Torsionless Conformal Killing tensors and cofactor pair systems

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1. Introduction.

Let $\mathbf{L}$ and $\mathbf{G}$ be two contravariant symmetric 2-tensors on a manifold Q_n. Let us consider the characteristic equation

(1) $$\det(\mathbf{L} - u\,\mathbf{G}) \equiv \det\left[L^{ij} - u\,G^{ij}\right] = 0.$$

We call the n roots of this algebraic equation the **eigenvalues of L w.r. to G**. In the following we shall examine the case in which:

(a) $\mathbf{G}$ is a metric tensor (of any signature) i.e., $\det\mathbf{G} \neq 0$.

(b) $\mathbf{L}$ has simple and real eigenvalues (u^i) w.r. to $\mathbf{G}$.

(c) $\mathbf{L}$ is a conformal Killing tensor w.r. to $\mathbf{G}$.

(d) $\mathbf{L}$ is torsionless w.r. to $\mathbf{G}$.

Let us call **L-system** a pair $(\mathbf{L}, \mathbf{G})$ of this kind, and **L-tensor** a tensor $\mathbf{L}$ satisfying the above conditions. The interest of considering such a system is due to the following

Theorem 1. *The symmetric 2-tensors* $\mathbf{K}_a$, $a = 0, 1, \dots, n-1$, *defined by the sequence*

(2) $$\mathbf{K}_0 = \mathbf{G}, \qquad \mathbf{K}_a = \tfrac{1}{a}\,\mathrm{tr}(\mathbf{K}_{a-1}\mathbf{L})\,\mathbf{G} - \mathbf{K}_{a-1}\,\mathbf{L}, \quad a \neq 0,$$

are independent Killing tensors in involution if and only if $\mathbf{L}$ *is a L-tensor.*

Since all these tensors have common eigenvectors, they define a **Stäckel system** [KM, 1980] [B, 1993].

This means that: (i) the eigenvectors are **normal** i.e., orthogonally integrable or surface-forming: each one admits an orthogonal foliation of hypersurfaces (submanifolds of codimension 1). The set of these n foliations forms an **orthogonal web** which we call, in this case, **separable orthogonal web** or **Stäckel web**. (ii) Any local parametrization of this web i.e., any coordinate system (q^i) such that each foliation is locally described by equation q^i = constant, is a **separable orthogonal coordinate system**: these coordinates separate the geodesic HJE.

This theorem summarizes results established in [B, 1992, 1993] in the case of positive-definite metrics. However, under the assumption that the eigenvalues of $\mathbf{L}$ are real, these results can be extended to indefinite metrics. This matter has been recently revisited in [B, 2004].

L-systems are only a special class of Stäckel (orthogonal separable) systems. However, they have the following nice property: *all the Killing tensors i.e., all the quadratic first integrals of the geodesic flow, which are related to the separation, can be constructed in a pure algebraic way by the sequence* (2) *starting from the tensor* $\mathbf{L}$. It is remarkable the fact that *this algebraic procedure* <u>*does not require*</u> *the knowledge of the eigenvalues of* $\mathbf{L}$. Hence, in this case, $\mathbf{L}$ plays the role of a *generator* of the involutive algebra of first integrals associated with the separation. Furthermore, for finding the separable coordinates we have only to examine the tensor $\mathbf{L}$, which contains all information: the web orthogonal to its eigenvectors is indeed a Stäckel web.

Note that we do not require the functional independence of the eigenvalues u^i of $\mathbf{L}$; some of them may be constant. The essential requirement is that they must be pointwise distinct, $u^i \neq u^j$. However,

Theorem 2. *If the eigenvalues* (u^i) *are independent functions, then they are separable coordinates.*

The functional independence of (u^i) is examined in [B, 2004]. It can be shown that,

Theorem 3. *The eigenvalues* (u^i) *of* $\mathbf{L}$ *are independent if and only if* $\mathbf{L}$ *is* <u>*not*</u> *invariant w.r. to a Killing vector of* $\mathbf{G}$.

In [B, 1992] it was shown that the definition (2) can be replaced by other equivalent definitions, which however, require the knowledge of the eigenvalues. Among them we recall the following one:

$$\mathbf{K}_a = \sum_{k=0}^{a} (-1)^k \, \sigma_{a-k} \, \mathbf{L}^k. \tag{3}$$

Here, $\sigma_p(\underline{u})$ denotes the elementary symmetric polynomial of order p in the variables $\underline{u} = (u^i)$.

In fact, sequences of the kind (2)-(3) appeared in the literature many years before within a completely different realm. In the *Ricci calculus* of Schouten [Sc, 1954] they are recalled from Souriau [So, 1950] and Fettis [F, 1950]: they are used for computing in a fast way the eigenvectors of a matrix $\mathbf{L}$ (knowing the eigenvalues). In the book it is also remarked that, in our notation and terminology, the tensor

$$\mathbf{Q}(x) = \mathrm{cof}(\mathbf{L} - x\,\mathbf{G}) \tag{4}$$

is a polynomial of degree $n-1$, whose coefficients are the tensors $\mathbf{K}_a$ defined in (3).

However, if we go further in the past, we find in a paper of the young Levi-Civita [LC, 1896] the construction of geodesic first integrals by a formula similar to (4), in connection with the problem of finding the most general metric tensor admitting a geodesic correspondent. Indeed, if two (positive-definite) metric tensors $\widetilde{\mathbf{G}}$ and $\mathbf{G}$ are such that the eigenvalues of $\widetilde{\mathbf{G}}$ w.r. to $\mathbf{G}$ are simple, then they have the same unparametrized geodesics if and only if $\mathbf{G}$ admits a L-tensor. This matter have been recently investigated by Bolsinov & Matveev [BM, 2003] and Crampin [C, 2003a, b]. Hence, *L-metric* may stand for *Levi-Civita-metric*: it is a metric admitting a L-tensor.

