However,

$$\{P^2(\mathbf{X}), P(\mathbf{G})\} = 2P(\mathbf{X})\,\{P(\mathbf{X}), P(\mathbf{G})\}$$

so that Eq. (5.2) becomes

$$P(\mathbf{X})\,\{P(\mathbf{X}), P(\mathbf{G})\} = -\,P(\mathbf{C})P(\mathbf{G}).$$

This polynomial equation shows that $P(\mathbf{X}) = -\,\psi\,P(\mathbf{C})$, where ψ is a function on Q, and consequently that Eq. (5.1) holds. □

THEOREM 5.3. *The one-form $\mathbf{X}^\flat = (X_i)$ is closed, $\partial_i X_j = \partial_j X_i$.*

Proof. This property is due to the torsionless condition $\mathbf{H}(\mathbf{N}) = \mathbf{H}(\mathbf{X}\otimes\mathbf{X}^\flat) = 0$. Indeed, by the definition of torsion (2.1), the tensor

$$S^k_{ij}(\mathbf{N}) \doteq N^h_i \nabla_h N^k_j - N^k_m \nabla_i N^m_j$$

must be symmetric in the lower indices. For $N^j_i = X_i X^j$ we have

$$\begin{aligned} S^k_{ij} &= X^h X_i \nabla_h (X_j X^k) - X_m X^k \nabla_i (X_j X^m) \\ &= X_i X^k X^h\,\nabla_h X_j + X_i X_j X^h\,\nabla_h X^k - X_m X^m X^k\,\nabla_i X_j \\ &\quad - X^k X_j X_m\,\nabla_i X^m \\ &= X^k\,[X_i X^h\,\nabla_h X_j - \nu_1\,\nabla_i X_j - X_j X^m\,\nabla_i X_m] + \dots \end{aligned}$$

where $\dots$ denote terms symmetric in the indices i, j. Now we recall that a CKV is also characterized by equation

$$\nabla_i X_j + \nabla_j X_i = \psi\, g_{ij}. \tag{5.3}$$

Thus,

$$\begin{aligned} S^k_{ij}(\mathbf{N}) &= X^k\,[X_i X^h\,\nabla_h X_j - \nu_1\,\nabla_i X_j + X_j X^m\,(\nabla_m X_i - \psi\, g_{im}] + \dots \\ &= -\,\nu_1\, X^k\,\nabla_i X_j. \end{aligned}$$

This shows that the torsionless condition is equivalent to $\nabla_i X_j - \nabla_j X_i = \partial_i X_j - \partial_j X_i = 0$. □

By summing this last equation to Eq. (5.3) we get ($\kappa \doteq \psi/2$)

$$\nabla_i X_j = \kappa\, g_{ij}. \tag{5.4}$$

Moreover, There exist local functions ρ such that

$$\mathbf{X} = \nabla\rho, \qquad X_i = \partial_i \rho. \tag{5.5}$$

As a consequence, we have proved that

LEMMA 5.1. *The vector field* $\mathbf{X}$ *is* **normal**.

Recall that a vector field $\mathbf{X}$ is called **normal** if it is orthogonal to a local web of submanifolds of codimension 1.[4] In this case the leaves of the web (which we call **X-web**) are defined by equation $\rho =$ constant. As a consequence of this fact, there exist local coordinates $(q^i) = (q^1, q^a)$ such that $q^1 = \rho$ and $g^{1a} = 0$. We call them **X-coordinates**. From Eq. (5.5) it follows that in these coordinates

$$X_1 = 1, \quad X_a = 0. \tag{5.6}$$

Hence,

$$\begin{gathered} X^1 = g^{1i}\, X_i = g^{11}, \qquad X^a = 0, \qquad g_{1a} = 0, \\ \nu_1 = X^i X_i = X^1, \qquad \mathbf{X} = \nu_1\, \partial_1. \end{gathered} \tag{5.7}$$

Now we prove Theorem 5.1.

Proof. In any coordinate system Eq. (5.1) reads

$$\{X^i\, p_i, g^{hk}\, p_h p_k\} = \psi\, g^{hk}\, p_h p_k. \tag{5.8}$$

In $\mathbf{X}$-coordinates,

$$\{X^1\, p_1, g^{11}\, p_1^2 + g^{ab}\, p_a p_b\} = \psi\,(g^{11}\, p_1^2 + g^{ab}\, p_a p_b),$$

$$\Longrightarrow \quad \begin{aligned} & X^1\,(\partial_1 g^{11}\, p_1^2 + \partial_1 g^{ab}\, p_a p_b) - \partial_i X^1 p_1 \partial^i (g^{11} p_1^2 + g^{ab}\, p_a p_b) \\ & = \psi\,(g^{11} p_1^2 + g^{ab} p_a p_b), \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} & X^1\,(\partial_1 g^{11}\, p_1^2 + \partial_1 g^{ab}\, p_a p_b) - 2\, p_1^2 \partial_1 X^1\, g^{11} - 2\, p_1\, \partial_a X^1\, g^{ab}\, p_b \\ & = \psi\,(g^{11}\, p_1^2 + g^{ab}\, p_a p_b). \end{aligned}$$

This polynomial equation in the momenta is equivalent to equations

$$\begin{aligned} & X^1\, \partial_1 g^{11} - 2\, g^{11} \partial_1 X^1 = \psi\, g^{11}, \\ & \partial_a X^1 = 0, \\ & X^1\, \partial_1 g^{ab} = \psi\, g^{ab}. \end{aligned} \tag{5.9}$$

Recall that $X^1 = g^{11} = \nu_1$. The first equation shows that

$$\psi = -\, \partial_1 \nu_1, \tag{5.10}$$

and the second one that ν_1 depends on the coordinate $q^1 = \rho$ only.

$$\nu_1 = \nu_1(q^1).$$

Eq. (5.10) shows that $\mathbf{X}$ is a KV i.e., $\psi = 0$ if and only if $\nu_1 =$ constant. Since $\nu_1 = \mathbf{X} \cdot \mathbf{X}$, if $\mathbf{X}$ is a KV, then it is a translation. In the case of

[4] It it also called **surface-forming** of **orthogonally integrable**.

a non-constant ν_1, it is convenient to replace the coordinate q^1 by a new coordinate r such that $\mathbf{X} = r\,\partial_r$ and $\partial_r \cdot \partial_r = 1$. These two conditions imply $\mathbf{X}\cdot\mathbf{X} = \nu_1 = r^2$. Hence, the coordinate transformation can be defined by

$$r = \sqrt{\nu_1(q^1)}.$$

Since

$$\mathbf{X} = r\,\partial_r = \nu_1 \partial_1, \tag{5.11}$$

we get equation

$$r\,\partial_r \nu_1 = \nu_1 \partial_1 \nu_1. \tag{5.12}$$

It follows that $\psi = -\partial_1 \nu_1 = -r\,\partial_r \log \nu_1 = -r\,\partial_r \log r^2 = -2$. This proves that $\mathbf{X}$ is a dilatation. □

Remark 5.1. Note that the third equation (5.9) is a further property of the vector field $\mathbf{X}$ not considered in Theorem 5.1. Due to (5.10) it can be written

$$\partial_1 g^{ab} = -\partial_1 \log \nu_1 \; g^{ab}.$$

It represents a law of evolution of the metric components g^{ab} along the flow of $\mathbf{X} = \nu_1 \partial_1$. It can be written

$$\partial_1 g^{ab} = \phi_1 \; g^{ab} \tag{5.13}$$

where $\phi_1 \doteq -\partial_1 \log \nu_1$ is a function of q^1 only. Eq. (5.13) is equivalent to

$$\partial_1 g_{ab} = \partial_1 \log \nu_1 \; g_{ab}, \tag{5.14}$$

and, for $g^{ab} \neq 0$, to

$$\partial_1 \log g^{ab} = \phi_1(q^1). \tag{5.15}$$

Note that the right hand side does not depend on the indices a, b. ♢

Remark 5.2. If $\nu_1 =$ constant, $\mathbf{X}$ is a translation, Eq. (5.13) reduces to $\partial_1 g^{ab} = 0$. If $\mathbf{X}$ is a dilatation, then from Eq. (5.12) we get

$$\partial_1 \log \nu_1 = \frac{r}{\nu_1}\,\partial_r \log \nu_1 = \frac{r}{r^2}\,\partial_r \log r^2 = \frac{2}{r^2}.$$

