The Lagrangian and Hamiltonian formulations for a special class of non-conservative systems

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Abstract

This is an outline of the major results contained in an extensive tutorial paper presented at SPT-2004 [B] and dedicated to a special kind of symmetric two-tensors which appear, in the recent and in the old literature, in connection with special kinds of mechanical systems and with the theory of the separation of variables in the Hamilton-Jacobi equation.

1 Posing a question

A **holonomic system** is a mechanical system whose configurations form a set Q endowed with a differentiable manifold structure with finite dimension n (the number of *degrees of freedom*).[1] This manifold is in turn endowed with a positive-definite metric tensor $\mathbf{g} = (g_{ij})$ determined by the expression of the kinetic energy $K = \frac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j$ w.r.to any natural coordinate system $(q^i, \dot{q}^i)$ on the tangent bundle TQ. The active forces are then represented by a vector field $\mathbf{F} = (F^i)$ on Q or by a one-form $F_i = g_{ij}\, F^j$.[2] For such a system, represented by the triple $(Q_n, \mathbf{g}, \mathbf{F})$, the dynamics is completely determined by the second-order Lagrange equations

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}^i} - \frac{\partial K}{\partial q^i} = F_i, \qquad F_i \doteq g_{ij}\, F^j, \tag{1}$$

which assume the **Newtonian form**

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{hj}\, \frac{dq^h}{dt}\, \frac{dq^j}{dt} = F^i \qquad \Longleftrightarrow \qquad \mathbf{a} = \mathbf{F}, \tag{2}$$

being

$$a^i \doteq \frac{d^2 q^i}{dt^2} + \Gamma^i_{hj}\, \frac{dq^h}{dt}\, \frac{dq^j}{dt}$$

the **absolute acceleration** of the dynamical system and Γ^i_{hj} the Christoffel symbols i.e., the coefficients of the Levi-Civita connection associated with the metric tensor. This last second-order system is equivalent to the first-order dynamical system

$$\mathbf{X} = \begin{cases} \frac{dq^i}{dt} & = v^i, \\ \frac{dv^i}{dt} & = -\,\Gamma^i_{hj}\, v^h v^j + F^i, \end{cases}$$

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[1] Here we consider only the case of time-independent constraints.

[2] The forces are here assumed to be time-independent and velocity-independent.

on the tangent bundle TQ, with coordinates $(\underline{q},\underline{v}) = (q^i, v^i) = (q^i, \dot{q}^i)$.

If the force $\mathbf{F}$ is **potential** (or "conservative") i.e., it is the gradient of a **potential energy** V (a real-valued smooth function on Q),

$$\mathbf{F} = -\nabla V \qquad \Longleftrightarrow \qquad F^i = \, - \, g^{ij}\, \partial_j V,$$

then we have a **Lagrangian system**. Eqs. (1) assume the well-known form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

where $L \doteq \frac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j - V$ is the **Lagrangian function** on TQ. In this case we can apply alternative methods of integration by passing to the **Hamiltonian function** on the cotangent bundle T^*Q, with coordinates $(\underline{q},\underline{p}) = (q^i, p_j)$, $H \doteq \frac{1}{2}\, g^{ij}\, p_i p_j + V$. This function gives rise to a **Hamiltonian system** (i.e., to first-order **Hamilton equations**) and to a **Hamilton-Jacobi equation**.

Let us pose the question: *may the Lagrangian and the Hamiltonian methods be extended to systems with non-potential forces?*

2 Equivalent systems

For posing this question in a more precise way we make use of the notion of equivalent systems introduced by Painlévé (1894) and Levi-Civita (1896).

Definition 2.1 *Let $(Q, \mathbf{g}, \mathbf{F})$ and $(Q, \bar{\mathbf{g}}, \bar{\mathbf{F}})$ be two holonomic systems with the same configuration manifold Q. Let*

$$\mathbf{X} = \begin{cases} \frac{dq^i}{dt} = v^i, \\ \frac{dv^i}{dt} = F^i - \Gamma^h_{ij}\, v^i v^j, \end{cases} \qquad \bar{\mathbf{X}} = \begin{cases} \frac{dq^i}{dt} = v^i, \\ \frac{dv^i}{dt} = \bar{F}^i - \bar{\Gamma}^h_{ij}\, v^i v^j. \end{cases} \tag{3}$$

be the corresponding dynamical systems of TQ with coordinates $(q^i, v^i = \dot{q}^i)$. They are said to be **equivalent** *or* **correspondent***, if there exists a function of $f\colon TQ \to$, such that, for any solution*

$$q^i = \varphi^i(t), \qquad v^i = \frac{d\varphi^i}{dt} = \dot{\varphi}^i(t)$$

of the first system (3), by a change of the time-parameter of the kind

$$\frac{d\bar{t}}{dt} = \frac{1}{f(\varphi^i(t), \dot{\varphi}^i(t))} \tag{4}$$

we get a solution of the second system,

$$q^i = \bar{\varphi}^i(\bar{t}), \qquad v^i = \frac{d\bar{\varphi}^i}{d\bar{t}}.$$