The aim of this lecture is to show how the scheme of L-systems fits with recent very interesting results of Rauch-Wojciechowski, Waksjö, Lundmark, Błaszak, Ibort, Magri, Marmo, Bolsinov, Matveev and Topalov. What illustrated here finds a more general setting based on the notion of **special conformal Killing tensor** introduced by Crampin, Sarlet and Thompson (see the references at the end of this paper). An extensive paper on this matter, entitled *Special tensors, equivalent dynamical systems and separation of variables*, is in preparation.

2. Notation and basic definitions.

If $\mathbf{L} = (L^{i\dots j})$ is a contravariant symmetric tensor on a manifold Q_n, then we denote by $P_{\mathbf{L}}$ the polynomial function on T^*Q defined by

$$P_{\mathbf{L}} = L^{i\dots j}\, p_i \cdots p_j.$$

Thus, we can define the symmetric tensor product $\odot$ between these tensors by setting

$$P_{\mathbf{L}\odot\mathbf{K}} = P_{\mathbf{L}}\, P_{\mathbf{K}}.$$

By means of the Poisson bracket, defined by

$$\{F, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i},$$

we define a Lie-bracket by setting

$$P_{[\mathbf{L},\mathbf{K}]} = \{P_{\mathbf{L}}, P_{\mathbf{K}}\}.$$

Two tensors are said to be **in involution** in $[\mathbf{L},\mathbf{K}] = 0$. If $\mathbf{G}$ is a metric tensor, then $\mathbf{L}$ is a **conformal Killing tensor** (CKT) if

$$[\mathbf{L},\mathbf{G}] = \mathbf{C} \odot \mathbf{G}.$$

If $\mathbf{C} = 0$ i.e., $[\mathbf{L},\mathbf{G}] = 0$, then $\mathbf{L}$ is a **Killing tensor**.
If $\mathbf{K}$, $\mathbf{L}$, ... are symmetric contravariant 2-tensors and if a metric tensor $\mathbf{G} = (G^{ij})$ is present, then we define their **algebraic product LK** by setting

$$(\mathbf{KL})^{ij} = K^{ih} L^{kj} g_{hk},$$

where $[g_{ij}] = [G^{ij}]^{-1}$ are the covariant components of the metric. We say that $\mathbf{K}$ and $\mathbf{L}$ **commute** when $\mathbf{KL} - \mathbf{LK} = 0$. This gives the meaning of some formula of §1.
The **torsion H** of a (1,1) tensor $\mathbf{T} = (T_i^j)$ is the (1,2)-tensor defined by

$$\boxed{\tfrac{1}{2}\, H^k_{ij}(\mathbf{T}) \doteq T^h_{[i}\partial_{|h|}L^k_{j]} - T^k_h \partial_{[i}T^h_{j]}}$$

This definition does not depend on the choice of the coordinates and the partial derivatives $\partial_i = \partial/\partial q^i$ may be replaced by the covariant derivatives ∇_i w.r.t. to any symmetric connection. When $\mathbf{H}(\mathbf{T}) = 0$, $\mathbf{T}$ is said to be **torsionless**. The torsion has been introduced by Nijenhuis and Haantjes [N, 1951] [H, 1955] [FN, 1956], in order to establish criteria for the normality of the eigenvectors of a $(1,1)$ tensor. If we start from a (symmetric) contravariant 2-tensor $\mathbf{L}$, then we can define the **torsion w.r. to a metric tensor** $[g_{ij}] = [G^{ij}]^{-1}$ by considering the associated (1,1)-tensor $L^j_i = L^{hj}\, g_{hi}$.

By the results of [N, 1951] [H, 1955] one can prove that,

Theorem 1. *If a symmetric tensor* $\mathbf{L}$ *has simple and real eigenvalues w.r. to a metric* $\mathbf{G}$*, then* $\mathbf{H}(\mathbf{L}) = 0$ *if and only if*

(i) *there are local coordinates* (q^i) *in which both* $\mathbf{L}$ *and* $\mathbf{G}$ *are diagonalized,*

(ii) $\partial_i u^j = 0$ *for* $i \neq j$*, being* u^i *the eigenvalue corresponding to the eigenvector* ∂_i*.*

3. Elliptic-parabolic tensors on $\mathbb{R}^n$.

We shall work on the manifold $Q_n = \mathbb{R}^n$, referred to the Cartesian coordinates $\underline{x} = (x^\alpha)$ centered at the origin $O = (0, \dots, 0)$. We denote by $\mathbf{r} = OP$ the position vector of the generic point P: its Cartesian components coincides with the coordinates of P.

We shall study contravariant symmetric 2-tensors of the form

$$\mathbf{E} = \mathbf{C} + m\,\mathbf{r} \otimes \mathbf{r} + \mathbf{w} \odot \mathbf{r}, \tag{1}$$

where $\mathbf{C} = (C^{\alpha\beta})$ is a constant symmetric contravariant 2-tensor, $\mathbf{w}$ is a constant vector and $m \in \mathbb{R}$. Here, $\odot$ denotes the symmetric tensor product (of vectors): $\mathbf{a} \odot \mathbf{b} \doteq \frac{1}{2}\,(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$. In the papers of S. Rauch and coworkers they are called **elliptic matrices** and denoted by $\mathbf{G}$ $(\alpha = m,\ \beta = \frac{1}{2}\mathbf{w})$. Here we prefer to use the symbol $\mathbf{G}$ for a generic metric although, as we shall see, we shall interpreted a tensor $\mathbf{E}$ as a metric. Tensors of this kind were introduced in [B, 1992] as **planar inertia tensors** of a system of points with (positive or negative) masses, and related to the separable webs of $\mathbb{R}^n$ endowed with the standard Euclidean metric. The scalar m is the total mass (it may be zero). It is remarkable the fact that a tensor $\mathbf{E}$ is a torsionless CKT w.r. to the standard Euclidean metric, so that, if it has simple eigenvalues, it is a L-tensor and it generates a L-system. In [B, 1992] it is shown that the case $m \neq 0$ corresponds to the **elliptic-hyperbolic web** (i.e., to the separation in confocal elliptic-hyperbolic coordinates) as well as the case $m = 0$ and $\mathbf{w} \neq 0$ corresponds to the **parabolic web** (i.e., to the separation in parabolic coordinates). In these two cases we call **E elliptic tensor** or **parabolic tensor**, respectively. The trivial case $m = 0$ and $\mathbf{w} = 0$ corresponds to the separation in Cartesian coordinates.

