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Variable-separation theory for the null Hamilton–Jacobi equation

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The theory of the separation of variables for the null Hamilton–Jacobi equation $\mathcal{H}=0$ is systematically revisited and based on Levi–Civita separability conditions with Lagrangian multipliers. The separation of the null equation is shown to be equivalent to the ordinary separation of the image of the original Hamiltonian under a generalized Jacobi–Maupertuis transformation. The general results are applied to the special but fundamental case of the orthogonal separation of a natural Hamiltonian with a fixed value of the energy. The separation is then related to conditions which extend those of Stäckel and Kalnins and Miller (for the null geodesic case) and it is characterized by the existence of conformal Killing two-tensors of special kind. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1862325]

I. INTRODUCTION

The aim of this paper is to propose a general approach to the theory of variable separation for the *null* Hamilton–Jacobi equation (HJE)

$$\mathcal{H}(\underline{q},\underline{p}) = 0, \quad \underline{q} = (q^i), \quad \underline{p} = (p_i), \quad p_i = \frac{\partial W}{\partial q^i}.$$

This approach is based on a suitable definition of separation (Sec. II), whose geometrical content (Sec. III) is related to special integrable Lagrangian distributions on the cotangent bundle T^*Q, of coordinates $(\underline{q},\underline{p})$. By the Hadamard lemma, the integrability conditions of these special distributions lead in Sec. IV to *Levi–Civita separability conditions with Lagrangian multipliers* (Theorem 4.1), which are a natural extension of the classical Levi–Civita conditions,[24] and from which we derive two characterizations of the separation of the null HJE. The first one (Theorem 4.2) asserts that the separation occurs in a given coordinate system if and only if the ordinary Levi–Civita equations are satisfied on the surface $\mathcal{H}=0$; the second one (Theorem 4.3) asserts that the separation occurs if and only if there exists a function $\Lambda(\underline{q},\underline{p}) \neq 0$ such that the ordinary Levi–Civita equations are satisfied by the *conformal Hamiltonian* $\mathcal{H}/\Lambda$. The passage from a Hamiltonian $\mathcal{H}$ to a conformal Hamiltonian $\mathcal{H}/\Lambda$ is an extension of the so-called *Jacobi* (or *Maupertuis*) *transformation* for natural Hamiltonians,[9,16,23,28,31,33] recalled and discussed in Sec. VI.

We apply these general results to the analysis of particular cases of Hamiltonians. In Sec. V we consider the so-called *homogeneous formalism* in time-dependent mechanics and get a rigorous proof of a known property of the separation in the time-dependent HJE.[15] In Sec. VI we consider a natural Hamiltonian in orthogonal coordinates, $H=\frac{1}{2}g^{ii}p_i^2+V(\underline{q})$ and the corresponding HJE with a fixed value E of the energy,

$$\mathcal{H}(\underline{q},\underline{p}) \doteq \tfrac{1}{2}g^{ii}p_i^2 + V(\underline{q}) - E = 0.$$

From a general theorem (Theorem 6.1) concerned with the separation of this equation, we derive three theorems (Theorems 6.3, 6.4, and 6.5) characterizing the separation for the following three special cases,

$$V=0,\quad E\neq 0,\quad \text{non-null geodesics},$$

$$V=0,\quad E=0,\quad \text{null geodesics},$$

$$V-E\neq 0,\quad \text{dynamical trajectories with total energy } E.$$

The case of null geodesics occurs, of course, for indefinite metric tensors. In Sec. VII we analyze the intrinsic framework of the theorems stated in Sec. VI, by considering special kinds of conformal Killing tensors and by introducing the notion of conformal involution. The first integrals and the separated equations are examined in Sec. VIII. In Sec. IX we consider the two-dimensional case. We renounce here to deal with the nonorthogonal separation for natural Hamiltonians, which is currently under investigation within the general framework presented in this paper. All this work is also made in the perspective of application to the theory of the ordinary multiplicative separation and of the R-separation for the second-order differential equations of mathematical physics (Laplace, Helmholtz, Poisson, Schrödinger equations).[20,22] As possible applications of the present theory we mention: (i) The integration of the equations of the null geodesics in general relativity theory,[11] (ii) The integration of dynamical systems which are Hamiltonian only on single hypersurfaces of the phase space.

II. DEFINITION OF SEPARATED COMPLETE SOLUTION OF THE NULL HJE

Let Q be a real n-dimensional differentiable manifold and $\mathcal{H}$ be a smooth real-valued function on the cotangent bundle T^*Q. Let $\underline{q}=(q^i)$ be local coordinates on an open subset $U\subseteq Q$ and $(\underline{q},\underline{p})=(q^i,p_i)$ the corresponding standard canonical coordinates on T^*Q. In the following, ∂_i and ∂^i will denote the partial derivative with respect to q^i and p_i, respectively.

We restrict our analysis to the open set $\mathcal{O}\subseteq T^*U\subseteq T^*Q$, where $\partial^i\mathcal{H}\neq 0$ for all $i=1,\ldots,n$, assuming that it is not empty. In this open set we have $d\mathcal{H}\neq 0$, so that any equation of the kind $\mathcal{H}(\underline{q},\underline{p})=h$, $h\in\mathbb{R}$, defines a set $\mathcal{E}_h$ that, if not empty, is a submanifold of codimension 1. In particular, we focus on the submanifold $\mathcal{E}_0$ described by equation $\mathcal{H}=0$.

We consider the HJE for $h=0$,

$$\mathcal{H}\left(\underline{q},\frac{\partial W}{\partial \underline{q}}\right)=0, \tag{2.1}$$

and two definitions of complete solution.

Definition 2.1: An *internal complete solution* of the HJE (2.1) is a solution $W^I(\underline{q},c_\alpha)$ depending on $n-1$ parameters (c_α) such that the following completeness condition is satisfied:

$$\operatorname{rank}\left[\frac{\partial^2 W^I}{\partial q^i\,\partial c_\alpha}\right]=n-1. \tag{2.2}$$

An *extended complete solution* of the HJE (2.1) is a function $W^E(\underline{q},\underline{c})$ depending on n real parameters $\underline{c}=(c_i)$ satisfying the completeness condition

$$\det\left[\frac{\partial^2 W^E}{\partial q^i\,\partial c_j}\right]\neq 0 \tag{2.3}$$

for all admissible values of $(\underline{q},\underline{c})$, and satisfying Eq. (2.1) for all $\underline{c}$ belonging to a suitable $n-1$-dimensional submanifold of $\mathbb{R}^n$ (see Remark 2.1 below) or (up to a transformation of $\underline{c}$) for $c_n=0$.