Moreover, due to Eq. (5.11), $\nu_1 \partial_1 r = r$, so that

$$\partial_1 g^{ab} = \partial_1 r\,\partial_r g^{ab} = \frac{r}{\nu_1}\,\partial_r g^{ab} = \frac{1}{r}\,\partial_r g^{ab},$$

and Eq. (5.13) becomes

$$\partial_r g^{ab} = -\frac{2}{r}\,g^{ab}. \qquad ♢$$

6. CKV in orthogonal coordinates. In orthogonal coordinates the characteristic equation (5.1) of a CKV reads

$$(X^i\,\partial_i g^{hh} - \psi\, g^{hh})\,p_h^2 - 2\,\partial_k X^h\ g^{kk}\,p_h p_k = 0. \tag{6.1}$$

The diagonal part of this equation gives

$$X^i\,\partial_i g^{hh} - \psi\, g^{hh} - 2\,\partial_h X^h\ g^{hh} = 0, \quad X^i\,\partial_i \log g^{hh} - \psi - 2\,\partial_h X^h = 0.$$

Hence,

$$\psi = \textstyle\sum_i\ X^i\,\partial_i \log g^{hh} - 2\,\partial_h X^h. \tag{6.2}$$

Note that this last equation holds for any choice of the index h, which is not summed. The non-diagonal part of Eq. (6.1) gives

$$g^{ii}\partial_i X^j + g^{jj}\partial_j X^i = 0, \quad j \neq i. \tag{6.3}$$

PROPOSITION 6.1. *If $\partial_i X_j = \partial_j X_i$, then Eq.* (6.3) *is equivalent to*

$$2\,\partial_i X_j + X_j \partial_i \log g^{jj} + X_i \partial_j \log g^{ii} = 0. \tag{6.4}$$

Proof. Let us translate Eq. (6.3) in covariant components of $\mathbf{X}$,

$$g_{jj}\partial_i(g^{jj}X_j) + g_{ii}\partial_j(g^{ii}X_i) = 0.$$

It follows that

$$X_j\partial_i \log g^{jj} + \partial_j X_i + X_i \partial_j \log g^{ii} + \partial_i X_j = 0.$$

Since $\partial_i X_j = \partial_j X_i$, we get Eq. (6.4). ∎

PROPOSITION 6.2. *Eq.* (6.2)*is equivalent to*

$$\begin{aligned}\psi &= \textstyle\sum_i\ X^i\,\partial_i \log g^{hh} - 2\,X^h\,\partial_h \log g^{hh} - 2\,g^{hh}\,\partial_h X_h \\ &= \textstyle\sum_{i\neq h}\ X^i\,\partial_i \log g^{hh} - X^h\,\partial_h \log g^{hh} - 2\,g^{hh}\,\partial_h X_h.\end{aligned} \tag{6.5}$$

Proof. Let us use to Eq. (6.2),

$$\begin{aligned}\psi &= \textstyle\sum_i\ X^i\,\partial_i \log g^{hh} - 2\,\partial_h X^h \\ &= \textstyle\sum_i\ X^i\,\partial_i \log g^{hh} - 2\,\partial_h(g^{hh}\,X_h) \\ &= \textstyle\sum_i\ X^i\,\partial_i \log g^{hh} - 2\,X_h\,\partial_h g^{hh} - 2\,g^{hh}\,\partial_h X_h \\ &= \text{etc.}\end{aligned}$$

∎

7. The condition $\mathbf{T}(\mathbf{M},\mathbf{N}) = 0$. In this section we prove the remarkable (and rather surprising) fact: *the differential condition* $\mathbf{T}(\mathbf{M},\mathbf{N}) = 0$ *is identically satisfied*. For this we shall use another special kind of coordinates associated with a L-pencil. Since $\mathbf{M}$ is a L-tensor, after the results of [5], there exist local coordinate systems, which we call **M-coordinates** in the following, such that

$$\begin{aligned} & g^{ij} = M^{ij} = 0, \quad i \neq j, \\ & M^{ii} = \mu^i \, g^{ii}, \\ & M_i^j = \mu^i \, \delta_i^j = \mu^j \, \delta_i^j, \end{aligned} \tag{7.1}$$

$$\begin{aligned} & \partial_i \mu^j = 0, \quad i \neq j, \\ & \partial_i \mu^i = (\mu^j - \mu^i) \partial_i \log g^{jj}, \end{aligned} \tag{7.2}$$

where the eigenvalues μ^i of $\mathbf{M}$ are all distinct. Note that the $\mathbf{M}$-coordinates are orthogonal and separable.

Due to the second equation (7.2), in $\mathbf{M}$-coordinates Eq. (6.4) reads

$$2\,\partial_i X_j = \, - X_j \, \partial_i \log g^{jj} - X_i \, \partial_j \log g^{ii} = -X_j \frac{\partial_i \mu^i}{\mu^j - \mu^i} - X_i \frac{\partial_j \mu^j}{\mu^i - \mu^j}.$$

Hence,

$$2\,\partial_i X_j = \frac{X_j \, \partial_i \mu^i - X_i \, \partial_j \mu^j}{\mu^i - \mu^j}. \tag{7.3}$$

Now we prove the following general statement.

THEOREM 7.1. *If* $\mathbf{M}$ *is a L-tensor and* $\mathbf{X}$ *is a CKV such that* $\mathbf{X}^\flat$ *is closed, then* $\mathbf{T}(\mathbf{M}, \mathbf{X} \otimes \mathbf{X}^\flat) = 0$.

LEMMA 7.1. *In* $\mathbf{M}$*-coordinates condition* $\mathbf{T}(\mathbf{M},\mathbf{N}) = 0$ *is equivalent to equations*

$$(\mu^i - \mu^k) \, \partial_i N_j^k = (\mu^j - \mu^k) \, \partial_j N_i^k, \qquad i, j, k \neq, \tag{7.4}$$

$$(\mu^j - \mu^i) \, \partial_j N_i^i + N_j^i \partial_i \mu^i = 0, \qquad i \neq j. \tag{7.5}$$

Proof. Due to the definition (2.3) this condition is equivalent to equation (of course with $i \neq j$)

$$\begin{aligned} & M_i^h \partial_h N_j^k - M_m^k \partial_i N_j^m + N_i^h \partial_h M_j^k - N_m^k \partial_i M_j^m \\ & = M_j^h \partial_h N_i^k - M_m^k \partial_j N_i^m + N_j^h \partial_h M_i^k - N_m^k \partial_j M_i^m, \end{aligned} \tag{7.6}$$

sum over h and m, in any coordinate systems. In M-coordinates, due to the third line of (7.1), this equation becomes

$$\begin{aligned} & M_i^i \partial_i N_j^k - M_k^k \partial_i N_j^k + N_i^h \partial_h M_j^k - N_j^k \partial_i M_j^j \\ & = M_j^j \partial_j N_i^k - M_k^k \partial_j N_i^k + N_j^h \partial_h M_i^k - N_i^k \partial_j M_i^i, \end{aligned} \qquad i \neq j,$$

and consequently

$$\begin{aligned}&\mu^i \partial_i N_j^k - \mu^k \partial_i N_j^k + N_i^h \partial_h M_j^k - N_j^k \partial_i \mu^j \\ &= \mu^j \partial_j N_i^k - \mu^k \partial_j N_i^k + N_j^h \partial_h M_i^k - N_i^k \partial_j \mu^i,\end{aligned} \qquad i \neq j.$$

We get a further simplification by the first line of (7.2):

$$(\mu^i - \mu^k)\, \partial_i N_j^k + N_i^h \partial_h M_j^k = (\mu^j - \mu^k)\, \partial_j N_i^k + N_j^h \partial_h M_i^k,$$

for $i \neq j$. For $i, j, k \neq$, taking into account (7.1) we get Eq. (7.4). For $k = i \neq j$,

$$0 = (\mu^j - \mu^i)\, \partial_j N_i^i + N_j^h \partial_h M_i^i,$$

and also Eq. (7.5) is proved. ☐

Let us apply this result to the case $N_i^j = X_i X^j$.