Our aim is to show that

Theorem 1. *If* $\mathbf{E}$ *and* $\widetilde{\mathbf{E}}$ *are two tensors of the kind* (1) *such that* $\det(\widetilde{\mathbf{E}}) \neq 0$ *and* $\mathbf{E}$ *has pointwise real simple eigenvalues w.r. to* $\widetilde{\mathbf{E}}$*, then the tensor*

$$\mathbf{L} \doteq \det(\widetilde{\mathbf{E}})\, \mathbf{E} \tag{2}$$

is a torsionless conformal Killing tensor w.r. to the metric tensor

$$\mathbf{G} \doteq \det(\widetilde{\mathbf{E}})\, \widetilde{\mathbf{E}}. \tag{3}$$

In other words, the pair $(\mathbf{L}, \mathbf{G})$ is a **L-system** on the manifold $\mathbb{R}^n$. For brevity we shall prove this theorem only in the case of an elliptic tensor $\widetilde{\mathbf{E}}$ $(\widetilde{m} \neq 0)$.

Going back to the definition (1), we observe that $\mathbf{E}$ *has an affine character* (it does not depend on a metric). Collecting results of [B, 1992] we can affirm that

Theorem 2. (i) *If $m \neq 0$, then there exists a unique point O' such that $\mathbf{E}$ assume the form*

$$\mathbf{E} = \mathbf{C}' + m\,\mathbf{r}' \otimes \mathbf{r}', \qquad \mathbf{r}' = O'P, \quad \mathbf{C}' = \mathbf{C} - \tfrac{1}{4m}\,\mathbf{w} \otimes \mathbf{w}. \tag{4}$$

(ii) *If $m = 0$, then there exists a unique point O' such that $\mathbf{E}_{O'}(\mathbf{w}) = 0$ and $\mathbf{w}$ is an eigenvector of $\mathbf{E}$ at all points P of the straight line parallel to $\mathbf{w}$ passing through O'.*

In item (ii) we refer to the standard Euclidean metric of $\mathbb{R}^n$.

Proof. (i) If $O' \neq O$ is any other fixed point, then $\mathbf{r} = \mathbf{v} + \mathbf{r}'$, with $\mathbf{v} = OO'$. Since

$$\begin{cases} \mathbf{r} \otimes \mathbf{r} = \mathbf{r}' \otimes \mathbf{r}' + 2\,\mathbf{r}' \odot \mathbf{v} + \mathbf{v} \otimes \mathbf{v}, \\ \mathbf{w} \odot \mathbf{r} = \mathbf{w} \odot \mathbf{r}' + \mathbf{w} \odot \mathbf{v}, \end{cases}$$

it follows that

$$\mathbf{E} = \mathbf{C}' + m\mathbf{r}' \otimes \mathbf{r}' + \mathbf{w}' \odot \mathbf{r}', \qquad \begin{cases} \mathbf{C}' = \mathbf{C} + \mathbf{v} \odot (m\mathbf{v} + \mathbf{w}), \\ \mathbf{w}' = \mathbf{w} + 2m\mathbf{v}. \end{cases}$$

For $m \neq 0$, by choosing

$$\mathbf{v} = -\frac{1}{2m}\,\mathbf{w},$$

we get $\mathbf{w}' = 0$ and (2). (ii) For $m = 0$ see [B, 1992]. ∎

As a consequence of this theorem we can always find a point O, in general different from the point $\mathbf{0} \in \mathbb{R}^n$, such that a tensor $\mathbf{E}$ has the form

$$\begin{cases} \mathbf{E} = \mathbf{C} + m\,\mathbf{r} \otimes \mathbf{r}, & \text{for } m \neq 0, \\ \mathbf{E} = \mathbf{C} + \mathbf{w} \odot \mathbf{r}, \quad \mathbf{C}(\mathbf{w}) = 0, & \text{for } m = 0. \end{cases} \tag{5}$$

Moreover, we can choose orthogonal Cartesian coordinates (x^α) with origin at O such that

$$\begin{cases} E^{\alpha\beta} = c^\alpha\,\delta^{\alpha\beta} + m\,x^\alpha\,x^\beta, & \text{for } m \neq 0, \\ E^{\alpha\beta} = c^\alpha\,\delta^{\alpha\beta} + \frac{w}{2}\,(x^\alpha\,\delta^{1\beta} + x^\beta\,\delta^{1\alpha}), \quad c^1 = 0, \;\; w = w^1, & \text{for } m = 0. \end{cases} \tag{6}$$

4. Commutation relations.

A first remarkable property of the elliptic-parabolic tensors is the commutation formula

$$\boxed{[\mathbf{E}, \mathbf{E}'] = 2\,(\mathbf{A}' \odot \mathbf{E} - \mathbf{A} \odot \mathbf{E}')} \qquad \begin{cases} \mathbf{A} \doteq 2m\mathbf{r} + \mathbf{w}, \\ \mathbf{A}' \doteq 2m'\mathbf{r} + \mathbf{w}'. \end{cases} \tag{1}$$

As we shall see below, the interest of this formula is that the Lie-commutator of $\mathbf{E}$ and $\mathbf{E}'$ is a sum of two terms which factorize in $\mathbf{E}$ and $\mathbf{E}'$ themselves. Note that the vectors $\mathbf{A}$, more precisely the vectors $\mathbf{N} = \frac{1}{2}\,\mathbf{A}$, have been already introduced in [RW, 2003].