The geometrical meaning of these two definitions is the following. An internal complete solution W^I defines a Lagrangian foliation $\mathcal{L}^I$ of the submanifold $\mathcal{E}_0$, Fig. 1(a), via equations $p_i=\partial_i W^I$. Each Lagrangian submanifold $L_{(c_\alpha)}\in\mathcal{L}^I$ is parametrized by the value of the $n-1$ parameters (c_α).

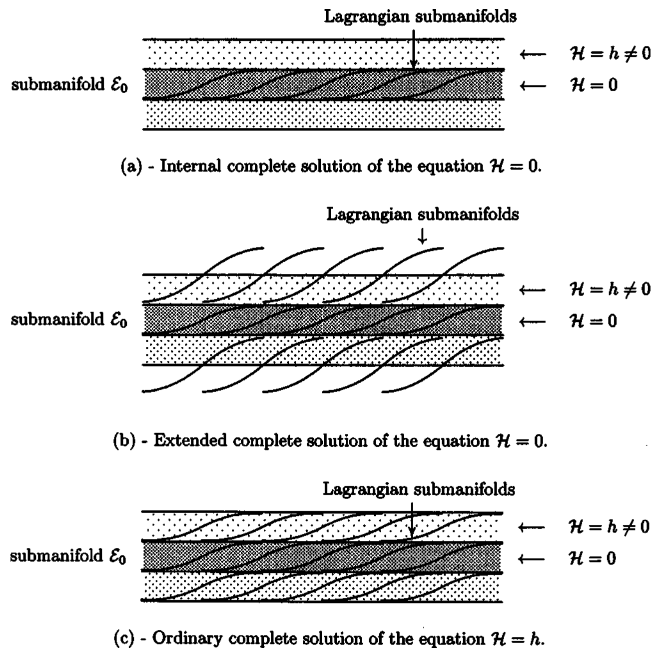

FIG. 1. The geometry of the three definitions of complete solution.

An extended complete solution W^E defines a Lagrangian foliation $\mathcal{L}^E$ on an open neighborhood of T^*Q via equations $p_i=\partial_i W^E$. Each Lagrangian submanifold $L_{(c_i)}\in\mathcal{L}^E$ is parametrized by the value of the n parameters $\underline{c}=(c_i)$. This foliation is compatible with the submanifold $\mathcal{E}_0$, in the sense that it is reducible to a foliation of $\mathcal{E}_0$, Fig. 1(b).

Remark 2.1: The quotient set $\mathcal{C}$ of the foliation $\mathcal{L}^E$ is locally a n-dimensional manifold with coordinates (c_i). The restriction $\mathcal{L}^I$ of $\mathcal{L}^E$ to $\mathcal{E}_0$ is a submanifold $\mathcal{S}\subset\mathcal{C}$ of dimension $n-1$. Then $\mathcal{S}$ is locally defined by an equation $h(c_i)=0$ with $\mathrm{d}h|\mathcal{S}\neq 0$. Up to a transformation $(c_i)\leftrightarrow(c_i')$ we can find coordinates adapted to $\mathcal{S}$ such that equation $h(c_i)=0$ is replaced by $c_n'=0$. This means that for a suitable choice of the parameters appearing in a W^E equation (2.1) is satisfied for $c_n=0$.

Remark 2.2: In general, the foliation generated by a W^E may be not reducible to the submanifolds $\mathcal{E}_h$ with $h\neq 0$. When it is reducible to each $\mathcal{E}_h$, then we have an *ordinary complete solution*, Fig. 1(c), of the HJE

$$\mathcal{H}\left(\underline{q},\frac{\partial W}{\partial \underline{q}}\right)=h. \tag{2.4}$$

The following proposition shows that the two definitions of internal and extended complete solutions are, in a sense, equivalent.

Proposition 2.1: Equation (2.1) *admits an internal complete solution W^I if and only if it admits an extended complete solution W^E.*

Proof: Let W^I be an internal complete solution. By the completeness condition (2.2), we know that there exist $n-1$ linearly independent columns in the matrix $\partial^2 W^I/\partial q^i\partial c_\alpha$. Thus, we can assume without loss of generality that

$$\det\left[\frac{\partial^2 W^I}{\partial q^\beta\,\partial c_\alpha}\right]\neq 0,\quad \alpha,\beta=1,\ldots,n-1.$$

Then, the function

$$W^E(\underline{q},\underline{c}) = W^I(\underline{q},c_\alpha) + c_n q^n, \quad \underline{c} = (c_i) = (c_\alpha, c_n) \tag{2.5}$$

is an extended complete solution: the completeness condition (2.3) is satisfied. Conversely, if W^E is an extended complete solution, then the function $W^I = W^E|_{c_n=0}$ is an internal complete solution. ■

Now we adapt the above-given definitions to a particular class of complete solutions:

Definition 2.2: An *internal separated solution* of the HJE (2.1) is an internal complete solution $W^I(\underline{q}, c_\alpha)$ of the form

$$W^I(\underline{q},c_\alpha) = \sum_{i=1}^n W_i^I(q^i,c_\alpha). \tag{2.6}$$

An *extended separated solution* of the HJE (2.1) is an extended complete solution $W^E(\underline{q}, \underline{c})$ of the form

$$W^E(\underline{q},\underline{c}) = \sum_{i=1}^n W_i^E(q^i,\underline{c}). \tag{2.7}$$

It is obvious that the additive separation is preserved in the passage from a W^E to a W^I. However, it is remarkable that it is also preserved in the inverse passage, from a W^I to a W^E, due to the particular form of formula (2.5). Hence, from Proposition 2.1 it follows that

Proposition 2.2: Equation (2.1) *admits an internal separated solution* W^I *in a coordinate system* $\underline{q} = (q^i)$ *if and only if it admits an extended separated solution* W^E *in the same coordinates.*

The equivalence proved in Proposition 2.2 leads to the following definition of separability for $\mathcal{H}=0$:

Definition 2.3: The null HJE (2.1) is *separable* in the coordinates $\underline{q} = (q^i)$ if it admits an internal separated solution or, equivalently, if it admits an extended separated solution.

Remark 2.3: The definition of *internal separated solution* given here, i.e., depending on $n-1$ constant parameters satisfying the completeness condition (2.2), is that commonly adopted in the literature. See, e.g., Ref. 17, p. 107. However, the use of a second, although equivalent, definition of separation (Proposition 2.2) is essential for a complete development of the present theory. Indeed, as will be shown in Sec. IV, the definition of *extended separated solution* allows the characterization of the separability for the null HJE (2.1) by means of Lagrangian multipliers.