Lemma 7.2. *The compatibility condition* $\mathbf{T}(\mathbf{M}, \mathbf{X} \otimes \mathbf{X}^\flat) = 0$ *is equivalent to the cyclic equation*

$$\begin{aligned}&X_k\,(X_j\, \partial_i \mu^i - X_i\, \partial_j \mu^j) + X_i\,(X_k\, \partial_j \mu^j - X_j\, \partial_k \mu^k) \\ &+ X_j\,(X_i\, \partial_k \mu^k - X_k\, \partial_i \mu^i) = 0, \qquad i, j, k \neq .\end{aligned} \tag{7.7}$$

Proof. For $N_i^j = X_i X^j$, Eqs. (7.4) and (7.5) read

$$(\mu^i - \mu^k)\, \partial_i (X_j X^k) - (\mu^j - \mu^k)\, \partial_j (X_i X^k) = 0, \tag{7.8}$$

$$(\mu^j - \mu^i)\, \partial_j (X_i X^i) + X^i X_j\, \partial_i \mu^i = 0, \tag{7.9}$$

respectively, for all distinct indices i, j, k. (i) Since

$$\begin{aligned}\partial_j (X_i X^i) &= X_i\, \partial_j X^i + X^i \partial_j X_i = X_i\, \partial_j (g^{ii} X_i) + X^i \partial_j X_i \\ &= (X_i)^2\, \partial_j g^{ii} + X_i g^{ii} \partial_j X_i + X^i \partial_j X_i \\ &= X^i\, (X_i\, \partial_j \log g^{ii} + 2\, \partial_j X_i),\end{aligned}$$

Eq. (7.9) becomes

$$(\mu^j - \mu^i)\, (X_i\, \partial_j \log g^{ii} + 2\, \partial_j X_i) + X_j\, \partial_i \mu^i = 0.$$

This implies

$$(\mu^j - \mu^i)\, (X_i\, \frac{\partial_j \mu^j}{\mu^i - \mu^j} + 2\, \partial_j X_i) + X_j\, \partial_i \mu^i = 0,$$

i.e.,

$$2\, (\mu^j - \mu^i)\, \partial_j X_i - X_i \partial_j \mu^j + X_j\, \partial_i \mu^i = 0.$$

This equation is identically satisfied, due to (7.3). (ii) Eq. (7.8) can be written

$$(\mu^i - \mu^k)\,[X^k\,\partial_i X_j + X_j\,\partial_i X^k] - (\mu^j - \mu^k)\,[X^k\,\partial_j X_i + X_i\,\partial_j X^k] = 0.$$

Since $\partial_i X_j = \partial_j X_i$, we get

$$(\mu^i - \mu^j)\,X^k\,\partial_i X_j + (\mu^k - \mu^j)\,X_i\,\partial_j X^k + (\mu^i - \mu^k) X_j\,\partial_i X^k = 0.$$

However,

$$\begin{aligned}\partial_i X^k \quad &= \partial_i(g^{kk} X_k) = g^{kk}\,\partial_i X_k + X_k\,\partial_i g^{kk}\\ &= g^{kk}\,\partial_i X_k + X^k\,\partial_i \log g^{kk}\\ &= g^{kk}\,\partial_i X_k + X^k\,\tfrac{\partial_i\mu^i}{\mu^k-\mu^i},\end{aligned}$$

and we get equation

$$\begin{aligned}&(\mu^i - \mu^j)\,X^k\,\partial_i X_j + (\mu^k - \mu^j) X_i\,\left(g^{kk}\,\partial_j X_k + X^k\,\tfrac{\partial_j\mu^j}{\mu^k-\mu^j}\right)\\ &+ (\mu^i - \mu^k) X_j\,\left(g^{kk}\,\partial_i X_k + X^k\,\tfrac{\partial_i\mu^i}{\mu^k-\mu^i}\right) = 0,\end{aligned}$$

which can be put in the form

$$\begin{aligned}&g^{kk}\,\big((\mu^i - \mu^j)\,X_k\,\partial_i X_j + (\mu^k - \mu^j) X_i\,\partial_j X_k + (\mu^i - \mu^k) X_j\,\partial_i X_k\big)\\ &+ X^k\,(X_i\,\partial_j\mu^j - X_j\,\partial_i\mu^i) = 0,\end{aligned}$$

Due to (7.3),

$$2\,\partial_i X_j = \frac{X_j\,\partial_i\mu^i - X_i\,\partial_j\mu^j}{\mu^i - \mu^j},$$

we get

$$\begin{aligned}&g^{kk}[(\mu^i - \mu^j)\,X_k\,\partial_i X_j + (\mu^k - \mu^j) X_i\,\partial_j X_k + (\mu^i - \mu^k) X_j\,\partial_i X_k]\\ &+ 2\,X^k\,(\mu^j - \mu^i)\,\partial_i X_j = 0,\end{aligned}$$

and finally

$$(\mu^j - \mu^i)\,X_k\,\partial_i X_j + (\mu^k - \mu^j) X_i\,\partial_j X_k + (\mu^i - \mu^k) X_j\,\partial_i X_k = 0.$$

This is an equivalent form of the cyclic equation (7.7). ☐

Proof of Theorem 7.1.

Proof. If we apply Eq. (7.3), $2\,(\mu^i - \mu^j)\,\partial_i X_j = X_j\,\partial_i\mu^i - X_i\,\partial_j\mu^j$, to the cyclic equation (7.7), we see that it is identically satisfied. ☐

Let us examine equation $\mathbf{T}(\mathbf{M}, \mathbf{X}\otimes\mathbf{X}^\flat) = 0$ in $\mathbf{X}$-coordinates.

THEOREM 7.2. *In* $\mathbf{X}$*-coordinates equation* $\mathbf{T}(\mathbf{M}, \mathbf{X}\otimes\mathbf{X}^\flat) = 0$ *is equivalent to the following equations,*

$$\partial_i M^1_j = \partial_j M^1_i, \quad i, j \neq 1, \tag{7.10}$$

$$\partial_1 M_j^k = 0, \quad j, k \neq 1, \tag{7.11}$$

$$M_j^1 \partial_1 \nu_1 = \nu_1 \partial_j M_1^1, \quad j \neq 1. \tag{7.12}$$

Proof. In any coordinate system, equation $\mathbf{T}(\mathbf{M}, \mathbf{X} \otimes \mathbf{X}^\flat) = 0$ is equivalent to Eq. (7.6), with $N_i^j = X_i X^j$:

$$\begin{aligned} &M_i^h \partial_h (X_j X^k) - M_m^k \partial_i (X_j X^m) \\ &+ X^h X_i \partial_h M_j^k - X_m X^k \partial_i M_j^m \\ &- M_j^h \partial_h (X_i X^k) + M_m^k \partial_j (X_i X^m) \\ &- X^h X_j \partial_h M_i^k + X_m X^k \partial_j M_i^m = 0. \end{aligned} \tag{7.13}$$

In $\mathbf{X}$-coordinates $X^1 = \nu_1$ and $X_1 = 1$ are the only non-vanishing components of $\mathbf{X}$, and they depend on the coordinate q^1 only. Thus, Eq. (7.13) becomes

$$\begin{aligned} &M_i^h \partial_h (X_j X^k) - M_1^k \partial_i (X_j X^1) \\ &+ X^1 X_i \partial_1 M_j^k - X_1 X^k \partial_i M_j^1 \\ &- M_j^h \partial_h (X_i X^k) + M_1^k \partial_j (X_i X^1) \\ &- X^1 X_j \partial_1 M_i^k + X_1 X^k \partial_j M_i^1 = 0. \end{aligned} \tag{7.14}$$

For $i, j \neq 1$, Eq. (7.14) reduces to

$$- X_1 X^k \partial_i M_j^1 + X_1 X^k \partial_j M_i^1 = 0.$$

This proves Eq. (7.10). For $i = 1$ Eq. (7.14) reduces to

$$\begin{aligned} &M_1^h \partial_h (X_j X^k) - M_1^k \partial_1 (X_j X^1) + X^1 X_1 \partial_1 M_j^k - X_1 X^k \partial_1 M_j^1 \\ &- M_j^h \partial_h (X_1 X^k) + M_1^k \partial_j (X_1 X^1) - X^1 X_j \partial_1 M_1^k + X_1 X^k \partial_j M_1^1 = 0. \end{aligned}$$

For $j = 1$ this equation is of course identically satisfied. For $j \neq 1$ it reads

$$\begin{aligned} &X^1 X_1 \partial_1 M_j^k - X_1 X^k \partial_1 M_j^1 - M_j^h \partial_h (X_1 X^k) + M_1^k \partial_j (X_1 X^1) \\ &+ X_1 X^k \partial_j M_1^1 = 0, \end{aligned}$$

Now we recall that $\nu_1(q^1) = X_1 X^1 > 0$:

$$\nu_1 \partial_1 M_j^k - X_1 X^k \partial_1 M_j^1 - M_j^h \partial_h (X_1 X^k) + X_1 X^k \partial_j M_1^1 = 0.$$

For $k \neq 1$ and $k = 1$ we find Eqs. (7.11) and (7.12), respectively. ∎

8. Further algebraic properties. Let us return to the explicit calculation of the first elements of a L-pencil. Of course, $\mathbf{K}_1$ is of first degree, Eq. (3.1),

$$\mathbf{K}_1 = \mu\,\mathbf{I} - \mathbf{M} + m\,(\nu\,\mathbf{I} - \mathbf{N}),$$

as well as $\mathbf{K}_2$, as shown in Section 3,

$$\mathbf{K}_2 = \mathbf{K}_{20} + m\,\mathbf{K}_{21},$$

with

$$\begin{aligned}\mathbf{K}_{20} &\doteq \tfrac{1}{2}\,(\mu^2 - \operatorname{tr}\mathbf{M}^2)\,\mathbf{I} - \mu\,\mathbf{M} + \mathbf{M}^2,\\ \mathbf{K}_{21} &\doteq (\mu\nu - \operatorname{tr}\mathbf{MN})\,\mathbf{I} - \nu\mathbf{M} - \mu\,\mathbf{N} + \mathbf{MN} + \mathbf{NM}.\end{aligned}$$

