Special Symmetric Two-Tensors, Equivalent Dynamical Systems, Cofactor and Bi-Cofactor Systems

Sergio Benenti*

Abstract

A general analysis of special classes of symmetric two-tensor on Riemannian manifolds is provided. These tensors arise in connection with special topics in differential geometry and analytical mechanics: geodesic equivalence and separation of variables. It is shown that they play an important role in the theory of correspondent (or equivalent) dynamical systems of Levi-Civita. By applying some new developments of this theory, it is shown that the recent notions of cofactor and cofactor-pair systems arise in a natural way, as non-Lagrangian systems having a Lagrangian equivalent. This circumstance extends the Hamiltonian methods, including the separation of variables of the Hamilton-Jacobi equation, to a special class of non-conservative systems. In this extension the case of indefinite metrics, may occur. Hence, it is shown that also pseudo-Riemannian geometry plays an important role also in classical mechanics.

1 Introduction

The aim of this tutorial paper is twofold: (i) To propose a unified and general introduction to the **symmetric two-tensors of special kind** which, in recent years, have been employed in the variable separation theory of the Hamilton-Jacobi and Schrödinger equations, in the geodesic equivalence theory, in the study of factor and bi-cofactor (or cofactor-pair) systems, bi-Hamiltonian systems, etc. (ii) To show how the notions of cofactor system and bi-cofactor system arise in a simple and natural way from the theory of **equivalent** (or **correspondent**) **dynamical systems**, whose foundation is due to Levi-Civita (1896) and in which special symmetric two-tensors play a significant fundamental role.
The leading ideas are explained in the following items.
(I) A holonomic mechanical system with time-independent ideal constraints and with n degrees of freedom is characterized by a triple $(Q, \mathbf{g}, \mathbf{F})$, where Q is the n-dimensional **configuration manifold** covered by local **Lagrangian coordinate systems** $\underline{q} = (q^i)$ and endowed with a metric tensor $\mathbf{g} = (g_{ij})$, representing the kinetic energy

$$K = \tfrac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j,$$

and where $\mathbf{F} = (F^i)$ is a vector field on Q representing the **Lagrangian force** i.e., all the active forces acting on the system (we consider the case of time-independent and velocity-independent forces). In the present paper by **dynamical system** we mean a triple of this kind.

*Research sponsored by the Dept. of Mathematics, University of Turin, and by INDAM-GNFM.

As shown by Lagrange, the motions are locally represented by the solutions $\underline{q}(t)$ of the second-order differential equations

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}^i} - \frac{\partial K}{\partial q^i} = F_i, \qquad F_i \doteq g_{ij}\, F^j, \tag{1}$$

which assume the **Newtonian form**

$$\frac{d^2 q^i}{dt^2} + \Gamma^i_{hj}\, \frac{dq^h}{dt}\,\frac{dq^j}{dt} = F^i \qquad \Longleftrightarrow \qquad \mathbf{a} = \mathbf{F}, \tag{2}$$

being

$$a^i \doteq \frac{d^2 q^i}{dt^2} + \Gamma^i_{hj}\, \frac{dq^h}{dt}\,\frac{dq^j}{dt}$$

the **absolute acceleration** of the dynamical system and Γ^i_{hj} the Christoffel symbols i.e., the coefficients of the Levi-Civita connection associated with the metric tensor. This last second-order system is equivalent to the first-order dynamical system

$$\mathbf{X} = \begin{cases} \dfrac{dq^i}{dt} &= v^i, \\ \dfrac{dv^i}{dt} &= -\,\Gamma^i_{hj}\, v^h v^j + F^i, \end{cases}$$

on the tangent bundle TQ, with coordinates $(\underline{q}, \underline{v}) = (q^i, v^i) = (q^i, \dot{q}^i)$.
(II) If the force $\mathbf{F}$ is conservative i.e., it is the gradient of a **potential energy** V (a real-valued smooth function on Q),

$$\mathbf{F} = -\nabla V \qquad \Longleftrightarrow \qquad F^i = \,-\,g^{ij}\,\partial_j V,$$

then we have a **Lagrangian system**. Eqs. (1) assume the well-known form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0,$$

where

$$L = \tfrac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j - V$$

is the **Lagrangian function** on TQ. In this case we can apply alternative methods of integration by passing to the **Hamiltonian function** on the cotangent bundle T^*Q, with coordinates $(\underline{q}, \underline{p}) = (q^i, p_j)$,

$$H = \tfrac{1}{2}\, g^{ij}\, p_i p_j + V.$$

This function gives rise to a **Hamiltonian system** (i.e., to first-order **Hamilton equations**) and to a **Hamilton-Jacobi equation**.
(III) Apparently, if the system is not Lagrangian, then the Hamiltonian methods of integration cannot be applied. In fact, this is not true in general. Indeed, although for very special cases, we can get a Hamiltonian description even for non-Lagrangian systems.
(IV) Let us call **equivalent** (or, according to Levi-Civita, **correspondent**) two dynamical systems $(Q, \mathbf{g}, \mathbf{F})$ and $(Q, \bar{\mathbf{g}}, \bar{\mathbf{F}})$ with the same configuration manifold Q, if their motions on Q locally coincide as unparameterized curves: more precisely, if a solution $\underline{q}(t)$ of Eqs. (1), or (2) (related to the first system) becomes a solution of the corresponding equations associated with the second system, up to a time-reparameterization. It is remarkable (but

not evident) the fact that the relationship between the two times involved, t and $\bar{t}$, is given by equation

$$\frac{dt}{d\bar{t}} = \mu, \tag{3}$$

where μ *depends only on the point of* Q *i.e., it is a nowhere-vanishing smooth function on* Q.
(V) As a consequence of the above-given definition, if we are able to characterize in a suitable way the equivalence of two systems, or the existence of a non-trivial equivalent of a given system, then we can consider the following process. If we have to deal with a non-Lagrangian system $(Q, \mathbf{g}, \mathbf{F})$, then we look for an equivalent one $(Q, \bar{\mathbf{g}}, \bar{\mathbf{F}})$ which is Lagrangian: $\bar{\mathbf{F}} = -\bar{\nabla} V$. If our search is successful, and we are able to integrate this new system, for instance by solving the Hamilton equations or the Hamilton-Jacobi equation, associated with the Hamiltonian

$$\bar{H} = \tfrac{1}{2}\, \bar{g}^{ij}\, p_i p_j + V.$$

then we determine the motions of the original dynamical system simply by changing the time-parameter according to Eq. (3).
As we shall see, this quite general process works very well and effectively for cases in which the metric tensors $\mathbf{g}$ and $\bar{\mathbf{g}}$ are themselves **equivalent**, in the sense that they have locally the same geodesics as unparameterized curves. Thus, the study of the geodesic equivalence becomes crucial. The result is surprising: a given dynamical system $(Q, \mathbf{g}, \mathbf{F})$ admits a Lagrangian equivalent system if and only if it is **quasi-Lagrangian** i.e., the force $\mathbf{F}$ is of the kind

$$\mathbf{F} = -\mathbf{A}^{-1} \nabla V,$$

where $\mathbf{A} = (A_i^j)$ is the **cofactor**,

$$\mathbf{A} = \mathrm{cof}\, \mathbf{J},$$

of a (1,1)-tensor $\mathbf{J}$ of a special kind, called **special conformal Killing tensor**. This kind of non-Lagrangian systems are the so-called **cofactor systems**. A dynamical system which is a cofactor system in two distinct ways,

$$\mathbf{F} = -\mathbf{A}^{-1} \nabla V = -\widetilde{\mathbf{A}}^{-1} \nabla \widetilde{V},$$

is called **cofactor-pair system** (or **bi-cofactor system**). Three facts are remarkable: (i) Under the condition that the eigenvalues of certain (1,1)-tensors are simple, the Hamilton-Jacobi equations of the equivalent systems are separable in orthogonal coordinates i.e., the equivalent Hamiltonian systems are Stäckel systems. (ii) These Stäckel systems are in fact of a special kind, here called **L-systems**, represented by a torsionless conformal Killing tensor with simple eigenvalues (here called **L-tensor**) introduced some years ago in [Benenti, 1992,1993]. (iii) The equivalent metric $\bar{\mathbf{g}}$ may not be positive-definite. Hence, it is shown that pseudo-Riemannian geometry is also involved in classical mechanics.
The notion of special conformal Killing tensor has been introduced by M. Crampin, W. Sarlet and G. Thompson (2000) (see Remark 3.1 below). Cofactor and cofactor-pair systems have been introduced (1999-2004) by S. Rauch-Wojciechowski, K. Marciniak, H. Lundmark and C. Waksjö (University of Linköping) (see the Bibliography for references). Most of their important and interesting results are here re-interpretable in a unified perspective.

2 Basic definitions

The Poisson bracket $\{F, G\}$ of two functions on a cotangent bundle T^*Q is defined by

$$\{F, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i}.$$

If $\mathbf{L} = (L^{i\ldots j})$ is a contravariant symmetric tensor on Q, then we denote by $P(\mathbf{L})$ the polynomial function on T^*Q defined by

$$P(\mathbf{L}) \doteq L^{i\ldots j}\, p_i \ldots p_j.$$

We define the symmetric tensor product $\odot$ and the Lie-Nijenhuis bracket $[\ ,\ ]$ of symmetric tensors by setting

$$P(\mathbf{L} \odot \mathbf{K}) = P(\mathbf{L})\, P(\mathbf{K}), \quad P([\mathbf{L}, \mathbf{K}]) = \{P(\mathbf{L}), P(\mathbf{K})\}.$$

Two tensors are said to be **in involution** if $[\mathbf{L}, \mathbf{K}] = 0$.
Our *arena* will be a n-dimensional Riemannian manifold $(Q, \mathbf{g})$, where the metric tensor is of any signature. Since we shall work with other metric tensors, $\mathbf{g} = (g_{ij})$ will play the role of **basic metric tensor**. The contravariant metric tensor will be denoted by $\mathbf{G} = (g^{ij})$: $[g^{ij}] = [g_{ij}]^{-1}$. The operation of rising and lowering indices will be performed by the basic metric tensor components, unless specified differently.
A vector field $\mathbf{X}$ is a **Killing vector** if

$$[\mathbf{X}, \mathbf{G}] = 0 \qquad \Longleftrightarrow \qquad \{P(\mathbf{X}), P(\mathbf{G})\} = 0.$$

An equivalent definition is

$$\nabla_{(i}X_{j)} = 0,$$

where ∇_i is the covariant derivative w.r.to the Levi-Civita connection. A symmetric two-tensor $\mathbf{K} = (K^{ij})$ is a **conformal Killing tensor** (CKT) if

$$[\mathbf{K}, \mathbf{G}] = \, -2\, \boldsymbol{\alpha} \odot \mathbf{G}, \qquad \Longleftrightarrow \qquad \{P(\mathbf{K}), P(\mathbf{G})\} = \, -2\, P(\boldsymbol{\alpha})\, P(\mathbf{G}),$$

and $\boldsymbol{\alpha}$ is called the **conformal factor**. An equivalent definition is

$$\nabla_{(h}K_{ij)} = \alpha_{(h}g_{ij)},$$

or

$$\nabla_h K_{ij}\, x^h x^i x^j = \alpha_h x^h\; g_{ij}\, x^i x^j, \qquad \forall (x^i) \in \mathbb{R}^n.$$

A CKT $\mathbf{K}$ is said to be of **gradient-type** if $\alpha_i = \partial_i f$, of **trace-type** if $\alpha_i = \partial_i \mathrm{tr}\, \mathbf{K}$, $\mathrm{tr}\, \mathbf{K} \doteq g_{ij}K^{ij} = K^i_i$. If $\boldsymbol{\alpha} = 0$ i.e.,

$$[\mathbf{K}, \mathbf{G}] = 0,$$

then $\mathbf{K}$ is a **Killing tensor** (KT). An equivalent definition is

$$\nabla_{(h}K_{ij)} = 0. \tag{4}$$

It can be shown that, if a symmetric two-tensor $\mathbf{K} = (K^{ij})$ is diagonalized in orthogonal coordinates (q^i),

$$g^{ij} = g^{ii}\, \delta^{ij}, \qquad K^{ij} = u^i\, g^{ii}\, \delta^{ij},$$

then **K** is a KT if and only if

$$\partial_i u^j = (u^i - u^j)\, \partial_i \log |g^{jj}|. \tag{5}$$

It is a CKT if and only if

$$\partial_i u^j = (u^i - u^j)\, \partial_i \log |g^{jj}| + \alpha_i, \qquad \alpha_i = \partial_i u^i. \tag{6}$$

Note that (u^i) are the eigenvalues of **K** i.e., the roots of equation $\det[K^{ij} - u\, g^{ij}] = 0$.

REMARK 2.1. The **Killing equations** (4), for a tensor of any order, and Eqs. (5) are written in [Levi-Civita, 1896a] and in [Levi-Civita, 1896b].

3 Special symmetric two-tensors

We consider <u>four</u> kinds of symmetric 2-tensors. The first three are defined by the following **characteristic equations**

$$\nabla_h A_{ij} = \mu_h\, A_{ij} - \tfrac{1}{2}\, (\mu_j\, A_{hi} + \mu_i\, A_{hj}) \tag{7}$$

$$\nabla_h B_{ij} = \, -\tfrac{1}{2}\, (\mu_j\, B_{hi} + \mu_i\, B_{hj}) \tag{8}$$

$$\nabla_h J_{ij} = \tfrac{1}{2}\, (\alpha_i\, g_{jh} + \alpha_j\, g_{ih}) \tag{9}$$

The fourth special symmetric two-tensor, called **L-tensor**, is defined as follows: *a L-tensor is a torsionless CKT with pointwise real simple eigenvalues.*

REMARK 3.1. A-tensors are Killing tensor and J-tensors are conformal Killing tensors. A B-tensor is neither a KT nor a CKT; however, B-tensors will play a remarkable role in our investigation. A-tensors have been introduced by Levi-Civita (1896), J-tensors by Crampin, Sarlet & Thompson (2000, 2001, 2003): they have been called **special conformal Killing tensors** in [Crampin, 2003a].[1] L-tensors have been introduced in [Benenti, 1992, 1993]. As we shall see, they form a special subclass of J-tensors.

NOTATION. In the following we denote by boldface letters **A**, **B**, **J** the (1,1)-form of these tensors, $\mathbf{A} = (A_i^j)$, $A_j^i = g^{ih} A_{hj}$, etc. In this form they are interpreted as endomorphisms on the space of vector fields **X** or on the space of one-forms on Q:

$$\begin{aligned} \mathbf{Y} = \mathbf{A}\mathbf{X} \qquad &\Longleftrightarrow \qquad Y^i = A_j^i X^j, \\ \phi = \mathbf{A}\psi \qquad &\Longleftrightarrow \qquad \phi_i = A_i^j \psi_j \end{aligned}$$

For defining the algebraic composition **BA** of these tensors we consider them as linear operators on vectors, so that

$$(\mathbf{BAX})^i = B_j^i\, A_h^j\, X^h, \qquad (\mathbf{BA})_h^i = B_j^i\, A_h^j.$$

The two main invariants, the determinant and the trace, are defined by $\det \mathbf{J} = \det[J_i^j]$ and $\operatorname{tr} \mathbf{J} = J_i^i = J^{ij} g_{ij}$. The cofactor (or adjoint) tensor $\operatorname{cof} \mathbf{J}$ is defined by $\mathbf{J}\,(\operatorname{cof} \mathbf{J}) =$

[1] According to V. Matveev (private communication), J-tensors and B-tensors seem to be present in the book by [Sinyukov, 1979] (not consulted). At least for J-tensors, in the survey by [Mikeš, 1996] this is confirmed.

$(\mathrm{cof}\,\mathbf{J})\mathbf{J} = (\det\mathbf{J})\,\mathbf{I}$ or by $\mathrm{cof}\,\mathbf{J} = \det\mathbf{J}\,\mathbf{J}^{-1}$ when $\det\mathbf{J} \neq 0$. The components of $\mathbf{A} = \mathrm{cof}\,\mathbf{J}$ are defined by

$$A_i^h \doteq \frac{\partial \det\mathbf{J}}{\partial J_h^i}.$$

with

$$\det\mathbf{J} = \frac{1}{n!}\,\delta^{i_1\ldots i_n}_{j_1\ldots j_n}\,J^{j_1}_{i_1}\ldots J^{j_n}_{i_n}$$

where

$$\delta^{i\ldots h}_{j\ldots k} = n!\,\delta^{[i}_j\ldots\delta^{h]}_k = n!\,\delta^i_{[j}\ldots\delta^h_{k]}$$

is the **generalized Kronecker symbol** of order n.

REMARK 3.2. Eqs. (7) (8) (9) are equivalent to

$$\begin{aligned}\nabla_h A_i^j &= \mu_h\,A_i^j - \tfrac{1}{2}\,(\mu^j\,A_{hi} + \mu_i\,A_h^j),\\ \nabla_h A^{ij} &= \mu_h\,A^{ij} - \tfrac{1}{2}\,(\mu^j\,A_h^i + \mu^i\,A_h^j),\end{aligned} \tag{10}$$

$$\begin{aligned}\nabla_h B_i^j &= -\tfrac{1}{2}\,(\mu^j\,B_{hi} + \mu_i\,B_h^j),\\ \nabla_h B^{ij} &= -\tfrac{1}{2}\,(\mu^j\,B_h^i + \mu^i\,B_h^j),\end{aligned} \tag{11}$$

$$\begin{aligned}\nabla_h J_i^j &= \tfrac{1}{2}\,(\alpha_i\,\delta_h^j + \alpha^j\,g_{ih}),\\ \nabla_h J^{ij} &= \tfrac{1}{2}\,(\alpha^i\,\delta_h^j + \alpha^j\,\delta_h^i),\end{aligned} \tag{12}$$

where $\mu^i = g^{ij}\,\mu_j$ and $\alpha^i = g^{ij}\,\alpha_j$.
In our investigation, J-tensors will play a prominent role, since:
(i) J-tensors on a Riemannian manifold form a linear space. This property does not hold for A-tensors and B-tensors.
(ii) Without loss of generality, we can assume that $\mathbf{J} = (J_i^j)$ is non-singular, except in a closed **singular subset** of Q. Indeed, if we consider the pencil

$$\mathbf{J}_x = \mathbf{J} + x\,\mathbf{I}, \quad x \in \mathbb{R},$$

then for all $x \in \mathbb{R}$, $\mathbf{J}_x$ is a J-tensor. If $u^i(\underline{q})$ are the eigenvalues of $\mathbf{J}$, then we choose $x \neq u^i(\underline{q}_0)$ at some point $\underline{q}_0$.
(iii) A non-singular J-tensor generates, in a pure algebraic way, a A-tensor and a B-tensor:

THEOREM 3.1. *If* $\mathbf{J}$ *is a non-singular J-tensor,* $\det\mathbf{J} \neq 0$*, then*

$$\mathbf{A} \doteq \mathrm{cof}\,\mathbf{J}, \qquad \mathbf{B} \doteq \mathbf{J}^{-1} = \tfrac{1}{\mu}\,\mathbf{A} \tag{13}$$

are a non-singular A-tensor and a non-singular B-tensor, respectively, with

$$\mu_i = \partial_i \log|\mu| = B_i^j\,\alpha_j, \qquad \alpha_i = J_i^j\,\mu_j, \qquad \mu \doteq \det\mathbf{J} \tag{14}$$

From (13) it follows that

$$\mathbf{A} = \det\mathbf{J}\,\mathbf{J}^{-1} = \mu\,\mathbf{J}^{-1}, \qquad \mathbf{AJ} = \mathbf{JA} = \mu\,\mathbf{I}, \qquad \mathbf{J} = \mu\,\mathbf{A}^{-1} \tag{15}$$

To prove this theorem we need two lemmas:

LEMMA 3.1. *A J-tensor is a conformal Killing tensor of* **trace-type***,*

$$\alpha_i = \partial_i \mathrm{tr}\,\mathbf{J},$$

and **torsionless**,

$$J^h_{[i}\nabla_{|h|}J^k_{j]} - J^k_l \nabla_{[i}J^l_{j]} = 0.$$

Proof. This follows, by a straightforward calculation, from Eqs. (3).

REMARK 3.3. The torsion of a (1,1)-tensor $\mathbf{X}$ is the (1,2)-tensor defined by

$$H^k_{ij}(\mathbf{X}) \doteq X^h_{[i}\partial_{|h|}X^k_{j]} - X^k_l \partial_{[i}X^l_{j]}.$$

However, the partial derivatives ∂_i can be replaced by the covariant derivative ∇_i w.r.to any symmetric connection (see Appendix A).

LEMMA 3.2. *For any torsionless tensor* $\mathbf{X}$ *equation*

$$\det\mathbf{X}\, d(\mathrm{tr}\,\mathbf{X}) = \mathbf{X}\, d(\det\mathbf{X}), \tag{16}$$

equivalent to $\det\mathbf{X}\,\partial_i \mathrm{tr}\,\mathbf{X} = X^h_i\,\partial_h \det\mathbf{X}$, *holds.*[2]

Proof. The torsionless condition means that

$$X^h_i\,\nabla_h X^k_j - X^k_l\,\nabla_i X^l_j = X^h_j\,\nabla_h X^k_i - X^k_l\,\nabla_j X^l_i$$

i.e.,

$$X^h_i\,\nabla_h X^k_j = X^k_l\,\nabla_i X^l_j + X^h_j\,\nabla_h X^k_i - X^k_l\,\nabla_j X^l_i.$$

Assume $\xi \doteq \det\mathbf{X} \neq 0$ and consider $\mathbf{C} \doteq \mathrm{cof}\,\mathbf{X} = \xi\,\mathbf{X}^{-1}$. We know that $\partial_k\xi = \nabla_k X^i_j\, C^j_i$. Then,

$$\begin{aligned} X^h_i\,\partial_h\xi &= X^h_i\,\nabla_h X^k_j\,C^j_k = (X^k_l\,\nabla_i X^l_j + X^h_j\,\nabla_h X^k_i - X^k_l\,\nabla_j X^l_i)\,C^j_k \\ &= (\delta^j_l\,\nabla_i X^l_j + \delta^h_k\,\nabla_h X^k_i - \delta^j_l\,\nabla_j X^l_i)\,\xi = \xi\,\nabla_i X^l_l. \qquad \square \end{aligned}$$

Proof of Theorem 3.1. If $\mathbf{A} \doteq \mathrm{cof}\,\mathbf{J}$, then $\mathbf{AJ} = \mathbf{JA} = \mu\,\mathbf{I}$, $\mu = \det\mathbf{J}$. Thus,

$$\nabla_h(J^i_j A^j_k) = A^j_k\,\nabla_h J^i_j + J^i_j\,\nabla_h A^j_k = \delta^i_k\,\partial_h\mu.$$

If we multiply the last equation by A^r_i, then we find

$$\begin{aligned} \mu\,\nabla_h A^r_k &= A^r_k\,\partial_h\mu - A^r_i\,A^j_k\,\nabla_h J^i_j = A^r_k\,\partial_h\mu - \tfrac{1}{2}\,A^r_i\,A^j_k\,(\alpha_j\,\delta^i_h + \alpha^i\,g_{jh}) \\ &= A^r_k\,\partial_h\mu - \tfrac{1}{2}\,\mu(\mu_k\,A^r_h + \mu^r\,A_{kh}). \end{aligned}$$

This proves that $\mathbf{A}$ is a A-tensor with $\mu_i = \partial_i \log|\mu|$. The proof that $\mathbf{B} = \mathbf{J}^{-1}$ is a B-tensor is similar. Moreover, since $\alpha_i = \partial_i \mathrm{tr}\,\mathbf{J}$, Eq. (16) for $\mathbf{X} = \mathbf{J}$ can be written in the following two equivalent forms:

$$\alpha_i = J^k_i\,\partial_k \log|\mu|, \qquad A^j_i\,\alpha_j = \partial_i\mu.$$

This proves Eqs. (14). $\square$

[2]See [Bolsinov & Matveev, 2003], §3, Lemma 1, and [Crampin & Sarlet, 2001], §4. Eq. (16) plays a remarkable role in several other papers.