III. SPECIAL DISTRIBUTIONS RELATED TO THE SEPARATION

In order to give necessary and sufficient conditions for the existence of separated solutions it is convenient to give a geometrical interpretation of the separation in terms of complete integrability of a special kind of first-order differential systems. This interpretation is related to the concept of separated connection on a cotangent bundle.[5]

With a coordinate system $\underline{q} = (q^i)$ on Q we associate n differential operators on functions on T^*Q of the kind

$$D_i = \partial_i + R_i(\underline{q},\underline{p})\partial^i, \tag{3.1}$$

where R_i are assigned functions on T^*Q. The vector fields D_i on T^*Q are pointwise independent and transversal to the fibers. Thus, they span a regular distribution $\Delta \subset TT^*U$ of rank n transversal to the fibers: this means that at each point $x \in T^*U$ they span an n-dimensional subspace $\Delta_x \subset T_x(T^*U)$ which is transversal to the vertical vectors of TT^*U. We say that the vector fields D_i are the generators of Δ.

With the same functions R_i entering (3.1) we associate the first-order normal differential system

$$\partial_i p_j = \delta_{ij} R_i(\underline{q},\underline{p}) \tag{3.2}$$

and we remark as follows,

(i) Any integral manifold L of Δ, i.e., a submanifold $L \subset T^*U$ such that $T_x L = \Delta_x$ for all $x \in L$, is locally described by equations

$$p_i = \varphi_i(\underline{q}), \tag{3.3}$$

where the functions φ_i are solutions of system (3.2). Indeed, since Δ is transversal to the fibers, any integral manifold L of Δ is an n-dimensional submanifold transversal to the fibers, thus locally described by equations of the kind (3.3). Moreover, by definition of integral manifold, the generators D_i are tangent to L, so that equations $D_j(p_i - \varphi_i(\underline{q})) = 0$ are satisfied on L. This implies $0 = D_j(p_i - \varphi_i(\underline{q})) = (\partial_j + R_j \partial^j)(p_i - \varphi_i(\underline{q})) = -\partial_j \varphi_i + R_j \delta_{ij}$. This shows that the functions (3.3) satisfy the differential system (3.2).

(ii) Any integral manifold L is a Lagrangian submanifold, since the distribution Δ spanned by D_i is Lagrangian with respect to the canonical symplectic form $\omega = \mathrm{d}p_k \wedge \mathrm{d}q^k$. Indeed, we have

$$\omega(D_i, D_j) = (\mathrm{d}p_k \wedge \mathrm{d}q^k)(D_i, D_j) = \langle D_i, \mathrm{d}p_k \rangle \langle D_j, \mathrm{d}q^k \rangle - \langle D_j, \mathrm{d}p_k \rangle \langle D_i, \mathrm{d}q^k \rangle = R_i \delta_j^i - R_j \delta_i^j = 0.$$

This shows that Δ is an isotropic distribution. Being of rank n, it is a Lagrangian distribution.

(iii) Any Lagrangian submanifold L transversal to the fibers of T^*Q admits local generating functions, i.e., functions $W(\underline{q})$ such that L is described by equations $p_i = \partial_i W$. If L is an integral manifold of the distribution Δ, then $p_i = \partial_i W$ must be a solution of system (3.2). It follows that for $i \neq j$, $\partial_i \partial_j W = 0$, i.e.,

$$W(\underline{q}) = \sum_{i=1}^{n} W_i(q^i). \tag{3.4}$$

(iv) The Lie brackets of the generators D_i are vertical vectors, i.e., vectors tangent to the fibers of T^*Q. Indeed,

$$\begin{aligned}[D_i, D_j] &= [\partial_i + R_i \partial^i, \partial_j + R_j \partial^j] \\ &= \partial_i \partial_j + \partial_i R_j \partial^j + R_j \partial_i \partial^j + R_i \partial^i \partial_j + R_i \partial^i R_j \partial^j + R_i R_j \partial^i \partial^j - \partial_j \partial_i - \partial_j R_i \partial^i - R_i \partial_j \partial^i - R_j \partial^j \partial_i \\ &\quad - R_j \partial^j R_i \partial^i - R_j R_i \partial^j \partial^i = (\partial_i R_j + R_i \partial^i R_j) \partial^j - (\partial_j R_i + R_j \partial^j R_i) \partial^i.\end{aligned}$$

It follows that

$$[D_i, D_j] = D_i R_j \partial^j - D_j R_i \partial^i, \tag{3.5}$$

being

$$\partial_i R_j + R_i \partial^i R_j = D_i R_j. \tag{3.6}$$

Hence, Δ is completely integrable if and only if the generators commute, $[D_i, D_j] = 0$, i.e., if and only if

$$D_i R_j = 0, \quad i \neq j. \tag{3.7}$$

So far we have no links with the HJE $\mathcal{H} = 0$. Now we introduce the function $\mathcal{H}$.

(v) The distribution Δ, when restricted to the points of $\mathcal{E}_0$, gives rise to a distribution Δ_0 on $\mathcal{E}_0$ if and only if the generators are tangent to $\mathcal{E}_0$, and this happens if and only if

$$D_i \mathcal{H}|_{\mathcal{E}_0} = (\partial_i \mathcal{H} + R_i \partial^i \mathcal{H})|_{\mathcal{E}_0} = 0. \tag{3.8}$$

In this case we say that Δ is reducible to $\mathcal{E}_0$ and a well-known property of the Lie bracket tells us that

$$[D_i|\mathcal{E}_0, D_j|\mathcal{E}_0] = [D_i, D_j]|\mathcal{E}_0. \tag{3.9}$$

It follows that the reduced distribution Δ_0 is integrable if and only if

$$D_i R_j|\mathcal{E}_0 = 0, \quad i \neq j. \tag{3.10}$$

Theorem 3.1: *The HJE* (2.1) *admits an extended separated solution in the coordinate system* $\underline{q}$ *if and only if there exist n functions* $R_i(\underline{q},\underline{p})$ *satisfying one of the following equivalent conditions:* (a) *The distribution* Δ *spanned by the generators* (3.1) *is completely integrable and reducible to a distribution* Δ_0 *on* $\mathcal{E}_0$. (b) *Conditions* (3.7) and (3.8) *are satisfied*:

$$\partial_i R_j + R_i \partial^i R_j = 0, \quad i \neq j, \quad (\partial_i \mathcal{H} + R_i \partial^i \mathcal{H})|_{\mathcal{E}_0} = 0. \tag{3.11}$$