To prove this formula we observe that, in Cartesian coordinates,

$$P_{\mathbf{E}} = C^{\alpha\beta}\, p_\alpha p_\beta + m\,(x^\alpha p_\alpha)^2 + w^\alpha x^\beta\, p_\alpha p_\beta,$$

where $C^{\alpha\beta}$ and w^α are constant. Then we can easily prove the following commutation relations:

$$\begin{cases} \{P_{\mathbf{C}}, P_{\mathbf{r}}\} = 2\,P_{\mathbf{C}}, \\ \{P_{\mathbf{C}}, (P_{\mathbf{r}})^2\} = 4\,P_{\mathbf{C}} P_{\mathbf{r}}, \\ \{P_{\mathbf{C}}, P_{\mathbf{w}} P_{\mathbf{r}}\} = 2\,P_{\mathbf{C}} P_{\mathbf{w}}, \\ \{P_{\mathbf{w}}, P_{\mathbf{r}}\} = P_{\mathbf{w}}, \\ \{P_{\mathbf{r}} P_{\mathbf{w}}, (P_{\mathbf{r}})^2\} = 2\,P_{\mathbf{w}} (P_{\mathbf{r}})^2. \end{cases}$$

Since $\{P_{\mathbf{C}}, P_{\mathbf{C}'}\} = 0$, $\{P_{\mathbf{C}}, P_{\mathbf{w}'}\} = 0$ and $\{P_{\mathbf{w}}, P_{\mathbf{w}'}\} = 0$, we have

$$\begin{aligned} &\{P_{\mathbf{E}}, P_{\mathbf{E}'}\} = \\ &= \{P_{\mathbf{C}} + m\,P_{\mathbf{r}}^2 + P_{\mathbf{w}}\,P_{\mathbf{r}}\,,\ P_{\mathbf{C}'} + m'\,P_{\mathbf{r}}^2 + P_{\mathbf{w}'}\,P_{\mathbf{r}}\} \\ &= \{P_{\mathbf{C}}, m'\,P_{\mathbf{r}}^2 + P_{\mathbf{w}'}\,P_{\mathbf{r}}\} + \{m P_{\mathbf{r}}^2\,,\ P_{\mathbf{C}'} + P_{\mathbf{w}'}\,P_{\mathbf{r}}\} \\ &\quad + \{P_{\mathbf{w}} P_{\mathbf{r}}\,,\ P_{\mathbf{C}'} + m'\,P_{\mathbf{r}}^2 + P_{\mathbf{w}'}\,P_{\mathbf{r}}\} - \dots \text{(similar terms with } ' \text{ interchanged)} \\ &= 2m' P_{\mathbf{r}} \{P_{\mathbf{C}}, P_{\mathbf{r}}\} + P_{\mathbf{w}'}\{P_{\mathbf{C}}, P_{\mathbf{r}}\} + 2m P_{\mathbf{r}}\{P_{\mathbf{r}}, P_{\mathbf{C}'}\} + m P_{\mathbf{r}}\{P_{\mathbf{r}}^2, P_{\mathbf{w}'}\} + P_{\mathbf{w}}\{P_{\mathbf{r}}, P_{\mathbf{C}'}\} \\ &\quad + m' P_{\mathbf{r}}\{P_{\mathbf{w}}, P_{\mathbf{r}}^2\} + \{P_{\mathbf{w}} P_{\mathbf{r}}, P_{\mathbf{w}'}\} P_{\mathbf{r}} + \{P_{\mathbf{w}} P_{\mathbf{r}}, P_{\mathbf{r}}\} P_{\mathbf{w}'} - \dots \\ &= 2(2m' P_{\mathbf{r}} + P_{\mathbf{w}'}) P_{\mathbf{C}} - 4m P_{\mathbf{r}} P_{\mathbf{C}'} - 2m P_{\mathbf{r}}^2 P_{\mathbf{w}'} - 2 P_{\mathbf{w}} P_{\mathbf{C}'} + 2m' P_{\mathbf{r}}^2 P_{\mathbf{w}} - \dots \\ &= 2(2m' P_{\mathbf{r}} + P_{\mathbf{w}'}) P_{\mathbf{C}} - 2(2m P_{\mathbf{r}} + P_{\mathbf{w}}) P_{\mathbf{C}'} + 2 P_{\mathbf{r}}^2 (m' P_{\mathbf{w}} - m P_{\mathbf{w}'}) - \dots . \end{aligned}$$

$$\boxed{\{P_{\mathbf{E}}, P_{\mathbf{E}'}\} = 2\Big[(2m' P_{\mathbf{r}} + P_{\mathbf{w}'}) P_{\mathbf{C}} - (2m P_{\mathbf{r}} + P_{\mathbf{w}}) P_{\mathbf{C}'} + P_{\mathbf{r}}^2 (m' P_{\mathbf{w}} - m P_{\mathbf{w}'})\Big]}$$

It is a surprising fact that, by introducing the vectors $\mathbf{A}$ we get

$$\begin{aligned} P_{\mathbf{A}'} P_{\mathbf{E}} - P_{\mathbf{A}} P_{\mathbf{E}'} &= (2m' P_{\mathbf{r}} + P_{\mathbf{w}'})\ (P_{\mathbf{C}} + m (P_{\mathbf{r}})^2 + P_{\mathbf{r}} P_{\mathbf{w}}) - \dots \\ &= (2m' P_{\mathbf{r}} + P_{\mathbf{w}'})\, P_{\mathbf{C}} + m\, P_{\mathbf{r}}^2\, P_{\mathbf{w}'} + 2m' P_{\mathbf{r}}^2 P_{\mathbf{w}} - \\ &\quad - (2m P_{\mathbf{r}} + P_{\mathbf{w}})\, P_{\mathbf{C}'} - m'\, P_{\mathbf{r}}^2\, P_{\mathbf{w}} - 2m P_{\mathbf{r}}^2 P_{\mathbf{w}'}, \end{aligned}$$

so that

$$\boxed{\{P_{\mathbf{E}}, P_{\mathbf{E}'}\} = 2\left(P_{\mathbf{A}'} P_{\mathbf{E}} - P_{\mathbf{A}} P_{\mathbf{E}'}\right)}$$

which is equivalent to (1).