REMARK 3.4. The mapping from non-singular J-tensors to A-tensors defined by $\mathbf{A} = \operatorname{cof}\mathbf{J}$ can be reversed in the following way:

$$\mathbf{J} \doteq \begin{cases} (\det \mathbf{A})^{\frac{1}{n-1}}\, \mathbf{A}^{-1} & \begin{cases} n \text{ even} \\ n \text{ odd and } \det \mathbf{A} > 0, \end{cases} \\ (-\det \mathbf{A})^{\frac{1}{n-1}}\, \mathbf{A}^{-1} & n \text{ odd and } \det \mathbf{A} < 0. \end{cases} \tag{17}$$

REMARK 3.5. The property that $\mathbf{A} = \operatorname{cof}\mathbf{J}$ is an A-tensor holds true also in the case of a singular J-tensor $\mathbf{J}$.

REMARK 3.6. The factor forms α_i and μ_i are exact. The vector field (α^i) is, up to a constant factor, the divergence of $\mathbf{J}$:

$$\alpha^i = \tfrac{2}{n+1}\, \nabla_j J^{ji}.$$

This follows from the expression of $\nabla_j J^{ji}$.

REMARK 3.7. Since $\operatorname{cof}\mathbf{B} = (\det \mathbf{B})\, \mathbf{B}^{-1} = (\det \mathbf{B})\, \mu\, \mathbf{A}^{-1}$ and $\det \mathbf{B} = \det(\mathbf{J}^{-1}) = \frac{1}{\mu}$, we have

$$\operatorname{cof}\mathbf{B} = \mathbf{A}^{-1}$$

THEOREM 3.2. *A B-tensor is torsionless:*

$$B^h_{[i} \nabla_{|h|} B^k_{j]} - B^k_l \nabla_{[i} B^l_{j]} = 0.$$

Proof. The tensor

$$B^h_i \nabla_h B^k_j - B^k_l \nabla_i B^l_j = \, -\tfrac{1}{2}\, (B^h_i\, (\mu^k\, B_{hj} + \mu_j\, B^k_h) - B^k_l\, (\mu^l\, B_{ij} + \mu_j\, B^l_i))$$

is symmetric w.r.to the indices (i, j). □

4 Equivalent metrics

Special tensors have a first important application in the geodesic equivalence problem, which dates back to [Painlévé, 1894] and [Levi-Civita, 1896b].

Let us recall that a **geodesic** on a Riemannian manifold $(Q, \mathbf{g})$ is a curve $q^i = q^i(\tau)$ (representing a *motion* of a point on Q with a time measured by the parameter τ) whose tangent vector (*velocity*) $dq^i/d\tau$ remains tangent under the parallel displacement along the curve itself. This is equivalent to say that the **acceleration** of such a motion is at each point (i.e., for each value of the parameter τ) parallel to the velocity:

$$\frac{d^2q^i}{d\tau^2} + \Gamma^i_{hj}\, \frac{dq^h}{d\tau}\, \frac{dq^j}{d\tau} = \alpha(\tau)\, \frac{dq^i}{d\tau}.$$

Up to a transformation of the parameter $t = t(\tau)$, given by the differential equation

$$\frac{d^2t}{d\tau^2} = \alpha(\tau)\, \frac{dt}{d\tau},$$

the acceleration vanishes, so that the differential equations of the geodesics becomes

$$\frac{d^2q^i}{dt^2} + \Gamma^i_{hj}\,\frac{dq^h}{dt}\,\frac{dq^j}{dt} = 0.$$

The new time-parameter t is called **affine parameter**; indeed it is defined up to an affine transformation $t' = at + b$ with constant coefficients. For non-null geodesics, among all possible affine parameters, we can chose those for which the velocity is a unit vector

$$g_{ij}\,\frac{dq^i}{dt}\,\frac{dq^j}{dt} = \pm 1.$$

In this case the affine parameter t is called **proper time** and it is defined up to a translation $t' = t + b$. For space-like geodesics (+1 in the last formula), this is just the arc-length of the geodesic, measured from a fixed point.

DEFINITION 4.1. Two metric tensors $\mathbf{g} = (g_{ij})$ and $\bar{\mathbf{g}} = (\bar{g}_{ij})$, on the same manifold Q, are said to be **equivalent** if their geodesics locally coincide as unparameterized curves. Any metric of the kind $c\mathbf{g}$, $c \neq 0 \in \mathbb{R}$, is equivalent $\mathbf{g}$. Such a metric is called **trivial equivalent metric**. The following statements always concern with non-trivial equivalent metrics.

THEOREM 4.1. *A metric* $\mathbf{g}$ *admits a equivalent metric* $\bar{\mathbf{g}}$ *if and only if it admits a non-singular B-tensor* (hence, *a non-singular J-tensor or a non-singular A-tensor*).

Proof. It is known [Schouten, 1954] [Eisenhart, 1949] that the Christoffel symbols of two equivalent metrics are related by a **projective transformation**,

$$\bar{\Gamma}^j_{hi} = \Gamma^j_{hi} - \tfrac{1}{2}\,(\delta^j_h\,\mu_i + \delta^j_i\,\mu_h), \tag{18}$$

with $\mu_i = \partial_i \log|\mu|$, where μ is a function on Q such that

$$\mu = \frac{dt}{d\bar{t}} \tag{19}$$

where t and $\bar{t}$ are two affine parameters of the geodesics of $\mathbf{g}$ and of $\bar{\mathbf{g}}$, respectively. Due to (18), for any two-tensor we have

$$\bar{\nabla}_h T_{ij} = \nabla_h T_{ij} + \mu_h\,T_{ij} + \tfrac{1}{2}\,(T_{hj}\,\mu_i + T_{ih}\,\mu_j).$$

Due to the uniqueness of a Levi-Civita connection, the connection $\bar{\nabla}_h$ is the Levi-Civita connection of a new metric $\bar{g}_{ij}$ if and only if $\bar{\nabla}_h \bar{g}_{ij} = 0$ i.e.,

$$\nabla_h \bar{g}_{ij} + \mu_h\,\bar{g}_{ij} + \tfrac{1}{2}\,(\bar{g}_{hj}\,\mu_i + \bar{g}_{ih}\,\mu_j) = 0.$$

Thus,

$$\mu\,\nabla_h \bar{g}_{ij} + \partial_h\mu\,\bar{g}_{ij} + \tfrac{1}{2}\,(\bar{g}_{hj}\,\partial_i\mu + \bar{g}_{ih}\,\partial_j\mu) = 0.$$

If we define

$$B_{ij} \doteq \mu\,\bar{g}_{ij}, \tag{20}$$

then this last equation is equivalent to:

$$\nabla_h B_{ij} = -\tfrac{1}{2}\,(B_{hj}\,\mu_i + B_{ih}\,\mu_j).$$

This is just the characteristic equation of a B-tensor. □
For alternative proofs of this property and further results on this topic, see [Bolsinov & Matveev, 2003], [Crampin, 2003b].

REMARK 4.1. The function μ relating two affine parameters according to Eq. (19), is defined up to a constant factor. It is remarkable the fact that we can *normalize* this function by setting

$$\mu = \det \mathbf{J} \tag{21}$$

To prove this, let us recall that

$$\Gamma^i_{ij} = \tfrac{1}{2}\, g^{ik}\, \partial_i g_{jk} = \tfrac{1}{2}\, \partial_j \log|g|, \qquad \overline{\Gamma}^i_{ij} = \tfrac{1}{2}\, \partial_j \log|\bar{g}|,$$

where

$$g \doteq \det[g_{ij}]\ , \qquad \bar{g} \doteq \det[\bar{g}_{ij}]\ .$$

Hence,

$$\Gamma^i_{ij} - \overline{\Gamma}^i_{ij} = \tfrac{1}{2}\, \partial_j \log\left|\frac{g}{\bar{g}}\right|.$$

But from (1) we get $\Gamma^i_{ij} - \overline{\Gamma}^i_{ij} = \frac{1}{2}\,(n+1)\,\mu_j$. As a consequence,

$$(n+1)\,\mu_i = (n+1)\,\partial_i \log \mu = \partial_i \log\left|\frac{g}{\bar{g}}\right|,$$

and

$$\mu^{n+1} = c\,\frac{g}{\bar{g}}.$$

On the other hand, from (20), $B^j_i = \mu\, \bar{g}_{hi}\, g^{hj}$, so that

$$\det \mathbf{J} = (\det \mathbf{B})^{-1} = \frac{g}{\bar{g}}\, \mu^{-n} = \tfrac{1}{c}\, \mu.$$

We normalize μ by choosing $c = 1$.
As a consequence of Eqs. (20) and (21), we can express the components of an equivalent metric as follows,

$$\bar{g}_{ij} = \tfrac{1}{\mu}\, B_{ij} = \tfrac{1}{\mu^2}\, A_{ij}, \qquad \bar{g}^{ij} = \mu\, J^{ij}, \qquad \mu = \det \mathbf{J} = \frac{dt}{d\bar{t}}$$

REMARK 4.2. If $\mathbf{g}$ admits an equivalent metric $\bar{\mathbf{g}}$, then it admits a one-parameter family of equivalent metrics $\bar{\mathbf{g}}^c$. Indeed, if $\mathbf{J}$ is the J-tensor generating $\bar{\mathbf{g}}$, then we can consider all equivalent metrics determined by the pencil $\mathbf{J}_c = \mathbf{J} + c\,\mathbf{I}$. Of course the parameter $c \in \mathbb{R}$ must not coincide with the eigenvalues of $\mathbf{J}$. This remark is in accordance with a statement of Levi-Civita, who was able to give the general form of an equivalent metric depending on a parameter [Levi-Civita, 1896b], p. 287-288.

5 Properties of the equivalent metrics

In this section we illustrate some basic properties of the geodesic equivalence, needed in the following discussion. The relation between geodesic equivalence and integrability was observed, in recent times, by [Matveev & Topalov, 1998] (within a global description of the structure of the geodesic equivalence) and later on in [Crampin, 2003b]. Further results and a bibliography of the wide literature on this topic can be found in [Mikeš, 1996], [Topalov & Matveev, 2003], [Bolsinov & Matveev, 2003], [Aminova, 2003] and, more recently, in [Matveev, 2004], [Burns & Matveev, 2004].
Let $\mathbf{J}$ be a non-singular J-tensor w.r.to a basic metric $\mathbf{g}$ generating an equivalent metric $\bar{\mathbf{g}}$. Since,

$$\bar{\Gamma}_{ij}^{h} = \Gamma_{ij}^{h} - \tfrac{1}{2}\,(\delta_i^h\,\mu_j + \delta_j^h\,\mu_i), \qquad \mu_i = \partial_i \log|\mu|, \qquad \mu = \det \mathbf{J},$$

for vector fields and 1-forms we have

$$\begin{aligned} \bar{\nabla}_h X^i &= \nabla_h X^i - \tfrac{1}{2}\,(\mu_h X^i + X^l \mu_l\, \delta_h^i), \\ \bar{\nabla}_h X_i &= \nabla_h X_i + \tfrac{1}{2}\,(\mu_h X_i + \mu_i X_h), \end{aligned}$$

and for 2-tensors,

$$\begin{aligned} \bar{\nabla}_h T_{ij} &= \nabla_h T_{ij} + \mu_h\, T_{ij} + \tfrac{1}{2}\,(T_{hj}\,\mu_i + T_{hi}\,\mu_j), \\ \bar{\nabla}_h T^{ij} &= \nabla_h T^{ij} - \mu_h\, T^{ij} - \tfrac{1}{2}\,\mu_l\,(\delta_h^i\, T^{lj} + \delta_h^j\, T^{il}), \\ \bar{\nabla}_h T_i^j &= \nabla_h T_i^j + \tfrac{1}{2}\,(\mu_i\, T_h^j - \delta_h^j\, T_i^l\, \mu_l). \end{aligned} \tag{22}$$

THEOREM 5.1. *The* $(1,1)$*-tensors* $\mathbf{B} = \mathbf{J}^{-1}$ *and* $\mathbf{A}^{-1} = (\operatorname{cof}\mathbf{J})^{-1}$ *are, respectively, a J-tensor and a A-tensor w.r.to the equivalent metric* $\bar{\mathbf{g}}$.
It follows that $\mathbf{B}$ is torsionless, in accordance with Theorem 3.3. Theorem 5.1 is a corollary of the more general

THEOREM 5.2. *Let* $\widetilde{\mathbf{J}}$ *be a* $(1,1)$*-J-tensor w.r.to the basic metric* $\mathbf{g}$*. Then*

$$\bar{\mathbf{J}} \doteq \widetilde{\mathbf{J}}\mathbf{J}^{-1} = \widetilde{\mathbf{J}}\mathbf{B}, \quad \bar{J}_i^j \doteq \widetilde{J}_k^j\, B_i^k, \tag{23}$$

$$\bar{\mathbf{A}} \doteq \operatorname{cof}\bar{\mathbf{J}} = \mathbf{A}^{-1}\widetilde{\mathbf{A}}, \tag{24}$$

where $\widetilde{\mathbf{A}} = \operatorname{cof}\widetilde{\mathbf{J}}$, $\mathbf{A} = \operatorname{cof}\mathbf{J}$, *are a J-tensor and a A-tensor w.r.to the equivalent metric* $\bar{\mathbf{g}}$, *respectively.*

Proof. (i) If $\widetilde{J}^{ij}$ is a J-tensor, then

$$\nabla_h \widetilde{J}^{ij} = \tfrac{1}{2}\,(\delta_h^i\, \widetilde{\alpha}^j + \delta_h^j\, \widetilde{\alpha}^i).$$

If we apply (22) to the tensor

$$\bar{J}^{ij} \doteq \mu\, \widetilde{J}^{ij}$$

then we get

$$\bar{\nabla}_h \bar{J}^{ij} = \tfrac{1}{2}\,(\delta_h^j\, \bar{C}^i + \delta_h^i\, \bar{C}^j)\,, \qquad \bar{\alpha}^i \doteq \mu\,(\widetilde{\alpha}^i - \widetilde{J}^{ij}\,\mu_j).$$

This proves that $\bar{J}^{ij}$ is a J-tensor w.r.to the equivalent metric $\bar{\mathbf{g}}$. Let us consider the (1,1)-form of this tensor, w.r.to the equivalent metric,

$$\bar{\mathbf{J}} = (\bar{J}_i^j), \qquad \bar{J}_i^j \doteq \bar{J}^{jk}\,\bar{g}_{ki}$$

Since $\bar{g}_{ij} = \frac{1}{\mu^2} A_{ij}$, we can write

$$\bar{J}_i^j = \bar{J}^{jk}\,\bar{g}_{ki} = \frac{1}{\mu}\,\widetilde{J}^{jk}\,A_{ki} = \frac{1}{\mu}\,\widetilde{J}_k^j\,A_i^k = \widetilde{J}_k^j\,B_i^k.$$

This proves Eq. (23), which implies the second one. (ii) Let us apply the above results to the fundamental metric tensor, $\widetilde{J}^{ij} = g^{ij}$, which is J-tensor. In this case $\widetilde{\mathbf{J}} = \widetilde{\mathbf{A}} = \mathbf{I}$ and Eqs. (23) (24) reduces to $\bar{\mathbf{J}} = \mathbf{J}^{-1}$ and $\bar{\mathbf{A}} = \mathbf{A}^{-1}$. □
For $\widetilde{\mathbf{J}} = \mu\,\mathbf{I}$ i.e., for $\bar{J}^{ij} = \mu\,g^{ij}$, we get Theorem 5.1.

REMARK 5.1. The metric $\mathbf{g}$ is the equivalent metric of $\bar{\mathbf{g}}$ generated by the J-tensor $\mathbf{B} = \mathbf{J}^{-1}$.

THEOREM 5.3. *The mapping*

$$K_{i_1 \ldots i_p} \longmapsto \bar{K}_{i_1 \ldots i_p} \doteq \mu^{-p}\,K_{i_1 \ldots i_p} \tag{25}$$

is linear isomorphism between the spaces of Killing tensors of order p w.r.to the metrics $\mathbf{g}$ *and* $\bar{\mathbf{g}}$*, respectively.*

Proof. A covariant symmetric two-tensor K_{ij} (we consider for simplicity the case $p = 2$) is a KT of $\mathbf{g}$ if and only if the function

$$F(\mathbf{K}) \doteq K_{ij}\,\frac{dq^i}{dt}\frac{dq^j}{dt} \tag{26}$$

is constant w.r.to the proper time t along any (non-null) geodesic: $\frac{d}{dt}F(\mathbf{K}) = 0$. Since $dt/d\bar{t} = \mu$, we have

$$\frac{dq^i}{dt} = \frac{dq^i}{d\bar{t}}\,\frac{d\bar{t}}{dt} = \mu^{-1}\,\frac{dq^i}{d\bar{t}}$$

thus,

$$F(\mathbf{K}) = \mu^{-2}\,K_{ij}\,\frac{dq^i}{d\bar{t}}\frac{dq^j}{d\bar{t}},$$

and along any geodesic

$$\frac{d}{d\bar{t}}\left(\mu^{-2}\,K_{ij}\,\frac{dq^i}{d\bar{t}}\frac{dq^j}{d\bar{t}}\right) = \frac{dt}{d\bar{t}}\,\frac{d}{dt}\left(K_{ij}\,\frac{dq^i}{dt}\frac{dq^j}{dt}\right) = 0.$$

This shows that $\mu^{-2}\,K_{ij}$ is a KT w.r.to the time $\bar{t}$. The mapping so defined is clearly reversible, due to the second part of Theorem 5.1. □

REMARK 5.2. Due to Theorem 5.3, the spaces of the Killing p-tensors of $\bar{\mathbf{g}}$ and $\mathbf{g}$ have the same dimension.

THEOREM 5.4. *If a metric* $\mathbf{g}$ *has constant curvature then any equivalent metric* $\bar{\mathbf{g}}$ *has constant curvature.*

Proof. In a space of constant curvature the dimension of the space of Killing vectors is maximal and equal to $\frac{1}{2}\,n(n+1)$. □

In other words, *the only spaces whose geodesics correspond to geodesics of a space of constant curvature are spaces of constant curvature*, [Beltrami, 1868]. See also [Eisenhart, 1926] p. 134, [Schouten, 1954] p. 293. The proof given here is much shorter than those give by these Authors. For $n = 2$ and $\mathbf{g}$ flat, see [Beltrami, 1895]. The argument used here for proving the Beltrami theorem 5.4 is similar to that used in [Matveev & Topalov, 1999] and [Topalov & Matveev, 2003].[3]

REMARK 5.3. As shown in [Schouten, 1954], p. 293, for $n > 2$ (in the notation of Schouten $p_i = -\frac{1}{2}\mu_i$), and in [Eisenhart, 1926] p. 133-134, the constant curvatures κ and $\bar{\kappa}$ of two equivalent metrics $\mathbf{g}$ and $\bar{\mathbf{g}}$ are related by equation

$$\bar{\kappa}\,\bar{g}_{ij} = \tfrac{1}{4}\,\mu_i\mu_j + \tfrac{1}{2}\,\nabla_i\mu_j + \kappa\,g_{ij}. \tag{27}$$

Since $\bar{g}^{ij} = \mu\,J^{ij}$, it follows that

$$\bar{\kappa} = \frac{\mu}{n}\,(J^{ij}\,(\mu_i\mu_j - \nabla_i\mu_j) + \kappa\,\mathrm{tr}\,\mathbf{J}). \tag{28}$$

Let us recall that for a CKT the function (26) is such that, along any geodesic,

$$\frac{dF}{dt} = -\,\alpha_h\dot{q}^h\,g_{ij}\,\dot{q}^i\dot{q}^j, \qquad g_{ij}\,\dot{q}^i\dot{q}^j = \pm 1.$$

Thus,

$$\frac{d}{d\bar{t}}\left(\mu^{-2}\,K_{ij}\,\frac{dq^i}{d\bar{t}}\frac{dq^j}{d\bar{t}}\right) = \frac{dt}{d\bar{t}}\,\frac{d}{dt}\left(K_{ij}\,\frac{dq^i}{dt}\,\frac{dq^j}{dt}\right) = \pm\,\mu\,\alpha_h\frac{dq^h}{dt} = \pm\,\alpha_h\frac{dq^h}{d\bar{t}}.$$

This proves

THEOREM 5.5. *The mapping* (25) *is a linear isomorphism between the spaces of conformal Killing tensors of order* p *w.r.to the metrics* $\mathbf{g}$ *and* $\bar{\mathbf{g}}$*, respectively.*

REMARK 5.4. For two-tensors the trace-free condition $K_i^i = K_{ij}g^{ij} = 0$ is not preserved by the mapping (25), since $\bar{K}_{ij}\,\bar{g}^{ij} = \mu^{-2}\,K_{ij}\,\mu J^{ij} = \mu^{-1}\,K_{ij}J^{ij}$. However, by the mapping

$$\bar{K}_{ij} \longmapsto \bar{K}^0_{ij} \doteq \bar{K}_{ij} - \tfrac{1}{n}\,\bar{K}_{hk}\,\bar{g}^{hk}\,\bar{g}_{ij}, \tag{29}$$

we get a trace-free tensor. Hence by composing the mapping (25) with the mapping (29) we get a linear mapping from the space of CKT's w.r.to $\mathbf{g}$ onto the space of trace free CKT's w.r.to $\bar{\mathbf{g}}$. It is known [Rani, Brian Edgar & Barnes, 2003] that the maximum number of linearly independent trace-free conformal Killing two-tensors is

$$d_n = \tfrac{1}{12}\,(n-1)(n+2)(n+3)(n+4), \qquad n > 2.$$

and it is attained in conformally flat spaces. Hence, if the equivalent metric $\bar{\mathbf{g}}$ is conformally flat then the same is for the basic metric, where the number of linearly independent trace-free conformal Killing two-tensors is necessarily $\geq d_n$. This conclusion is in agreement with Theorem 5.4, since a space of constant curvature is always conformally flat ([Schouten, 1954], p. 306).

[3] The same trick is used in the forthcoming papers by Matveev dedicated to the Lichnerowicz-Obata conjecture (private communication).