This means that the trajectories on the configuration manifold Q of the two systems coincide, up to a change of the time-parameters given by (4) (in other words, the trajectories are the same, but covered with different velocities).[3]

A special but fundamental case is that with $\mathbf{F} = \bar{\mathbf{F}} = 0$. It concerns with geodesics:

Definition 2.2 *Two metric tensors* $\mathbf{g}$ *and* $\bar{\mathbf{g}}$ *on the same manifold* Q *are said to be* **equivalent** *if they have the same unparametrized geodesics.*

After this definition our question can be reformulated as follows: *which non-potential systems are equivalent to Lagrangian systems?* However, as we shall see in the next section, at the state-of-the-art we are able to answer this question only by adding a further condition on the definition of equivalence.

3 The equivalence theorem of Levi-Civita

Levi-Civita was able to prove [LC] (p. 272)

Theorem 3.1 *Two systems* $(\mathbf{g}, \mathbf{F})$ *and* $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ *are equivalent if and only if there exist functions* μ *and* c_{ij}*, depending on the coordinates only, such that the following equations are satisfied*

$$\begin{cases} \mu^2 F^i = \bar{F}^i, \\ \\ v^h\,(v^i\,\partial_i \log\mu + c_{ij}\,F^i\,v^j) + (\bar{\Gamma}^h_{ij} - \Gamma^h_{ij} + F^h\,c_{ij})\,v^i v^j = 0, \\ \\ \frac{d\log|f|}{dt} = \frac{d\log|\mu|}{dt} + c_{ij}\,F^i\,v^j, \qquad f^2 \doteq \frac{\mu^2}{1 - c_{ij}\,v^i v^j}. \end{cases} \tag{5}$$

Remark 3.1 *In the special case* $c_{ij} = 0$ *conditions (5) reduce to*

$$\mu^2 F^i = \bar{F}^i, \qquad \bar{\Gamma}^h_{ij} = \Gamma^h_{ij} - \tfrac{1}{2}\,(\delta^h_i \mu_j + \delta^h_j \mu_i).$$

This last equation shows that the two metrics are equivalent and suggests the notion of **geodesically equivalent dynamical systems***: they are equivalent holonomic systems whose underlying Riemannian metrics are also equivalent.*

4 Main theorems

Theorem 4.1 *A dynamical system* $(Q, \mathbf{g}, \mathbf{F})$ *is geodesically equivalent to a Lagrangian system i.e., to a system* $(Q, \bar{\mathbf{g}}, \bar{\mathbf{F}})$ *where* $\bar{\mathbf{g}}$ *is an equivalent metric and* $\bar{\mathbf{F}} = -\bar{\nabla} V$, if and only if *the fundamental metric* $\mathbf{g}$ *admits a non-singular* <u>*special conformal Killing tensor*</u> $\mathbf{J}$ *such that*

$$\mathbf{F} = -\mathbf{A}^{-1}\nabla V, \quad \mathbf{A} \doteq \mathbf{J}. \tag{6}$$

[3] In the definition of Levi-Civita the function f is considered depending also on t. However, he proves that in fact this function is independent of time, under the assumption that the forces depend only on the coordinates [LC] (p. 269).

Definition 4.1 *A* **special conformal Killing tensor** *(SCKT) is a symmetric two-tensor J_{ij} satisfying the equation*

$$\nabla_h J_{ij} = \tfrac{1}{2}\,(\alpha_i\, g_{jh} + \alpha_j\, g_{ih}), \tag{7}$$

where α_i are the components of a suitable one-form.[4] *We denote by a boldface letter the corresponding (1,1)-tensor, $\mathbf{J} = (J^i_j)$, $J^i_j = g^{ih} J_{hj}$.*

Such a tensor is strictly connected with other **special symmetric two-tensors**, here denoted by **A**, **B** and **L** and called for simplicity A-tensor, B-tensor and L-tensor (then a SCKT will be also called J-tensor). The first two-tensors are defined by the differential equations

$$\begin{aligned} \nabla_h A_{ij} &= \mu_h\, A_{ij} - \tfrac{1}{2}\,(\mu_j\, A_{hi} + \mu_i\, A_{hj}), \\ \nabla_h B_{ij} &= -\tfrac{1}{2}\,(\mu_j\, B_{hi} + \mu_i\, B_{hj}), \end{aligned} \tag{8}$$

where μ_i are the components of a suitable one-form. The definition of L-tensor will be given in §5 below.