5. Elliptic metric tensors.

Let us consider an elliptic tensor $\mathbf{E}$, $m \neq 0$. As we have seen, §1, (2), we can always find orthogonal Cartesian coordinates (x^α) such that

(1) $$\boxed{E^{\alpha\beta} = \delta^{\alpha\beta}\, c^\alpha + m\, x^\alpha\, x^\beta}$$

Note that the constants c^α are the eigenvalues of $\mathbf{C}$ i.e., of $\mathbf{E}$ at $\mathbf{r} = 0$, w.r. to the standard Euclidean metric. Let us consider a contravariant symmetric tensor of the kind

$$e_{\alpha\beta} = \delta_{\alpha\beta}\, \varepsilon^\alpha - \xi\, \varepsilon^\alpha \varepsilon^\beta\, x^\alpha\, x^\beta.$$

We have

$$\begin{aligned} E^{\gamma\beta} e_{\alpha\beta} &= \left(\delta^{\gamma\beta}\, c^\gamma + m\, x^\gamma\, x^\beta\right)\left(\delta_{\alpha\beta}\, \varepsilon^\alpha - \xi\, \varepsilon^\alpha \varepsilon^\beta\, x^\alpha\, x^\beta\right) \\ &= \delta^\gamma_\alpha\, c^\gamma \varepsilon^\alpha + m\, x^\gamma\, x^\alpha\, \varepsilon^\alpha - \xi\, \left(\varepsilon^\gamma\, c^\gamma\, x^\gamma\, \varepsilon^\alpha\, x^\alpha + m\, x^\gamma\, \varepsilon^\alpha\, x^\alpha \textstyle\sum_\beta \varepsilon^\beta\, (x^\beta)^2\right) \\ &= \delta^\gamma_\alpha\, c^\gamma \varepsilon^\alpha + x^\gamma\, x^\alpha\, \varepsilon^\alpha\, \left[m - \xi\left(\varepsilon^\gamma c^\gamma + m\, \textstyle\sum_\beta \varepsilon^\beta\, (x^\beta)^2\right)\right]. \end{aligned}$$

Equation $E^{\gamma\beta} e_{\alpha\beta} = \delta^\gamma_\alpha$ is satisfied if and only if

$$c^\alpha \varepsilon^\alpha = 1, \qquad \xi\left(\varepsilon^\gamma c^\gamma + m\, \textstyle\sum_\beta \varepsilon^\beta\, (x^\beta)^2\right) = m.$$

This shows that

Theorem 1. *The tensor* $\mathbf{E}$ *(with* $m \neq 0$*) is a metric tensor if and only if all* $c^\alpha \neq 0$*. The covariant metric* $e_{\alpha\beta}$ *is*

$$e_{\alpha\beta} = \varepsilon^\alpha\, \delta_{\alpha\beta} - \xi\, \varepsilon^\alpha\, \varepsilon^\beta x^\alpha\, x^\beta, \qquad \xi = \left(\frac{1}{m} + \textstyle\sum_\beta \varepsilon^\beta\, (x^\beta)^2\right)^{-1}, \qquad \varepsilon^\alpha = \frac{1}{c_\alpha} \tag{2}$$

The metric is not defined on the hyperquadric

$$1 + m\, \textstyle\sum_\beta \varepsilon^\beta\, (x^\beta)^2 = 0. \tag{3}$$

Besides ξ it is convenient to consider the quantity

$$\Sigma \doteq \textstyle\sum_\alpha \varepsilon^\alpha (x^\alpha)^2\ ,$$

so that

$$\xi = \frac{m}{1 + m\Sigma}, \qquad \Sigma = \tfrac{1}{\xi} - \tfrac{1}{m}, \qquad 1 - \xi\, \Sigma = \tfrac{\xi}{m}. \tag{4}$$

The tensor $e_{\alpha\beta}$ has a remarkable geometrical interpretation, which will be of help in the following. In $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} = (x^\alpha, x^{n+1})$, with the canonical basis $(\mathbf{c}_\alpha, \mathbf{t})$, we consider the surface $\mathbb{H}_n$ described by the vector-parametric equation

$$\mathbf{q} = x^\alpha\, \mathbf{c}_\alpha + \sqrt{\frac{1}{m} + \textstyle\sum_\alpha \varepsilon^\alpha (x^\alpha)^2}\ \mathbf{t} = \mathbf{r} + z\, \mathbf{t}, \qquad z \doteq \xi^{-\frac{1}{2}}, \tag{6}$$

with domain $D_+ \subset \mathbb{R}^n$ where $\boxed{\xi > 0}$. Since $\partial_\alpha z = z^{-1}\, \varepsilon^\alpha\, x^\alpha$, the natural basis of the tangent spaces of $\mathbb{H}_n$ is

$$\mathbf{e}_\alpha \doteq \partial_\alpha \mathbf{q} = \mathbf{c}_\alpha + z^{-1}\, \varepsilon^\alpha\, x^\alpha\, \mathbf{t} = \mathbf{c}_\alpha \xi^{\frac{1}{2}}\, \varepsilon^\alpha\, x^\alpha\, \mathbf{t}.$$

If in $\mathbb{R}^{n+1}$ we introduce the metric tensor

$$\mathbf{g}(\mathbf{u},\mathbf{v}) = \textstyle\sum_\alpha \varepsilon^\alpha u^\alpha v^\alpha - u^{n+1} v^{n+1}, \tag{7}$$

then

$$\begin{cases} \mathbf{g}(\mathbf{c}_\alpha,\mathbf{c}_\beta) = \delta_{\alpha\beta}\,\varepsilon^\alpha, \\ \mathbf{g}(\mathbf{c}_\alpha,\mathbf{t}) = 0, \end{cases} \qquad \begin{cases} \mathbf{g}(\mathbf{t},\mathbf{t}) = -1, \\ \mathbf{g}(\mathbf{q},\mathbf{q}) = -1/m. \end{cases} \tag{8}$$