Let us introduce the vector field

$$\mathbf{Y} \doteq \mathbf{M}\,\mathbf{X}.$$

Then Eqs. (3.6) become

$$\begin{aligned}\mathbf{H}_1 &= \nu_1\,\mathbf{I} - \mathbf{X}\otimes\mathbf{X}^\flat,\\ \mathbf{H}_2 &= (\mu\nu_1 - \mathbf{Y}\cdot\mathbf{X})\,\mathbf{I} - \nu_1\mathbf{M} - \mu\,\mathbf{X}\otimes\mathbf{X}^\flat + \mathbf{MN} + \mathbf{NM}.\end{aligned}$$

In any coordinate system,

$$\begin{aligned}(\mathbf{MN})_i^j &= M_k^j N_i^k = M_k^j X^k X_i = Y^j X_i,\\ (\mathbf{NM})_i^j &= N_k^j M_i^k = X^j X_k M_i^k = X^j Y_i.\end{aligned}$$

Hence,

$$\mathbf{MN} = \mathbf{Y}\otimes\mathbf{X}^\flat, \qquad \mathbf{NM} = \mathbf{X}\otimes\mathbf{Y}^\flat, \qquad \operatorname{tr}(\mathbf{MN}) = \mathbf{X}\cdot\mathbf{Y}.$$

This shows that

PROPOSITION 8.1. *In a L-pencil the two tensors $\mathbf{H}_1$ and $\mathbf{H}_2$ have the following expressions:*

$$\begin{aligned}\mathbf{H}_1 &= \nu_1\,\mathbf{I} - \mathbf{X}\otimes\mathbf{X}^\flat,\\ \mathbf{H}_2 &= (\mu\nu_1 - \mathbf{Y}\cdot\mathbf{X})\,\mathbf{I} - \nu_1\mathbf{M} - \mu\,\mathbf{X}\otimes\mathbf{X}^\flat + \mathbf{Y}\otimes\mathbf{X}^\flat + \mathbf{X}\otimes\mathbf{Y}^\flat.\end{aligned} \tag{8.1}$$

As a consequence,

PROPOSITION 8.2. *The vector field $\mathbf{X}$ is an eigenvector of both $\mathbf{H}_1$ and $\mathbf{H}_2$, with zero eigenvalue.*

Proof. Since $\mathbf{X}\cdot\mathbf{X} = \nu_1$, we have $\mathbf{H}_1\mathbf{X} = \nu_1\,\mathbf{X} - \mathbf{X}\nu_1 = 0$, and $\mathbf{H}_2\mathbf{X} = (\mu\nu_1 - \mathbf{Y}\cdot\mathbf{X})\,\mathbf{X} - \nu_1\mathbf{Y} - \mu\,\nu_1\mathbf{X} + \nu_1\,\mathbf{Y} + \mathbf{Y}\cdot\mathbf{X}\,\mathbf{X} = 0$. ∎

REMARK 8.1. In any coordinate system Eqs. (8.1) read

$$\begin{aligned}(\mathbf{H}_1)_i^j &= \nu_1\,\delta_i^j - X_i X^j,\\ (\mathbf{H}_2)_i^j &= (\mu\nu_1 - \eta)\,\delta_i^j - \nu_1\,M_i^j - \mu\,X_i X^j + X_i Y^j + Y_i X^j,\end{aligned} \tag{8.2}$$

where $\eta \doteq \mathbf{Y} \cdot \mathbf{X} = X^i Y_i$. Recall that in $\mathbf{X}$-coordinates Eqs. (5.6) and (5.7) hold. so that

$$\begin{aligned}
&Y^1 = g^{11} Y_1 = M_i^1 X^i = M_1^1 X^1 = \nu_1 M_1^1, \\
&\eta = X_i Y^i = X_1 Y^1 = \nu_1 M_1^1 = Y^1, \\
&Y^a = M_i^a X^i = \nu_1 M_1^a, \\
&Y_a = M_{ai} X^i = \nu_1 M_{a1} = g^{11} M_{a1} = M_a^1.
\end{aligned}$$

Hence, from Eqs. (8.2) it follows that $(a, b = 2, \dots, n)$

$$\begin{aligned}
&(\mathbf{H}_1)_1^1 = \nu_1 - X_1 X^1 = 0, \\
&(\mathbf{H}_2)_1^1 = (\mu\nu_1 - \eta) - \nu_1 M_1^1 - \mu X_1 X^1 + X_1 Y^1 + Y_1 X^1, \\
&= (\mu\nu_1 - Y^1) - \nu_1 M_1^1 - \mu \nu_1 + Y^1 + Y_1 g^{11} = 0.
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{H}_1)_1^a = 0, \\
&(\mathbf{H}_1)_a^1 = 0, \\
&(\mathbf{H}_1)_a^b = \nu_1 \delta_a^b.
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{H}_2)_1^a = \ -\nu_1 M_1^a + \nu_1 M_1^a = 0, \\
&(\mathbf{H}_2)_a^1 = \ -\nu_1 M_a^1 + \nu_1 Y_a = 0, \\
&(\mathbf{H}_2)_a^a = \nu_1 (\mu - M_1^1 - M_a^a), \\
&(\mathbf{H}_2)_a^b = \ -\nu_1 M_a^b, \quad a \neq b.
\end{aligned}$$

In matrix form,

$$\mathbf{H}_1 = \nu_1 \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathbf{I}} \end{bmatrix}, \qquad \mathbf{H}_2 = \nu_1 \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathbf{H}}_1 \end{bmatrix}, \tag{8.3}$$

where

$$\begin{aligned}
&\bar{\mathbf{I}} = \mathbf{I}_{n-1}, \qquad \bar{\mathbf{H}}_1 \doteq \bar{\mu}\, \bar{\mathbf{I}} - \bar{\mathbf{M}}, \\
&\bar{\mathbf{M}} \doteq [M_a^b], \qquad \bar{\mu} \doteq \operatorname{tr} \bar{\mathbf{M}} = \mu - M_1^1. \qquad \diamondsuit
\end{aligned} \tag{8.4}$$

The matrix forms (8.3) of $\mathbf{H}_1$ and $\mathbf{H}_2$ show once more the property expressed by Proposition 8.2: $\mathbf{H}_1 \mathbf{X} = \mathbf{H}_2 \mathbf{X} = 0$. For proving this we observe that

Proposition 8.3. *If $\mathbf{H}_a \mathbf{X} = 0$, then all $\mathbf{K}_a$ are of first degree in m,*

$$\mathbf{K}_a = \mathbf{K}_{a0} + m\, \mathbf{K}_a.$$

Proof. Recall the recursive formula (1.1) of a L-chain,

$$\mathbf{K}_{a+1} = \tfrac{1}{a+1}\,\mathrm{tr}\,(\mathbf{K}_a\mathbf{L}_m)\,\mathbf{I} - \mathbf{K}_a\,\mathbf{L}_m$$

Assume that $\mathbf{K}_a$ is first-degree, $\mathbf{K}_a = \mathbf{K}_{a0} + m\,\mathbf{H}_a$. Let us denote by a prime $'$ the derivative w.r.to m. For instance, $\mathbf{L}'_m = \mathbf{N}$. Note that $\mathbf{K}'_a = \mathbf{H}_a$ is of zero-degree. Then,

$$\begin{aligned}(\mathbf{K}_a\mathbf{L}_m)' &= \mathbf{K}'_a\mathbf{L}_m + \mathbf{K}_a\mathbf{N} = \mathbf{H}_a(\mathbf{M} + m\,\mathbf{N}) + (\mathbf{K}_{a0} + m\mathbf{H}_a)\mathbf{N}\\ &= \mathbf{H}_a\mathbf{M} + \mathbf{K}_{a0}\mathbf{N} + 2m\,\mathbf{H}_a\mathbf{N} = \mathbf{H}_a\mathbf{M} + \mathbf{K}_{a0}\mathbf{N}.\\ (\mathbf{K}_a\mathbf{L}_m)'' &= 2\,\mathbf{H}_a\mathbf{N} = 2\,\mathbf{H}_a\mathbf{X}\cdot\mathbf{X} = 0.\end{aligned}$$

It follows that $\mathbf{K}_{a+1}$ is first-degree. ☐

9. The induced L-systems. What we are going to do now is suggested by the remarkable formulas (8.4). Indeed, the $(n-1)\times(n-1)$ matrix

$$\bar{\mathbf{H}}_1 = \bar{\mu}\,\bar{\mathbf{I}} - \bar{\mathbf{M}}, \qquad \bar{\mu} = \mathrm{tr}\,\bar{\mathbf{M}},$$

has the same form of the first step of a L-sequence. To interpret this analogy in a right way, let us consider any leaf of the $\mathbf{X}$-web i.e., any $n-1$-dimensional submanifold S orthogonal to $\mathbf{X}$. Such a submanifold is defined by an equation $q^1 =$ constant (in $\mathbf{X}$-coordinates). Then the bar-tensors like $\bar{\mathbf{I}}$, $\bar{\mathbf{M}}$, $\bar{\mathbf{H}}_1$, etc., can be interpreted as $(1,1)$-tensors on any S.