Proof: Conditions (a) and (b) are clearly equivalent because of the remarks above. Let $W^E(\underline{q},\underline{c})$ be an extended separated solution. The completeness condition (2.3) means that $(\underline{q},\underline{c})$ are local noncanonical coordinates of T^*Q. Thus, the functions $R_i = \partial_i^2 W^E(\underline{q},\underline{c})$ are well defined on an open subset of T^*Q. Then we consider the separable connection Δ with generators

$$D_i = \partial_i + \partial_i^2 W^E \partial^i.$$

The integral manifolds of Δ are locally described by equations $p_i = \partial_i W^E(\underline{q},\underline{c})$ and, due to the completeness condition (2.3), we get the complete integrability of Δ. Moreover, since for $c_n = 0$, W^E is a solution of (2.1), the generators are tangent to $\mathcal{E}_0$. Conversely, assume that functions R_i exist satisfying (3.11). Then the distribution Δ is completely integrable and the integral manifolds are generated by a separated solution $W^E(\underline{q},\underline{c})$ parametrized by n parameters $\underline{c} = (c_i)$. Since the integral manifolds form a foliation, there is only one integral manifold $L_{\underline{c}}$ containing a given point. This means that equations $p_i = \partial_i W^E$ must be solvable with respect to $\underline{c}$. This is equivalent to the completeness condition (2.3). ■

Theorem 3.2: *The HJE* (2.1) *admits an internal separated solution in the coordinate system* $\underline{q}$ *if and only if there exist functions* $R_i(\underline{q},\underline{p})$ *such that* (a) *the distribution* Δ *spanned by the generators* (3.1) *is reducible to a distribution* Δ_0 *on* $\mathcal{E}_0$ *and this reduced distribution is completely integrable, i.e., such that* (b) *conditions* (3.8) and (3.10) *are satisfied*:

$$(\partial_i \mathcal{H} + R_i \partial^i \mathcal{H})|_{\mathcal{E}_0} = 0, \quad (\partial_i R_j + R_i \partial^i R_j)|_{\mathcal{E}_0} = 0, \quad i \neq j \tag{3.12}$$

Proof: The equivalence between conditions (a) and (b) follows from the above remarks. Let W^I be an internal separated solution. Then, by Proposition 2.2, we can construct an extended separated solution W^E in the same coordinates. Hence, by Theorem 3.1, there exist functions R_i satisfying (3.11). It is clear that these functions R_i satisfy (3.12). Conversely, we consider functions R_i satisfying (3.12) which are associated with a distribution Δ reducible to a completely integrable distribution Δ_0 on $\mathcal{E}_0$. The integral manifolds of Δ_0 are generated by a separated solution $W^I(\underline{q}, c_\alpha)$ parametrized by $n-1$ parameters (c_α). Since they form a foliation $\mathcal{L}^I \subseteq \mathcal{E}_0$, there is only one integral manifold containing a given point $p \in \mathcal{E}_0$. This means that equations $p_i = \partial_i W^I$ are solvable with respect to c_α so that the completeness condition (2.2) holds. ■

Remark 3.1: We know from Proposition 2.2 that the existence of a separated solution W^I is equivalent to the existence of a separated solution W^E. However, this does not mean that the functions R_i entering Theorem 3.1 and Theorem 3.2 are the same functions. Indeed, it is obvious that functions R_i satisfying (3.11) also satisfy (3.12); but functions R_i satisfying (3.12) may not satisfy (3.11).

IV. THE LEVI–CIVITA SEPARABILITY CONDITIONS WITH LAGRANGIAN MULTIPLIERS

The ordinary separation of the HJE $\mathcal{H} = h$ (2.4) is characterized by the Levi–Civita equations,[24]

$$\partial_i\mathcal{H}\partial_j\mathcal{H}\partial^i\partial^j\mathcal{H} + \partial^i\mathcal{H}\partial^j\mathcal{H}\partial_i\partial_j\mathcal{H} - \partial_i\mathcal{H}\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial^i\mathcal{H}\partial_j\mathcal{H}\partial_i\partial^j\mathcal{H} = 0, \quad i \neq j. \tag{4.1}$$

We want to write similar differential equations characterizing the separation of the null HJE $\mathcal{H}=0$ (2.1). For this reason, we look for differential systems equivalent to the conditions (3.11) and (3.12), respectively. Since these conditions are differential conditions restricted to a submanifold, we base our approach on the Hadamard lemma (cf. Ref. 1, for the one-dimensional case).

Lemma 4.1: Let $f(\underline{x},y)$ *be a smooth* (C^∞) *function on an open subset* U of $\mathbb{R}^{m+1}$ *such that* $f(\underline{x},0)=0$. *Then there exists a smooth function* $\lambda(\underline{x},y)$ *such that* $f(\underline{x},y)=y\lambda(\underline{x},y)$.

Proof: The function $\lambda(\underline{x},y)$ is defined by

$$\lambda(\underline{x},y) = \int_0^1 \frac{\partial f}{\partial y}(\underline{x},ty)\mathrm{d}t. \tag{4.2}$$

Indeed we have

$$y\int_0^1 \frac{\partial f}{\partial y}(\underline{x},ty)\mathrm{d}t = \int_0^1 \frac{\partial f}{\partial y}(\underline{x},ty)y\,\mathrm{d}t = \int_0^y \frac{\partial f}{\partial y}(\underline{x},u)\mathrm{d}u = f(\underline{x},y) - f(\underline{x},0) = f(\underline{x},y).$$

The function defined by (4.2) is smooth. ■

We use Hadamard's lemma in the following form.

Lemma 4.2: A smooth function F *on* T^*Q *which vanishes on the submanifold* $\mathcal{E}_0$ *defined by equation* $\mathcal{H}=0$ *is of the form*

$$F = \mathcal{H}\lambda, \tag{4.3}$$

where λ *is a suitable function on* T^*Q.

Proof: Since $\mathcal{E}_0$ is a submanifold of codimension 1, and $\mathrm{d}\mathcal{H}\neq 0$, we can consider local coordinates $(\underline{x},y)$ on T^*Q, such that $\mathcal{E}_0$ is locally defined by equation $y=\mathcal{H}=0$. Then, by applying Lemma 4.1 to the function $F=F(\underline{x},y)$, there exists a smooth function $\lambda(\underline{x},y)$ such that $F=y\lambda$, i.e., a smooth function $\lambda(\underline{q},\underline{p})$ such that (4.3) holds. ■

Due to Lemma 4.2, and recalling our assumption $\partial^i\mathcal{H}\neq 0$, we can reformulate Theorems 3.1 and 3.2 as follows.