It follows that the metric tensor $\mathbf{g}$ induces on $\mathbb{H}_n$ the metric tensor

$$e_{\alpha\beta} \doteq \mathbf{g}(\mathbf{e}_\alpha,\mathbf{e}_\beta) = \delta_{\alpha\beta}\,\varepsilon^\alpha - \frac{1}{z^2}\,\varepsilon^\alpha\varepsilon^\beta\,x^\alpha x^\beta.$$

This metric tensor, reduced to $\mathbb{R}^n$, is exactly that defined in (2). The conclusion is that *we can work on the covariant metric tensor* $\mathbf{E}$ *by interpreting it as the metric tensor induced on* $\mathbb{H}_n$ *by the pseudo-Euclidean metric* (7)

Theorem 2. *Let* (p,q) *be the number of the positive and negative constants* c^α*, respectively. Then, for* $m > 0$ *the signature of* $\mathbf{E}$ *is* (p,q) *and for* $m < 0$ *the signature is* $(p-1,q+1)$.

Proof. The vector $\mathbf{q}$ is orthogonal to $\mathbb{H}_n$,

$$\mathbf{g}(\mathbf{q},\mathbf{e}_\alpha) = 0.$$

The signature of $\mathbf{g}$ is $(p,q+1)$. From (8) we see that if $m>0$, $\mathbf{q}$ is timelike. Since it is orthogonal to $\mathbb{H}_n$, it follows that the signature of the metric induced on $\mathbb{H}_n$ hence, the signature of $\mathbf{E}$ is (p,q). If $m<0$, $\mathbf{q}$ is spacelike, so that the signature is $(p-1,q+1)$. ∎

Theorem 3. *The metric* $\mathbf{E}$ *has constant negative curvature.*

Proof. $\partial_\beta \mathbf{e}_\alpha = \varepsilon^\alpha\,\partial_\beta \frac{x^\alpha}{z}\,\mathbf{t} = \varepsilon^\alpha \left(\frac{1}{z}\,\delta_{\alpha\beta} - \frac{x^\alpha}{z^2}\,\partial_\beta z\right)\mathbf{t} = \frac{\varepsilon^\alpha}{z}\left(\delta_{\alpha\beta} - \frac{\varepsilon^\beta\,x^\alpha\,x^\beta}{z^2}\right)\mathbf{t}.$

$$\partial_\beta \mathbf{e}_\alpha = \frac{1}{z}\,e_{\alpha\beta}\,\mathbf{t}. \tag{9}$$

We consider the unit vector orthogonal to $\mathbb{H}_n$, $\mathbf{u} = |m|^{\frac{1}{2}}\,\mathbf{q}$, for computing the second fundamental form of $\mathbb{H}_n$,

$$B_{\alpha\beta} = \mathbf{g}(\partial_\alpha \mathbf{e}_\beta,\mathbf{u}) = -|m|^{\frac{1}{2}}\,e_{\alpha\beta}.$$

The eigenvalues of $B_{\alpha\beta}$ w.r. to the metric $e_{\alpha\beta}$ are all equal to $-|m|^{\frac{1}{2}}$. This shows that $\mathbb{H}_n$ has a constant intrinsic curvature. The sign of this curvature is a matter of convention. We note that the eigenvalues do not depend on the sign of the ε^α. In the case where they are all positive, the metric $\mathbf{g}$ is a Minkowski metric, and $\mathbb{H}_n$ is the hyperboloid of the unit timelike vectors oriented in the future. Its constant curvature is known to be, by convention, negative (as well as for a unit sphere the curvature is assumed to be positive). ∎

Theorem 4. *The Christoffel symbols of the metric* $e_{\alpha\beta}$ *are*

$$\boxed{\Gamma^\gamma_{\alpha\beta} = -m\,x^\gamma\,e_{\alpha\beta}}$$

Proof. By definition, $\Gamma_{\alpha\beta,\mu} = \mathbf{E}(\partial_\alpha \mathbf{e}_\beta, \mathbf{e}_\mu) = \mathbf{g}(\partial_\alpha \mathbf{e}_\beta, \mathbf{e}_\mu)$. Because of (9) and (6),

$$\Gamma_{\alpha\beta,\mu} = \frac{1}{z}\, e_{\alpha\beta}\, \mathbf{g}(\mathbf{e}_\mu, \mathbf{t}) = \frac{1}{z}\, e_{\alpha\beta} \cdot \big(-\xi^{\frac{1}{2}}\, \varepsilon^\mu\, x^\mu \big) = -\,\xi\, \varepsilon^\mu\, x^\mu\, e_{\alpha\beta}.$$

$$\begin{aligned}\Gamma^\gamma_{\alpha\beta} &= E^{\gamma\mu}\, \Gamma_{\alpha\beta,\mu} = -\,(c^\gamma\, \delta^{\gamma\mu} + m\, x^\gamma\, x^\mu)\, (\xi\, \varepsilon^\mu\, x^\mu\, e_{\alpha\beta}) \\ &= -\,\xi\, e_{\alpha\beta}\, \big(c^\gamma \varepsilon^\gamma x^\gamma + m\, x^\gamma \textstyle\sum_\mu \varepsilon^\mu (x^\mu)^2\big) = -\,m\,\xi\, e_{\alpha\beta}\, x^\gamma\, \big(\frac{1}{m} + \textstyle\sum_\mu \varepsilon^\mu (x^\mu)^2\big). \quad \blacksquare\end{aligned}$$

For any contravariant 2-tensor $T^{\alpha\beta}$,

$$(10) \qquad \nabla_\alpha T^{\beta\gamma} = \partial_\alpha T^{\beta\gamma} + \Gamma^\beta_{\alpha\rho}\, T^{\rho\gamma} + \Gamma^\gamma_{\alpha\rho} T^{\beta\rho} = \partial_\alpha T^{\beta\gamma} - m\big(x^\beta\, T_\alpha^{\cdot\gamma} + x^\gamma\, T^\beta_{\cdot\alpha}\big).$$