THEOREM 9.1. *On each leaf S the tensor $\bar{\mathbf{M}} = [M^b_a]$ is torsionless.*

Proof. The tosionless condition of $\mathbf{M}$ is equivalent to the symmetry of

$$S^k_{ij}(\mathbf{M}) \doteq M^h_i\partial_h M^k_j - M^k_m\partial_i M^m_j.$$

A special case is

$$\begin{aligned}0 &= S^c_{ab}(\mathbf{M}) = M^h_a\partial_h M^c_b - M^c_m\partial_a M^m_b\\ &= M^d_a\partial_d M^c_b - M^c_d\partial_a M^d_b + M^1_a\partial_1 M^c_b - M^c_1\partial_a M^1_b\\ &= S^c_{ab}(\bar{\mathbf{M}}) + M^1_a\partial_1 M^c_b - M^c_1\partial_a M^1_b.\end{aligned}$$

Due to Eq. (7.11), $\partial_1 M^c_b = 0$. Due to Eq. (7.10), $\partial_a M^1_b = \partial_b M^1_a$. This shows that $S^c_{ab}(\bar{\mathbf{M}})$ is symmetric. ☐

THEOREM 9.2. *On each leaf S the tensor $\bar{\mathbf{M}}$ is a trace-type CKT.*

Proof. Since $M^{ab} = g^{bi}M^a_i = g^{bc}M^a_c$, the contravariant components of $\mathbf{M}'$ are just the contravariant components M^{ab} (with $a,b>1$) of $\mathbf{M}$. The tensor $\mathbf{M}$ is a trace-type CKT i.e., the polynomial equation

$$\{M^{ij}\,p_ip_j, g^{hk}\,p_hp_k\} = -2\,C^ip_i\;g^{hk}\,p_hp_k$$

holds with

$$C^i \doteq g^{ij}\;\partial_i\mu.$$

In **X**-coordinates, we get

$$\begin{aligned}&\{M^{11}\,p_1^2 + 2M^{1a}\,p_1p_a + M^{ab}\,p_ap_b, g^{11}\,p_1^2 + g^{ab}\,p_ap_b\}\\&= -2\,(C^1p_1 + C^ap_a)(g^{11}\,p_1^2 + g^{ab}\,p_ap_b).\end{aligned} \tag{9.1}$$

The left hand side is

$$\begin{aligned}&L \doteq \{M^{11}\,p_1^2 + 2M^{1a}\,p_1p_a + M^{ab}\,p_ap_b, g^{11}\,p_1^2 + g^{ab}\,p_ap_b\} =\\&= 2\,(M^{11}\,p_1 + M^{1a}\,p_a)(\partial_1 g^{11}\,p_1^2 + \partial_1 g^{bc}\,p_bp_c)\\&+ 2\,(M^{1a}\,p_1 + M^{ab}\,p_b)(\partial_a g^{11}\,p_1^2 + \partial_a g^{bc}\,p_bp_c)\\&- 2\,(\partial_1 M^{11}\,p_1^2 + 2\partial_1 M^{1a}\,p_1p_a + \partial_1 M^{ab}\,p_ap_b)\,g^{11}\,p_1\\&- 2\,(\partial_a M^{11}\,p_1^2 + 2\partial_a M^{1b}\,p_1p_b + \partial_a M^{bc}\,p_bp_c)\,g^{ab}\,p_b.\end{aligned}$$

This is a homogeneous third-degree polynomial, which can be written in the following form:

$$\begin{aligned}&\tfrac{1}{2}\,L = p_1^3\,[M^{11}\partial_1 g^{11} + M^{1a}\,\partial_a g^{11} - g^{11}\,\partial_1 M^{11}]\\&+ p_1^2p_a\,[M^{1a}\,\partial_1 g^{11} + M^{ab}\,\partial_b g^{11} - g^{ab}\partial_b M^{11} - 2\,g^{11}\,\partial_1 M^{1a}]\\&+ p_1p_ap_b\,[M^{11}\,\partial_1 g^{ab} + M^{1c}\,\partial_c g^{ab} - g^{11}\,\partial_1 M^{ab} - 2\,g^{ac}\partial_c M^{1b}]\\&+ p_ap_bp_c\,[M^{1a}\,\partial_1 g^{bc} + M^{da}\,\partial_d g^{bc} - g^{da}\,\partial_d M^{bc}].\end{aligned}$$

A further evolution of this expression is due to equations $g^{11} = \nu_1$ and $\partial_a g^{11} = \partial_a \nu_1 = 0$,

$$\begin{aligned}&\tfrac{1}{2}\,L = p_1^3\,[M^{11}\partial_1\nu_1 - \nu_1\,\partial_1 M^{11}]\\&+ p_1^2p_a\,[M^{1a}\,\partial_1\nu_1 - g^{ab}\partial_b M^{11} - 2\,\nu_1\,\partial_1 M^{1a}]\\&+ p_1p_ap_b\,[M^{11}\,\partial_1 g^{ab} + M^{1c}\,\partial_c g^{ab} - \nu_1\,\partial_1 M^{ab} - 2\,g^{ac}\partial_c M^{1b}]\\&+ p_ap_bp_c\,[M^{1a}\,\partial_1 g^{bc} + M^{da}\,\partial_d g^{bc} - g^{da}\,\partial_d M^{bc}].\end{aligned}$$

For the right hand side R we have

$$\begin{aligned}&-\tfrac{1}{2}\,R = (C^1p_1 + C^ap_a)\,(g^{11}\,p_1^2 + g^{ab}\,p_ap_b)\\&= p_1^3\,C^1\,g^{11} + p_1^2p_a\,C^a\,g^{11} + p_1p_ap_b\,C^1\,g^{ab} + p_ap_bp_c\,C^ag^{bc}\\&= p_1^3\,C^1\,\nu_1 + p_1^2p_a\,C^a\,\nu_1 + p_1p_ap_b\,C^1\,g^{ab}\\&+ p_ap_bp_c\,C^ag^{bc}.\end{aligned}$$

Hence, Eq. (9.1) is equivalent to the following equations:

$$\begin{aligned}&\text{(I)}\quad M^{11}\partial_1\nu_1 - \nu_1\,\partial_1 M^{11} + C^1\nu_1 = 0,\\&\text{(II)}\quad M^{1a}\,\partial_1\nu_1 - g^{ab}\partial_b M^{11} - 2\,\nu_1\,\partial_1 M^{1a} + C^a\nu_1 = 0,\\&\text{(III)}\quad p_ap_b\,[M^{11}\,\partial_1 g^{ab} + M^{1c}\,\partial_c g^{ab}\\&\qquad - \nu_1\,\partial_1 M^{ab} - 2\,g^{ac}\partial_c M^{1b} - C^1\,g^{ab}] = 0,\\&\text{(IV)}\quad p_ap_bp_c\,[M^{1a}\,\partial_1 g^{bc} + M^{da}\,\partial_d g^{bc} - g^{da}\,\partial_d M^{bc}\\&\qquad + C^a\,g^{bc}] = 0.\end{aligned} \tag{9.2}$$

Let us develop these equations taking into account that in **X**-coordinates

$$\begin{aligned}
&M^{11} = g^{1i}\, M_i^1 = g^{11}\, M_1^1 = \nu_1\, M_1^1,\\
&M^{1a} = g^{ai} M_i^1 = g^{ab}\, M_b^1,\\
&M^{1a} = g^{1i} M_i^a = g^{11}\, M_1^a = \nu_1\, M_1^a,\\
&C^1 = g^{11}\, C_1 = \nu_1\, \partial_1 \mu = \nu_1\, \partial_1(\bar\mu + M_1^1),\\
&C^a = g^{ab}\, \partial_b \mu = g^{ab}\, \partial_b(\bar\mu + M_1^1).
\end{aligned}$$