6 B-metrics

As we shall see, it is useful to consider as a metric the (non-singular) B-tensor **B** associated with a J-tensor **J**,

$$\overset{*}{g}_{ij} \doteq B_{ij} = \mu\,\bar{g}_{ij}$$

This new metric is conformal to the equivalent metric $\bar{g}_{ij}$ and its contravariant components are the contravariant components of the related J-tensor,

$$\overset{*}{g}{}^{ij} = J^{ij}$$

The Christoffel symbols of the B-metric are

$$\overset{*}{\Gamma}_{ijk} = \tfrac{1}{2}\,(\partial_i B_{jk} + \partial_j B_{ik} - \partial_k B_{ij})$$

Since the characteristic equations of a B-tensor are

$$\begin{aligned} \nabla_k B_{ij} &= \,-\tfrac{1}{2}\,(\mu_i\,B_{jk} + \mu_j\,B_{ik}) \\ \Longleftrightarrow \quad \partial_k B_{ij} &= \Gamma^l_{ki}\,B_{lj} + \Gamma^l_{kj}\,B_{il} - \tfrac{1}{2}\,(\mu_i\,B_{jk} + \mu_j\,B_{ik}), \end{aligned} \tag{30}$$

we have

$$\begin{aligned} 2\,\overset{*}{\Gamma}_{ijk} &= \Gamma^l_{ij}\,B_{lk} + \Gamma^l_{ik}\,B_{jl} - \tfrac{1}{2}\,(\mu_j B_{ik} + \mu_k B_{ij}) \\ &\quad + \Gamma^l_{ji}\,B_{lk} + \Gamma^l_{jk}\,B_{il} - \tfrac{1}{2}\,(\mu_i B_{jk} + \mu_k B_{ji}) \\ &\quad - \Gamma^l_{ki}\,B_{lj} - \Gamma^l_{kj}\,B_{il} + \tfrac{1}{2}\,(\mu_i\,B_{jk} + \mu_j\,B_{ik}). \end{aligned}$$

Thus,

$$\overset{*}{\Gamma}_{ijk} = \Gamma^l_{ij}\,B_{lk} - \tfrac{1}{2}\,\mu_k\,B_{ij}, \qquad \overset{*}{\Gamma}{}^h_{ij} = \Gamma^h_{ij} - \tfrac{1}{2}\,\alpha^h\,B_{ij}.$$

For a vector field X^i and a one-form φ_i we have

$$\begin{aligned} \overset{*}{\nabla}_i X^k &= \partial_i X^k + \overset{*}{\Gamma}{}^k_{ij}\,X^j = \nabla_i X^k - \tfrac{1}{2}\,\alpha^k\,B_{ij} X^j, \\ \overset{*}{\nabla}_i \varphi_k &= \partial_i \varphi_k - \overset{*}{\Gamma}{}^j_{ik}\,\varphi_j = \nabla_i \varphi_k + \tfrac{1}{2}\,\alpha^j\,\varphi_j\,B_{ik}, \end{aligned} \tag{31}$$

In particular,

$$\begin{aligned} \overset{*}{\nabla}_i\,\alpha^k &= \nabla_i \alpha^k - \tfrac{1}{2}\,\alpha^k\,\alpha^j\,B_{ij} = \nabla_i \alpha^k - \tfrac{1}{2}\,\alpha^k\,\mu_i, \\ \overset{*}{\nabla}_i\,\mu_k &= \nabla_i \mu_k + \tfrac{1}{2}\,\alpha^j\,\mu_j\,B_{ik} = \nabla_i \mu_k + \tfrac{1}{2}\,\sigma\,B_{ik}, \end{aligned} \tag{32}$$

where

$$\sigma \doteq \alpha^i \mu_i = J^{ij}\,\mu_i \mu_j = B_{ij}\,\alpha^i \alpha^j.$$

For the Riemann tensor of a metric g_{ij} we use the definition of [Eisenhart, 1949],

$$R^h{}_{ijk} \doteq \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^m_{ik}\,\Gamma^h_{mj} - \Gamma^m_{ij}\,\Gamma^h_{mk}, \qquad R_{hijk} \doteq g_{hl}\,R^l{}_{ijk}.$$

Then the metric is of constant curvature $\kappa \in \mathbb{R}$ if and only if

$$R_{hijk} = \kappa\,(g_{hj}\,g_{ik} - g_{hk}\,g_{ij}).$$

Let $\overset{*}{R}{}^h{}_{ijk}$ be the Riemann tensor of $\overset{*}{g}_{ij}= B_{ij}$. By setting

$$\overset{*}{\Gamma}{}^h_{ij} = \Gamma^h_{ij} + T^h_{ij}, \qquad T^h_{ij} \doteq -\tfrac{1}{2}\,\alpha^h\,B_{ij}$$

it follows that

$$\begin{aligned}\overset{*}{R}{}^h{}_{ijk} &= R^h{}_{ijk} + \nabla_j T^h_{ik} - \nabla_k T^h_{ij} + T^m_{ik}\,T^h_{mj} - T^m_{ij}\,T^h_{mk} \\ &= R^h{}_{ijk} - \tfrac{1}{2}\,(\nabla_j\alpha^h\,B_{ik} + \alpha^h\,\nabla_j B_{ik} - \nabla_k\alpha^h\,B_{ij} - \alpha^h\nabla_k B_{ij}) \\ &\quad + \tfrac{1}{4}\,\alpha^h\,\alpha^m\,(B_{ik}\,B_{mj} - B_{ij}\,B_{mk}) \\ &= R^h{}_{ijk} - \tfrac{1}{2}\,(\nabla_j\alpha^h\,B_{ik} - \nabla_k\alpha^h\,B_{ij}) - \tfrac{1}{2}\,\alpha^h\,(\nabla_j B_{ik} - \nabla_k B_{ij}) \\ &\quad + \tfrac{1}{4}\,\alpha^h\,(\mu_j\,B_{ik} - \mu_k\,B_{ij}).\end{aligned}$$

Due to (30),

$$\begin{aligned}\nabla_j B_{ik} - \nabla_k B_{ij} &= -\tfrac{1}{2}\,(\mu_i B_{jk} + \mu_k B_{ji} - \mu_i B_{jk} - \mu_j B_{ik}) \\ &= \tfrac{1}{2}\,(\mu_j B_{ik} - \mu_k B_{ji}).\end{aligned}$$

Thus,

$$\overset{*}{R}{}^h{}_{ijk} = R^h{}_{ijk} - \tfrac{1}{2}\,(\nabla_j\alpha^h\,B_{ik} - \nabla_k\alpha^h\,B_{ij}).$$

Due to (32),

$$\overset{*}{R}{}^h{}_{ijk} = R^h{}_{ijk} - \tfrac{1}{2}\,(\overset{*}{\nabla}_j\,\alpha^h B_{ik} - \overset{*}{\nabla}_k\,\alpha^h B_{ij}) - \tfrac{1}{4}\,\alpha^h\,(\mu_j B_{ik} - \mu_k B_{ij}).$$

Hence, due to (30) and (32),

$$\begin{aligned}\overset{*}{R}_{hijk} &\doteq \overset{*}{R}{}^l{}_{ijk}\;\overset{*}{g}_{lh} = R^l{}_{ijk}\,B_{lh} - \tfrac{1}{2}\,(\overset{*}{\nabla}_j\,\mu_h\,B_{ik} - \overset{*}{\nabla}_k\,\mu_h\,B_{ij}) \\ &\quad -\tfrac{1}{4}\,\mu_h\,(\mu_j\,B_{ik} - \mu_k\,B_{ij}) \\ &= R^l{}_{ijk}\,B_{lh} - \tfrac{1}{2}\,(\nabla_j\mu_h\,B_{ik} - \nabla_k\mu_h\,B_{ij}) \\ &\quad -\tfrac{1}{4}\,\sigma\,(B_{jh}B_{ik} - B_{hk}B_{ij}) - \tfrac{1}{4}\,\mu_h\,(\mu_j\,B_{ik} - \mu_k\,B_{ij}).\end{aligned}$$

This proves

THEOREM 6.1. *Let B_{ij} be a non-singular B-tensor w.r.to the fundamental metric* $\mathbf{g}$. *Then the Riemann tensor of the metric $\overset{*}{g}_{ij}= B_{ij}$ is*

$$\overset{*}{R}_{hijk}= R^l{}_{ijk}\,B_{lh} + H_{hj}\,B_{ik} - H_{hk}\,B_{ij} \tag{33}$$

with

$$H_{ij} \doteq -\tfrac{1}{4}\,(2\,\nabla_i\mu_j + \sigma\,B_{ij} + \mu_i\,\mu_j). \tag{34}$$

Note that H_{ij} is symmetric since μ_i is a gradient.

7 L-tensors, L-sequences and L-systems.

DEFINITION 7.1. We call **L-tensor** a conformal Killing two-tensor L^{ij} on a Riemannian manifold (Q, g_{ij}), such that $\mathbf{L} = (L^j_i)$, $L^j_i = g_{hi}L^{jh}$, is torsionless and with real[4] simple eigenvalues.
It follows that (see Appendix A, §16): (i) The eigenvectors of $\mathbf{L}$ are **normal** i.e., orthogonally integrable, so that they are orthogonal to an orthogonal web of submanifolds of codimension 1. (ii) There exist local coordinates $\underline{q} = (q^i)$, which we call **adapted** to $\mathbf{L}$, such that the partial derivatives $\partial_i = \partial/\partial q^i$ are eigenvectors of $\mathbf{L}$ with eigenvalues u^i; this means that these coordinates are orthogonal and diagonalize the tensor L^{ij}:

$$\begin{aligned} &g^{ij} = L^{ij} = 0, \quad i \neq j, \\ &L^{ii} = u^i\, g^{ii}, \\ &L^j_i = u^i\, \delta^j_i = u^j\, \delta^j_i. \end{aligned}$$

(iii) Equations

$$\partial_i u^j = 0, \quad \partial_i u^i = \alpha_i = (u^j - u^i)\partial_i \log|g^{jj}|, \quad i \neq j \tag{35}$$

hold; the first equation (which follows from the torsionless condition) shows that the eigenvalues depend on the corresponding coordinate only, $u^i = u^i(q^i)$. The second equation follows form (2.3). (iv) The diagonal components of the metric tensor have the form

$$g^{ii} = \frac{\phi_i(q^i)}{\prod_{j\neq i}(u^i - u^j)} \tag{36}$$

The introduction of such a kind of tensor [Benenti, 1992, 1993] is motivated by the following theorem:[5]

THEOREM 7.1. *Let $\mathbf{L} = (L^j_i)$ be a symmetric 2-tensor with eigenvalues (u^i). Then the tensors $(\mathbf{K}_a) = (\mathbf{K}_0, \mathbf{K}_1, \ldots, \mathbf{K}_{n-1})$ defined by*

$$\mathbf{K}_0 = \mathbf{I}, \quad \mathbf{K}_a = \tfrac{1}{a}\,\mathrm{tr}\,(\mathbf{K}_{a-1}\mathbf{L})\,\mathbf{I} - \mathbf{K}_{a-1}\,\mathbf{L}, \quad a > 1 \tag{37}$$

are n independent Killing tensors with common normal eigenvectors if and only if $\mathbf{L}$ *is a L-tensor.*

REMARK 7.1. A n-dimensional space of Killing 2-tensors $\mathcal{K}$ is called **Killing-Stäckel space** (KS-space) or **KS-algebra**, if its elements (i) have n common eigenvectors, (ii) are in involution, (iii) the eigenvectors are normal. It is known that: (a) conditions (i) and (ii) imply (iii), while conditions (i) and (iii) imply (ii); (b) the identity $\mathbf{I}$ (which corresponds to the metric) belong to any KS-space; (c) if a KT $\mathbf{K}$ has the same eigenvectors of $\mathcal{K}$, then it belongs to $\mathcal{K}$; (d) a KS-space is determined by a single KT $\mathbf{K}$ with simple eigenvalues and normal eigenvectors. Such a tensor is called **characteristic Killing tensor**. Note that it is not uniquely determined by the KS-space.

REMARK 7.2. The foliations orthogonal to the eigenvectors of a KS-space form an **orthogonal separable web** or a **Stäckel web**. This means that: (i) there exist local

[4] This assumption is obviously fulfilled if the metric is positive (or negative)-definite. The case of simple complex eigenvalues has not yet been investigated. Hereafter this assumption will be understood.

[5] See [Benenti, 2004b] for a proof and further details.

coordinate systems $\underline{q} = (q^i)$ such that ∂_i are eigenvectors (or, equivalently, each foliation of the web is described by equations $q^i =$ constant); (ii) these coordinates are orthogonal and separable: the geodesic Hamilton-Jacobi equation is separable in these coordinates.

REMARK 7.3. A potential V is said to be **separable** w.r.to a KS-space if the Hamilton-Jacobi equation $\frac{1}{2}\, g^{ij} p_i p_j + V = E$ is separable in the orthogonal coordinates associated to the corresponding web. A necessary and sufficient condition for this is expressed by the **characteristic equation**

$$d(\mathbf{K}\, dV) = 0, \qquad d(K_i^j\, \partial_j V\, dq^i) = 0, \tag{38}$$

which locally is equivalent to

$$\mathbf{K}\nabla V = \nabla U, \tag{39}$$

where $\mathbf{K}$ is any characteristic Killing tensor. In turn, this equation means that

$$F = \tfrac{1}{2}\, K^{ij}\, p_i p_j + U$$

is a first integral. Then, if $\mathbf{K}_a$ is a basis of the KS-space, then for each one of these tensors an equation similar to (6) holds,

$$\mathbf{K}_a \nabla V = \nabla V_a,$$

so that

$$F_a = \tfrac{1}{2}\, K_a^{ij}\, p_i p_j + V_a$$

are independent first integrals in involution.

REMARK 7.4. Going back to Theorem 7.1, the sequence (37) defines a basis of a KS-space if and only if $\mathbf{L}$ is a L-tensor. What is remarkable in this statement is that this basis is determined in a pure algebraic way, starting from the tensor $\mathbf{L}$, by the **L-sequence** (37). We call **L-system** a Stäckel system generated in this way. L-systems form a special subclass of Stäckel systems. For testing if a potential V is separable, we can apply formula (38) by using the tensor $\mathbf{K}_1$ in the L-sequence,

$$\mathbf{K}_1 \doteq (\mathrm{tr}\,\mathbf{L})\,\mathbf{I} - \mathbf{L}.$$

Then we get the second-order differential equation on the potential V,

$$d((\mathrm{tr}\,\mathbf{L})dV - \mathbf{L}\, dV) = 0.$$

REMARK 7.5. The L-sequence (37) has two equivalent definitions:

$$\begin{aligned} \mathbf{K}_a &= \textstyle\sum_{k=0}^{a} (-1)^k \sigma_{a-k} \mathbf{L}^k, \\ \mathbf{K}_a &= \sigma_a\, \mathbf{G} - \mathbf{K}_{a-1}\mathbf{L} \quad (\mathbf{K}_{-1} = \mathbf{0}), \end{aligned} \tag{40}$$

where $\sigma_a(\underline{u})$ denotes the elementary symmetric polynomial of order a of the eigenvalues $\underline{u} = (u^i)$. Note that in applying these formulas we need to know the eigenvalues of $\mathbf{L}$. On the contrary, the sequence (37) is more effective than (40) since it does dot require the knowledge of the eigenvalues of $\mathbf{L}$. The first formula (40) shows that $\mathbf{K}_a = 0$ for $a \geq n$, since for $a = n$ the right-hand side vanishes due to the Hamilton-Cayley theorem:

$$\textstyle\sum_{k=0}^{n} (-1)^k \sigma_{n-k} \mathbf{L}^k = \sigma_n\, \mathbf{I} - \sigma_{n-1}\, \mathbf{L} + \sigma_{n-2}\, \mathbf{L}^2 - \ldots + (-1)^n\, \mathbf{L}^n = 0.$$

REMARK 7.6. The L-sequence (4) appeared in the literature within a quite different context. It was firstly introduced in [Fettis, 1950] and [Souriau, 1950] for computing the eigenvectors of a matrix $\mathbf{L}$ without solving systems of linear equations (if the eigenvalues are known and simple). This fact is mentioned in [Schouten, 1954], p. 30. As observed by Schouten (p. 30, 38), the tensor $\mathbf{A}(x) = \operatorname{cof}(\mathbf{L} - x\,\mathbf{I})$ is a polynomial of degree $n-1$ in x,

$$\begin{aligned} \mathbf{A}(x) &= \operatorname{cof}(\mathbf{L} - x\,\mathbf{I}) \\ &= (-1)^n\,(\mathbf{I}\,x^{n-1} - \mathbf{K}_1\,x^{n-2} + \mathbf{K}_2\,x^{n-3} - \ldots - (-1)^n\,\mathbf{K}_{n-1}), \end{aligned} \tag{41}$$

where $\mathbf{K}_a$ are the tensors of the L-sequence (37). Since $x\,\mathbf{A}(x) = \det(\mathbf{L} - x\,\mathbf{I})\,\mathbf{I} + \mathbf{L}\,\mathbf{A}(x)$, we get, for each eigenvalue u^i of $\mathbf{L}$,

$$\mathbf{L}\,\mathbf{A}(u^i) = u^i\,\mathbf{A}(u^i).$$

This shows that $\mathbf{A}(u^i)\,\mathbf{v}$ is an eigenvector of $\mathbf{L}$ belonging to the eigenvalue u^i, whatever $\mathbf{v}$. However, if the eigenvalue u^i has a multiplicity greater than 1, then $\mathbf{A}(u^i)$ may vanish. Formula (41) shows that the L-sequence (37) defines, up to the sign, the same Killing tensors of a cofactor system.

THEOREM 7.2. (i) *The eigenvectors of a L-tensor are Ricci-principal directions.* (ii) *A L-tensor* $\mathbf{L} = (L_i^j)$ *and the Ricci tensor* $\mathbf{R} = (R_i^j)$ *commute as linear operators.* (iii) *The Ricci tensor is diagonalized in the orthogonal coordinates adapted to a L-tensor.*

Proof. Since the eigenvalues of $\mathbf{L}$ are distinct, these three conditions are equivalent. Due to the expression (36) of the metric components, in adapted coordinates the **contracted Christoffel symbols** $\Gamma_i \doteq g^{hj}\,\Gamma_{hj,i}$ assume the simple form

$$\Gamma_i = -\tfrac{1}{2}\;\phi_i\,\phi_i'.$$

On the other hand, in separable orthogonal coordinates [Benenti, Chanu & Rastelli, 2002a]

$$R_{ij} = \tfrac{3}{2}\,\partial_i\Gamma_j, \qquad i \neq j.$$

Thus, $R_{ij} = 0$ for $i \neq j$. □

REMARK 7.7. For a L-system the so-called Robertson condition $R_{ij} = 0$ for $i \neq j$ is satisfied. Thus, for a L-system also the Schrödinger equation is separable and the operators associated with the first integrals in involution commute [Benenti, Chanu & Rastelli, 2002b].

8 The eigenvalues of a L-tensor

THEOREM 8.1. *If an eigenvalue* u^α *of a L-tensor* $\mathbf{L}$ *is constant, then* $\mathbf{L}$ *is invariant w.r.to a Killing vector* $\mathbf{X}_\alpha$*, which is an eigenvector of* $\mathbf{L}$ *belonging to* u^α*:*

$$[\mathbf{X}_\alpha, \mathbf{L}] = d_{\mathbf{X}_\alpha}\mathbf{L} = 0, \qquad \mathbf{L}\mathbf{X}_\alpha = u^\alpha\,\mathbf{X}_\alpha.$$

Conversely, if there exists a Killing vector $\mathbf{X}$ *such that* $[\mathbf{X}, \mathbf{L}] = 0$*, then* $\mathbf{X}$ *is a linear combination with constant coefficients,*

$$\mathbf{X} = \textstyle\sum_{\alpha=1}^r\, c^\alpha \mathbf{X}_\alpha,$$

of commuting Killing eigenvectors $\mathbf{X}_\alpha$ *whose eigenvalues* u^α *are constant.* [Benenti, 2004]

Proof. (i) Assume u^α = constant. In coordinates adapted to $\mathbf{L}$ we have, cf. (7.3),

$$g_{\alpha\alpha} = \frac{\prod_{j\neq\alpha}(u^\alpha - u^j)}{\phi_\alpha(q^\alpha)}, \qquad g_{ii} = \frac{\prod_{j\neq i}(u^i - u^j)}{\phi_i(q^i)}, \quad i \neq \alpha.$$

By a rescaling of each q^α, we introduce new coordinates $(\bar{q}^\alpha)$ such that

$$d\bar{q}^\alpha = \sqrt{|\phi_\alpha(q^\alpha)|^{-1}}\, dq^\alpha.$$

Thus,

$$g_{\alpha\alpha}\, dq^\alpha \otimes dq^\alpha = g_{\alpha\alpha}\, |\phi_\alpha|\, d\bar{q}^\alpha \otimes d\bar{q}^\alpha = e_\alpha\, \textstyle\prod_{j\neq\alpha}(u^\alpha - u^j)\, d\bar{q}^\alpha \otimes d\bar{q}^\alpha,$$

where $e_\alpha \doteq \text{sign}(\phi_\alpha) = \pm 1$. This shows that all the new coordinates $(\bar{q}^\alpha)$ are ignorable. Then, by setting $\mathbf{X}_\alpha = \partial/\partial\bar{q}^\alpha$, we get a vector field with the required property. Indeed, the two equations $[\mathbf{X}, \mathbf{G}] = 0$ and $[\mathbf{X}, \mathbf{L}] = 0$ are, in orthogonal and adapted coordinates, equivalent to equations

$$\textstyle\sum_i X^i \partial_i g^{kk} - 2\, g^{kk}\, \partial_k X^k = 0, \tag{42}$$

$$g^{jj}\, \partial_j X^k + g^{kk}\, \partial_k X^j, \quad j \neq k, \tag{43}$$

$$\textstyle\sum_i X^i \partial_i (u^k\, g^{kk}) - 2\, u^k\, g^{kk}\, \partial_k X^k = 0, \tag{44}$$

$$u^j\, g^{jj}\, \partial_j X^k + u^k\, g^{kk}\, \partial_k X^j, \qquad j \neq k, \tag{45}$$

respectively. For $\mathbf{X} = \mathbf{X}_\alpha$ these equations are satisfied. (ii) Conversely, let $\mathbf{X} = (X^i)$ be a vector field satisfying these equations. Eqs. (44) and (45) imply $(u^j - u^k)\, g^{jj}\, \partial_j X^k = 0$ for $j \neq k$. Since $u^j \neq u^k$ (by the definition of L-tensor) we conclude that $\partial_j X^k = 0$ for $j \neq k$, which means that $X^i = X^i(q^i)$. Since $\partial_i u^k = 0$ for $i \neq k$ (by the torsionless condition of a L-tensor) from the third equation it follows that $X^k\, \partial_k u^k\, g^{kk} + u^k \sum_i X^i\, \partial_i g^{kk} - 2\, u^k\, g^{kk}\, \partial_k X^k = 0$. Due to the first of (1), this last equation implies (†) $X^k\, \partial_k u^k = 0$ (<u>no</u> <u>summation</u> over the index k). Up to a reordering of the coordinates, let us assume that $X^a = 0$ and $X^\alpha \neq 0$ for $a = 1, \ldots, m$ and $\alpha = m+1, \ldots, n$. Then, $\mathbf{X} = X^\alpha\, \partial_\alpha$. From (†) it follows that u^α = constant. At the beginning of this proof we have seen that if u^α = constant, then q^α are ignorable (up to a rescaling). Thus, $\mathbf{X}_\alpha = \partial_\alpha$ are Killing vectors in involution and eigenvectors of $\mathbf{L}$. Since $\mathbf{X} = X^\alpha\, \partial_\alpha$ is a Killing vector, the components X^α must be constant, $X^\alpha = c^\alpha$. □

THEOREM 8.2. *The eigenvalues of a L-tensor* $\mathbf{L}$ *are independent if and only if* $\mathbf{L}$ *is not invariant w.r.to a Killing vector (there is no Killing vector* $\mathbf{X}$ *such that* $d_{\mathbf{X}}\mathbf{L} = 0$*).*

Proof. Since in adapted coordinates (q^i), $\partial_i u^j = 0$ for $i \neq j$, we have $\det[\partial_i u^j] \neq 0$ if and only if $\partial_i u^i \neq 0$ i.e., if and only if no eigenvalues is constant. Then we apply Theorem 8.1. □

Assume that the eigenvalues u^α $(u^\alpha = m+1, \ldots, n)$ of a L-tensor are constant and that the remaining (u^a) $(a = 1, \ldots, m)$ are independent functions. Then, due to Theorem 8.1, we can consider adapted orthogonal coordinates $(q^i) = (q^a, q^\alpha)$ such that $q^a = u^a$ and q^α are ignorable. In these coordinates Eqs. (35) read

$$(u^b - u^a)\, \partial_a \ln g^{bb} = 1, \qquad (u^\alpha - u^a)\, \partial_a \ln g^{\alpha\alpha} = 1,$$

being the remaining equations identically satisfied. Thus, we are faced with three cases: (I) $m = 0$, all u^i = constant i.e, all q^i are ignorable: the manifold Q is locally flat,

the coordinates (q^i) are orthogonal Cartesian coordinates, g^{ii} = constant and $\mathbf{L}$ is a constant tensor. (II) $0 < m < n$: in this case $g^{\alpha\alpha} \neq$ constant due to Eq. (43). Since $\mathbf{g}(\partial_\alpha, \partial_\alpha) = g_{\alpha\alpha} = (g^{\alpha\alpha})^{-1}$, condition $g^{\alpha\alpha} \neq$ constant means that the Killing vectors $\mathbf{X}_\alpha = \partial_\alpha$ are not **translations** [Eisenhart, 1933], §52. (III) $m = n$, all eigenvalues are independent: this is the case examined in Theorem 8.3.
Due to Theorem 8.1, cases (I) and (II) shows that

THEOREM 8.3. *Let* $\mathbf{L}$ *be a L-tensor.* (i) *If* $\mathbf{L}$ *has all constant eigenvalues, then the manifold* Q *is locally flat and* $\mathbf{L} =$ *constant* (in the sense that all its components in Cartesian coordinates are constant). (ii) *If* $\mathbf{L}$ *is invariant w.r.to* $m < n$ *Killing vectors, then these vectors are not translations.* [Benenti, 2004b]

9 Stäckel systems which are not L-systems

L-systems form a special but important subclass of Stäckel systems. A simple criterion for recognizing if a Stäckel system is a L-system is given by the following theorem, whose proof is obvious,

THEOREM 9.1. *A Stäckel system is a L-system if and only if in any orthogonal separable coordinate system* (q^i) *the metric tensor has the form* (36) *with* $u^i = u^i(q^i)$.