For any $(1,1)$ tensor T^β_α,

$$\nabla_\alpha T^\gamma_\beta = \partial_\alpha T^\gamma_\beta + \Gamma^\gamma_{\alpha\rho} T^\rho_\beta - \Gamma^\sigma_{\alpha\beta} T^\gamma_\sigma = \partial_\alpha T^\gamma_\beta - m\,\big(x^\gamma\, T_{\alpha\beta} - x^\sigma\, T^\gamma_\sigma\, e_{\alpha\beta}\big), \qquad e_{\alpha\rho} T^\rho_\beta = T_{\alpha\beta}.$$

We consider the case

$$T^{\alpha\beta} = \widetilde{E}^{\alpha\beta} = \widetilde{C}^{\alpha\beta} + \widetilde{m}\, x^\alpha\, x^\beta + \tfrac{1}{2}\, (\widetilde{w}^\alpha\, x^\beta + \widetilde{w}^\beta\, x^\alpha),$$

where $\widetilde{C}^{\alpha\beta}$ and $\widetilde{w}^\alpha$ are constant. From (10),

$$\nabla_\alpha T^{\beta\gamma} = \widetilde{m}\, (\delta^\beta_\alpha\, x^\gamma + \delta^\gamma_\alpha\, x^\beta) + \tfrac{1}{2}\, (\widetilde{w}^\beta\, \delta^\gamma_\alpha + \widetilde{w}^\gamma\, \delta^\beta_\alpha) - m\big(x^\beta\, T^\gamma_\alpha + x^\gamma\, T^\beta_\alpha\big).$$

We observe that

$$e_{\alpha\beta}\, x^\alpha = \delta_{\alpha\beta}\, \varepsilon^\alpha\, x^\alpha - \xi\, \varepsilon^\alpha\, \varepsilon^\beta\, x^\beta\, (x^\alpha)^2 = \varepsilon^\beta\, x^\beta \big(1 - \xi\Sigma\big) = \tfrac{\xi}{m}\, \varepsilon^\beta\, x^\beta.$$

Thus,

$$\begin{aligned}\nabla_\alpha T^\beta_\sigma &= \nabla_\alpha T^{\beta\gamma}\, e_{\gamma\sigma} \\ &= e_{\gamma\sigma}\big(\widetilde{m}\, (\delta^\beta_\alpha\, x^\gamma + \delta^\gamma_\alpha\, x^\beta) + \tfrac{1}{2}\, (\widetilde{w}^\beta\, \delta^\gamma_\alpha + \widetilde{w}^\gamma\, \delta^\beta_\alpha) - m\big(x^\beta\, T^\gamma_\alpha + x^\gamma\, T^\beta_\alpha\big)\big) \\ &= \widetilde{m}\, \big(\delta^\beta_\alpha\, \tfrac{\xi}{m}\, \varepsilon^\sigma\, x^\sigma + e_{\alpha\sigma}\, x^\beta\big) + \tfrac{1}{2}\, (\widetilde{w}^\beta\, e_{\alpha\sigma} + \widetilde{w}_\sigma\, \delta^\beta_\alpha) - m\, x^\beta\, T_{\alpha\sigma} - \xi\, \varepsilon^\sigma\, x^\sigma\, T^\beta_\alpha.\end{aligned}$$

This proves the useful formula

$$(11) \qquad \boxed{\nabla_\alpha \widetilde{E}^\beta_\sigma = x^\beta\, \big(\widetilde{m}\, e_{\alpha\sigma} - m\, \widetilde{E}_{\alpha\sigma}\big) + \tfrac{\xi}{m}\, \varepsilon^\sigma\, x^\sigma\, \big(\widetilde{m}\, \delta^\beta_\alpha - m\, \widetilde{E}^\beta_\alpha\big) + \tfrac{1}{2}\, (\widetilde{w}^\beta\, e_{\alpha\sigma} + \widetilde{w}_\sigma\, \delta^\beta_\alpha)}$$

6. The torsion of $\mathbf{E}$ w.r. to $\widetilde{\mathbf{E}}$.

Let $\mathbf{E}$ be an elliptic-parabolic tensor. Assume that $\widetilde{\mathbf{E}}$ is an elliptic metric tensor ($\widetilde{m} \neq 0$). We apply to it all the results of the preceding section. We recall the last formula (by interchanging $\mathbf{E}$ with $\widetilde{\mathbf{E}}$):

$$(1) \qquad \widetilde{\nabla}_\sigma E^\gamma_\beta = x^\gamma\, \big(m\, \widetilde{e}_{\sigma\beta} - \widetilde{m}\, E_{\sigma\beta}\big) + \frac{\widetilde{\xi}}{\widetilde{m}}\, \widetilde{\varepsilon}^\beta\, x^\beta\, \big(m\, \delta^\gamma_\sigma - \widetilde{m}\, E^\gamma_\sigma\big) + \tfrac{1}{2}\, (w^\gamma\, \widetilde{e}_{\sigma\beta} + w_\beta\, \delta^\gamma_\sigma).$$

Hence,

$$E^\sigma_\alpha\,\widetilde{\nabla}_\sigma E^\gamma_\beta =$$

$$= E^\sigma_\alpha \left[x^\gamma \left(m\,\widetilde{e}_{\sigma\beta} - \widetilde{m}\,E_{\sigma\beta}\right) + \frac{\widetilde{\xi}}{\widetilde{m}}\,\widetilde{\varepsilon}^\beta\,x^\beta\left(m\,\delta^\gamma_\sigma - \widetilde{m}\,E^\gamma_\sigma\right) + \tfrac{1}{2}\,(w^\gamma\,\widetilde{e}_{\sigma\beta} + w_\beta\,\delta^\gamma_\sigma)\right]$$

$$= x^\gamma\,(m\,E_{\alpha\beta} - \widetilde{m}\,E^2_{\alpha\beta}) + \frac{\widetilde{\xi}}{\widetilde{m}}\,\widetilde{\varepsilon}^\beta\,x^\beta\,(m\,E^\gamma_\alpha - \widetilde{m}\,(E^2)^\gamma_\alpha) + \tfrac{1}{2}\,(w^\gamma\,E_{\alpha\beta} + w_\beta\,E^\gamma_\alpha).$$