In fact, for proving the theorem it is sufficient to develop Eq. (IV), which reads

$$p_a p_b p_c\, [M^{1a}\, \partial_1 g^{bc} + M^{da}\, \partial_d g^{bc} - g^{da}\, \partial_d M^{bc} + g^{ad}\, \partial_d(\bar\mu + M_1^1)\, g^{bc}] = 0.$$

It implies

$$\begin{aligned}
&p_a p_b p_c\, [M^{da}\, \partial_d g^{bc} - g^{da}\, \partial_d M^{bc} + g^{ad}\, \partial_d \bar\mu\, g^{bc}]\\
&+ p_a p_b p_c\, [M^{1a}\, \partial_1 g^{bc} + g^{ad}\, \partial_d M_1^1\, g^{bc}] = 0,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\tfrac{1}{2}\, \{P(\bar{\mathbf{M}}), P(\bar{\mathbf{G}}\}_S + P(\nabla\bar\mu) P(\bar{\mathbf{G}})\\
&+ p_a p_b p_c\, [M^{1a}\, \partial_1 g^{bc} + g^{ad}\, \partial_d M_1^1\, g^{bc}] = 0.
\end{aligned}$$

Hence, $\bar{\mathbf{M}}$ is a trace-type CKT on S if and only if

$$p_a p_b p_c\, [M^{1a}\, \partial_1 g^{bc} + g^{ad}\, \partial_d M_1^1\, g^{bc}] = 0,$$

i.e.,

$$p_a p_b p_c\, g^{ad}\, [M_d^1\, \partial_1 g^{bc} + \partial_d M_1^1\, g^{bc}] = 0.$$

Due to (5.13) it follows that

$$p_a p_b p_c\, g^{ad}\, g^{bc}\, [M_d^1\, \phi_1 + \partial_d M_1^1] = 0,$$

where $\phi_1(q^1) \doteq -\partial_1 \log \nu_1$. But Eq. (7.12) shows that$M_b^1 \partial_1 \nu_1 - \nu_1 \partial_b M_1^1 = 0$ i.e., $M_b^1 \phi_1 + \partial_b M_1^1 = 0$. ▯

Remark 9.1. Equations (I), (II), (III) in (9.2) have not been used. They provide further necessary conditions on **M**, whose analysis is here omitted. ♢

According to the two theorems above, $\bar{\mathbf{M}}$ is a torsionless CKT on each submanifold S orthogonal to **X**. Its eigenvalues are distinct (see Remark 9.3 below) with the exception of a closed subset (may be empty) of S. Thus, $\bar{\mathbf{M}}$ is a L-tensor on any S and the recursive formula

$$\bar{\mathbf{H}}_a = \tfrac{1}{a}\, \mathrm{tr}\, (\bar{\mathbf{H}}_{a-1} \bar{\mathbf{M}})\, \bar{\mathbf{I}} - \bar{\mathbf{H}}_{a-1}$$

for $a = 1, 2, \ldots, n-2$ and $\bar{\mathbf{H}}_0 = \bar{\mathbf{I}}$, form a L-system on S (whose dimension is $n-1$).

Now we return to the whole space Q_n and consider the $n-1$ tensors

$$\mathbf{H}_a = \nu_1 \begin{bmatrix} 0 & 0 \\ 0 & \bar{\mathbf{H}}_{a-1} \end{bmatrix} \tag{9.3}$$

with $a = 1, \ldots, n-1$. Note that for $a = 1$ and $a = 2$ we get the matrices (8.3). The $n-1$ (1,1)-tensors $\mathbf{H}_a$ are linearly independent, commute and have common eigenvectors tangent to S, since the same properties hold for $\bar{\mathbf{H}}_a$. But they have also $\mathbf{X}$ as common eigenvector, with zero eigenvalue.

In order to get a basis for a KS-space we have to add to them the identity

$$\mathbf{H}_0 = \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\mathbf{I}} \end{bmatrix}$$

and to prove that

THEOREM 9.3. *The tensors $\mathbf{H}_a$ are Killing tensors.*

Proof. The $\mathbf{X}$-components of $\mathbf{H}_*$ are $H^{11} = 0$, $H^{1a} = 0$, $H^{bc} = \nu_1\, g^{bd}\, \bar{H}^c_d$. Hence, $P(\mathbf{H}_*) = \nu_1\, \bar{H}^{bc}\, p_b p_c$. We have to show that

$$\{P(\mathbf{H}_*), g^{ij} p_i p_j\} = 0.$$

In $\mathbf{X}$-coordinates $g^{11} = \nu_1$ is a function of q^1 only, so that

$$\begin{aligned}
&\{P(\mathbf{H}_*)\,,\, g^{ij} p_i p_j\} = \{\nu_1\, \bar{H}^{bc}\, p_b p_c\,,\, g^{11} p_1^2 + g^{ab}\, p_a p_b\} \\
&= 2\,\nu_1\, \bar{H}^{bc}\, p_c\, \partial_b g^{ad}\, p_a p_d - 2 p_1 \nu_1\, \partial_1(\nu_1\, \bar{H}^{bc}\, p_b p_c) \\
&- 2 g^{ab} p_b\, \partial_a(\nu_1\, \bar{H}^{bc}\, p_b p_c) \\
&= 2\,\nu_1\, \bar{H}^{bc}\, p_c\, \partial_b g^{ad}\, p_a p_d - 2\,\nu_1\, g^{ab} p_b\, \partial_a(\bar{H}^{bc}\, p_b p_c) \\
&- 2 p_1 \nu_1\, \partial_1(\nu_1\, \bar{H}^{bc}\, p_b p_c) \\
&= \nu_1\, \{P(\bar{\mathbf{H}}_*), P(\bar{\mathbf{G}})\} - 2 p_1 \nu_1\, \partial_1(\nu_1\, \bar{H}^{bc}\, p_b p_c) \\
&= \; - 2 p_1 \nu_1\, \partial_1(\nu_1\, \bar{H}^{bc}\, p_b p_c) = \ldots
\end{aligned}$$

Since $\bar{H}^{bc} = g^{ba} \bar{H}^c_a$, we have $\partial_1 \bar{H}^{bc} = \partial_1 g^{ba} \bar{H}^c_a + g^{ba}\, \partial_1 \bar{H}^c_a$. However, $\partial_1 \bar{H}^c_a = 0$, since $\bar{H}^c_a$ is constructed by an algebraic process from M^b_a and $\partial_1 M^b_a = 0$, Eq. (7.11). Hence, due to Eqs. (5.13) and (5.14),

$$\partial_1 \bar{H}^{bc} = \partial_1 g^{ba} \bar{H}^c_a = \phi_1\; g^{ba} \bar{H}^c_a = \phi_1\; \bar{H}^{bc}.$$

It follows that

$$\partial_1(\nu_1 \bar{H}^{bc}) = \partial_1 \nu_1\; \bar{H}^{bc} + \nu_1 \phi_1\; \bar{H}^{bc} = 0.$$

Thus, $\ldots = 0$. ∎

REMARK 9.2. This theorem proves that $\mathbf{H}_0 = \mathbf{I}$ and the $n-1$ Killing tensors $\mathbf{H}_a$ defined in (9.3) form a basis of a KS-space on Q. Since $\mathbf{H}_a\mathbf{X} = 0$ $(a > 0)$, due to Proposition 8.3, the tensors $\mathbf{K}_a$ are of first degree in m. $\diamondsuit$

REMARK 9.3. Let us consider in $\mathbb{R}^n$ a diagonal matrix

$$\mathbf{M} = \mathrm{diag}\,[\mu_1, \mu_2, \ldots, \mu_n]$$

with all distinct $\mu_i \neq 0$, and a vector $\mathbf{X} = [X^1, X^2, \ldots, X^n]$. The matrix

$$\bar{\mathbf{M}} \doteq \mathbf{M} - \alpha\,\mathbf{M}(\mathbf{X}) \otimes \mathbf{X}^\flat,$$

with $\alpha^{-1} \doteq \langle \mathbf{X}, \mathbf{X}^\flat \rangle = X^i X_i$, satisfies equation $\bar{\mathbf{M}}(\mathbf{X}) = 0$. Its components are (we consider for simplicity $n = 4$)

For a vector $\mathbf{X}$ with only a non-zero component, say $X^1 \neq 0$, we get the diagonal form

$$\bar{\mathbf{M}} = \mathrm{diag}\,[0, \mu_2, \ldots, \mu_n]$$

which shows that the eigenvalues are distinct. Then, in an open cone around $\mathbf{X}$ this property is preserved. We have n open cones generated in this way. On the other hand it is known (after Sylvester) that the condition that $\bar{\mathbf{M}}$ has non-simple eigenvalues is expressed by an algebraic equation of order $2n + 2$ in the variables X^i. This equation defines a surface (or the union of surfaces) in the space (X^i), or the empty set. $\diamondsuit$

10. Conclusion. The necessity of the conditions listed in Theorems 1.3 and 1.4 are proved:

- $\mathbf{M}$ is a L-tensor: Theorem 2.1.
- $\mathbf{N} = \mathbf{X} \otimes \mathbf{X}^\flat$: Theorem 4.1.
- $\mathbf{X}$ is a conformal vector field: Theorem 5.2.
- $\mathbf{X}$ is a translation or a dilatation: Theorem 5.1.
- All tensors $\mathbf{K}_a$ of the L-sequence are linear in m: Remark 9.2.
- $\mathbf{H}_a\mathbf{X} = 0$: Eq. (9.3).
- The restrictions of $\mathbf{H}_a$ to any leaf of the foliation orthogonal to $\mathbf{X}$ form a L-system on that leaf: Section 9.