Conversely, the following theorems can be useful for testing if a Stäckel system is not a L-system. All of them have an intrinsic character.

THEOREM 9.2. *If a characteristic tensor* $\mathbf{K}$ *does not commute (as linear operator) with the Ricci tensor* $\mathbf{R}$*, then it generates a Stäckel system which is not a L-system.*

This criterion follows from Theorem 7.2. It does not work on spaces of constant curvature, in Einstein spaces, in Ricci-flat spaces, where the Robertson condition is satisfied.

THEOREM 9.3. *Let* $\mathbf{K}$ *be a characteristic Killing tensor. The following two conditions are incompatible:* (i) $\mathbf{K}$ *generates a L-system;* (ii) $\mathbf{K}$ *admits an eigenvectors which is a proper conformal Killing vector.*

A **proper** CKV is a CKV which is not a Killing vector [Rani, Brian Edgar & Barnes, 2003]. This property is also proved in [Chanu & Rastelli, 2004] within a different context.

Proof. A characteristic KT $\mathbf{K}$ generates local orthogonal (separable) coordinates (q^i). (i) Assume that there exists a L-tensor $\mathbf{L}$ with the same eigenvectors of $\mathbf{K}$ (this means that the separable system generated by $\mathbf{K}$ is a L-system). Then Eqs. (35) hold in these coordinates. (ii) Assume that an eigenvector $\mathbf{X}$ of $\mathbf{K}$, say $\mathbf{X} = f\partial_1$, $f \neq 0$, is a proper CKV. This means that

$$[f\,\partial_1, \mathbf{G}] = c\,\mathbf{G}, \quad c \neq 0,$$

where c is a function on Q and $\mathbf{G} = (g^{ij})$ i.e., that

$$\{f\,p_1, g^{hk}p_hp_k\} = c\,g^{ij}\,p_ip_j.$$

This last equation is equivalent to equations

$$f\,\partial_1 g^{11} - 2\,\partial_1 f\,g^{11} = c\,g^{11}, \qquad \partial_1 g^{kk} = c\,g^{kk}, \qquad \partial_k f\,g^{kk} = 0,$$

for $k \neq 1$ i.e., to

$$\partial_1 \log g^{11} - 2\,\partial_1 \log f = \tfrac{c}{f}, \qquad \partial_1 \log g^{kk} = c, \qquad f = f(q^i).$$

In virtue of (35), the second equation becomes

$$\frac{\partial_1 u^1}{u^k - u^1} = c, \qquad k \neq 1.$$

Since $c \neq 0$, we get $u^k - u^1 = u^h - u^1$, thus $u^k = u^h$ for $h, k \neq 1$, which is an absurd, since the eigenvalues of $\mathbf{L}$ are all distinct, by definition of L-tensor. □

A geometrical version of Theorem 9.3 is

THEOREM 9.4. *If a Stäckel web has a foliation which is orthogonal to a proper conformal Killing vector, then it is not a L-web.*

REMARK 9.1. In the Euclidean three-space the spherical webs (i.e., the web associated with polar-spherical coordinates and the asymmetric spherical conical webs) are not L-webs, since the foliation of spheres is orthogonal to the **dilatation** field $\mathbf{X} = \mathbf{r}$, which is a CKV with $c = -2$.

From Theorem 8.3 it follows that

THEOREM 9.5. *If a characteristic tensor* $\mathbf{K}$ *has* $m < n$ *Killing eigenvectors which are translations, then it generates a Stäckel system which is not a L-system.*

The geometrical version of this theorem is

THEOREM 9.6. *If a Stäckel web has* $m < n$ *foliations which are orthogonal to translations, then it is not a L-web.*

REMARK 9.2. The condition $m < n$ is crucial. All translational Stäckel webs in the Euclidean spaces, different from the Cartesian webs (for which $m = n$), are not L-webs (for instance, the cylindrical webs in the three-space).

10 Equivalent L-tensors

DEFINITION 10.1. Two L-tensors $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ are said to be **equivalent** if their L-sequences generate the same KS-space.

A necessary condition is that $\mathbf{L}$ and $\widetilde{\mathbf{L}}$ have equivalent eigenframes (i.e., the same eigenvectors, up to factors). A sufficient condition is that $\widetilde{\mathbf{L}} = a\mathbf{L} + b\mathbf{I}$, $a, b \in \mathbb{R}$.

THEOREM 10.1. *Two L-tensors* $\mathbf{L}$ *and* $\widetilde{\mathbf{L}}$ *are equivalent if and only if*

$$\widetilde{\mathbf{L}} = a\,\mathbf{L} + b\,\mathbf{I} + \textstyle\sum_\alpha c^\alpha \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha,$$

where c^α *are constants,* $\mathbf{X}_\alpha$ *are commuting Killing eigenvectors of* $\mathbf{L}$ *such that*

$$[\mathbf{X}_\alpha, \mathbf{L}] = 0, \tag{46}$$

and $\boldsymbol{\xi}^\alpha$ *are the one-forms belonging to the dual frame* $(\boldsymbol{\xi}^i)$ *of an eigenframe* $(\mathbf{X}_i)$ *containing the* $\mathbf{X}_\alpha$*.*

The dual frame is defined by $\langle \mathbf{X}_i, \boldsymbol{\xi}^j \rangle = \delta_i^j$. Note that $\sum_\alpha c^\alpha \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha$ is a Killing (1,1)-tensor.

Proof. Let (u^i) and $(\widetilde{u}^i)$ be the eigenvalues of $\mathbf{L}$ and $\widetilde{\mathbf{L}}$, respectively. (i) Assume that they are equivalent. In coordinates (q^i) adapted to the KS-space, from the second equation (35) we get

$$\partial_i \log |g^{jj}| = \frac{\partial_i u^i}{u^j - u^i} = \frac{\partial_i \widetilde{u}^i}{\widetilde{u}^j - \widetilde{u}^i}, \quad i \neq j, \tag{47}$$

thus,

$$(\widetilde{u}^j - \widetilde{u}^i)\, \partial_i u^i = (u^j - u^i)\, \partial_i \widetilde{u}^i. \tag{48}$$

Being $\partial_i u^j = \partial_i \widetilde{u}^j = 0$, by applying ∂_j to both sides of this equation, we get

$$\partial_j \widetilde{u}^j\, \partial_i u^i = \partial_j u^j\, \partial_i \widetilde{u}^i.$$

Assume (†) $\partial_i u^i \neq 0$ and $\partial_j u^j \neq 0$ (notice that, due to (47), we have also $\partial_i \widetilde{u}^i \neq 0$ and $\partial_j \widetilde{u}^j \neq 0$). Then (48) can be written

$$\frac{\partial_j \widetilde{u}^j}{\partial_j u^j} = \frac{\partial_i \widetilde{u}^i}{\partial_i u^i}.$$

Since these two fractions are functions of distinct coordinates, we have

$$\frac{\partial_j \widetilde{u}^j}{\partial_j u^j} = \frac{\partial_i \widetilde{u}^i}{\partial_i u^i} = \text{constant} = a \neq 0.$$

Thus, for any index i, $\widetilde{u}^i = a\, u^i + b_i$, where b_i is a constant (apparently) depending on the index. However, if we insert this last result into (48), we get $b_i = b_j = b$ thus, $\widetilde{u}^i = a\, u^i + b$. Assume (‡) $\partial_i u^i = 0$. From (47) it follows that also $\partial_i \widetilde{u}^i = 0$. Thus, $u^i = \text{constant} = c^i$ and $\widetilde{u}^i = \text{constant} = \widetilde{c}^i$. In virtue of Theorem 10.1, we can write (no summation over the index i)

$$\mathbf{L} = \mathbf{L}_i + c^i\, \partial_i \otimes dq^i, \qquad \widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}_i + \widetilde{c}^i\, \partial_i \otimes dq^i,$$

where $\mathbf{L}_i$ and $\widetilde{\mathbf{L}}_i$ have eigenvector ∂_i with zero eigenvalue. Let us split the indices into two classes, $a = 1, \ldots, m$ and $\alpha = m+1, \ldots, n$, such that for $i = a$ and $i = \alpha$ conditions (†) or (‡) hold, respectively. Then

$$\widetilde{u}^a = a\, u^a + b \tag{49}$$

and

$$\mathbf{L} = \mathbf{L}_* + \textstyle\sum_\alpha c^\alpha\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha, \qquad \widetilde{\mathbf{L}} = \widetilde{\mathbf{L}}_* + \textstyle\sum_\alpha \widetilde{c}^\alpha\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha, \tag{50}$$

where $\mathbf{X}_\alpha = \partial_\alpha$ and $\boldsymbol{\xi}^\alpha = dq^\alpha$. Due to (49), we have

$$\widetilde{\mathbf{L}}_* = a\, \mathbf{L}_* + b\, \mathbf{I}_* \tag{51}$$

where $\mathbf{I}_*$ is the identity on the spaces orthogonal to the vectors $\mathbf{X}_\alpha$. Since

$$\mathbf{I}_* = \mathbf{I} - \textstyle\sum_\alpha \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha, \tag{52}$$

from (50)-(51)-(52) we derive

$$\begin{aligned}
\widetilde{\mathbf{L}} &= a\, \mathbf{L}_* + b\, \mathbf{I}_* + \textstyle\sum_\alpha \widetilde{c}^\alpha\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha \\
&= a\, (\mathbf{L} - \textstyle\sum_\alpha c^\alpha\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha) + b\, (\mathbf{I} - \textstyle\sum_\alpha \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha) + \textstyle\sum_\alpha \widetilde{c}^\alpha\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha \\
&= a\, \mathbf{L} + b\, \mathbf{I} + \textstyle\sum_\alpha (\widetilde{c}^\alpha - a c^\alpha - b)\, \mathbf{X}_\alpha \otimes \boldsymbol{\xi}^\alpha.
\end{aligned}$$

By replacing the constants $\widetilde{c}^\alpha - ac^\alpha - b$ with c^α we get (10) and (46). (ii) Conversely, if (10) and (46) hold, then $\widetilde{\mathbf{L}}$ and $\mathbf{L}$ have the same eigenvectors; then they define the same KS-space. □

11 L-tensors versus J-tensors

Let us examine the relationship between L-tensors and J-tensors. Since a J-tensor is a torsionless CKT, *a J-tensor with simple eigenvalues is a L-tensor*. Conversely,

THEOREM 11.1. *A L-tensor is a J-tensor.*

Proof. Let us recall the characteristic equation of a J-tensor,

$$\partial_h L_i^j - \Gamma_{hi}^l\, L_l^j + \Gamma_{hl}^j\, L_i^l = \tfrac{1}{2}\,(C_i\,\delta_h^j + C^j\,g_{ih}),$$

and write it in adapted coordinates, $L_i^j = u^i\,\delta_i^j$, $u^i = u^i(q^i)$:

$$\delta_i^j\,\partial_h u^i - \Gamma_{hi}^j\,(u^j - u^i) = \tfrac{1}{2}\,(C_i\,\delta_h^j + C^j\,g_{ih}), \tag{53}$$

In orthogonal coordinates,

$$\Gamma_{hi}^j = 0, \quad h,i,j \neq, \qquad \Gamma_{ij}^i = \tfrac{1}{2}\,g^{ii}\partial_j g_{ii} = -\tfrac{1}{2}\,\partial_j \log|g^{ii}|, \quad i \neq j,$$

$$\Gamma_{ii}^j = -\tfrac{1}{2}\,g^{jj}\,\partial_j g_{ii}, \quad i \neq j.$$

For $k,i,j \neq$ and $h \neq i = j$, Eq. (53) is identically satisfied. For the other cases it reduces to equations:

$$\begin{array}{lll}
h = i = j & \Rightarrow & \partial_i u^i = C_i. \\
j \neq i = h & \Rightarrow & \Gamma_{ii}^j (u^i - u^j) = \tfrac{1}{2}\,C^j\,g_{ii} \\
 & \Rightarrow & \partial_j \log|g^{ii}|\,(u^i - u^j) = C_j. \\
i \neq j = h & \Rightarrow & \Gamma_{ji}^i\,(u^i - u^j) = \tfrac{1}{2}\,C_i \\
 & \Rightarrow & \partial_i \log|g^{jj}|\,(u^j - u^i) = C_i.
\end{array}$$

This proves that if Eqs. (35) hold, then (53) is satisfied. □
Theorem 11.1 is a special case of a more general statement proved in Appendix A (Theorem 19.3): *any trace-type torsionless CKT is a J-tensor*. The converse is also true (Lemma 3.1).
As a consequence: (I) *the space of L-tensors is the subspace of J-tensors with simple eigenvalues* and (II) *the space of J-tensors is the subspace of the trace-type torsionless CKT's with eigenframes* (i.e., admitting a frame of eigenvectors).
In the same section of Appendix A (Theorem 19.2) it is also proved that (III) *a torsionless CKT is of gradient type; if the eigenvalues are simple, then it is of trace-type.*
Furthermore, Theorems 11.2 and 11.3 below show that (IV) *a torsionless CKT with independent eigenvalues is a L-tensor.*
All the statements (I)-(IV) are summarized in Figure 1.

THEOREM 11.2. *If the eigenvalues of a torsionless CKT of trace type are independent, then they are pointwise distinct.*

Proof. This follows from Theorem 19.1 of Appendix A: if there is a non-simple eigenvalue, then this eigenvalue is constant. □

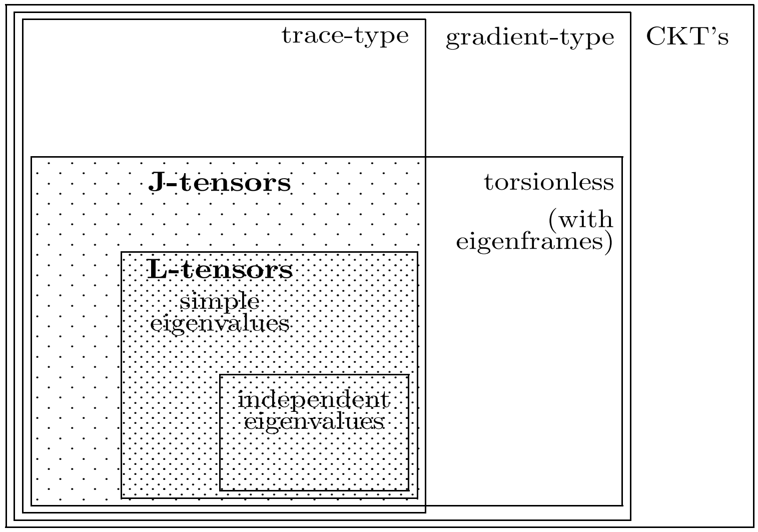

Figure 1:

This is the case of a J-tensor:

THEOREM 11.3. *If a J-tensor has independent eigenvalues, then it is a L-tensor (the eigenvalues are simple).*

12 Equivalent dynamical systems

In this section we illustrate the basic facts concerning the theory of the equivalent dynamical system. At the beginning, we shall follow the approach of Levi-Civita [Levi-Civita 1896b], with minor changes and improvements. Then, we shall propose new results, in a way suitable for our needs. The work of Levi-Civita has been continued by [Agostinelli, 1937] and [Thomas, 1946]. The theory of equivalence has been extended to systems with time-dependent constraints by [Lichnerowicz, 1946] and [Lichnerowicz & Aufenkamp, 1952].

As we said in the Introduction, by **dynamical system** we mean a holonomic mechanical system with time-independent constraints and forces, characterized by a triple $(Q, \mathbf{g}, \mathbf{F})$, where Q is the **configuration manifold**, $\mathbf{g} = (g_{ij})$ is the metric tensor on Q determined by the kinetic energy $K = \frac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j$ of the system, and $\mathbf{F} = (F^i)$ is the **force**, a vector field on Q.

Let us consider two dynamical systems with the same configuration manifold Q_n, but with different kinetic energies, $K = \frac{1}{2}\, g_{ij}\, \dot{q}^i \dot{q}^j$ and $\bar{K} = \frac{1}{2}\, \bar{g}_{ij}\, \dot{q}^i \dot{q}^j$, and Lagrangian forces F^i and $\bar{F}^i$, respectively. Then the configuration manifold Q is endowed with two distinct metric tensors, $\mathbf{g} = (g_{ij})$ and $\bar{\mathbf{g}} = (\bar{g}_{ij})$. The dynamics of the two systems $(\mathbf{g}, \mathbf{F})$ and $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ is represented in the tangent bundle TQ, with coordinates $(q, v) = (q^i, v^i)$, by the vector fields $\mathbf{X}$ and $\bar{\mathbf{X}}$, whose first-order equations are

$$\mathbf{X} = \begin{cases} \dfrac{dq^i}{dt} = v^i, \\ \dfrac{dv^i}{dt} = F^i - \Gamma^h_{ij}\, v^i v^j, \end{cases} \qquad \bar{\mathbf{X}} = \begin{cases} \dfrac{dq^i}{d\bar{t}} = v^i, \\ \dfrac{dv^i}{d\bar{t}} = \bar{F}^i - \bar{\Gamma}^h_{ij}\, v^i v^j. \end{cases} \tag{54}$$

Two dynamical systems $(\mathbf{g}, \mathbf{F})$ and $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ are said to be **equivalent** or **correspondent**, if there exists a function $f: TQ \to \mathbb{R}$, such that, for any solution

$$q^i = \varphi^i(t), \qquad v^i = \frac{d\varphi^i}{dt} = \dot{\varphi}^i(t)$$

of the first system (54), by a change of the time-parameter of the kind

$$\frac{d\bar{t}}{dt} = \frac{1}{f(\varphi^i(t), \dot{\varphi}^i(t))} \tag{55}$$

we get a solution of the second system

$$q^i = \bar{\varphi}^i(\bar{t}), \qquad v^i = \frac{d\bar{\varphi}^i}{d\bar{t}}.$$

This means that the trajectories on the configuration manifold Q of the two systems coincide, up to the reparameterization of the time-parameters given by (55) (in other words, the trajectories are the same, but covered with different velocities). In the definition of Levi-Civita the function f is considered depending also on t. However, he proves that in fact this function is independent of time, under the assumption that the forces depend only on the coordinates [Levi-Civita 1986b, p. 269].
Since the two vector fields $\mathbf{X}$ and $\bar{\mathbf{X}}$ have no singular points, we have $f \neq 0$. Let us consider the Newton-Lagrange (second-order) equations,

$$\frac{d^2q^h}{dt^2} + \Gamma^h_{ij}\,\frac{dq^i}{dt}\,\frac{dq^j}{dt} = F^h, \qquad \frac{d^2q^h}{d\bar{t}^2} + \bar{\Gamma}^h_{ij}\,\frac{dq^i}{d\bar{t}}\,\frac{dq^j}{d\bar{t}} = \bar{F}^h. \tag{56}$$

Since $f\,d\bar{t} = dt$, the second equation, divided by f^2, gives

$$\frac{1}{f^2}\,\frac{d^2q^h}{d\bar{t}^2} + \bar{\Gamma}^h_{ij}\,\frac{dq^i}{dt}\,\frac{dq^j}{dt} = \frac{1}{f^2}\,\bar{F}^h.$$

About the first term we observe that

$$\frac{1}{f^2}\,\frac{d^2q^h}{d\bar{t}^2} = \frac{1}{f}\,\frac{d}{dt}\left(\frac{dt}{d\bar{t}}\,\frac{dq^h}{dt}\right) = \frac{1}{f}\,\frac{d}{dt}\left(f\,\frac{dq^h}{dt}\right) = \frac{1}{f}\,\frac{df}{dt}\,\frac{dq^h}{dt} + \frac{d^2q^h}{dt^2}.$$

Hence, the second equation (56) becomes

$$\frac{d^2q^h}{dt^2} + \frac{1}{f}\,\frac{df}{dt}\,\frac{dq^h}{dt} + \bar{\Gamma}^h_{ij}\,\frac{dq^i}{dt}\,\frac{dq^j}{dt} = \frac{1}{f^2}\,\bar{F}^h.$$

The two systems are equivalent if and only if this last equation coincides with the first equation (56). This proves that

THEOREM 12.1. [Levi-Civita, 1986b, p. 268] *Two systems* $(\mathbf{g}, \mathbf{F})$ *and* $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ *are equivalent if and only if there exists a function* $f(q^i, v^i)$ *such that equations*

$$F^h - \frac{1}{f^2}\,\bar{F}^h + \frac{1}{f}\,\frac{df}{dt}\,v^h + (\bar{\Gamma}^h_{ij} - \Gamma^h_{ij})\,v^i v^j = 0 \tag{57}$$

are identically satisfied along any motion.

In Eq. (57), and in the following, d/dt is the derivative w.r.to t along any integral curve of the first system. Following Levi-Civita we write

$$f = \mu\,(1 + c_i\,v^i + \tfrac{1}{2}\,c_{ij}\,v^iv^j + \mathbf{3}) \tag{58}$$

where c_i, $c_{ij} = c_{ji}$ and μ are functions of the coordinates only, and $\mathbf{3}$ stands for terms of order ≥ 3 in the velocities. It follows that

$$\begin{aligned} f^{-1} &= \mu^{-1}\,(1 - c_i\,v^i + \mathbf{2}\,), \\ f^{-2} &= \mu^{-2}\,(1 - 2\,c_i\,v^i - c_{ij}\,v^iv^j + 3\,(c_i\,v^i)^2 + \mathbf{3}\,). \end{aligned} \tag{59}$$

Moreover,

$$\begin{aligned} \frac{d}{dt}\log f \; &= \frac{d}{dt}\log\mu + \frac{d}{dt}(1 + c_i\,v^i + \tfrac{1}{2}\,c_{ij}\,v^iv^j + \mathbf{3}) \\ &= v^i\,\partial_i\log\mu + c_i\,F^i + c_{ij}\,F^i\,v^j + \mathbf{2}. \end{aligned}$$

As a consequence, Eq. (57) assumes the form

$$\begin{aligned} 0 \; &= F^h - \mu^{-2}\,\bar{F}^h\,(1 - 2\,c_i\,v^i - c_{ij}\,v^iv^j + 3\,(c_i\,v^i)^2\,) \\ &+ v^h\,(v^i\,\partial_i\log\mu + c_i\,F^i + c_{ij}\,F^i\,v^j) \\ &+ (\bar{\Gamma}^h_{ij} - \Gamma^h_{ij})\,v^iv^j + \mathbf{3}. \end{aligned} \tag{60}$$

It follows that

THEOREM 12.2. [Levi-Civita, 1986b], p. 272] *Two systems* $(\mathbf{g}, \mathbf{F})$ *and* $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ *are equivalent if and only if there exist functions* μ *and* c_{ij}*, depending on the coordinates only, such that the following equations are satisfied*

$$\begin{cases} \mu^2\,F^i = \bar{F}^i, \\ v^h\,(v^i\,\partial_i\log\mu + c_{ij}\,F^i\,v^j) + (\bar{\Gamma}^h_{ij} - \Gamma^h_{ij} + F^h\,c_{ij})\,v^iv^j = 0, \\ \dfrac{d\log|f|}{dt} = \dfrac{d\log|\mu|}{dt} + c_{ij}\,F^i\,v^j, \qquad f^2 \doteq \dfrac{\mu^2}{1 - c_{ij}\,v^iv^j}. \end{cases} \tag{61}$$

Proof. In Eq. (60) the terms which do not contain velocities yield the first equation (61). The terms linear in the velocities give rise to equation

$$v^h\,c_i\,F^i + 2\,\bar{F}^h\,c_i\,v^i = 0.$$

This equation implies $c_i = 0$. As a consequence, the quadratic terms yield the second equation (61), while the second equation (58) and Eqs. (59) reduce to the third equation (61). This proves that Eqs. (61) are necessary conditions. Conversely, if we assume that Eqs. (61) are fulfilled, then a straightforward calculation shows that Eqs. (57), which characterize the equivalence in the general case, are identically satisfied. □

REMARK 12.1. A dynamical system $(\mathbf{g}, \mathbf{F})$ has a two-parameter family of **trivial equivalent systems** $(c\,\mathbf{g}, \mu^2\,\mathbf{F})$, with $c, \mu \neq 0 \in \mathbb{R}$. Indeed, Eqs. (61) are satisfied for $c_{ij} = 0$ and $\mu =$ constant. In the special case $c_{ij} = 0$ conditions (61) reduce to

$$\mu^2 F^i = \bar{F}^i, \qquad \bar{\Gamma}^h_{ij} = \Gamma^h_{ij} - \tfrac{1}{2}\,(\delta^h_i\mu_j + \delta^h_j\mu_i). \tag{62}$$

This last equation shows that the two metrics are equivalent and suggests the notion of **geodesic equivalence** of dynamical systems (see the next section).