All these tensors are related by several equations, used for proving Theorem 4.1 and other theorems. For instance,

$$\begin{gathered} \mathbf{A} = \mathbf{J} = \mu\, \mathbf{J}^{-1}, \quad \mathbf{J} = \mu\, \mathbf{A}^{-1}, \\ \mathbf{B} = \mathbf{J}^{-1} = \tfrac{1}{\mu}\,\mathbf{A}, \quad \mathbf{B} = \mathbf{A}^{-1}, \\ \mu \doteq \det \mathbf{J}. \end{gathered} \tag{9}$$

Hence, to look for a J-tensor with $\mu \neq 0$ is the same as to look for a non-singular A-tensor or B-tensor.

It is rather surprising that systems satisfying the condition (6) of Theorem 4.1 have been recently introduce in the literature: they have been called **cofactor systems**.[5] Hence, we can restate Theorem 4.1 as follows: *A dynamical system $(Q, \mathbf{g}, \mathbf{F})$ is geodesically equivalent to a Lagrangian system i.e., if and only if it is a cofactor system.*

As a consequence of Theorem 4.1 it is clear that when we have a cofactor system then we can apply to the equivalent Lagrangian system the Hamiltonian methods, including the integration (possibly by separation of variables) of the Hamilton-Jacobi equation. Then we describe the motions of the original system simply by changing the time-parameter according to the formula

$$\frac{dt}{d\bar{t}} = \mu, \tag{10}$$

which follows from (4) and (5). The components of the new metric tensor $\bar{\mathbf{g}}$, with which we can write the Lagrangian and Hamiltonian function of the equivalent system,

$$\bar{L} = \tfrac{1}{2}\, \bar{g}_{ij}\, \bar{v}^i \bar{v}^j - V, \qquad \bar{H} = \tfrac{1}{2}\, \bar{g}^{ij}\, p_i p_j + V, \qquad \bar{v}^i \doteq \tfrac{dq^i}{d\bar{t}}. \tag{11}$$

[4] This kind of tensor has been introduced and studied by Crampin and Sarlet (2000-2003).

[5] Rauch-Wojciechowski, Marciniak, Lundmark, 1999, et al.

are given by

$$\bar{g}_{ij} = \frac{1}{\mu} B_{ij}, \qquad \bar{g}^{ij} = \mu J^{ij}. \tag{12}$$

Note that the operation of raising and lowering indices is always performed by the basic metric $\mathbf{g}$.

Note that the equivalent metric (12) may not be positive-definite. So, we are led to consider pseudo-Riemannian manifolds also in connection with problems of classical mechanics.

A test for finding a cofactor-system-structure is given by the following

Theorem 4.2 *A dynamical system $(Q, \mathbf{g}, \mathbf{F})$ is a cofactor system if and only if $\mathbf{g}$ admits a non-singular A-tensor $\mathbf{A} = (A_{ij})$ such that*

$$d(A_{ij}\, F^j\, dq^i) = 0. \tag{13}$$

This theorem has a local character. For a global meaning, the vector field $\mathbf{AF} = (A^i_j F^j)$ must be a gradient. Hence, Eq. (13) must be replaced by

$$\mathbf{AF} = -\nabla V. \tag{14}$$

This formula has the advantage of giving the potential V of the equivalent system.

A remarkable fact concerning the cofactor systems (or the bi-cofactor systems, see below) is that, under certain condition on the eigenvalues of the special tensors involved, the Hamilton system defined by $\bar{H}$ (11) is an *orthogonal separable system of special kind*, called L-system.[6] To see this we need the notion of L-tensor.

5 L-tensors, L-sequences, L-systems

Definition 5.1 *A* **L-tensor L** *on a Riemannian (or pseudo-Riemannian) manifold is a torsionless conformal Killing two-tensor with pointwise simple and real eigenvalues.*

The following theorem shows the interest of this definition.

Theorem 5.1 *Let $\mathbf{L} = (L^j_i)$ be a symmetric two-tensor. Then the tensors $(\mathbf{K}_a) = (\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_{n-1})$ defined by the* **L-sequence**

$$\mathbf{K}_0 = \mathbf{I}, \quad \mathbf{K}_a = \frac{1}{a}\,(\mathbf{K}_{a-1}\mathbf{L})\,\mathbf{I} - \mathbf{K}_{a-1}\,\mathbf{L}, \quad a>1 \tag{15}$$

are n independent Killing tensors with common normal[7] *eigenvectors if and only if $\mathbf{L}$ is a L-tensor.*

Indeed, n independent Killing tensors with common normal eigenvectors define a **Killing-Stäckel space** $\mathcal{K}$, containing the metric tensor and whose elements commute in the Schouten brackets of symmetric tensors; in other

[6] Orthogonal separable systems are also called *Stäckel systems*.