It remains to prove the sufficiency in Theorem 1.3: *If* $\mathbf{M}$ *is a L-tensor and* $\mathbf{X}$ *is a CKV such that* $d\mathbf{X}^\flat = 0$, *then* $\mathbf{L}_m = \mathbf{M} + m\,\mathbf{X} \otimes \mathbf{X}^\flat$ *is an effective L-pencil.*

Proof. Since $\mathbf{X}$ is a CKV, $\mathbf{N} = \mathbf{X} \otimes \mathbf{X}^\flat$ is a CKT. Since $d\mathbf{X}^\flat = 0$, $\mathbf{N}$ is torsionless: see the proof of Theorem 5.3. Then apply Theorems 7.1 and 2.1. ∎

11. Final comments. About the existence of L-pencils, we recall the following properties (to be applied to the L-tensor $\mathbf{M}$ of a L-pencil).

- *If a Stäckel web has a foliation orthogonal to a proper CKV, then it is not a L-web* (Theorem 9.4 in [5]).
- *If a Stäckel web has* $m < n$ *foliations orthogonal to translations, then it is not a L-web* (Theorem 9.6 in [5]).

As a consequence,

- *All translational Stäckel webs in a Euclidean n-space, different from the Cartesian web (for which $m = n$) are not L-webs* (Remark 9.2 in [5]).

Moreover, let us recall that ([9], p. 249):

- *If a Killing vector is normal then the lines of curvature of the orthogonal submanifolds are indeterminate.*
- *The orbits of a translation form a flat submanifold.*
- *If a translation is normal, then the orthogonal submanifolds are totally geodesic.*

In [9] – formula (69.5) – it is proved that for a CKV $\mathbf{X}$ equation

$$\Delta_2\psi = \frac{2}{n-1}\left(X_m\nabla_i R^{mi} + \nabla_i X_m\, R^{mi}\right)$$

holds. Thus, for a Killing vector ($\psi = 0$) and for a dilatation ($\psi =$ constant),

$$X_m\nabla_i R^{mi} + \nabla_i X_m\, R^{mi} = 0.$$

For a manifold of constant curvature $K_0 \neq 0$ this equation reduces to

$$\nabla_i\nabla_j\psi + K_0\, g_{ij}\,\psi = 0.$$

For a Killing vector this is identically satisfied, but for a dilatation it gives $K_0 = 0$: absurd. Thus,

Theorem 11.1. *A manifold with non-zero constant curvature does not admit dilatations.*

But it can also be shown that a manifold with non-zero constant curvature does not admit translations. So, for instance, the sphere $\mathbb{S}_n$ and the pseudosphere $\mathbb{H}_n$, does not have L-pencils.

All statements listed above represent strong obstructions for the existence of translations and dilatations, hence, for the existence of L-pencils. This list is certainly incomplete, since further results should be present in the ancient–may be also recent–literature.

As said in Remark 1.1, the basic example of L-pencil is the spherical-conical (asymmetric) web in $\mathbf{r}^n$, and, according to the above remarks, it is the only L-pencil existing in $\mathbb{R}^n$. In spite to this rather restrictive result, futher arguments of research arise:

- To find examples of L-pencils form Riemannian manifolds with non-constant curvature.
- To extend the notion of L-pencil to pseudo-Euclidean spaces. Indeed, we know the general form of a L-tensor in these spaces (Appendix B in [5], see also [8]).
- To introduce and study the notion of **multipencil**. A 2-pencil is for instance

$$\mathbf{L}_{m_1,m_2} = \mathbf{M} + m_1\,\mathbf{X}_1\otimes\mathbf{X}_1^\flat + m_2\,\mathbf{X}_2\otimes\mathbf{X}_2^\flat, \quad m_1 \neq m_2.$$

Since the spherical-conical webs play an important role in the diagrammatic classification of Stäckel systems due to Kalnins and Miller [11, 10], the notion of multipencil should be useful for this classification.
- To extend the notions of L-tensor and of L-pencil by dropping out the requirement of 'simple eigenvalues', and including the cases of S-webs invariant w.r.to Killing tensors.

12. Appendices.

12.1. Induced L-systems. It is known that a S-system induces a S-system on each leaf of its web [3, 7].

In Section 9 we have seen that a LP-system induces a L-system on each leaf of its web. A similar property holds for L-systems:

THEOREM 12.1. *A L-system induces a L-system on each leaf of its web.*

Proof. Let (u^i) be the eigenvalues of a L-tensor $\mathbf{L}$. Let us order the indices in such a way that

$$u^1 < u^2 < \ldots < u^n,$$

and set

$$\Delta_i \doteq \prod_{k=1}^{i-1}(u^i - u^k) \prod_{k=i+1}^{n}(u^k - u^i).$$

Then $\Delta_i > 0$ for each index i, and the metric tensor components in $\mathbf{L}$-coordinates can be written

$$g^{ii} = \frac{\phi_i(q^i)}{\Delta_i}$$

with $\phi_i(q^i) > 0$. As a consequence, up to a rescaling of the coordinates, we get

$$g^{ii} = \frac{1}{\Delta_i}, \qquad g_{ii} = \Delta_i.$$

These components are all positive. For the special case $n = 3$:

$$\begin{aligned} \Delta_1 &= (u^2 - u^1)(u^3 - u^1), \\ \Delta_2 &= (u^2 - u^1)(u^3 - u^2), \\ \Delta_3 &= (u^3 - u^1)(u^3 - u^2). \end{aligned} \tag{12.1}$$

For $n = 4$:

$$\begin{aligned} \Delta_1 &= (u^2 - u^1)(u^3 - u^1)(u^4 - u^1), \\ \Delta_2 &= (u^2 - u^1)(u^3 - u^2)(u^4 - u^2), \\ \Delta_3 &= (u^3 - u^1)(u^3 - u^2)(u^4 - u^3), \\ \Delta_4 &= (u^4 - u^1)(u^4 - u^2)(u^4 - u^3). \end{aligned} \tag{12.2}$$

Let us consider for simplicity the case $n = 4$ and the leaf S defined by $u^4 = 1$. The components g_{11}, g_{22}, g_{33} restricted to S become

$$\begin{aligned}\Delta_1 &= (u^2 - u^1)(u^3 - u^1)(1 - u^1),\\ \Delta_2 &= (u^2 - u^1)(u^3 - u^2)(1 - u^2),\\ \Delta_3 &= (u^3 - u^1)(u^3 - u^2)(1 - u^3).\end{aligned}$$

They can be written in the form

$$\begin{aligned}\Delta_1 &= (u^2 - u^1)(u^3 - u^1)\phi_1(q^1),\\ \Delta_2 &= (u^2 - u^1)(u^3 - u^2)\phi_2(q^2),\\ \Delta_3 &= (u^3 - u^1)(u^3 - u^2)\phi_3(q^3),\end{aligned}$$

with $\phi_i(q^i) > 0$. Then the coordinates can be normalized in order to get

$$\begin{aligned}\Delta_1 &= (u^2 - u^1)(u^3 - u^1),\\ \Delta_2 &= (u^2 - u^1)(u^3 - u^2),\\ \Delta_3 &= (u^3 - u^1)(u^3 - u^2).\end{aligned}$$

This is a form of the kind (12.1). $\square$

REMARK 12.1. For $n = 4$ we get

$$\begin{aligned}\Delta_1 &= (u^2 - u^1)(u^3 - u^1)(u^4 - u^1)\\ &= (u^2 - u^1)(u^3u^4 - u^3u^1 - u^1u^4 + (u^1)^2)\\ &= u^2u^3u^4 - u^2u^3u^1 - u^2u^1u^4 + u^2(u^1)^2\\ &\quad - u^1u^3u^4 + u^3(u^1)^2 + (u^1)^2u^4 - (u^1)^3)\\ &= \sigma_3^1 - u^1\,\sigma_2^1 + (u^1)^2\,\sigma_1^1 - (u^1)^3.\end{aligned}$$