REMARK 12.2. The first equation (61) shows that for the equivalence of two mechanical systems the forces must be parallel and equioriented,

$$\mu^2\,\mathbf{F} = \bar{\mathbf{F}}.$$

Hence, in particular, they must have the same singular points.

13 Geodesically equivalent systems and cofactor systems

DEFINITION 13.1. We call **geodesically equivalent** two equivalent systems $(\mathbf{g}, \mathbf{F})$ and $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ such that also the two metrics $\mathbf{g}$ and $\bar{\mathbf{g}}$ are equivalent.

THEOREM 13.1. *Two systems* $(\mathbf{g}, \mathbf{F})$ *and* $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ *are geodesically equivalent if and only if the metric* $\mathbf{g}$ *admits a (non-singular) J-tensor* $\mathbf{J}$ *such that*

$$\bar{g}_{ij} = \tfrac{1}{\mu}\, B_{ij}, \qquad \bar{\mathbf{F}} = \mu^2\,\mathbf{F}, \qquad \mu = \det \mathbf{J}, \tag{63}$$

where $\mathbf{B}$ *is the B-tensor associated with* $\mathbf{J}$.

REMARK 13.1. Due to Theorem 5.1 this is equivalent to say that the metric $\bar{\mathbf{g}}$ admits a J-tensor $\bar{\mathbf{J}}$ such that $\mathbf{F} = (\det \bar{\mathbf{J}})^2\,\bar{\mathbf{F}}$. Hence, this statement is *symmetric* w.r.to the two systems.

Proof. We know (Theorem 4.1) that two metrics $\mathbf{g}$ and $\bar{\mathbf{g}}$ are equivalent if and only if the metric $\mathbf{g}$ admits a (non-singular) J-tensor $\mathbf{J}$ such that

$$\bar{g}^{ij} = \mu\, J^{ij}, \qquad \mu = \det \mathbf{J} = \frac{dt}{d\bar{t}}.$$

Let us apply Theorem 12.2: Eqs. (61) are satisfied with $c_{ij} = 0$ and $\mu = \det \mathbf{J}$ (cf. Remark 4.1 and Remark 12.1, Eq. (62)). □

Theorem 13.1 leads in a natural way to consider the case in which the equivalent system is Lagrangian (the force $\bar{\mathbf{F}}$ is conservative):

$$\bar{\mathbf{F}} = -\,\bar{\nabla} V.$$

Here, $\bar{\nabla}$ is the gradient operator w.r.to the equivalent metric, $(\bar{\nabla} V)^i = \bar{g}^{ij}\,\partial_j V$.

THEOREM 13.2. *A dynamical system* $(\mathbf{g}, \mathbf{F})$ *is geodesically equivalent to a Lagrangian system i.e., to a system* $(\bar{\mathbf{g}}, \bar{\mathbf{F}})$ *where* $\bar{\mathbf{g}}$ *is an equivalent metric and* $\bar{\mathbf{F}} = -\,\bar{\nabla} V$*, if and only if the fundamental metric* $\mathbf{g}$ *admits a non-singular J-tensor* $\mathbf{J}$ *such that*

$$\mathbf{F} = -\mathbf{A}^{-1}\nabla V, \quad \mathbf{A} \doteq \operatorname{cof} \mathbf{J}. \tag{64}$$

Proof. We have only to prove that $\bar{\mathbf{F}} = -\,\bar{\nabla} V$ is equivalent to $\mathbf{F} = -\,\mathbf{A}^{-1}\nabla V$, being, according to (63), $\bar{\mathbf{F}} = \mu^2 \mathbf{F}$. We observe that, being $\bar{g}^{ij} = \mu\, J^{ij}$ and $\mathbf{J} = \mu \mathbf{A}^{-1}$, we have $\bar{\nabla} V = \mu\,\mathbf{J}\nabla V = \mu^2 \mathbf{A}^{-1}\nabla V$. Then, $\mathbf{F} = \frac{1}{\mu^2}\,\bar{\mathbf{F}} = -\frac{1}{\mu^2}\,\bar{\nabla} V = -\mathbf{A}^{-1}\nabla V$. □

DEFINITION 13.2. A dynamical system whose force is of the kind (64) is called **cofactor system**. [Rauch-Wojciechowski, Marciniak & Lundmark, 1999] *et al.*

THEOREM 13.3. *A dynamical system* $(Q, \mathbf{g}, \mathbf{F})$ *is a cofactor system if and only if* $\mathbf{g}$ *admits a non-singular A-tensor* $\mathbf{A} = (A_{ij})$ *such that*

$$d(\mathbf{A}\mathbf{F}) = 0, \qquad d(A_{ij}\, F^j\, dq^i) = 0. \tag{65}$$

Proof. Eq. (64) is equivalent to $\mathbf{A}\mathbf{F} = -\nabla V$, hence to $A_{ij}\, F^j = \partial_i V$. This means that the 1-form $A_{ij}\, F^j\, dq^i$ is exact. Since our considerations are local, this is equivalent to (65). □

REMARK 13.2. Since the components of the equivalent metric are $\bar{g}^{ij} = \mu\, J^{ij}$ and $\bar{g}_{ij} = \frac{1}{\mu}\, B_{ij}$, the **equivalent Hamiltonian** and the **equivalent Lagrangian** functions of a cofactor system are

$$\bar{H} = \tfrac{1}{2}\, \mu\, J^{ij}\, p_i p_j + V, \qquad \bar{L} = \tfrac{1}{2\mu}\, B_{ij}\, \bar{v}^i \bar{v}^j - V,$$

where $\bar{v}^i = dq^i/d\bar{t}$. Hence, we shall denote by

$$(Q, \bar{\mathbf{g}}, \bar{H}), \qquad (Q, \bar{\mathbf{g}}, \bar{L})$$

the **equivalent Hamiltonian system** and the **equivalent Lagrangian system** of a cofactor system $(Q, \mathbf{g}, \mathbf{F})$, respectively.

REMARK 13.3. Since $\bar{g}_{ij} = \frac{1}{\mu}\, B_{ij}$ and $\bar{g}^{ij} = \mu\, J^{ij}$, the signature of the equivalent metric $(\bar{p}, \bar{q})$ is the signature of the quadratic forms B_{ij} of J^{ij} times the sign of μ. This means that pseudo-Riemannian geometry plays a significant role also in classical mechanics.

THEOREM 13.4. *If* $\mathbf{J}$ (or $\mathbf{A}$) *in Theorem 13.2 has real and pointwise simple eigenvalues in an open subset of* Q*, then in this subset the tensor*

$$\bar{\mathbf{J}} = (\bar{J}_i^j) = \mathbf{B} = \mathbf{J}^{-1} = \tfrac{1}{\mu}\, \mathbf{A} \tag{66}$$

is a L-tensor w.r.to the equivalent metric $\bar{\mathbf{g}}$.

Proof. Due to Theorem 5.1 and Remark 5.1, $\bar{\mathbf{J}}$ is a J-tensor w.r.to $\bar{\mathbf{g}}$, hence, it is a torsionless CKT. If the eigenvalues are simple, then it is a L-tensor. □
It follows that the L-sequence (37), with $\mathbf{L} = \bar{\mathbf{J}}$, define a KS-space $\bar{\mathcal{K}}$ of the metric $\bar{\mathbf{g}}$, and the orthogonal coordinates adapted to $\bar{\mathbf{J}}$ separate the geodesic Hamilton-Jacobi equation.

REMARK 13.4. The eigenvalues (u^i) of $\bar{\mathbf{J}}$ are the roots of the algebraic equation

$$\det(\bar{\mathbf{J}} - u\, \mathbf{I}) = 0.$$

Due to (66), this equation is equivalent to

$$\det(\mathbf{B} - u\, \mathbf{I}) = 0, \qquad \det(\mathbf{B} - \mu\, u\, \mathbf{I}) = 0, \qquad \det(\mathbf{I} - u\, \mathbf{J}) = 0.$$

Further equivalent equations are

$$\det[B_{ij} - \mu\, u\, g_{ij}] = 0, \qquad \det[\bar{g}_{ij} - u\, g_{ij}] = 0.$$

This last equation shows that (u^i) are the eigenvalues of the equivalent metric w.r.to the basic metric.

REMARK 13.5. As we have seen, when the eigenvalues (u^i) are simple, $\bar{\mathbf{J}}$ is a L-tensor w.r.to $\bar{\mathbf{g}}$. If (u^i) are independent functions, then they are orthogonal coordinates w.r.to $\bar{\mathbf{g}}$ and in general non-orthogonal w.r.to the basic metric $\mathbf{g}$. It is important to observe that, even in the case of a positive-definite basic metric, the equivalent metric $\bar{\mathbf{g}}$ may not be positive-definite.[6]

THEOREM 13.5. *Let* $(Q, \mathbf{g}, \mathbf{F})$ *be a cofactor system,*

$$\mathbf{F} = -(\operatorname{cof}\mathbf{J})^{-1}\nabla V,$$

such that the J-tensor $\mathbf{J}$ *has pointwise simple eigenvalues. Then the equivalent Hamiltonian system* $(Q, \bar{\mathbf{g}}, \bar{H})$ *is separable in the coordinates adapted to* $\bar{\mathbf{J}} = \mathbf{B}$ *if and only if it is a Lagrangian system i.e.,*

$$\mathbf{F} = -\nabla W.$$

Proof. The tensor $\bar{\mathbf{A}} = \operatorname{cof}\bar{\mathbf{J}} = \mathbf{A}^{-1}$ is a A-tensor w.r.to $\bar{\mathbf{g}}$ (Theorem 5.2). It has the same normal eigenvectors of the L-tensor $\bar{\mathbf{J}}$ and simple eigenvalues as well as $\bar{\mathbf{J}}$. Thus, it is a characteristic Killing tensor of the KS-space $\bar{\mathcal{K}}$ generated by $\bar{\mathbf{J}}$. Then the potential V is separable in the orthogonal coordinates determined by $\bar{\mathbf{J}}$ (orthogonal w.r.to the metric $\bar{\mathbf{g}}$) if and only if (see Remark 7.3)

$$\bar{\mathbf{A}}\, dV = dW, \qquad \bar{A}_i^j\, \partial_j V = \partial_i W.$$

Since $\bar{\mathbf{A}} = \mathbf{A}^{-1}$, this is equivalent to say that the one-form φ corresponding to the force $\mathbf{F} = -\mathbf{A}^{-1}\nabla V$ is exact. □

14 Bi-cofactor systems

DEFINITION 14.1. A **bi-cofactor system** (or **cofactor-pair system**) is a dynamical system $(\mathbf{g}, \mathbf{F})$ which is a cofactor-system in two distinct ways:

$$\mathbf{F} = -\mathbf{A}^{-1}\nabla V = -\widetilde{\mathbf{A}}^{-1}\nabla\widetilde{V}, \tag{67}$$

where $\mathbf{A} = \operatorname{cof}\mathbf{J}$ and $\widetilde{\mathbf{A}} = \operatorname{cof}\widetilde{\mathbf{J}}$, being $\mathbf{J} = (J_i^j)$ and $\widetilde{\mathbf{J}} = (\widetilde{J}_i^j)$ two non-singular (and non-trivial) J-tensors w.r.to the metric $\mathbf{g}$.
Let us consider the equivalent metric $\bar{\mathbf{g}}$ determined by the first J-tensor $\mathbf{J}$. Due to Theorems 5.2 and 5.3, the tensor

$$\bar{\mathbf{A}} = \mathbf{A}^{-1}\widetilde{\mathbf{A}}$$

is a A-tensor (hence, a Killing tensor) in this metric. It follows from (67) and from (57), that

$$\bar{\mathbf{A}}\,\bar{\nabla} V = \bar{\nabla}\widetilde{V}. \tag{68}$$

Indeed, $\bar{\nabla} V = \mu^2\,\mathbf{A}^{-1}\,\nabla V = -\mu^2\mathbf{F} = \mu^2\,\widetilde{\mathbf{A}}^{-1}\,\nabla\widetilde{V} = \widetilde{\mathbf{A}}^{-1}\mathbf{A}\,\bar{\nabla}\widetilde{V} = \bar{\mathbf{A}}^{-1}\,\bar{\nabla}\widetilde{V}$. This formula shows a very remarkable fact:

[6]The nice figures contained in [Waksjö, 2003] can be interpreted in this sense.

THEOREM 14.1. *If a bi-cofactor system is such that the tensor*

$$\bar{\mathbf{J}} = \widetilde{\mathbf{J}}\,\mathbf{J}^{-1}$$

has pointwise real simple eigenvalues, then the equivalent Hamiltonian system $(Q, \bar{\mathbf{g}}, \bar{H})$, *where* $\bar{\mathbf{g}}$ *is the equivalent metric determined by* $\mathbf{J}$, *is a L-system generated by the L-tensor* $\bar{\mathbf{J}}$.
In this case the equivalent system is separable in orthogonal coordinates and the KS-space is determined by a L-sequence.

Proof. Under the above assumptions, $\bar{\mathbf{J}}$ is a L-tensor (Theorems 5.1 and 5.2). Since $\bar{\mathbf{A}} = \mathrm{cof}\,\bar{\mathbf{J}}$ is a Killing tensor and (as well as $\bar{\mathbf{J}}$) with simple eigenvalues and normal eigenvectors, it is a characteristic Killing tensor. Eq. (68) shows that the potential V is separable in this web (see Remark 7.3). □

REMARK 14.1. The eigenvalues (u^i) of $\bar{\mathbf{J}}$ are the roots of equation

$$\det(\bar{\mathbf{J}} - u\,\mathbf{I}) = 0.$$

Since $\bar{\mathbf{J}} = \widetilde{\mathbf{J}}\mathbf{J}^{-1}$, this equation is equivalent to

$$\det(\widetilde{\mathbf{J}} - u\,\mathbf{J}) = \det[\widetilde{J}_i^j - u\,J_i^j] = 0.$$

Equivalent equations are

$$\det[\widetilde{J}^{ij} - u\,J^{ij}] = 0, \qquad \det[\widetilde{J}^{ij} - \tfrac{u}{\mu}\,\bar{g}^{ij}] = 0, \qquad \det[\widetilde{J}^{ij} - u\,\overset{*}{g}{}^{ij}] = 0.$$

This shows that (u^i) are the eigenvalues of $\widetilde{J}^{ij}$ w.r.to the B-metric $\overset{*}{g}{}^{ij}$.
Remark 13.5 can be extended to this case.

Appendix A - Torsionless tensors

15 The Haantjes-Nijenhuis torsion

Let $\mathcal{X}(\mathcal{Q})$ be the space of vector fields on a differentiable manifold Q_n. The **torsion** of a (1,1)-tensor $\mathbf{L} = (L_i^j)$ on Q, interpreted as a linear endomorphism on $\mathcal{X}(\mathcal{Q})$, is the skew-symmetric mapping

$$\mathbf{H}\colon \mathcal{X}(\mathcal{Q}) \times \mathcal{X}(\mathcal{Q}) \to \mathcal{X}(\mathcal{Q})$$

defined by ([Nijenhuis, 1951] [Haantjes, 1955] [Frölicher & Nijenhuis, 1956])

$$\mathbf{H}(\mathbf{X}, \mathbf{Y}) \doteq [\mathbf{LX}, \mathbf{LY}] - \mathbf{L}[\mathbf{X}, \mathbf{LY}] - \mathbf{L}[\mathbf{LX}, \mathbf{Y}] + \mathbf{L}^2[\mathbf{X}, \mathbf{Y}] \tag{69}$$

In components,

$$\begin{aligned}
[\mathbf{LX}, \mathbf{LY}]^k &= L_l^i X^l\,\partial_i(L_h^k Y^h) - \ldots \\
&= L_l^i X^l\,Y^h \partial_i L_h^k + L_l^i X^l L_h^k\,\partial_i Y^h - \ldots \\
&= 2\,L_{[l}^i \partial_{|i|} L_{h]}^k X^l Y^h + L_l^i L_h^k\,(X^l \partial_i Y^h - Y^l \partial_i X^h).
\end{aligned}$$

$$[\mathbf{X}, \mathbf{LY}]^h = X^i \partial_i (L_l^h Y^l) - L_l^i Y^l \partial_i X^h = X^i L_l^h \partial_i Y^l + X^i Y^l \partial_i L_l^h - L_l^i Y^l \partial_i X^h.$$

$$\begin{aligned} L_h^k([\mathbf{X},\mathbf{LY}]^h-[\mathbf{Y},\mathbf{LX}]^h) &= L_h^k(L_l^h[\mathbf{X},\mathbf{Y}]^l+(X^iY^l-Y^iX^l)\,\partial_i L_l^h \\ &-L_l^i(Y^l\partial_i X^h-X^j\partial_i Y^h)). \end{aligned}$$

It follows that

$$(\mathbf{H}(\mathbf{X},\mathbf{Y}))^k = 2(L^i_{[l}\partial_{|i|}L^k_{h]} - L^k_m\partial_{[l}L^m_{h]})\;X^lY^h.$$

This shows a first remarkable fact:

PROPOSITION 15.1. *The mapping* (69) *is* $\mathcal{F}(\mathcal{Q})$*-bilinear, so that it defines a* $(1,2)$*-tensor with components*

$$H^k_{ij} = 2(L^h_{[i}\partial_{|h|}L^k_{j]} - L^k_m\partial_{[i}L^m_{j]}). \tag{70}$$

This is formula (3.1) in [Nijenhuis,1951]. The torsion is denoted by $\mathbf{H}$ by Nijenhuis. It is very often denoted by $\mathbf{N}$ and called the **Nijenhuis torsion**. In the following we shall denote by $\mathbf{H}(\mathbf{L})$ the torsion of $\mathbf{L}$. If $\mathbf{H}(\mathbf{L})=0$, then $\mathbf{L}$ is said to be **torsionless**.

REMARK 15.1. The definition (70) does not depend on the choice of the coordinates and the partial derivatives $\partial_i=\partial/\partial q^i$ may be replaced by the covariant derivatives ∇_i w.r. to any symmetric connection,

$$H^k_{ij} = 2(L^h_{[i}\nabla_{|h|}L^k_{j]} - L^k_m\nabla_{[i}L^m_{j]}).$$

REMARK 15.2. Let $\mathbf{X}_i$ and $\mathbf{X}_j$ be two eigenvectors of $\mathbf{L}$, $\mathbf{LX}_i=u^i\,\mathbf{X}_i$ and $\mathbf{LX}_j=u^j\,\mathbf{X}_j$. Let us introduce the (1,1) tensor

$$\mathbf{T}^{(u^i,u^j)} \doteq (\mathbf{L}-u^i\,\mathbf{I})(\mathbf{L}-u^j\,\mathbf{I}) = \mathbf{L}^2-u^i\,\mathbf{L}-u^j\,\mathbf{L}+u^iu^j\,\mathbf{I}$$

with $\mathbf{I}=$ identity. If $\mathbf{LX}_h=u^h\,\mathbf{X}_h$, then

$$\mathbf{T}^{(u^i,u^j)}\mathbf{X}_h = (u^h-u^i)(u^h-u^j)\;\mathbf{X}_h \tag{71}$$

REMARK 15.3. From the definition (69) it follows that,

$$\begin{aligned} \mathbf{H}(\mathbf{X}_i,\mathbf{X}_j) &= [u^i\mathbf{X}_i,u^j\mathbf{X}_j]-\mathbf{L}[\mathbf{X}_i,u^j\mathbf{X}_j]-\mathbf{L}[u^i\mathbf{X}_i,\mathbf{X}_j] \\ &\quad+\mathbf{L}^2[\mathbf{X}_i,\mathbf{X}_j] \\ &= u^i\,\mathbf{X}_iu^j\,\mathbf{X}_j-u^j\,\mathbf{X}_ju^i\,\mathbf{X}_i+u^iu^j\,[\mathbf{X}_i,\mathbf{X}_j] \\ &\quad-\mathbf{L}([\mathbf{X}_i,\mathbf{X}_j]\,u^j+\mathbf{X}_iu^j\,\mathbf{X}_j)-\mathbf{L}([\mathbf{X}_i,\mathbf{X}_j]\,u^i \\ &\quad-\mathbf{X}_ju^i\,\mathbf{X}_i)+\mathbf{L}^2[\mathbf{X}_i,\mathbf{X}_j] \\ &= u^i\,\mathbf{X}_iu^j\,\mathbf{X}_j-u^j\,\mathbf{X}_ju^i\,\mathbf{X}_i-\mathbf{X}_iu^j\,\mathbf{LX}_j+\mathbf{X}_ju^i\,\mathbf{LX}_i \\ &\quad+T^{(u^i,u^j)}[\mathbf{X}_i,\mathbf{X}_j] \\ &= u^i\,\mathbf{X}_iu^j\,\mathbf{X}_j-u^j\,\mathbf{X}_ju^i\,\mathbf{X}_i-u^j\,\mathbf{X}_iu^j\,\mathbf{X}_j+u^i\,\mathbf{X}_ju^i\,\mathbf{X}_i \\ &\quad+T^{(u^i,u^j)}[\mathbf{X}_i,\mathbf{X}_j] \\ &= T^{(u^i,u^j)}[\mathbf{X}_i,\mathbf{X}_j]+(u^i\,\mathbf{X}_iu^j-u^j\,\mathbf{X}_iu^j)\,\mathbf{X}_j \\ &\quad-(u^j\,\mathbf{X}_ju^i-u^i\,\mathbf{X}_ju^i)\,\mathbf{X}_i. \end{aligned}$$

This shows that[7]

$$\mathbf{H}(\mathbf{X}_i,\mathbf{X}_j) = \mathbf{T}^{(u^i,u^j)}[\mathbf{X}_i,\mathbf{X}_j]+(u^i-u^j)(\mathbf{X}_iu^j\,\mathbf{X}_j+\mathbf{X}_ju^i\,\mathbf{X}_i) \tag{72}$$

[7] Cf. formula (6.13) of [Frölicher & Nijenhuis, 1956].