[7] This means that each eigenvector field is orthogonal to a family of surfaces (a foliation) of submanifolds of codimension 1. All these foliations form an **orthogonal web**.

words, they give rise to a n-space of geodesic quadratic first integrals in involution. The geodesic flow is then completely integrable. Furthermore, the orthogonal web determined by the common eigenvectors is a **separable web**, in the sense that any orthogonal coordinate system $\underline{q} = (q^i)$ **adapted** to this web is separable (it separates, additively, the geodesic Hamilton-Jacobi equation): these coordinates are such that the web is locally represented by equations $q^i = constant$ or equivalently, the vector fields $\partial_i = \partial/\partial q^i$ are common eigenvectors (the one-forms dq^i are common eigenforms). Let us call **L-system** any Stäckel system of this kind.

It is remarkable the fact that for a L-system all the Killing tensors underlying the separation, and forming the Killing-Stäckel space $\mathcal{K}$, are all constructed <u>in</u> <u>an</u> <u>algebraic</u> <u>way</u> by means of the L-sequence (15).

The general theory of the orthogonal separable systems shows that a potential V is separable w.r.to a Killing-Stäckel space $\mathcal{K}$ if and only if for any arbitrary element $\mathbf{K} \in \mathcal{K}$ with simple eigenvalues [8] the one-form $\mathbf{K}\, dV$ is closed (may be exact),

$$d(\mathbf{K}\, dV) = 0, \qquad \mathbf{K}\, dV = dU.$$

If this condition is satisfied, then by taking a basis $(\mathbf{K}_a)$ of $\mathcal{K}$ and the functions V_a such that $\mathbf{K}_a\, dV = dV_a$,[9] then the functions

$$H_a \doteq \tfrac{1}{2} K_a^{ij}\, p_i p_j + V_a$$

form a system of independent first integrals in involution.

All the results above applied to a L-system, show that, for instance, $\mathbf{K}_1$ in the L-sequence is a characteristic Killing tensor, so that a potential V is separable in a L-system if and only if [10]

$$d((\mathbf{L})\, dV - \mathbf{L}\, dV) = 0. \tag{16}$$

6 Cofactor and bi-cofactor systems are L-systems

The sentence which is taken as a title of this last section is true under certain conditions.

Theorem 6.1 *Let $(Q, \mathbf{g}, \mathbf{F})$ be a cofactor system, whose J-tensor $\mathbf{J}$ has pointwise simple eigenvalues. Then the geodesically equivalent Hamiltonian system is a L-system generated by the L-tensor $\mathbf{B}$, the B-tensor associated with $\mathbf{J}$, if and only if $\mathbf{F} = -\nabla W$ (i.e., the cofactor system is itself a Lagrangian system).*

Definition 6.1 *A* **bi-cofactor system** *(or* **cofactor-pair system***) is a holonomic system $(Q, \mathbf{g}, \mathbf{F})$ which is a cofactor-system in two distinct ways:*

$$\mathbf{F} = -\mathbf{A}^{-1}\nabla V = -A^{-1}\nabla V, \tag{17}$$

[8] It can be proved that such a tensor, called **characteristic tensor**, always exists.

[9] Indeed, it can be proved that if $\mathbf{K}\, dV$ is closed or exact, the same happens for all elements of $\mathcal{K}$.

[10] For $Q =^n$ and in Cartesian coordinates, this is known as the *Bertrand-Darboux equation.*

where $\mathbf{A} = \mathbf{J}$ *and* $A = J$*, being* $\mathbf{J} = (J_i^j)$ *and* $J = (J_i^j)$ *two non-singular (and non-trivial) J-tensors w.r.to the metric* $\mathbf{g}$.

Theorem 6.2 *If a bi-cofactor system is such that the tensor* $\bar{\mathbf{J}} = J\,\mathbf{J}^{-1}$ *has pointwise real and simple eigenvalues, then the equivalent Hamiltonian system* $(Q, \bar{\mathbf{g}}, \bar{H})$*, where* $\bar{\mathbf{g}}$ *is the equivalent metric determined by* $\mathbf{J}$*, is a L-system generated by the L-tensor* $\bar{\mathbf{J}}$*.*

References

[1] S. Benenti, *Special Symmetric two tensors, Equivalent Dynamical Systems, Cofactor and Bi-cofactor systems*, Tutorial Papers SPT-2004, Acta Applicandae Mathematicae, to appear.

[2] Levi-Civita, *Sulle trasformazioni delle equazioni dinamiche*, Ann. di Matem. **24** (1896).