Proposition 4.1: The HJE (2.1) *admits an extended separated solution in the coordinates* $\underline{q}=(q^i)$ *if and only if there exist n functions* $\lambda_i(\underline{q},\underline{p})$ *such that equations*

$$\partial_i R_j + R_i\partial^i R_j = 0, \quad i \neq j, \tag{4.4}$$

are satisfied for

$$R_i \doteq -\frac{\partial_i\mathcal{H} - \lambda_i\mathcal{H}}{\partial^i\mathcal{H}}. \tag{4.5}$$

Proof: It is sufficient to examine the second equation (3.11). Due to Lemma 4.2, this is equivalent to the existence of functions λ_i on T^*Q such that

$$\partial_i\mathcal{H} + R_i\partial^i\mathcal{H} = \lambda_i\mathcal{H}. \tag{4.6}$$

Then we get for R_i the expression (4.5). ■

Proposition 4.2: The HJE (2.1) *admits an internal separated solution in the coordinates* $\underline{q}=(q^i)$ *if and only if there exist functions* $\lambda_i(\underline{q},\underline{p})$ *and* $\mu_{ij}(\underline{q},\underline{p})$ *such that equations*

$$\partial_i R_j + R_i\partial^i R_j = \mu_{ij}\mathcal{H}, \quad i \neq j, \tag{4.7}$$

are satisfied for R_i *given by* (4.5),

$$R_i \doteq -\frac{\partial_i\mathcal{H} - \lambda_i\mathcal{H}}{\partial^i\mathcal{H}}.$$

Proof: By recalling the proof of Proposition 4.1, the first equation (3.12) means that R_i must have the form (4.5). Moreover, due to Lemma 4.2, the second equation (3.12) is equivalent to the existence of functions μ_{ij} on T^*Q such that (4.7) holds

.■

We remark that the additional functions λ_i and μ_{ij} play the role of Lagrangian multipliers. Now we are able to state three effective criteria for the separability (Definition 2.3) of the null HJE.

Theorem 4.1: The HJE $\mathcal{H}=0$ (2.1) *is separable in the coordinates* $\underline{q}=(q^i)$ *if and only if there exist n functions* $\underline{\lambda}=(\lambda_k)$ *on* T^*Q *such that equations*

$$L_{ij}(\mathcal{H};\underline{\lambda}) \doteq L_{ij}(\mathcal{H}) + \mathcal{H}[\lambda_i(\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial_j\mathcal{H}\partial^i\partial^j\mathcal{H}) + \lambda_j(\partial^i\mathcal{H}\partial_i\partial^j\mathcal{H} - \partial^i\partial^j\mathcal{H}\partial_i\mathcal{H}) + \lambda_i\lambda_j(\mathcal{H}\partial^i\partial^j\mathcal{H} - \partial^i\mathcal{H}\partial^j\mathcal{H}) - \partial_j\lambda_i\partial^i\mathcal{H}\partial^j\mathcal{H} + \partial^j\lambda_i(\partial_j\mathcal{H}\partial^i\mathcal{H} - \lambda_j\mathcal{H}\partial^i\mathcal{H})] = 0 \quad (4.8)$$

are satisfied for $i \neq j$, *where*

$$L_{ij}(\mathcal{H}) \doteq \partial_i\mathcal{H}\partial_j\mathcal{H}\partial^i\partial^j\mathcal{H} + \partial^i\mathcal{H}\partial^j\mathcal{H}\partial_i\partial_j\mathcal{H} - \partial_i\mathcal{H}\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial^i\mathcal{H}\partial_j\mathcal{H}\partial_i\partial^j\mathcal{H}. \quad (4.9)$$

We call equations (4.8), $L_{ij}(\mathcal{H},\underline{\lambda})=0$, the *Levi–Civita conditions with Lagrangian multipliers.*

Proof: By Proposition 4.1, the HJE (2.1) is separable if and only if there exist functions λ_i on T^*Q such that equations (4.4) and (4.5) hold. Due to the expression of R_i given by (4.5), the left-hand side D_iR_j of (4.4) becomes

$$D_iR_j = -\frac{1}{\partial^i\mathcal{H}(\partial^j\mathcal{H})^2}L_{ij}(\mathcal{H};\underline{\lambda}), \quad (4.10)$$

so that equations (4.4) are equivalent to equations (4.8). ■

Remark 4.1: For all $\lambda_i=0$, the expression (4.5) reduces to

$$R_i = -\frac{\partial_i\mathcal{H}}{\partial^i\mathcal{H}}, \quad (4.11)$$

and (4.8) become the usual Levi–Civita conditions (4.1). Conditions (4.8) are more general than (4.1). Indeed, equations (4.1) are satisfied if and only if for R_i given by (4.11) we have $D_iR_j=0$, i.e., if and only if the associated distribution Δ is completely integrable on an open subset of T^*Q and reducible to every $\mathcal{E}_h$, that is if and only if there exists an ordinary complete solution of equation $\mathcal{H}=h$ (2.4).

Remark 4.2: Equations (4.8) can be written in the form $L_{ij}(\mathcal{H})=P_{ij}(\underline{q},\underline{p})\mathcal{H}$. This is just a special case [corresponding to a first order PDE $\mathcal{H}(\underline{q},\underline{p})=0$] of a general equation written by Kalnins and Miller in their theory of the variable separation for partial differential equations [Eq. (1.25) of Ref. 26, or Eq. (1.17) of Ref. 21]. In fact, equation $L_{ij}(\mathcal{H})=\mathcal{H}P_{ij}$ is considered as a *definition* of a so-called regular separation of equation $\mathcal{H}=0$. Instead, in our approach it is a consequence of the definitions of separation given in Sec. II and rigorously proved by means of the Hadamard lemma, as shown by the following.

Theorem 4.2: *The HJE* $\mathcal{H}=0$ (2.1) *is separable in the coordinates* $\underline{q}=(q^i)$ *if and only if the Levi–Civita conditions* $L_{ij}(\mathcal{H})=0$ *are satisfied for* $\mathcal{H}=0$, *that is on the submanifold* $\mathcal{E}_0$,

$$L_{ij}(\mathcal{H})|_{\mathcal{E}_0} = 0. \quad (4.12)$$

Proof: The Levi–Civita conditions with Lagrangian multipliers (4.8) obviously imply $L_{ij}(\mathcal{H})=0$ for $\mathcal{H}=0$. Conversely, assume that (4.12) holds. Then, by Lemma 4.2, there exist functions ν_{ij} such that $L_{ij}(\mathcal{H})=\nu_{ij}\mathcal{H}$. If we set

$$\mu_{ij} = -\frac{1}{\partial^i\mathcal{H}(\partial^j\mathcal{H})^2}\nu_{ij},$$

and $\lambda_i = 0$, then due to (4.10) we get that equations (4.5) and (4.7) are satisfied, and due to Proposition 4.2 the HJE admits an internal separated solution. ■

We remark that conditions (4.12) do not involve Lagrangian multipliers.