Thus, in this special case,

$$g_{11} = \sigma_3^1 - u^1\,\sigma_2^1 + (u^1)^2\,\sigma_1^1 - (u^1)^3.$$

We get the general formula

$$g_{ii} = \sigma_3^i - u^i\,\sigma_2^i + (u^i)^2\,\sigma_1^i - (u^i)^3.$$

Similar formulas can be obtained for any dimension n. $\diamondsuit$

12.2. An example of type-2 L-pencil. In $\mathbb{R}^3$ the parabolic web is determined by the L-tensor [2]

$$\mathbf{L}_m = \mathbf{M} + m\,\mathbf{u}\odot\mathbf{r}, \qquad \mathbf{u}^2 = 1,$$

where $\mathbf{u}$ is a constant unit vector. Let us look at this tensor as a L-pencil with

$$\mathbf{N} = \mathbf{u}\odot\mathbf{r} = \tfrac{1}{2}(\mathbf{u}\otimes\mathbf{r} + \mathbf{r}\otimes\mathbf{u}).$$

Since $\operatorname{tr}\mathbf{L}_m = \mu + m\,\mathbf{u}\cdot\mathbf{r}$, $\nu = \mathbf{u}\cdot\mathbf{r}$, we get

$$\mathbf{K}_{22} = \tfrac{1}{2}\,(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2.$$

Moreover,

$$\begin{aligned}\mathbf{N}^2 &= \tfrac{1}{4}(\mathbf{u}\otimes\mathbf{r} + \mathbf{r}\otimes\mathbf{u})(\mathbf{u}\otimes\mathbf{r} + \mathbf{r}\otimes\mathbf{u})\\ &= \tfrac{1}{4}\,(\mathbf{r}\cdot\mathbf{u}\,\mathbf{u}\otimes\mathbf{r} + r^2\,\mathbf{u}\otimes\mathbf{u} + \mathbf{r}\otimes\mathbf{r} + \mathbf{u}\cdot\mathbf{r}\,\mathbf{r}\otimes\mathbf{u})\\ &= \tfrac{1}{4}\,(2\,\mathbf{r}\cdot\mathbf{u}\,\mathbf{u}\odot\mathbf{r} + r^2\,\mathbf{u}\otimes\mathbf{u} + \mathbf{r}\otimes\mathbf{r}),\end{aligned}$$

and

$$\operatorname{tr}\mathbf{N}^2 = \tfrac{1}{4}\,[2(\mathbf{r}\cdot\mathbf{u})^2 + 2r^2] = \tfrac{1}{2}\,[(\mathbf{r}\cdot\mathbf{u})^2 + r^2],$$

so that

$$\tfrac{1}{2}\,(\nu^2 - \operatorname{tr}\mathbf{N}^2) = \tfrac{1}{2}\,[(\mathbf{u}\cdot\mathbf{r})^2 - \tfrac{1}{2}\,((\mathbf{r}\cdot\mathbf{u})^2 + r^2)] = \tfrac{1}{2}\,[\tfrac{1}{2}\,((\mathbf{r}\cdot\mathbf{u})^2 + r^2)],$$

and

$$\begin{aligned}\mathbf{N}^2 - \nu\,\mathbf{N} &= \tfrac{1}{4}\,(2\,\mathbf{r}\cdot\mathbf{u}\,\mathbf{u}\odot\mathbf{r} + r^2\,\mathbf{u}\otimes\mathbf{u} + \mathbf{r}\otimes\mathbf{r}) - (\mathbf{u}\cdot\mathbf{r})(\mathbf{u}\odot\mathbf{r})\\ &= \tfrac{1}{4}\,(r^2\,\mathbf{u}\otimes\mathbf{u} - 2\,\mathbf{r}\cdot\mathbf{u}\,\mathbf{u}\odot\mathbf{r} + \mathbf{r}\otimes\mathbf{r}).\end{aligned}$$

We observe that

$$\begin{aligned}&(\mathbf{r}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{r})(\mathbf{r}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{r})\\ &= \mathbf{u}\cdot\mathbf{r}\,\mathbf{r}\otimes\mathbf{u} - \mathbf{r}\otimes\mathbf{r} - r^2\,\mathbf{u}\otimes\mathbf{u} + \mathbf{r}\cdot\mathbf{u}\,\mathbf{u}\otimes\mathbf{r}\\ &= 2\mathbf{u}\cdot\mathbf{r}\,\mathbf{r}\odot\mathbf{u} - \mathbf{r}\otimes\mathbf{r} - r^2\,\mathbf{u}\otimes\mathbf{u},\end{aligned}$$

and we get the final expression

$$\mathbf{N}^2 - \nu\,\mathbf{N} = -\tfrac{1}{4}\,(\mathbf{r}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{r})^2,$$

which allows the computation of $\mathbf{K}_{22}$:

$$\begin{aligned}\mathbf{K}_{22} &= \tfrac{1}{2}\,(\nu^2 - \operatorname{tr}\mathbf{N}^2)\,\mathbf{I} - \nu\,\mathbf{N} + \mathbf{N}^2\\ &= \tfrac{1}{2}\,[\tfrac{1}{2}\,((\mathbf{r}\cdot\mathbf{u})^2 + r^2)]\,\mathbf{I} - \tfrac{1}{4}\,(\mathbf{r}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{r})^2 \neq 0.\end{aligned}$$

This shows that $\mathbf{L}_m$ is a L-pencil of type 2.

Let us check the validity of Remark 3.1. The first element of the L-sequence generated by $\mathbf{L}_m$ is

$$\mathbf{K}_1 = (\mu + m\,\mathbf{u}\cdot\mathbf{r})\,\mathbf{I} - \mathbf{M} - m\,\mathbf{u}\odot\mathbf{r}.$$

Thus, $\mathbf{H}_1 = \mathbf{u}\cdot\mathbf{r}\,\mathbf{I} - \mathbf{u}\odot\mathbf{r}$. For computing $\mathbf{K}_2$ we need to compute $\mathbf{K}_1\mathbf{L}_m$,

$$\mathbf{K}_1\mathbf{L}_m = [(\mu + m\,\mathbf{u}\cdot\mathbf{r})\,\mathbf{I} - \mathbf{M} - m\,\mathbf{u}\odot\mathbf{r}]\,[\mathbf{M} + m\,\mathbf{u}\odot\mathbf{r}].$$

After a straightforward calculation we get

$$\begin{aligned}\mathbf{K}_1\mathbf{L}_m &= \mu\,\mathbf{M} - \mathbf{M}^2 + m\,(\mathbf{u}\cdot\mathbf{r}\,\mathbf{M} - \mathbf{u}\odot\mathbf{M}\mathbf{r} - \mathbf{r}\odot\mathbf{M}\mathbf{u} + \mu\,\mathbf{u}\odot\mathbf{r} \\ &\quad - \tfrac{1}{4}\,r^2\,\mathbf{u}\otimes\mathbf{u} - \tfrac{1}{4}\,\mathbf{r}\otimes\mathbf{r}) + \tfrac{1}{2}\,m^2\,\mathbf{u}\cdot\mathbf{r}\,\mathbf{u}\odot\mathbf{r}.\end{aligned}$$

The coefficient of m^2 is $\frac{1}{2}\,\mathbf{u}\cdot\mathbf{r}\,\mathbf{u}\odot\mathbf{r}$. Since

$$\mathbf{K}_2 = \tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{K}_1\mathbf{L}_m)\,\mathbf{I} - \mathbf{K}_1\mathbf{L}_m,$$

we find

$$\mathbf{H}_2 = \tfrac{1}{4}\,(\mathbf{u}\cdot\mathbf{r})^2\,\mathbf{I} - \tfrac{1}{2}\,\mathbf{u}\cdot\mathbf{r}\,\mathbf{u}\odot\mathbf{r} = \tfrac{1}{4}\,(\mathbf{u}\cdot\mathbf{r})\,(\mathbf{u}\cdot\mathbf{r}\,\mathbf{I} - 2\,\mathbf{u}\odot\mathbf{r}).$$

For $n = 3$, the tensors

$$\mathbf{H}_0 = \mathbf{I}, \quad \mathbf{H}_1 = \mathbf{u}\cdot\mathbf{r}\,\mathbf{I} - \mathbf{u}\odot\mathbf{r}, \quad \mathbf{H}_2 = \tfrac{1}{4}\,(\mathbf{u}\cdot\mathbf{r})\,(\mathbf{u}\cdot\mathbf{r}\,\mathbf{I} - 2\,\mathbf{u}\odot\mathbf{r}).$$

are linearly dependent: $\mathbf{H}_2 = \frac{1}{4}\,(\mathbf{u}\cdot\mathbf{r})\,(2\,\mathbf{H}_1 - \mathbf{u}\cdot\mathbf{r}\,\mathbf{H}_0)$.

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