16 Frames of eigenvectors

A **frame** on the manifold Q_n is a set $(\mathbf{X}_i)$ of n vector fields such that at each point $q \in Q$ they form a basis of the tangent space T_qQ.[8] Two frames $(\mathbf{X}_i)$ and $(\mathbf{Y}_i)$ are said to be **equivalent** if there exist nowhere-vanishing functions f_i such that $\mathbf{Y}_i = f_i\mathbf{X}_i$. A frame is called **integrable** or **holonomic** if it is equivalent to a **natural frame** (∂_i) made of the partial derivatives (interpreted as vector fields) w.r.to a coordinate system (q^i). It can be proved that

THEOREM 16.1. *The following three conditions are equivalent:*
(a) *The frame* $(\mathbf{X}_i)$ *is integrable;*
(b) *For each index i the distribution Δ_i' spanned by all the vectors of the frame except $\mathbf{X}_i$ is completely integrable;*
(c) *For each pair of distinct indices $i \neq j$ the distribution Δ_{ij} spanned by the vectors $(\mathbf{X}_i, \mathbf{X}_j)$ is completely integrable.*
Hereafter we shall consider (1,1)-tensors $\mathbf{L}$ satisfying the following
Fundamental assumption: *The eigenvalues u^i of $\mathbf{L}$ are real and there exists a frame $(\mathbf{X}_i)$ made of eigenvectors of $\mathbf{L}$. In this case we say that $\mathbf{L}$ admits an* **eigenframe***.*
As a consequence we can write

$$[\mathbf{X}_i, \mathbf{X}_j] = \Omega_{ij}^h\, \mathbf{X}_h.$$

REMARK 16.1. Due to Theorem 15.1 and the Frobenius theorem, the frame $(\mathbf{X}_i)$ is integrable if and only if $\Omega_{ij}^h = 0$ for all distinct indices, $h, i, j \neq$.
Due to (71) and (72), we have

$$\begin{aligned}\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j) &= (u^i - u^j)(\mathbf{X}_j u^i\, \mathbf{X}_i + \mathbf{X}_i u^j\, \mathbf{X}_j) \\ &+ \textstyle\sum_h (u^h - u^i)(u^h - u^j)\, \Omega_{ij}^h\, \mathbf{X}_h\end{aligned}$$

Note that in this formula the sum $\sum_h$ can be replaced by $\sum_{h \neq i,j}$. Then,

THEOREM 16.2. *The torsionless condition $\mathbf{H}(\mathbf{L}) = 0$ is equivalent to equations*

$$(u^k - u^i)(u^k - u^j)\, \Omega_{ij}^k = 0, \quad i, j, k \neq, \qquad (u^i - u^j)\, \mathbf{X}_j u^i = 0.$$

As a corollary we have

THEOREM 16.3. *If the eigenvalues of $\mathbf{L}$ are simple, then $\mathbf{H}(\mathbf{L}) = 0$ if and only if the frame $\mathbf{X}_i$ is integrable and for each index i the eigenvalues u^i is constant on the integral submanifolds of Δ_i':*

$$\mathbf{H}(\mathbf{L}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \Omega_{ij}^k = 0, & i, j, k \neq, \\ X_j u^i = 0, & i \neq j. \end{cases}$$

In other words,

THEOREM 16.4. *If the eigenvalues of $\mathbf{L}$ are simple, then $\mathbf{H}(\mathbf{L}) = 0$ if and only if there exist coordinates (q^i) such that the vectors ∂_i are eigenvectors of $\mathbf{L}$ i.e.,*

$$L_i^j = u^i\, \delta_i^j, \qquad u^i = u^i(q^i).$$

[8] Frames exist only locally, unless Q is parallelizable.

Note that the last condition means that each eigenvalue is a function of the corresponding coordinate only (or a constant).[9]
Now we consider the case of $\mathbf{L}$ with non-simple real eigenvalues. We denote the N distinct eigenvalues by $(u^A) = (u^1, \ldots, u^N)$. Then we order all the n eigenvalues u^i by an index i such that $i \in A$ means $u^i = u^A$. We use the notation $i \simeq j$ for indices i, j belonging to the same group A; otherwise we write $i \not\simeq j$.

REMARK 16.2. The **algebraic multiplicity** $\mathrm{am}(u)$ of an eigenvalue u is its multiplicity as a root of the characteristic equation. If u is real, then all the corresponding eigenvectors are real and form (at each point) a linear subspace $E_q(u) \subseteq T_qQ$. The dimension of this space is the **geometrical multiplicity** $\mathrm{gm}(u)$ of the eigenvalue. It is well known that

$$\mathrm{am}(u) \geq \mathrm{gm}(u)$$

and

$$u^1 \neq u^2 \quad \Longrightarrow \quad E(u^1) \cap E(u^2) = 0.$$

Thus, the fundamental assumption implies

$$m_A \doteq \mathrm{am}(u^A) = \mathrm{gm}(u^A)$$

Notation: Δ_A is the distribution (of rank m_A) spanned by the eigenvectors X_A of u^A. Δ_{AB} $(A \neq B)$ is the distribution (of rank $m_A + m_B$) spanned by the eigenvectors (X_A, X_B) of u^A and u^B. Δ'_A is the distribution (of rank $n - m_A$) spanned by all eigenvectors X_B of u^B with $B \neq A$. Note that $LX_A = u^A X_A$ means that on the distribution Δ_A, L reduce to the multiplication by u^A.
A corollary of Theorem 16.2 is

THEOREM 16.5.

$$\mathbf{H}(\mathbf{L}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \Omega^C_{AB} = 0, & A, B, C \ \neq, \\ X_A u^B = 0, & A \neq B. \end{cases}$$

It follows that

THEOREM 16.6. $\mathbf{H} = 0$ *if and only if* (i) *all* Δ_A, Δ_{AB}, Δ'_A *are completely integrable and* (ii) u^A *is constant on each integral submanifold of* Δ'_A.

THEOREM 16.7. $\mathbf{H}(\mathbf{L}) = 0$ *if and only if there are coordinates* $\underline{q} = (q^i)$ *such that* ∂_A *span* Δ_A, *so that* $L^B_A = 0$ *for* $A \neq B$ *and* $L^j_i = \delta^j_i u^A$ *for* $i, j \in A$, *and* $u^A = u^A(q^A)$.

REMARK 16.3. From this last theorem it follows that:
(i) There exist N integrable distributions $\Delta_A = \Delta(u^A)$ of rank m_A and N transversal foliations U_A of submanifolds of dimension m_A such that u^B is locally constant on the leaves U_A with $A \neq B$.

[9] Theorems 16.3 and 16.4 do not appear as statements and in this synthetic form in [N]. Condition $\mathbf{H} = 0$ implies condition (4.5) in [Nijenhuis,1951], which is proved to be necessary and sufficient for $\Omega^k_{ij} = 0$ with $k \neq i, j$ (the eigenvalues are assumed to be all distinct). In §5, condition $\mathbf{H} = 0$ is seen to be only sufficient for $X_i u^j = 0$ with $u^i \neq u^j$. Actually, the proofs given here are only based on the fundamental assumption.

(ii) there exist local coordinates $\underline{q} = (q^i)$ **adapted** to this web: each ∂_i is tangent to U_A, for $i \in A$, and it is eigenvector of $\mathbf{L}$, $\mathbf{L}\partial_i = u^A \partial_i$, and

$$\mathbf{L} = \textstyle\sum_i u^i dq^i \otimes \partial_i, \qquad L_i^j = u^i \delta_i^j.$$

This means that

$$\begin{aligned} L_i^j &= 0, \quad \text{for } i \not\simeq j \\ L_i^j &= \delta_i^j\, u^A, \quad \text{for } i \simeq j \in A. \end{aligned}$$

Thus, $\mathbf{L}$ admits a holonomic frame of eigenvectors.

17 Torsionless conformal Killing tensors

Let us study the torsionless condition $\mathbf{H}(\mathbf{L}) = 0$ for a conformal Killing two-tensor $\mathbf{L}$ satisfying the fundamental assumption: $\mathbf{L}$ admits a real (local) eigenframe. Let us recall that a CKT is characterized by equation

$$\{P(\mathbf{L}), P(\mathbf{G})\} = -2\, P(\boldsymbol{\alpha})\, P(\mathbf{G}). \tag{73}$$

The eigenspaces $E(u^i)$ and $E(u^j)$ associated with two distinct eigenvalues $u^i \neq u^j$ are orthogonal. It follows from Remark 16.6 that the two foliations U_A and U_B are orthogonal if $A \neq B$. Hence, the metric $\mathbf{G}$ is the direct sum of metrics $\mathbf{G}_A$, each other orthogonal:

$$g^{ij} = 0, \quad i \in A,\ j \in B,\ A \neq B,$$

in coordinates adapted to $\mathbf{L}$. Let us use the notation $P(\mathbf{G}_A) = P_A$. Then,

$$P(\mathbf{G}) = \sum_A P_A, \qquad P(\mathbf{L}) = \sum_A u^A P_A, \qquad \{u^A, P_B\} = 0, \quad A \neq B.$$

It follows that

$$\begin{aligned} \{P(\mathbf{L}), P(\mathbf{G})\} &= \{\textstyle\sum_A u^A P_A, \sum_B P_B\} = \sum_{A,B}\{u^A P_A, P_B\} \\ &= \textstyle\sum_{A,B}\Big(u^A\{P_A, P_B\} + P_A\{u^A, P_B\}\Big) \\ &= \textstyle\sum_{A<B}(u^A - u^B)\{P_A, P_B\} + \sum_A\{u^A, P_A\}\ P_A. \end{aligned}$$

Since u^A are functions of the q^A only, $\partial_B u^A = 0$ for $A \neq B$,

$$\begin{aligned} \{u^A, P_A\} &= -\textstyle\sum_B \sum_{i\in B} \partial_i u^A \partial^i P_A = -\sum_{i\in A} \partial_i u^A\, \partial^i P_A \\ &= -2\ \textstyle\sum_{i,k\in A} \partial_i u^A\, g^{ik} p_k. \end{aligned}$$

Moreover,

$$\begin{aligned} \{P_A, P_B\} &= \textstyle\sum_{i\in B} \partial_i P_A\, \partial^i P_B - \sum_{i\in A} \partial_i P_B\, \partial^i P_A \\ &= 2\textstyle\sum_{i,k\in B} \partial_i P_A\, g^{ik}\, p_k - 2\sum_{i,k\in A} \partial_i P_B\, g^{ik}\, p_k \end{aligned}$$

Thus,

$$\begin{aligned} \{P(\mathbf{L}) &, P(\mathbf{G})\} = 2\textstyle\sum_{A<B}(u^A - u^B)\,(\sum_{i,k\in B} \partial_i P_A\, g^{ik}\, p_k \\ &- \textstyle\sum_{i,k\in A} \partial_i P_B\, g^{ik}\, p_k) - 2\ \sum_A \sum_{i,k\in A} \partial_i u^A\, g^{ik}\, p_k\ P_A, \end{aligned}$$

and Eq. (73) is equivalent to

$$\begin{aligned}&\textstyle\sum_{A<B}(u^A-u^B)\,(\sum_{i,k\in B}\partial_i P_A\, g^{ik}\, p_k-\sum_{i,k\in A}\partial_i P_B\, g^{ik}\, p_k)\\&-\textstyle\sum_A\sum_{i,k\in A}\partial_i u^A\, g^{ik}\, p_k\; P_A=\,-\,\alpha^i p_i\,\sum_A P_A.\end{aligned}\tag{74}$$

This is a homogeneous polynomial equation of degree 3 in the momenta $\underline{p}$. From the coefficients of $\underline{p}$ of a same group A we get equation

$$\textstyle\sum_A\, P_A\,(\sum_{a,b\in A} g^{ab}\partial_a u^A\, p_b-\alpha^b\, p_b)=0.$$

This is equivalent to

$$\textstyle\sum_{a,b\in A}(g^{ab}\partial_a u^A-\alpha^b)\, p_b=0$$

i.e., to

$$g^{ab}\partial_a u^A=\alpha^b,\qquad \partial_a u^A=\alpha_a,\quad a\in A.\tag{75}$$

The remaining part of (74) is

$$\begin{aligned}\textstyle\sum_{A<B}(u^A-u^B)\quad&(\textstyle\sum_{i,k\in B}\partial_i P_A\, g^{ik}\, p_k-\sum_{i,k\in A}\partial_i P_B\, g^{ik}\, p_k)\\&=\,-\,\alpha^i p_i\,\textstyle\sum_A P_A.\end{aligned}$$

We can write it in the form

$$\begin{aligned}\textstyle\sum_{A<B}(u^A-u^B)\quad&(\textstyle\sum_{\alpha,\beta\in B}\partial_\alpha P_A\, g^{\alpha\beta}\, p_\beta-\sum_{a,b\in A}\partial_a P_B\, g^{ab}\, p_b)\\&=\,-\,(\alpha^a p_a+\alpha^\alpha p_\alpha)\,\textstyle\sum_A P_A.\end{aligned}$$

This equation is equivalent to

$$\begin{aligned}&(u^B-u^A)\textstyle\sum_{a\in A} g^{ab}\,\partial_a g^{\alpha\beta}=\alpha^b\, g^{\alpha\beta},\\&b\in A,\quad \alpha,\beta\in B,\quad A\neq B.\end{aligned}\tag{76}$$

Eqs. (75) and (76) prove

THEOREM 17.1. *Let* $\mathbf{L}=(L^{ij})$ *be symmetric tensor such that the tensor* L_i^j *is torsionless and satisfies the fundamental assumption. Then* $\mathbf{L}$ *is a CKT if and only if in adapted coordinates equations*

$$\begin{aligned}&\partial_a u^A=\alpha_a,\quad a\in A,\\&(u^B-u^A)\,\partial_a g^{\alpha\beta}=\alpha_a\, g^{\alpha\beta},\quad a\in A,\quad \alpha,\beta\in B,\quad A\neq B.\end{aligned}\tag{77}$$

are satisfied.

REMARK 17.1. The fundamental assumption is satisfied by any symmetric tensor in a positive-definite metric.

18 The canonical form of the metric tensor

THEOREM 18.1. *If there exists a torsionless CKT* $\mathbf{L}$*, then in adapted coordinates the metric tensor components assume the canonical block-form*

$$g_A^{ab} = \frac{f_A^{ab}(q^A)}{\prod_{B\neq A}(u^A - u^B)}, \quad a, b \in A, \tag{78}$$

$$g^{a\beta} = 0, \quad a \in A,\ \beta \in B, \quad A \neq B \tag{79}$$

$$g^A_{ab} = f^A_{ab}(q^A) \prod_{B\neq A} (u^A - u^B), \quad a, b \in A, \tag{80}$$

$$g_{a\beta} = 0, \quad a \in A,\ \beta \in B, \quad A \neq B, \tag{81}$$

where

$$\sum_{c\in A} f^A_{ac}\, f_A^{bc} = \delta^b_a, \qquad a, b \in A.$$

Proof. From Eqs. (77) we get

$$\partial_a u^A\, g^{\alpha\beta} = (u^B - u^A)\, \partial_a g^{\alpha\beta}, \quad a \in A, \quad \alpha, \beta \in B, \quad A \neq B.$$

Hence,

$$\frac{\partial_a u^A}{(u^B - u^A)}\, g^{\alpha\beta} = \partial_a g^{\alpha\beta}, \quad a \in A, \quad \alpha, \beta \in B.$$

Assume for the moment that $g^{\alpha\beta} \neq 0$ and $B > A$ (so that $u^B > u^A$). Then,

$$\partial_a \log(u^B - u^A) + \partial_a \log g^{\alpha\beta} = 0,$$

i.e.,

$$\partial_a((u^B - u^A)\, g^{\alpha\beta}) = 0, \quad a \in A, \quad \alpha, \beta \in B.$$

It follows that $(u^B - u^A)\, g^{\alpha\beta}$ is a function of all q^i, with $i \notin A$, so that

$$g^{\alpha\beta} = \frac{h^{\alpha\beta}_{AB}(q^{\bar{A}})}{u^B(q^B) - u^A(q^A)}, \qquad \alpha, \beta \in B. \tag{82}$$

Here and in the following, by $q^{\bar{A}}$ we mean all the coordinates except those which belong to the group A. Thus,

$$\frac{h^{\alpha\beta}_{AB}(q^{\bar{A}})}{u^B(q^B) - u^A(q^A)} = \frac{h^{\alpha\beta}_{CB}(q^{\bar{C}})}{u^B(q^B) - u^C(q^C)}, \quad \alpha, \beta \in B, \quad A, B, C \neq .$$

Let us derive w.r.to $i \in C$:

$$\frac{\partial_i h^{\alpha\beta}_{AB}(q^{\bar{A}})}{u^B(q^B) - u^A(q^A)} = \frac{h^{\alpha\beta}_{CB}(q^{\bar{C}})\, \partial_i u^C}{(u^B(q^B) - u^C(q^C))^2}, \quad \alpha, \beta \in B.$$

$$\frac{(u^B(q^B) - u^C(q^C))^2}{\partial_i u^C}\, \partial_i h^{\alpha\beta}_{AB}(q^{\bar{A}}) = (u^B(q^B) - u^A(q^A))\, h^{\alpha\beta}_{CB}(q^{\bar{C}}),$$

$$A, B, C \neq .$$

The left hand side is a function of all coordinates except (q^A), the right hand side of all coordinates except (q^C). This implies that both are functions of all coordinates except (q^A, q^C). Hence,

$$(u^B(q^B) - u^A(q^A))\, h^{\alpha\beta}_{CB}(q^{\bar{C}}) = f^{\alpha\beta}_{ABC}(q^{\bar{A}}, q^{\bar{C}}), \quad \alpha, \beta \in B, \quad A, B, C \neq .$$

It follows from (82) that

$$g^{\alpha\beta} = \frac{h^{\alpha\beta}_{CB}(q^{\bar{C}})}{u^B - u^C} = \frac{f^{\alpha\beta}_{ABC}(q^{\bar{A}}, q^{\bar{C}})}{(u^B - u^C)(u^B - u^A)}, \qquad \alpha, \beta \in B.$$

By iterating the same process above, and changing the indices, we find Eqs. (79) and (81). They have been proved for $g^{ab} \neq 0$, but of course they hold true also for $g^{ab} = 0$. □

REMARK 18.1. In the case of simple eigenvalues we find

$$g^{ii} = \frac{\phi_i(q^i)}{\prod_{j \neq i}(u^i - u^j)}, \quad g^{ij} = 0, \quad i \neq j.$$

19 The trace-type condition

THEOREM 19.1. *For a torsionless CKT* $\mathbf{L}$ *of trace-type,* $m_A > 1$ *implies* $u^A = \text{constant}$ *and* $\alpha_i = 0$ *for* $i \in A$.

Proof. $\alpha_i = \partial_i \mathrm{tr}\,(\mathbf{L}) = \partial_i \sum_A m_A u^A(q^A) = m_A\, \partial_i u^A$ with $i \in A$. Being $\alpha_i = \partial_i u^A(q^A)$ with $i \in A$ (see the first equation (17)), it follows that $(m_A - 1)\partial_i u^A = 0$, $i \in A$. □

THEOREM 19.2. *A torsionless CKT* $\mathbf{L}$ *is of gradient type:* $\alpha_i = \partial_i U$, $U \doteq \sum_A u^A$. *If the eigenvalues are simple, then it is of trace-type:* $\alpha_i = \partial_i \mathrm{tr}\,(\mathbf{L})$.

Proof. Due to the first equation (17), $\alpha_a = \partial_a u^A = \partial_a U$, with $a \in A$. Since $\mathrm{tr}\,(L) \doteq \sum_A m_A u^A$, we have $U = \mathrm{tr}\,(L)$ if all eigenvalues are simple, $m_A = 1$. □

THEOREM 19.3. *A trace-type torsionless CKT is a J-tensor.*

Proof. The assumption that the tensor admits eigenframes is understood. The existence of such a tensor implies the existence of adapted coordinates in which the metric tensor assumes the canonical form (79), (81). In these coordinates

$$\begin{aligned} &\Gamma^j_{hi} = 0, \quad \text{for } h, i, j \not\simeq\ , \\ &g_{hi} = 0, \quad L^i_h = 0, \quad \text{for } h \not\simeq i, \\ &L^j_i = \delta^j_i\, u^A(q^A), \quad i \simeq j \in A, \end{aligned}$$

where $i \simeq j$ means that i and j belong to the same group of indices A. Let us recall the characteristic equation of a J-tensor,

$$\partial_h L^j_i - \Gamma^l_{hi}\, L^j_l + \Gamma^j_{hl}\, L^l_i = \tfrac{1}{2}\,(\alpha_i\, \delta^j_h + \alpha^j\, g_{ih}), \tag{83}$$

and write it in adapted coordinates (q^i). For $h, i, j \not\simeq\ $, Eq. (83) is identically satisfied, $0 = 0$. For $h \simeq i \simeq j \in A$, Eq. (83) reduces to $\delta^j_i\, \partial_h u^A = \frac{1}{2}\,(\alpha_i\, \delta^j_h + \alpha^j\, g^A_{ih})$, which is equivalent to

$$g^A_{ij}\, \partial_h u^A = \tfrac{1}{2}\,(\alpha_i\, g^A_{hj} + \alpha_j\, g^A_{ih}). \tag{84}$$

For $A \ni h \simeq i \not\simeq j \in B$, Eq. (83) reduces to $\Gamma^j_{hi}\,(u^A - u^B) = \frac{1}{2}\,\alpha^j\, g_{ih}$, which is equivalent to

$$\Gamma_{hi,j}\,(u^A - u^B) = \tfrac{1}{2}\,\alpha_j\, g_{ih}. \tag{85}$$

For $A \ni h \not\simeq i \simeq j \in B$, Eq. (83) is identically satisfied. For $A \ni h \simeq j \not\simeq i \in B$,

$$\Gamma^j_{hi}\,(u^B - u^A) = \tfrac{1}{2}\,\alpha_i\,\delta^j_h. \tag{86}$$

We conclude that Eq. (83), in coordinates adapted to a torsionless CKT, is equivalent to Eqs. (84), (85), (86).
Due to (81), Eq. (84) is equivalent to

$$f^A_{ij}\,\partial_h u^A = \tfrac{1}{2}\,(\alpha_i\, f^A_{hj} + \alpha_j\, f^A_{ih}), \quad h \simeq j \simeq i \in A.$$

Theorem 19.1 shows that for $m_A > 1$ this equation is identically satisfied. For studying Eq. (85), let us compute the Christoffel symbols for $A \ni h \simeq i \not\simeq j \in B$:

$$\begin{aligned}
\Gamma_{hi,j} &= \tfrac{1}{2}\,(\partial_h g_{ij} + \partial_i g_{jh} - \partial_j g_{hi}) = -\,\tfrac{1}{2}\,\partial_j g^A_{hi} \\
&= -\,\tfrac{1}{2}\,\partial_j (f^A_{hi}(q^A) \textstyle\prod_{I\neq A}(u^A - u^I)) \\
&= -\,\tfrac{1}{2}\, f^A_{hi}(q^A)\,\partial_j (\textstyle\prod_{I\neq A}(u^A - u^I)) \\
&= \tfrac{1}{2}\, f^A_{hi}(q^A)\, \textstyle\prod_{I\neq A,B}(u^A - u^I)\,\partial_j u^B = \tfrac{1}{2}\, g^A_{hi}\,\dfrac{\partial_j u^B}{u^A - u^B}.
\end{aligned}$$

Then (85) reduces to

$$\partial_j u^B = \alpha_j, \qquad j \in B. \tag{87}$$

Also this equation is identically satisfied, in virtue of (17). For Eq. (86), let us compute the Christoffel symbols for $A \ni h \simeq j \not\simeq i \in B$

$$\begin{aligned}
\Gamma^j_{hi} &= \tfrac{1}{2}\,\textstyle\sum_{l\in A}\, g^{jl}\,(\partial_h g_{il} + \partial_i g_{lh} - \partial_l g_{hi}) = \tfrac{1}{2}\,\textstyle\sum_{l\in A}\, g^{jl}_A\,\partial_i g^A_{hl} \\
&= \tfrac{1}{2}\,\textstyle\sum_{l\in A}\, \dfrac{f^{jl}_A}{\prod_{I\neq A}(u^A - u^I)}\,\partial_i\Big(f^A_{hl}\,\textstyle\prod_{J\neq A}(u^A - u^J)\Big) \\
&= \tfrac{1}{2}\,\textstyle\sum_{l\in A}\, \dfrac{f^{jl}_A\, f^A_{hl}}{\prod_{I\neq A}(u^A - u^I)}\,\partial_i\Big(\textstyle\prod_{J\neq A}(u^A - u^J)\Big) \\
&= -\,\tfrac{1}{2}\,\dfrac{\delta^j_h}{\prod_{I\neq A}(u^A - u^I)}\,\textstyle\prod_{J\neq A,B}(u^A - u^J)\,\partial_i u^B = \tfrac{1}{2}\,\dfrac{\partial_i u^B}{u^B - u^A}\,\delta^j_h.
\end{aligned}$$

Hence, also (86) reduces to (87) and it is identically satisfied. □

Appendix B - Special tensors in pseudo-Euclidean spaces

20 J-tensors on $\mathbb{R}^n$

Let us list all metrics which are involved in our discussion:

$$\begin{array}{rcl}
\text{basic metric} & \mathbf{g} & \left\{\begin{array}{l} g_{ij} \\ g^{ij} \end{array}\right. \\
\text{equivalent metric} & \bar{\mathbf{g}} & \left\{\begin{array}{l} \bar{g}_{ij} = \frac{1}{\mu}\, B_{ij} = \frac{1}{\mu^2} A_{ij} \\ \bar{g}^{ij} = \mu\, J^{ij} = \mu\, \overset{*}{g}{}^{ij} \end{array}\right. \\
\text{B-metric} & \overset{*}{\mathbf{g}} & \left\{\begin{array}{l} \overset{*}{g}_{ij} = B_{ij} = \mu\, \bar{g}_{ij} \\ \overset{*}{g}{}^{ij} = J^{ij} \end{array}\right.
\end{array}$$

Let us study the special tensors and the associated metrics in the manifold $\mathbb{R}^n$ endowed with natural Cartesian coordinates (x^i). Let us consider as **basic metric** the flat metric $\mathbf{g}$ with covariant components

$$g_{ij} = e_i\,\delta_{ij}, \quad e_i = \pm 1. \tag{88}$$

THEOREM 20.1. *The most general J-tensor J^{ij} in $\mathbb{R}^n = (x^i)$ endowed with the metric* (88) *is*

$$J^{ij} = m\,x^i\,x^j + \beta^i\,x^j + \beta^j\,x^i + C^{ij}, \tag{89}$$

where $m \in \mathbb{R}$, $\boldsymbol{\beta} = (\beta^i)$ is a constant vector, and $[C^{ij}]$ is a constant symmetric matrix.