Remark 4.3: When the functions R_i are given by (4.5), the generators D_i become

$$D_i = \partial_i - \frac{\partial_i\mathcal{H} - \lambda_i\mathcal{H}}{\partial^i\mathcal{H}}\,\partial^i. \tag{4.13}$$

Then, the Levi–Civita conditions with Lagrangian multipliers (4.8) can be written in the form

$$L_{ij}(\mathcal{H};\lambda_k) = L_{ij}(\mathcal{H}) + \mathcal{H}[\lambda_i(\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial_j\mathcal{H}\partial^i\partial^j\mathcal{H}) + \lambda_j(\partial^i\mathcal{H}\partial_i\partial^j\mathcal{H} - \partial^i\partial^j\mathcal{H}\partial_i\mathcal{H}) + \lambda_i\lambda_j(\mathcal{H}\partial^i\partial^j\mathcal{H} - \partial^i\mathcal{H}\partial^j\mathcal{H}) - D_j\lambda_i\partial^i\mathcal{H}\partial^j\mathcal{H}] = 0. \tag{4.14}$$

It is a remarkable fact that the Levi–Civita separability conditions with Lagrangian multipliers λ_i are equivalent to differential conditions involving a single undetermined function Λ on T^*Q and that these new conditions are the ordinary Levi–Civita separability conditions, but with respect to a modified Hamiltonian, $\mathcal{J} = \mathcal{H}/\Lambda$.

Theorem 4.3: *The HJE* $\mathcal{H} = 0$ (2.1) *is separable in the coordinates* $\underline{q} = (q^i)$ *if and only if there exists a nowhere vanishing function* $\Lambda = \Lambda(\underline{q},\underline{p})$ *such that for any* $i \neq j$,

$$L_{ij}\left(\frac{\mathcal{H}}{\Lambda}\right) = 0. \tag{4.15}$$

Proof: The Levi–Civita equations with Lagrangian multipliers (4.14) are not symmetric in the indices (i,j), due to the last term. By taking their skew-symmetric part, we obtain equations

$$\partial^i\mathcal{H}\partial^j\mathcal{H}D_j\lambda_i = \partial^i\mathcal{H}\partial^j\mathcal{H}D_i\lambda_j, \tag{4.16}$$

which are necessary conditions for the solvability of (4.14). Since D_i and D_j commute, it follows that (4.16) are locally equivalent to the existence of a function $F(\underline{q},\underline{p})$ such that

$$\lambda_i = D_iF. \tag{4.17}$$

Indeed, the commutation condition $[D_i, D_j] = 0$ is equivalent to the existence of local coordinates (x^i, y^j) such that $D_i = \partial/\partial y^i$. Hence, equations (4.16) become equivalent to $\partial\lambda_i/\partial y^j = \partial\lambda_j/\partial y^i$. By considering the coordinates (x^i) as independent parameters, this is locally equivalent to the existence of a function $F(\underline{x},\underline{y})$ such that $\lambda_i = \partial F/\partial y^i$, and we get (4.17). However, it turns out to be more convenient to replace the function F in (4.17) with $F = \ln|\Lambda|$, where $\Lambda(\underline{q},\underline{p})$ is a nowhere vanishing function, so that

$$\lambda_i = \tfrac{1}{\Lambda}D_i\Lambda. \tag{4.18}$$

Then, by inserting (4.18) in (4.5), we obtain

$$R_i = -\frac{\partial_i\mathcal{H}}{\partial^i\mathcal{H}} + \frac{1}{\Lambda}D_i\Lambda\frac{\mathcal{H}}{\partial^i\mathcal{H}},$$

and, by (4.13),

$$R_i = -\frac{\partial_i\mathcal{H}}{\partial^i\mathcal{H}} + \frac{1}{\Lambda}(\partial_i\Lambda + R_i\partial^i\Lambda)\frac{\mathcal{H}}{\partial^i\mathcal{H}}.$$

By solving this equation with respect to R_i, we find

$$R_i = -\frac{\Lambda\partial_i\mathcal{H} - \mathcal{H}\partial_i\Lambda}{\Lambda\partial^i\mathcal{H} - \mathcal{H}\partial^i\Lambda}, \tag{4.19}$$

that is

$$R_i = -\frac{\partial_i(\mathcal{H}/\Lambda)}{\partial^i(\mathcal{H}/\Lambda)}. \tag{4.20}$$

Hence, for the functions (4.20) the complete integrability conditions (4.4) become the Levi–Civita conditions (4.15) for the new Hamiltonian $\mathcal{J}=\mathcal{H}/\Lambda$. ■

Remark 4.4: The explicit expression of the n Lagrangian multipliers λ_i in terms of Λ is

$$\lambda_i = \frac{\{\mathcal{H},\Lambda\}_i}{\Lambda\partial^i\mathcal{H} - \mathcal{H}\partial^i\Lambda}, \tag{4.21}$$

where $\{\mathcal{H},\Lambda\}_i \doteq \partial^i\mathcal{H}\partial_i\Lambda - \partial^i\Lambda\partial_i\mathcal{H}$. Indeed, by (4.20), the generators D_i become

$$D_i = \partial_i - \frac{\partial_i(\mathcal{H}/\Lambda)}{\partial^i(\mathcal{H}/\Lambda)}\partial^i.$$

In particular, using (4.19), the Lagrangian multipliers (4.18) become

$$\lambda_i = \frac{\partial^i\mathcal{H}\partial_i\Lambda - \partial^i\Lambda\partial_i\mathcal{H}}{\Lambda\partial^i\mathcal{H} - \mathcal{H}\partial^i\Lambda},$$

that is (4.21).

We call the function

$$\mathcal{J} = \frac{\mathcal{H}}{\Lambda} \tag{4.22}$$

the *conformal Hamiltonian* associated to $\mathcal{H}$ and the function Λ the conformal factor. The link between the two Hamiltonian vector fields $\mathbf{X}_{\mathcal{H}}$ and $\mathbf{X}_{\mathcal{J}}$ generated by the Hamiltonians $\mathcal{H}$ and $\mathcal{J}$, respectively, is given by the following.