This proves that

$$(2)\qquad E^\sigma_\alpha\,\widetilde{\nabla}_\sigma E^\gamma_\beta = \frac{\widetilde{\xi}}{\widetilde{m}}\,\widetilde{\varepsilon}^\beta\,x^\beta\,(m\,E^\gamma_\alpha - \widetilde{m}\,(E^2)^\gamma_\alpha) + \tfrac{1}{2}\,w_\beta\,E^\gamma_\alpha + \ldots$$

$\ldots =$ terms symmetric in (α,β). On the other hand, we can write (1) in the form

$$\widetilde{\nabla}_\alpha E^\rho_\beta = \frac{\widetilde{\xi}}{\widetilde{m}}\,\widetilde{\varepsilon}^\beta\,x^\beta\left(m\,\delta^\rho_\alpha - \widetilde{m}\,E^\rho_\alpha\right) + \tfrac{1}{2}\,w_\beta\,\delta^\rho_\alpha + \ldots,$$

so that

$$E^\gamma_\rho\,\widetilde{\nabla}_\alpha E^\rho_\beta = \frac{\widetilde{\xi}}{\widetilde{m}}\,\varepsilon^\beta\,x^\beta\,(m\,E^\gamma_\alpha - \widetilde{m}\,(E^2)^\gamma_\alpha) + \tfrac{1}{2}\,w_\beta\,E^\gamma_\alpha + \ldots.$$

The comparison with (2) shows that

$$\boxed{\mathbf{H}(\mathbf{E}) = 0}$$

Theorem 1. *The elliptic-parabolic tensor* $\mathbf{E}$ *is torsionless w.r.t. to the elliptic metric* $\widetilde{\mathbf{E}}$.

7. Elliptic-parabolic tensors as conformal Killing tensors.

It can be shown [B,1992] that for $E^{\alpha\beta} = c^\alpha\,\delta^{\alpha\beta} + m\,x^\alpha\,x^\beta$,

$$(1)\qquad E \doteq \det[E^{\alpha\beta}] = \sigma_n(\underline{c}) + m\,\textstyle\sum_\alpha \sigma^\alpha_{n-1}(\underline{c})(x^\alpha)^2.$$

It follows that

$$(2)\quad \partial_\alpha E = 2\,m\,\sigma^\alpha_{n-1}\,x^\alpha,\qquad E^{\alpha\beta}\partial_\alpha E = 2\,m\,\sigma^\alpha_{n-1}\,x^\alpha(c^\alpha\,\delta^{\alpha\beta} + m\,x^\alpha\,x^\beta) = 2\,m\,E\,x^\beta.$$

Let us recall the commutation relation

$$[\mathbf{E},\widetilde{\mathbf{E}}] = 2\,(\widetilde{\mathbf{A}}\odot\mathbf{E} - \mathbf{A}\odot\widetilde{\mathbf{E}}),\qquad \mathbf{A} = 2m\mathbf{r} + \mathbf{w},\qquad \widetilde{\mathbf{A}} = 2\widetilde{m}\,\mathbf{r} + \widetilde{\mathbf{w}}.$$

For any function f on $\mathbb{R}^n$ we have

$$\begin{aligned}[f\mathbf{E}, f\widetilde{\mathbf{E}}] &= f^2\,[\mathbf{E},\widetilde{\mathbf{E}}] + f\,[\mathbf{E},f]\odot\widetilde{\mathbf{E}} + f\,[f,\widetilde{\mathbf{E}}]\odot\mathbf{E}\\ &= 2\,f^2\,(\widetilde{\mathbf{A}}\odot\mathbf{E} - \mathbf{A}\odot\widetilde{\mathbf{E}}) + f\,[\mathbf{E},f]\odot\widetilde{\mathbf{E}} + f\,[f,\widetilde{\mathbf{E}}]\odot\mathbf{E}.\end{aligned}$$

Thus, an equation of the kind $\big[f\mathbf{E}, f\widetilde{\mathbf{E}}\big] = \mathbf{V} \odot \widetilde{\mathbf{E}}$ is fulfilled iff

$$2f\,\widetilde{\mathbf{A}} + [f, \widetilde{\mathbf{E}}] = 0, \qquad \text{i.e.,} \quad [\widetilde{\mathbf{E}}, f] = 2\,(2\widetilde{m}\,\mathbf{r} + \widetilde{\mathbf{w}})\,f.$$

For $\widetilde{m} \neq 0$ we can consider $\widetilde{\mathbf{w}} = 0$ without loss of generality: $[\widetilde{\mathbf{E}}, f] = 4\widetilde{m}\,f\,\mathbf{r}$.
This equation is equivalent to

$$\widetilde{E}^{\alpha\beta}\partial_\alpha f = 2\widetilde{m} f\, x^\beta.$$

Due to (2), this equation is solved by $f = \det(\widetilde{\mathbf{E}}) = \det(\widetilde{E}^{\alpha\beta})$. Thus,

Theorem 1. *If the tensor* $\mathbf{G} = \det(\widetilde{\mathbf{E}})\,\widetilde{\mathbf{E}}$ *is a metric tensor, then* $\mathbf{L} = \det(\widetilde{\mathbf{E}})\,\mathbf{E}$ *is a conformal Killing tensor.*

Theorem 2. *The tensor* $\mathbf{L}$ *is torsionless w.r. to the metric* $\mathbf{G}$.

Proof. $\mathbf{E}$ is torsionless w.r. to $\widetilde{\mathbf{E}}$. The eigenvalues and the eigenvectors of $\mathbf{E}$ w.r. to $\widetilde{\mathbf{E}}$ are the same of $\mathbf{L}$ w.r. to $\mathbf{G}$. If **the eigenvalues of E are simple and real**, then also $\mathbf{L}$ is torsionless. ∎

This proves Theorem 1 of §3.

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