REMARK 20.1. For the basic Euclidean metric $g_{ij} = \delta_{ij}$ (i.e., for $e_i = 1$) this theorem is proved in [Crampin, Sarlet, Thompson, 2000]. The tensor $\mathbf{J}$ is the sum of the two tensors introduced in [Benenti, 1992, 1993], as *planar inertia tensors* and used for constructing the fundamental Stäckel systems of $\mathbb{R}^n$. By the Linköping school $[J^{ij}]$ is denoted by $[G^{ij}]$ and called **elliptic matrix**. The constants m and C^{ij} are respectively denoted by α and γ^{ij}.
It follows that the space of the J-tensors has dimension

$$\tfrac{1}{2}\,(n+1)(n+2).$$

Proof. Since we use Cartesian coordinates (x^i), which are the components of the generic point-vector $\mathbf{x}$, the Christoffel symbols vanish and the differential equation to solve is

$$\partial_k J_{ij} = \tfrac{1}{2}\,(\alpha_i\,g_{jk} + \alpha_j\,g_{ik}) = \tfrac{1}{2}\,(\alpha_i\,e_j\,\delta_{jk} + \alpha_j\,e_i\,\delta_{ik}).$$

This is equivalent to the following system,

$$\begin{cases} \partial_i J_{ii} = e_i\,\alpha_i, \\ \partial_k J_{ii} = 0, \quad k \neq i, \\ \partial_k J_{ij} = 0 k, i, j \neq, \\ \partial_j J_{ij} = \tfrac{1}{2}\,\alpha_i\,e_j, \quad i \neq j. \end{cases} \tag{90}$$

The second equation shows that J_{ii} is a function of the corresponding coordinate x^i only,

$$J_{ii} = J_{ii}(x^i). \tag{91}$$

Then, from the first equation we get

$$\alpha_i = \alpha_i(x^i).$$

From the third equation it follows that for $i \neq j$

$$J_{ij} = J_{ij}(x^i, x^j).$$

Due to these results, from the fourth equation it follows that

$$J_{ij} = \tfrac{1}{2}\,e_j\alpha_i(x^i)\,x^j + f_i(x^i), \tag{92}$$

where $f_i(x^i)$ is any function depending on x^i only. Due to the symmetry we have

$$J_{ij} = \tfrac{1}{2}\,e_i\alpha_j(x^j)\,x^i + f_j(x^j),$$

and

$$\tfrac{1}{2}\, e_j \alpha_i(x^i)\, x^j + f_i(x^i) = \tfrac{1}{2}\, e_i \alpha_j(x^j)\, x^i + f_j(x^j). \tag{93}$$

Let us derive this equation twice w.r.to x^i:

$$\tfrac{1}{2}\, e_j\, \alpha_i'\, x^j + g_i' = \tfrac{1}{2}\, e_i\, \alpha_j, \qquad \tfrac{1}{2}\, e_j\, \alpha_i''\, x^j + g_i'' = 0.$$

Let us derive once more w.r.to x^i: $e_j\, \alpha_i'' = 0$. From the last two equations we see that $\alpha_i'' = 0$ and $f_i'' = 0$. Thus,

$$\alpha_i = 2(m_i\, x^i + \beta_i), \qquad f_i = h_i\, x^i + k_i,$$

where m_i, β_i, h_i, k_i are arbitrary constants. From the first equation (90) and from (91) we get

$$J_{ii} = e_i\, (m_i\, (x^i)^2) + 2\, \beta_i\, x^i) + c_{ii}, \qquad c_{ii} \in \mathbb{R}. \tag{94}$$

From Eq. (92) we get

$$J_{ij} = e_j\, (m_i\, x^i + \beta_i)\, x^j + h_i\, x^i. \tag{95}$$

Thus, Eq. (93) becomes

$$e_j\, (m_i\, x^i + \beta_i)\, x^j + h_i\, x^i = e_i\, (m_j\, x^j + \beta_j)\, x^i + h_j\, x^j.$$

This implies

$$e_j m_i = e_i m_j, \qquad h_j = e_j\, \beta_i, \qquad h_i = e_i\, \beta_j \qquad k_i = k_j = c_{ij}.$$

The first equation implies

$$\frac{m_i}{e_i} = \frac{m_j}{e_j} = m,$$

thus $m_i = m\, e_i$, $\quad m \in \mathbb{R}$, and (94) and (95) are equivalent to the single equation

$$J_{ij} = m\, e_i e_j\, x^i x^j + \beta_i\, e_j\, x^j + \beta_j\, e_i\, x^i + c_{ij}$$

and we get Eq. (89) by setting

$$J^{ij} = e_i e_j\, J_{ij}, \qquad \beta^i = e_i\, \beta_i, \qquad C^{ij} = e_i e_j\, c_{ij}. \qquad \square$$

REMARK 20.2. Since in the above case the Christoffel symbols vanish, from (89) we derive

$$\nabla_k J^{ij} = m\, (\delta^i_k\, x^j + \delta^j_k\, x^i) + \beta^i\, \delta^j_k + \beta^j\, \delta^i_k.$$

Since, by definition, $\nabla_k J^{ij} = \frac{1}{2}\, (\delta^i_k\, \alpha^j + \delta^j_k\, \alpha^i)$, we get

$$\alpha^i = 2\, (m x^i + \beta^i), \qquad \alpha_i = 2 e_i\, (m x^i + \beta^i). \tag{96}$$

Furthermore, the trace of J_i^j is

$$\tau = g_{ij}\, J^{ij} = e_i\, \delta_{ij}\, J^{ij} = e_i\, [m\, (x^i)^2 + 2\, \beta^i\, x^i].$$

This agrees with $\alpha_i = \partial_i \tau$.

It is convenient to examine separately (as it was done in [Benenti, 1992, 1993]) the two cases $m \neq 0$ and $m = 0$, called **elliptic-hyperbolic case** and **parabolic case**, respectively.

For brevity, we shall analyze in detail the elliptic case only, for a basic flat metric of any signature (p, q), although the interesting case for classical mechanics is that of a positive definite metric, $p = n$, $q = 0$. Indeed, the parabolic case for a metric of arbitrary signature requires a much longer discussion, since the vector $\boldsymbol{\beta}$ may be space-like or time-like or light-like. For all relevant objects defined under the assumption $m \neq 0$ we shall use the attribute **elliptic**.

21 Elliptic J-tensors and B-tensors

If $m \neq 0$ we can perform the coordinate translation

$$\bar{x}^i \doteq x^i + b^i, \qquad b^i \doteq \tfrac{1}{m}\,\beta^i.$$

It follows that

$$\begin{aligned} J^{ij} &= m\,(\bar{x}^i - b^i)\,(\bar{x}^j - b^j) + \beta^i\,(\bar{x}^j - b^j) + \beta^j\,(\bar{x}^i - b^i) + C^{ij} \\ &= m\,\bar{x}^i\bar{x}^j - m\,b^i\bar{x}^j - m\,b^j\bar{x}^i + \beta^i\bar{x}^j + \beta^j\bar{x}^i + C^{ij} + m\,b^i b^j \\ &\quad - \beta^i b^j - \beta^j b^i \\ &= m\,\bar{x}^i\bar{x}^j + C^{ij} - m\,b^i b^j. \end{aligned}$$

Thus, in the new coordinates the components of the most general elliptic J-tensor assume the **standard form**

$$J^{ij} = m\,\bar{x}^i\bar{x}^j + C^{ij} - m\,b^i b^j = m\,\bar{x}^i\bar{x}^j + \bar{C}^{ij}, \tag{97}$$

where

$$\bar{C}^{ij} \doteq C^{ij} - m\,b^i b^j = C^{ij} - \tfrac{1}{m}\,\beta^i\beta^j$$

is a constant symmetric matrix.

THEOREM 21.1. *The covariant components B_{ij} of a non-singular elliptic B-tensor on $\mathbb{R}^n = (x^i)$ endowed with a flat metric of any signature have the form*

$$B_{ij} = D_{ij} - \frac{m}{\Omega}\,D_{ih}D_{jk}\,\bar{x}^h\bar{x}^k, \qquad \Omega \doteq 1 + m\,D_{ij}\,\bar{x}^i\bar{x}^j, \tag{98}$$

where $[D_{ij}]$ is a non-singular symmetric constant matrix and $\bar{x}^i = x^i + b^i$, $b^i \in \mathbb{R}$.

Proof. A straightforward calculation shows that if $[D_{ij}]$ is the inverse matrix of $[\bar{C}^{ij}]$,

$$D_{hi}\,\bar{C}^{hj} = \delta_i^j,$$

then $B_{hi}J^{hj} = \delta_i^j$. $\square$

REMARK 21.1. It follows that the J-tensor (97) is singular when $\det[\bar{C}^{ij}] = 0$. For $\det[\bar{C}^{ij}] \neq 0$ it is singular on the hyperquadric $\mathcal{S}_\prime \subset \mathbb{R}^\setminus$ of equation $\Omega = 0$,

$$m\,D_{ij}\,\bar{x}^i\bar{x}^j + 1 = 0.$$

In the following we shall denote by (r, s) the signature of the quadratic form associated with the symmetric non-singular matrix $[D_{ij}]$. Note that $r + s = n$. For $r = 0$ (the quadratic form $[D_{ij}]$ is negative-definite) and $m < 0$, or for $s = 0$ and $m > 0$, we have $\mathcal{S}_\prime = \emptyset$. In these two cases the J-tensor is everywhere non-singular. As a consequence, the signature of B_{ij} is everywhere the signature (r, s) of D_{ij} (i.e., the signature at the point $\bar{x}^i = 0$). In all other cases the quadric $\mathcal{S}_\prime$ is the boundary of two open subsets $\mathcal{S}_+$ and $\mathcal{S}_-$ where $\Omega > 0$ and $\Omega < 0$, respectively. As we shall see (Remark 21.2 below), the signature of B_{ij} is different from (r, s) in these two regions.

THEOREM 21.2. *For a J-tensor* (97) *and a B-tensor* (98),

$$\alpha^i = 2\,m\,\bar{x}^i, \qquad \mu_i = 2\,\xi\,y_i, \qquad \sigma \doteq \alpha^i\mu_i = 4\,m\,\xi\,\Sigma, \tag{99}$$

where

$$\begin{aligned} y_i \doteq D_{ij}\,\bar{x}^j, \qquad \xi \doteq \frac{m}{1+m\,\Sigma} = \frac{m}{\Omega}, \\ \Sigma \doteq D_{ij}\,\bar{x}^i\bar{x}^j = \bar{x}^i\,y_i = \bar{C}^{ij}\,y_iy_j. \end{aligned} \tag{100}$$

Proof. The first equation (99) follows from (96). Since $\nabla_k\bar{x}^i = \partial_k\bar{x}^i = \delta^i_k$, we find

$$\begin{aligned} \partial_k\Sigma &= \nabla_k\Sigma = \nabla_k(D_{ij}\bar{x}^i\bar{x}^j) = D_{ij}\,\nabla_k(\bar{x}^i\bar{x}^j) \\ &= D_{ij}\,(\delta^i_k\bar{x}^j + \delta^j_k\bar{x}^i) = 2\,D_{ik}\,\bar{x}^i = 2y_k. \end{aligned}$$

Thus,

$$\partial_k\xi = -\,\frac{m^2}{(1+m\Sigma)^2}\,\partial_k\Sigma = -\,2\,\xi^2\,y_k. \tag{101}$$

Now we can compute the covariant derivative of B_{ij}. Since $\nabla_k y_i = \nabla_k(D_{ij}\bar{x}^j) = D_{ij}\nabla_k\bar{x}^j = D_{ik}$, we obtain:

$$\begin{aligned} \nabla_k B_{ij} &= \nabla_k(D_{ij} - \xi\,y_iy_j) = 2\,\xi^2\,y_iy_jy_k - \xi\,(D_{ik}\,y_j + D_{jk}\,y_i) \\ &= 2\,\xi^2\,y_iy_jy_k - \xi\,(B_{ik}\,y_j + B_{jk}\,y_i) - 2\,\xi^2\,y_iy_jy_k \\ &= -\xi\,(B_{ik}\,y_j + B_{jk}\,y_i). \end{aligned}$$

By definition of B-tensor, we must have

$$-\,\xi\,(B_{ik}\,y_j + B_{jk}\,y_i) = -\,\tfrac{1}{2}\,(\mu_i B_{jk} + \mu_j B_{ik}).$$

This proves the second equation (99). The last equation follows from the first two. □
Let us remark that, with the notation (100), we can write

$$B_{ij} = D_{ij} - \xi\,y_iy_j. \tag{102}$$

THEOREM 21.3. *On a pseudo-Euclidean space (of any signature) any B-metric* $\overset{*}{g}_{ij} = B_{ij}$ *determined by a non singular elliptic J-tensor* ($m \neq 0$) *has constant curvature* $\kappa = -\,m$.

Proof. Since

$$\begin{aligned} \nabla_i\mu_j &= 2\,\nabla_i(\xi y_j) = 2\,\partial_i\xi\,y_j + 2\,\xi\,\nabla_i y_j = -\,4\,\xi^2\,y_iy_j + 2\,\xi\,D_{ij} \\ &= 2\,\xi\,(D_{ij} - 2\,\xi\,y_iy_j) = 2\,\xi\,(B_{ij} - \xi y_iy_j), \end{aligned} \tag{103}$$

we have (recall the definition (34))

$$\begin{aligned} -\,H_{ij} &\doteq \tfrac{1}{4}\,(2\,\nabla_i\mu_j + \sigma\,B_{ij} + \mu_i\,\mu_j) \\ &= \tfrac{1}{4}\,(4\,\xi\,B_{ij} - 4\,\xi^2\,y_iy_j + \sigma\,B_{ij} + 4\,\xi^2\,y_iy_j) = (\xi + \tfrac{1}{4}\,\sigma)\,B_{ij}. \end{aligned}$$

Since $\xi + \frac{1}{4}\,\sigma = \xi\,(1+m\Sigma) = m$, we find $H_{ij} = -\,m\,B_{ij}$. Hence, by applying Eq. (33) we get

$$\overset{*}{R}_{hijk} = -\,m(B_{hj}\,B_{ik} - B_{hk}\,B_{ij}). \qquad \square$$

REMARK 21.2. There is an interesting geometrical description of the B-metric, from which we can derive its signature. (i) On the space $\mathbb{R}^{1+n} = (\bar{x}^0, \bar{x}^i)$, with the canonical basis

$(\mathbf{c}_\alpha) = (\mathbf{c}_0, \mathbf{c}_i)$, let us consider the hyperquadric $\mathbb{H}_n^+$ described by the vector-parametric equation

$$\mathbf{q} = \bar{x}^i\,\mathbf{c}_i + \sqrt{\tfrac{1}{m} + D_{ij}\,\bar{x}^i\bar{x}^j}\;\mathbf{c}_0 = \mathbf{r} + \sqrt{\tfrac{\Omega}{m}}\,\mathbf{c}_0 = \mathbf{r} + \xi^{-\frac{1}{2}}\,\mathbf{c}_0,$$

with parameters $(\bar{x}^i)$ running in the domain $\mathcal{S}_+ \subset \mathbb{R}^\backslash$ for $m > 0$ and in the domain $\mathcal{S}_-$ for $m < 0$. Due to (101), the frame tangent to $\mathbb{H}_n^+$ associated with these parameters is given by the vectors

$$\mathbf{e}_i \doteq \partial_i \mathbf{q} = \mathbf{c}_i + \xi^{\frac{1}{2}}\,y_i\,\mathbf{c}_0. \tag{104}$$

Let us consider on $\mathbb{R}^{1+n}$ the flat metric

$$\widehat{\mathbf{g}} = [\widehat{g}_{\alpha\beta}] \doteq [\,-1 \quad 00 \quad D_{ij}\,]\,, \tag{105}$$

such that

$$\widehat{\mathbf{g}}(\mathbf{c}_i, \mathbf{c}_j) = D_{ij}, \qquad \widehat{\mathbf{g}}(\mathbf{c}_0, \mathbf{c}_0) = -1, \qquad \widehat{\mathbf{g}}(\mathbf{c}_i, \mathbf{c}_0) = 0.$$

If (r, s) is the signature of D_{ij}, then the signature of $\widehat{\mathbf{g}}$ is $(r, s+1)$. Due to (102), the covariant metric tensor induced on $\mathbb{H}_n^+$ is

$$\hat{\mathbf{g}}(\mathbf{e}_i, \mathbf{e}_j) = D_{ij} - \xi\,y_i y_j = B_{ij}. \tag{106}$$

This is exactly the covariant B-tensor on $\mathbb{R}^n$. Moreover,

$$\begin{aligned} \hat{\mathbf{g}}(\mathbf{q}, \mathbf{e}_i) &= \hat{\mathbf{g}}(\mathbf{r} + \xi^{-\frac{1}{2}}\,\mathbf{c}_0, \mathbf{c}_i + \xi^{\frac{1}{2}}\,y_i\,\mathbf{c}_0) \\ &= \hat{\mathbf{g}}(\mathbf{r}, \mathbf{c}_i) - y_i = D_{ih}\,\bar{x}^h - y_i = 0, \\ \hat{\mathbf{g}}(\mathbf{q}, \mathbf{q}) &= D_{ij}\,\bar{x}^i\bar{x}^j - \xi^{-1} = \Sigma - \xi^{-1} = -\tfrac{1}{m}. \end{aligned} \tag{107}$$

This shows that $\mathbf{q}$ is orthogonal to $\mathbb{H}_n^+$ and that it is time-like or space-like for $m > 0$ or $m < 0$, respectively. Hence, the signature of the metric induced on $\mathbb{H}_n^+$ by the metric (105) is (r, s) for $m > 0$ and $(r-1, s+1)$ for $m < 0$. This second case is meaningless for $r = 0$, in accordance with the fact that for $m < 0$ and $r = 0$ we are in the case $\mathcal{S}_\prime = \emptyset$ (Remark 21.1).

(ii) Let us consider the hyperquadric $\mathbb{H}_2^-$ described by the parametric equation

$$\mathbf{q} = \bar{x}^i\,\mathbf{c}_i - \sqrt{-\tfrac{1}{m} - D_{ij}\,\bar{x}^i\bar{x}^j}\;\mathbf{c}_0 = \mathbf{r} - \sqrt{-\tfrac{\Omega}{m}}\,\mathbf{c}_0 = \mathbf{r} - (-\xi)^{-\frac{1}{2}}\,\mathbf{c}_0,$$

with parameters $(\bar{x}^i)$ running in the domain $\mathcal{S}_+ \subset \mathbb{R}^\backslash$ for $m < 0$ and in the domain $\mathcal{S}_-$ for $m > 0$. Then, instead of (104), we get

$$\mathbf{e}_i = \mathbf{c}_i - (-\xi)^{\frac{1}{2}}\,y_i\,\mathbf{c}_0.$$

However, if instead of the metric (105) we consider the metric

$$\widehat{\mathbf{g}} = [\widehat{g}_{\alpha\beta}] \doteq [\,1 \quad 00 \quad D_{ij}\,]\,,$$

such that

$$\widehat{\mathbf{g}}(\mathbf{c}_i, \mathbf{c}_j) = D_{ij}, \qquad \widehat{\mathbf{g}}(\mathbf{c}_0, \mathbf{c}_0) = 1, \qquad \widehat{\mathbf{g}}(\mathbf{c}_i, \mathbf{c}_0) = 0,$$

then we get again Eq. (106): the metric induced on $\mathbb{H}_n^-$ by (14′) is B_{ij}. Moreover, in analogy with (107) we get

$$\begin{aligned}\hat{\mathbf{g}}(\mathbf{q},\mathbf{e}_i) &= \hat{\mathbf{g}}(\mathbf{r}-(-\xi)^{-\frac{1}{2}}\,\mathbf{c}_0, \mathbf{c}_i-(-\xi)^{\frac{1}{2}}\,y_i\,\mathbf{c}_0) = \hat{\mathbf{g}}(\mathbf{r},\mathbf{c}_i)-y_i \\ &= D_{ih}\,\bar{x}^h - y_i = 0,\end{aligned}$$

$$\hat{\mathbf{g}}(\mathbf{q},\mathbf{q}) = D_{ij}\,\bar{x}^i\bar{x}^j + (-\xi)^{-1} = \Sigma - \xi^{-1} = -\tfrac{1}{m}.$$

This shows that $\mathbf{q}$ is orthogonal to $\mathbb{H}_n^-$ and that it is time-like or space-like for $m>0$ or $m<0$, respectively. Now the metric $\widehat{\mathbf{g}}$ has signature $(r+1,s)$ thus, the signature of the metric induced on $\mathbb{H}_n^-$ is (r,s) for $m<0$ and $(r+1,s-1)$ for $m>0$. This second case is meaningless for $s=0$, in accordance with the fact that for $m>0$ and $s=0$ we are in the case $\mathcal{S}_\prime=\emptyset$ (Remark 21.1). As a result of this discussion we can state

THEOREM 21.4. *Let* (r,s) *be the signature of the non-singular symmetric matrix* $[D_{ij}]$. *Then the signature* $(\overset{*}{p},\overset{*}{q})$ *of the B-metric (4) is*

$$\mathcal{S}_0=\emptyset \qquad \begin{cases} (\overset{*}{p},\overset{*}{q})=(n,0) & \text{for } s=0 \text{ and } m>0 \\ (\overset{*}{p},\overset{*}{q})=(0,n) & \text{for } r=0 \text{ and } m<0 \end{cases}$$

$$\mathcal{S}_0\neq\emptyset \qquad \begin{cases} (\overset{*}{p},\overset{*}{q})=(r,s) & \text{on } \mathcal{S}_+ \\ (\overset{*}{p},\overset{*}{q})=(r-1,s+1) & \text{on } \mathcal{S}_- \text{ for } m<0 \\ (\overset{*}{p},\overset{*}{q})=(r+1,s-1) & \text{on } \mathcal{S}_- \text{ for } m>0. \end{cases}$$

REMARK 21.3. It is remarkable the fact that, in all cases, this signature does not depend on the signature (p,q) of the basic metric.