Proposition 4.3: On the submanifold $\mathcal{E}_0$ the vector fields $\mathbf{X}_{\mathcal{H}}$ and $\mathbf{X}_{\mathcal{J}}$ are parallel and differ by the factor Λ,

$$(\Lambda X_{\mathcal{J}})|_{\mathcal{E}_0} = X_{\mathcal{H}}|_{\mathcal{E}_0} \tag{4.23}$$

so that the corresponding affine parameters t and $\bar{t}$ are related by equation

$$\mathrm{d}\bar{t} = \Lambda\,\mathrm{d}t. \tag{4.24}$$

Proof: Let ω be the symplectic form on T^*Q. Then,

$$i_{\mathbf{X}_{\mathcal{J}}}\omega = -\,\mathrm{d}\mathcal{J} = \tfrac{1}{\Lambda^2}\mathcal{H}\,\mathrm{d}\Lambda - \tfrac{1}{\Lambda}\mathrm{d}\mathcal{H}, \quad i_{\mathbf{X}_{\mathcal{H}}}\omega = -\,\mathrm{d}\mathcal{H}. \tag{4.25}$$

By eliminating $\mathrm{d}\mathcal{H}$ in these two equations we get the single equation $\Lambda i_{\mathbf{X}_{\mathcal{J}}}\omega - d\ln|\Lambda|\mathcal{H} = i_{\mathbf{X}_{\mathcal{H}}}\omega$, which is equivalent to

$$i_{(\mathbf{X}_{\mathcal{H}}-\Lambda\mathbf{X}_{\mathcal{J}})}\omega = \frac{\mathcal{H}}{\Lambda}\mathrm{d}\Lambda. \tag{4.26}$$

By (4.26), for $\mathcal{H}=0$ the Hamiltonian vector field $(\mathbf{X}_{\mathcal{H}} - \Lambda\mathbf{X}_{\mathcal{J}})$ vanishes and we get (4.23). If t and $\bar{t}$ are the affine parameters of $\mathbf{X}_{\mathcal{H}}$ and $\mathbf{X}_{\mathcal{J}}$, respectively, then by (4.23), we find that (4.24) holds on $\mathcal{E}_0$. ■

Theorem 4.4: *If we know a complete solution of the HJE $\mathcal{J}=h$ for the conformal Hamiltonian*

(4.22), *then for* $h=0$ *we get the orbits on* $\mathcal{E}_0$ *of the Hamiltonian vector field* $\mathbf{X}_{\mathcal{H}}$.

Proof: According to Proposition 4.3, on $\mathcal{E}_0$, the integral curves of the vector fields $\mathbf{X}_{\mathcal{H}}$ and $\mathbf{X}_{\mathcal{J}}$ coincide, up to the reparametrization given by (4.24). Since $\mathcal{H}=0$ means $\mathcal{J}=0$, by inserting the condition $h=0$ in a complete solution of the HJE $\mathcal{J}=h$, we get the orbits of the field $\mathbf{X}_{\mathcal{H}}$ on the hypersurface $\mathcal{H}=0$. ■

Remark 4.5: We recall that a first integral of a Hamiltonian $\mathcal{H}$ is a function F on T^*Q which is constant on the integral curves of $\mathbf{X}_{\mathcal{H}}$ and that this is equivalent to the condition $\mathbf{X}_{\mathcal{H}}F=0$ or $\{\mathcal{H},F\}=0$. We call isoenergetic first integral of a function $\mathcal{H}$ any function F which is constant on all the integral curves contained in a submanifold $\mathcal{H}=h$ for some values of $h\in\mathbb{R}$. Due to the Hadamard lemma, this is equivalent to the existence of a function ϕ such that $\{\mathcal{H},F\}=\phi(\mathcal{H}-h)$. Of course, any ordinary first integral is a special kind of isoenergetic first integral. If $h=0$ we call F a *null first integral* of $\mathcal{H}$: it is characterized by equation

$$\{\mathcal{H},F\} = \phi\mathcal{H}. \tag{4.27}$$

By (4.23), it follows that any first integral F of $\mathbf{X}_{\mathcal{J}}$ is a null first integral of $\mathbf{X}_{\mathcal{H}}$. In Sec. VIII we shall use this definition.

Remark 4.6: Let the Hamiltonian $\mathcal{H}$ be of the form $\mathcal{H}=H+\Lambda$. In this case we can consider a particular conformal Hamiltonian $J=H/\Lambda$. We call the Hamiltonian J the *generalized Jacobi transform* of $\mathcal{H}$. According to Theorem 4.4, we get that the orbits of $\mathcal{H}$ on the hypersurface $\mathcal{H}=0$ coincide with the orbits of J on $J=1$. Moreover, by (4.24), the generalized Jacobi transform can be considered[33] as a transformation on the cotangent bundle $T^*\bar{Q}$ of the extended configuration manifold $\bar{Q}=\mathbb{R}\times Q$ which is a canonical transformation only on the hypersurface $p_0+\mathcal{H}=0$.

V. A FIRST APPLICATION: THE SEPARATION FOR TIME-DEPENDENT HAMILTONIANS

Let $H(t,q)$ be a time-dependent Hamiltonian, that is a function on the $(n+1)$-dimensional manifold $\bar{Q}=\mathbb{R}\times Q$ (the extended configuration manifold). The well-known HJE associated with a time-dependent system is

$$\frac{\partial W}{\partial t} + H\left(t,q^i,\frac{\partial W}{\partial q^i}\right) = 0. \tag{5.1}$$

In the so-called homogeneous formalism, this is equivalent to consider on the cotangent bundle $T^*\bar{Q}$, with coordinates $(q^A,p_A)=(q^0,q^i,p_0,p_i)$, the function

$$\mathcal{H}(q^A,p_A) = p_0 + H(q^A,p_i), \tag{5.2}$$

whose corresponding equation $\mathcal{H}=0$ is (5.1) (with $q^0=t$). We have the separation of variables of (5.1) on the hypersurface $\mathcal{H}=0$ if and only if the Levi–Civita conditions $L_{AB}(\mathcal{H})=0(A\neq B=0,\ldots,n)$ are satisfied on $\mathcal{E}_0$, that is for

$$p_0 = -H(q^A,p_i). \tag{5.3}$$

The Levi–Civita equations $L_{AB}(\mathcal{H})=0$ for the Hamiltonian (5.2) become

$$L_{ij}(\mathcal{H}) = L_{ij}(H), \quad L_{i0}(\mathcal{H}) = \partial^i H\partial_i\partial_0 H - \partial_i H\partial^i\partial_0 H \quad (i,j=1,\ldots,n). \tag{5.4}$$