22 Elliptic A-tensors

THEOREM 22.1. *Let* (p,q) *be the signature of the basic metric* g_{ij} ($p=$ the number of positive e_i and $q=n-p$). *Then for an elliptic J-tensor in standard form*

$$J^{ij} = m\,\bar{x}^i\bar{x}^j + C^{ij} - m\,b^ib^j = m\,\bar{x}^i\bar{x}^j + \bar{C}^{ij}$$

we have

$$\mu \doteq \det[J_i^j] = (-1)^q\,\det[\bar{C}^{ij}]\,\Omega. \tag{108}$$

Proof. Let us set

$$\gamma \doteq \det[\bar{C}^{ij}] = \frac{1}{\det[D_{ij}]}, \qquad [\gamma_{ij}] = \operatorname{cof}[\bar{C}^{ij}],$$

so that $\gamma_{ij}=\gamma\,D_{ij}$. It can be proved that $\det[\bar{C}^{ij}+m\,\bar{x}^i\bar{x}^j]=\gamma+m\,\gamma_{ij}\,\bar{x}^i\bar{x}^j$. Since $J_i^j=e_iJ^{ij}$,

$$\begin{aligned}\mu=\det[J_i^j] &= \textstyle\prod_h e_h\,\det[J^{ij}] = (-1)^q\,(\gamma+m\,\gamma_{ij}\,\bar{x}^i\bar{x}^j) \\ &= (-1)^q\,\gamma\,(1+D_{ij}\,\bar{x}^i\bar{x}^j) = (-1)^q\,\gamma\,\Omega. \qquad \square\end{aligned}$$

Eq. (108) can also be written

$$\mu = (-1)^q\,\gamma\,(1+m\,\Sigma). \tag{109}$$

It follows that

$$\begin{aligned} A_{ij} &= \mu\, B_{ij} = (-1)^q\, \gamma\, (1+m\Sigma)(D_{ij} - \xi\, y_i y_j) \\ &= (-1)^q\, \gamma\, ((1+m\Sigma)\, D_{ij} - m\, y_i y_j). \end{aligned}$$

This proves

THEOREM 22.2. *The covariant components A_{ij} of a non-singular elliptic A-tensor on $\mathbb{R}^n = (x^i)$ endowed with a flat metric of signature (p,q) have the form*

$$A_{ij} = \frac{(-1)^q}{\det[D_{ij}]}\, [(1 + m\, D_{ij}\, \bar{x}^i \bar{x}^j)\, D_{ij} - m\, D_{hi} D_{kj}\, \bar{x}^h \bar{x}^k], \tag{110}$$

where $[D_{ij}]$ is a constant non-singular symmetric matrix and $\bar{x}^i = x^i + b^i$, $b^i \in \mathbb{R}$.

23 Elliptic equivalent metrics

Since the components of the equivalent metric associated with a J-tensor are $\bar{g}_{ij} = \frac{1}{\mu}\, B_{ij}$ and $\bar{g}^{ij} = \mu\, J^{ij}$, from (97), (98) and (108) or (109) it follows that

THEOREM 23.1. *The components of the equivalent metric associated with a J-tensor of $\mathbb{R}^n = (x^i)$ endowed with a flat metric of signature (p,q) are*

$$\begin{aligned} \bar{g}_{ij} &= \frac{(-1)^q}{\gamma\, (1 + m\, D_{hk}\, \bar{x}^h \bar{x}^k)} \times \\ &\quad \times \left(D_{ij} - \frac{m}{1 + m\, D_{hk}\, \bar{x}^h \bar{x}^k}\, D_{ih} D_{ik}\, \bar{x}^h \bar{x}^k \right), \end{aligned} \tag{111}$$

$$\bar{g}^{ij} = (-1)^q\, \gamma\, (1 + m\, D_{hk}\, \bar{x}^h \bar{x}^k)\, (m\, \bar{x}^i \bar{x}^j + \bar{C}^{ij}),$$

where $[\bar{C}^{ij}]$ is a constant non-singular symmetric matrix, $[D_{ij}]$ is its inverse matrix, $\gamma = \det[\bar{C}^{ij}]$, and $\bar{x}^i = x^i + b^i$, $b^i \in \mathbb{R}$.

THEOREM 23.2. *Let (p,q), (r,s) and $(\bar{p},\bar{q})$ be the signatures of the basic metric g_{ij}, of D_{ij} and of the equivalent metric $\bar{\mathbf{g}}$, respectively. Then,*

$$q+s \text{ odd} \quad \Rightarrow \quad (\bar{p},\bar{q}) = \begin{cases} (r-1, s+1) & \text{on } \mathcal{S}_- \text{ for } m<0 \\ (r+1, s-1) & \text{on } \mathcal{S}_- \text{ for } m>0 \\ (s,r) & \text{on } \mathcal{S}_+ \\ (0,n) & \text{for } s=0 \text{ and } m>0 \\ (n,0) & \text{for } s=n \text{ and } m<0. \end{cases}$$

$$q+s \text{ even} \quad \Rightarrow \quad (\bar{p},\bar{q}) = \begin{cases} (s+1, r-1) & \text{on } \mathcal{S}_- \text{ for } m<0 \\ (s-1, r+1) & \text{on } \mathcal{S}_- \text{ for } m>0 \\ (r,s) & \text{on } \mathcal{S}_+ \\ (n,0) & \text{for } s=0 \text{ and } m>0 \\ (0,n) & \text{for } s=n \text{ and } m<0. \end{cases}$$

Proof. If $(\overset{*}{p},\overset{*}{q})$ is the signature of the B-metric B_{ij}, then

$$(\bar{p},\bar{q}) = (\overset{*}{p},\overset{*}{q}) \quad \text{for } \mu>0, \qquad (\bar{p},\bar{q}) = (\overset{*}{q},\overset{*}{p}) \quad \text{for } \mu<0.$$

Let (r, s) be the signature of $[D_{ij}]$. Since $\gamma = \det[\bar{C}^{ij}] = 1/\det[D_{ij}]$, we have $\mathrm{sign}(\gamma) = (-1)^s$. Eq. (108) shows that $\mathrm{sign}(\mu) = (-1)^{q+s}\,\varepsilon$, with $\varepsilon \doteq \mathrm{sign}(\Omega) = \pm 1$. This means that

$$\mathrm{sign}(\mu) = \begin{cases} (-1)^{q+s} & \text{on } \mathcal{S}_+ \\ \\ (-1)^{q+s+1} & \text{on } \mathcal{S}_- \end{cases} \qquad \text{for } \mathcal{S}_\prime \neq \emptyset.$$

For $\mathcal{S}_\prime = \emptyset$ i.e., for $s = 0$ and $m > 0$ or for $s = n$ and $m < 0$, the function μ has everywhere the same sign, equal to that assumed at the point $\bar{x}^i = 0$. Thus,

$$\mathrm{sign}(\mu) = (-1)^q\,\mathrm{sign}(\gamma) = (-1)^{q+s} \qquad \text{for } \mathcal{S}_\prime = \emptyset.$$

Consequently,

$$q + s \text{ odd} \quad \Longrightarrow \quad \begin{cases} \mu > 0 & \text{on } \mathcal{S}_- \\ \mu < 0 & \text{on } \mathcal{S}_+ \\ \mu < 0 & \text{if } \mathcal{S}_\prime = \emptyset \end{cases}$$

$$q + s \text{ even} \quad \Longrightarrow \quad \begin{cases} \mu > 0 & \text{on } \mathcal{S}_+ \\ \mu < 0 & \text{on } \mathcal{S}_- \\ \mu > 0 & \text{if } \mathcal{S}_\prime = \emptyset. \end{cases}$$

By these results and Theorem 21.4, the theorem is proved. □

THEOREM 23.3. *The constant curvature of the equivalent elliptic metric* (111) *is*

$$\bar{\kappa} = (-1)^q\, m\, \det[\bar{C}^{ij}]$$

Proof. Let us recall Theorem 5.4. By applying Eqs. (99), (101) and (102) we get

$$\mu_i \mu_j = 4\,\xi^2\, y_i y_j, \qquad \nabla_i \mu_j = 2\xi\,(B_{ij} - \xi\, y_i y_j).$$

Thus, in the present case Eq. (27) reads, since $\kappa = 0$,

$$\bar{\kappa}\,\bar{g}_{ij} = \xi^2\, y_i y_j + \xi\,(B_{ij} - \xi\, y_i y_j) = \xi\, B_{ij} = \xi\,\mu\,\bar{g}_{ij}.$$

Due to the expressions (109), (100) and (98) of μ, ξ and Ω, respectively, it follows that

$$\bar{\kappa} = \xi\,\mu = \frac{m}{1 + m\,\Sigma}\,(-1)^q\,\gamma\,\Omega = (-1)^q\, m\,\gamma. \qquad \square$$

24 Elliptic cofactor systems

If $(\mathbf{g}, \mathbf{F})$ is a cofactor system on $\mathbb{R}^n = (x^i)$, endowed with a flat metric of signature (p, q), associated with an elliptic J-tensor, then we call $\mathbf{F}$ a **quasi-potential elliptic force**. Eqs. (15), $\mathbf{A}^{-1} = \frac{1}{\mu}\,\mathbf{J}$, (97) and (109) show that the (1,1)-components of $\mathbf{A}^{-1}$ are

$$\breve{A}_i^j = \tfrac{1}{\mu}\, e_i\, J^{ij} = \frac{(-1)^q}{\gamma\,(1 + m\, D_{hk}\bar{x}^h \bar{x}^k)}\, e_i\,(m\,\bar{x}^i \bar{x}^j + \bar{C}^{ij}). \tag{112}$$

Let us recall that $[\bar{C}^{ij}]$ is a constant symmetric non-singular matrix, $[D_{ij}]$ is its inverse, $\gamma = \det[\bar{C}^{ij}]$, $\bar{x}^i = x^i + b^i$ and $m \neq 0$.

THEOREM 24.1. *Any quasi-potential elliptic-force* $\mathbf{F} = (F^i)$ *is of the kind*

$$F^i = \frac{(-1)^{q+1}}{\gamma\,(1 + m\,D_{hk}\bar{x}^h\bar{x}^k)}\,(m\,\bar{x}^i\bar{x}^j + \bar{C}^{ij})\,\frac{\partial V}{\partial x^j},$$

where $V(x^i)$ *is a function on* $\mathbb{R}^n$.

Proof. In virtue of Theorem 13.2, the most general quasi-potential elliptic force $\mathbf{F} = (F^i)$ is of the kind $F^i = -\check{A}^i_j g^{jh}\,\partial_h V$. In the present case $g^{ij} = e_i\,\delta_{ij}$. □
Conversely, due to Theorem 13.3, a given force $\mathbf{F}$ is quasi-potential if and only if there exists a non-singular A-tensor the one-form of components $\phi_i = A_{ij}F^j$ is closed. In the present case we have, see Eq. (110),

$$\phi_i = \alpha\,[(1 + m\,D_{ij}\,\bar{x}^i\bar{x}^j)\,D_{ij} - m\,D_{hi}D_{kj}\,\bar{x}^h\bar{x}^k]\,F^j.$$

where $\alpha = (-1)^q\,\gamma$ is a constant. Hence,

THEOREM 24.2. *A force* $\mathbf{F} = (F^i)$ *is a quasi-potential elliptic force if and only if there exist a constant symmetric non-singular matrix* $[D_{ij}]$*, a constant vector* $\mathbf{b} = (b^i)$ *and a number* $m \neq 0$ *such that*

$$d\,[((1 + m\,D_{ij}\,\bar{x}^i\bar{x}^j)\,D_{ij} - m\,D_{hi}D_{kj}\,\bar{x}^h\bar{x}^k)\,F^j\,dx^i] = 0. \tag{113}$$

where $\bar{x}^i = x^i + b^i$.

REMARK 24.1. This formula does not depend on the signature of the basic flat metric on $\mathbb{R}^n$.

REMARK 24.2. When the constants D_{ij}, b^i and m are determined, then we can compute the components of the equivalent metric $\bar{\mathbf{g}}$ by applying formulas (111). Local potentials V can be determined by integrating the closed one-form in (113). The tensor B_{ij} are given by (98). The tensor $\mathbf{B} = (B^j_i)$ is a L-tensor (so that the elliptic cofactor system is equivalent to a L-system) when its eigenvalues u^i are pointwise simple (Theorem 13.4; they may be complex in some regions of $\mathbb{R}^n$). Due to Eq. (102), this happens in an open subset containing the point $\bar{x}^i = 0$, if and only if the $(1,1)$ constant tensor $\mathbf{D} = [D^j_i] \doteq [e_j\,D_{ij}]$ has simple eigenvalues. Due to Eqs. (66) and (112), this is equivalent to say that the constant matrix $[e_i\bar{C}^{ij}] = [e_i(C^{ij} - m\,b^ib^j)]$ has simple eigenvalues (see [Waksjö, 2003], Eq. (2.4), for $e_i = 1$).

Acknowledgements

The idea of writing an extensive paper on special tensors and applications grew up in occasion of two visits at the Department of Mathematics of the University of Linköping, the first one as opponent of Claes Waksjö's thesis, the second one for attending the meeting *State-of-the-art of classical separability theory for differential equations*, January 6-11, 2004. During these visits I had many interesting conversations with Stefan Rauch, Krzysztof Marciniak, Hans Lundmark, Maciej Błaszak, Claes Waksjö, Vladimir Matveev, Mike Crampin, Willy Sarlet and Ernie Kalnins, after which I felt the necessity of resetting all the matter concerning special symmetric tensors within a general unified picture, by taking as a starting point the paper of Levi-Civita on the corresponding dynamical systems. I hope to have accomplished this aim, even if this matter knows a continuous and fast development. I wish to express my gratitude to these colleagues, including my collaborators Claudia Chanu and Giovanni Rastelli, for suggestions and comments. I am grateful to Jens Jonasson, who found a mistake in the last version of my manuscript, and to Willy Sarlet and Vladimir Matveev, which gave me precise and useful information about the bibliography. Nevertheless, erroneous and missing references may be still present: I apologize for this. Finally, special thanks are due to Beppe Gaeta and to all the organizers of SPT-2004, for making my aim realizable.

References

[Agostinelli, 1937] Agostinelli, C., *Sui sistemi dinamici corrispondenti*. Mem. Ist. Lombardo, **23**, 191-240 (1937).

[Aminova, 2003] Aminova, A.V. *Projective transformations of pseudo-Riemannian manifolds*. J. Math. Sci. **113**(03), 367-469 (2003).

[Beltrami, 1865] Beltrami, E., *Risoluzione del problema: riportare i punti di una superficie sopra un piano in modo che le linee geodetiche vengano rappresentate da linee rette*. Ann. Mat. Pura ed Appl. **1** (7), 185-204 (1865).

[Beltrami, 1868] Beltrami, E., *Teoria fondamentale degli spazii a curvatura costante*. Ann. Mat. Pura ed Appl. **???** (2), 232-255 (1865).

[Benenti, 1992] Benenti, S., *Inertia tensors and Stäckel systems in the Euclidean spaces*. Rend. Semin. Mat. Univ. Polit. Torino **50**, 315-341 (1992).

[Benenti, 1993] Benenti, S., *Orthogonal separable dynamical systems*. In *Proceedings of the 5th International Conference on Differential Geometry and Its Applications*, Silesian University at Opava, August 24-28, 1992, O.Kowalski and D.Krupka Eds., Differential Geometry and Its Applications **1**, 163-184 (1993). - Web edition: ELibEMS, http://www.emis.de/proceedings.

[Benenti, 2003] Benenti, S., *An outline of the geometrical theory of the separation of variables in the Hamilton-Jacobi and Schrödinger equations*. In *Symmetry and Perturbation Theory - SPT 2002* (proceedings of the conference held in Cala Gonone, 19-26 May 2002), S. Abenda, G. Gaeta, S. Walcher Eds., World Scientific (2003).

[Benenti, 2004a] Benenti, S., *Hamiltonian Optics and Generating Families*. Napoli Series on Physics and Astrophysics, Bibliopolis (Napoli, 2004).

[Benenti, 2004b] Benenti, S., *Separability in Riemannian manifolds*. Royal Society, Phil. Trans. A (forthcoming).

[Benenti, Chanu & Rastelli, 2002a] Benenti, S., Chanu, C., Rastelli, G., *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and the Robertson conditions*. J. Math. Phys. **43** (11), 5183-5222 (2002).

[Benenti, Chanu & Rastelli, 2002b] Benenti, S., Chanu, C., Rastelli, G., *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators*. J. Math. Phys. **43** (11), 5223-5253 (2002).

[Błaszak & Ma, 2003] Błaszak, M., Ma, W.-X., *Separable Hamiltonian equations on Riemann manifolds and related integrable hydrodynamic systems*. J. Geom. Phys. **47**, 21-42 (2003).

[Błaszak, 2003] Błaszak, M., *Separability theory of Gel'fand-Zakharevic systems on Riemannian manifolds*. Preprint, A. Mickiewics University, Poznań (2003).

[Błaszak & Badowski, 2003] Błaszak, M., Badowski, L., *From separable geodesic motion to bihamiltonian dispersionless chains*. Preprint, A. Mickiewics University, Poznań (2003).

[Bolsinov & Matveev, 2003] Bolsinov, A. V., Matveev, V. S., *Geometrical interpretation of Benenti systems*. J. Geom. Phys. **44**, 489-506 (2003).

[Burns & Matveev, 2004] Burns, K., Matveev, V.S., *On the rigidity of magnetic systems with the same magnetic geodesics*, preprint (2004).

[Chanu & Rastelli, 2004] Chanu, C., Rastelli, G., *Eigenvalues of Killing tensors and separable webs in Riemannian manifolds*. Preprint (2004).

[Crampin, 2003a] Crampin, M., *Conformal Killing tensors with vanishing torsion and the separation of variables in the Hamilton-Jacobi equation*. Diff. Geom. Appl. **18**, 87-102 (2003).

[Crampin, 2003b] Crampin, M., *Projectively Equivalent Riemannian Spaces as Quasi-bi-Hamiltonian Systems*. Acta Appl. Math. **77** (3), 237-248 (2003).

[Crampin & Sarlet, 2001] Crampin., M., Sarlet, W., *A class of non-conservative Lagrangian systems on Riemannian manifolds*. J. Math. Phys. **42** (9), 4313-4326 (2001).

[Crampin, Sarlet & Thompson, 2000] Crampin., M., Sarlet, W., Thompson, G., *Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors*. J. Phys. A: Math. Gen. **33**, 8755-8770 (2000).

[Eisenhart, 1933] Eisenhart, L. P.,*Continuous Groups of Transformations*. Princeton University Press (1933).

[Eisenhart, 1949] Eisenhart, L. P.,*Riemannian geometry*. Princeton: Princeton University Press (1949)

[Fettis, 1950] Fettis, H.E., *A method for obtaining the characteristic equation of a matrix and computing the associated modal columns*. Quart. Appl. Math. **8**, 206-212 (1950).

[Frölicher & Nijenhuis, 1956] Frölicher, A., Nijenhuis, A., *Theory of vector-valued differential forms*. Proc. Kon. Ned. Ak. Wet. Amsterdam **59-A**, 338-359 (1956).

[Fubini, 1903] Fubini G., *Sui gruppi di trasformazioni geodetiche*. Mem. Acc. Torino **2**, 261-313 (1903).

[Haantjes, 1955] Haantjes, J., *On X_m-forming sets of eigenvectors*. Proc. Kon. Ned. Ak. Wet. Amsterdam **58-A**, 158-162 (1955).

[Ibort, Magri & Marmo, 2000] Ibort, A., Magri, F., Marmo, G., *Bihamiltonian structures and Stäckel separability*. J. Geom. Phys. **33**, 210-228 (2000).

[Levi-Civita, 1896a] Levi-Civita, T., *Sugli integrali algebrici delle equazioni dinamiche*, Atti della Accademia Reale delle Scienze di Torino, **31** (1896).

[Levi-Civita, 1896b] Levi-Civita, T., *Sulle trasformazioni delle equazioni dinamiche*. Ann. di Matem. **24**, 255-300 (1896).

[Lichnerowicz, 1946] Lichnerowicz, A., *Sur la transformation des équations de la dynamique*. C. R. Acad. Sci. Paris. **223**, 649-651 (1946).

[Lichnerowicz & Aufenkamp, 1952] Lichnerowicz, A., Aufenkamp, D. *The general problem of the transformation the equations of dynamics*. J. rational Mech. Anal. **1**, 499-520 (1952).

[Lundmark, 2001a] Lundmark, H., *A new class of integrable Newton systems*. J. Nonlin. Math. Phys., **8**, Supplement, 195-199 (2001).

[Lundmark, 2001b] Lundmark, H., *Newton Systems of Cofactor type in Euclidean and Riemannian Spaces*. Linköping Studies in Sciences and Technology, **719** (2001).

[Lundmark, 2001c] Lundmark, H., *Multiplicative structure of cofactor pair system in Riemannian spaces*. Preprint (2001), in [Lundmark, 2001b].

[Lundmark, 2003] Lundmark, H., *Higher-dimensional integrable Newton systems with quadratic integrals of motion*. Studies in Appl. Math. **110**, 257-296 (2003).

[Marciniak & Błaszak, 2002] Marciniak, K., Błaszak, M., *Separation of variables in quasi-potential systems of bi-cofactor form*. J. Phys. A: Math. Gen. **35**, 2947-2964 (2002).

[Marciniak & Rauch-Wojciechowski, 1998] Marciniak, K., Rauch-Wojciechowski, S., it Two families of non-standard Poisson structures for Newton equations. J. Math. Phys. **39**, 5292-5306 (1998).

[Marshall & Wojciechowski, 1988] Marshall, I., Wojciechowski, S., *When is a Hamiltonian system integrable ?*. J. Math. Phys. **29**, 1338-1346 (1988).

[Matveev, 2004] Matveev, V.S., *Lichnerowicz-Obata conjecture in dimension two*, preprint (2004).

[Matveev & Topalov, 1998] Matveev, V.S., Topalov, P.I., *Trajectory equivalence and corresponding integrals*, Regul. Chaotic Dyn. **3** (2), 30-45 (1998).

[Matveev & Topalov, 1999] Matveev, V.S., Topalov, P.I., *A metric on a sphere that is geodesically equivalent to a metric of constant curvature is itself a metric of constant curvature*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **5**, 53-55 (1999). Translation in Moscow Univ. Math. Bull. **53** (5), 31-33 (1999).

[Mikeš, 1996] Mikeš J., *Geodesic mappings of affine-connected and Riemannian spaces*. J. Math. Sci. New Jork **78** (3), 311-333 (1996).

[Nijenhuis, 1951] Nijenhuis, A., *X_{n-1}-forming sets of eigenvectors*. Nederl. Akad. Wetensch. Proc. **54A**, 200-212 (1951).

[Painlévé, 1894] Painlévé, P., *Sur la transformation des èquations de la dynamique*. J. de Liouville, **10** (1894).

[Rani, Brian Edgar & Barnes, 2003] Rani, R., Brian Edgar, S., Barnes, A., *Killing tensors and conformal Killing tensors from conformal Killing vectors*. Preprint (2003).

[Rauch-Wojciechowski, 2004] Rauch-Wojciechowski, S., *From Jacobi problem of separation of variables to theory of quasipotential Newton equations*. Royal Society, Phil. Trans. A (forthcoming).

[Rauch-Wojciechowski & Waksjö, 2003] Rauch-Wojciechowski, S., Waksjö, C., *Stäckel separability for Newton systems of cofactor type*. Preprint, University of Linköping, Linköping (2003).

[Rauch-Wojciechowski, Marciniak & Lundmark, 1999] Rauch-Wojciechowski S., Marciniak, K. & Lundmark, H., *Quasi-Lagrangian systems of Newton equations*. J. Math. Phys. **40** (12), 6366-6398 (1999).

[Schouten, 1954] Schouten, J. A., *Ricci Calculus*. Springer (Berlin, 1954).

[Souriau, 1950] Souriau, J.M., *Le calcul spinoriel et ses applications*. Recherche Aéronautique **14**, 3-8 (1950).

[Sinyukov, 1979] Sinyukov, N.S., *Geodesic Mappings of Riemannian Spaces* [in Russina]. Nauka, Moskow (1979).

[Thomas, 1946] Thomas, T.Y., *On the transformation of the equations of dynamics*, J. of Math. and Phys. **25**, 191-208 (1946).

[Topalov, 2002] Topalov, P., *Hierarchies of cofactor systems*. J. Phys. A **35**, L175-L179 (2002).

[Topalov & Matveev, 2003] Topalov, P., Matveev, V. S. *Geodesic equivalence via integrability*. Geom. Dedic. **96**, 91-115 (2003).

[Waksjö, 2000] Waksjö, C., *Stäckel Multipliers in Euclidean Spaces*. Linköping Studies in Sciences and Technology, **833** (2000).

[Waksjö, 2003] Waksjö, C., *Determination of Separation Coordinates for Potential and Quasi-potential Newton Systems*. Linköping Studies in Sciences and Technology, **845** (2003).

[Waksjö & Rauch-Wojciechowski, 2003] Waksjö, C., Rauch-Wojciechowski, S., *How to find separation coordinates for the Hamilton-Jacobi equation: a criterion of separability for natural Hamiltonian systems*. Math. Phys. Anal. Geom. (to appear).

Contents