It is remarkable the fact that, due to (5.3) and since equations (5.4) do not contain p_0, we have

$$L_{AB}(\mathcal{H})|\mathcal{E}_0 = 0 \quad\Leftrightarrow\quad L_{AB}(\mathcal{H}) = 0.$$

Thus, in this case we have the perfect equivalence between the separation of the HJE $\mathcal{H}=h$ of the kind (2.4) and the separation of variables for the single equation $\mathcal{H}=0$ of the kind (2.1). Then, in order to have the separability for the HJE (5.1) we need that the following conditions be satisfied:

$$L_{ij}(H) = 0, \quad \partial^i H\partial_i\partial_0 H - \partial_i H\partial^i\partial_0 H = 0 \quad (i \text{ n.s.}). \tag{5.5}$$

Remark 5.1: Equations (5.5) implies that the T^*Q-Poisson bracket of the functions H and $\partial_0 H$ vanishes,

$$\{H,\partial_0 H\}_{T^*Q} = \partial^i H \partial_i \partial_0 H - \partial_i H \partial^i \partial_0 H = 0.$$

Remark 5.2: If conditions (5.5) hold, then the Levi–Civita conditions with $n+1$ Lagrangian multipliers $L_{AB}(\mathcal{H};\lambda_C)=0$ are satisfied for $\lambda_C=0(C=0,\ldots,n)$ or, equivalently, conditions (4.15) hold for $\Lambda=1$.

Remark 5.3: Equations (5.5) are the Levi–Civita separability conditions for the time-dependent case proposed by Forbat.[15] However, the proof given by Forbat is unsatisfactory. Indeed, it is based on the fact that, assuming that the equation (5.1) admits a complete solution of the form $W=W_0(t,\underline{c})+\Sigma_i W_i(q^i,\underline{c})$, by differentiating (5.1) with respect to a coordinate q^i, we get equations

$$\partial^i H \partial_i p_i + \partial_i H = 0 \tag{5.6}$$

(no summation on the index i). By solving (5.6) with respect to $\partial_i p_i$, we obtain the system

$$\partial_j p_i = \delta_{ij} R_i, \quad R_i = -\frac{\partial_i H}{\partial^i H}, \tag{5.7}$$

whose integrability conditions are (5.5). However, equations (5.6) are derivable also from equation $\mathcal{H}=h$, where h is any constant, not only from equation $\mathcal{H}=0$. In other words, in considering the integrability conditions of system (5.7) one is actually considering the separation of all equations $\mathcal{H}=h=\text{const}$, which is not in general equivalent to the separation of the single equation $\mathcal{H}=0$, as we have seen in the preceding sections.

VI. THE ORTHOGONAL SEPARATION FOR NATURAL HAMILTONIANS

Let us apply the general theory so far developed to the special but fundamental case of a natural Hamiltonian $H=G+V$ in orthogonal coordinates,

$$H(\underline{q},\underline{p}) = \tfrac{1}{2} g^{ii} p_i^2 + V(\underline{q}).$$

With an orthogonal metric $\mathbf{G}=(g^{ii})$ we associate differential operators $S_{ij}(A)$ on functions $A(\underline{q})$,

$$S_{ij}(A) \doteq \partial_i\partial_j A - \partial_j \ln|g^{ii}|\partial_i A - \partial_i \ln|g^{jj}|\partial_j A = \partial_i\partial_j A - \frac{1}{g^{ii}}\partial_j g^{ii}\partial_i A - \frac{1}{g^{jj}}\partial_i g^{jj}\partial_j A,$$

which we call *Stäckel operators*. The indices (i,j) are assumed to be distinct and not summed (n.s.). In the following the condition "$i \neq j$ n.s." referred to an operator S_{ij} will be understood. We know (see, e.g., Ref. 3) that g^{kk} is a Stäckel metric if and only if

$$S_{ij}(g^{kk}) = 0, \tag{6.1}$$

and that a potential V is separable in these coordinates if and only if $S_{ij}(V)=0$. Indeed, for a natural Hamiltonian in orthogonal coordinates the Levi–Civita equations become

$$L_{ij}(H) = g^{ii}g^{jj}p_i p_j\left(\tfrac{1}{2}S_{ij}(g^{kk})p_k^2 + S_{ij}(V)\right) = 0, \tag{6.2}$$

and they are satisfied if and only if $\frac{1}{2}S_{ij}(g^{kk})p_k^2+S_{ij}(V)=0$. For the operators S_{ij} the following rules hold:

$$S_{ij}(c) = 0, \quad c \in \mathbb{R},$$

$$S_{ij}(A+B) = S_{ij}(A) + S_{ij}(B),$$

$$S_{ij}(AB) = AS_{ij}(B) + BS_{ij}(A) + \partial_i A \partial_j B + \partial_j A \partial_i B,$$

$$S_{ij}(A^{-1}) = 2A^{-3}\partial_i A \partial_j A - A^{-2} S_{ij}(A). \tag{6.3}$$

The Stäckel operators $\bar{S}_{ij}$ corresponding to a conformal orthogonal metric $\bar{g}^{ii} = (1/\sigma) g^{ii}$ obey rules (6.3) and

$$\bar{S}_{ij}(A) = S_{ij}(A) - \sigma^{-3}(\partial_i \sigma \partial_j A + \partial_i A \partial_j \sigma),$$

$$\bar{S}_{ij}(\bar{g}^{kk}) = \sigma^{-1} S_{ij}(g^{kk}) - g^{kk}\sigma^{-2} S_{ij}(\sigma) = \frac{g^{kk}}{\sigma}\left(\frac{1}{g^{kk}} S_{ij}(g^{kk}) - \frac{1}{\sigma} S_{ij}(\sigma)\right). \tag{6.4}$$

Remark 6.1: From the second equation (6.4), by setting $\sigma = g^{11}, \ldots, g^{nn}$, we derive the following theorem due to Kalnins and Miller, Ref. 20, Lemma 1: *If* (g^{ii}) *is a Stäckel metric then all metrics* $(g^{ii}/g^{11}), \ldots, (g^{ii}/g^{nn})$ *are Stäckel metrics*. We recall also Lemma 2 of Ref. 20: *An orthogonal metric* (g^{ii}) *is conformal to a Stäckel metric if and only if* g^{ii}/g^{jj}, *for any fixed value of the index* j, *is a Stäckel metric*. Indeed, this follows from Lemma 1 and the fact that for any conformal metric $\bar{g}^{ii}$ we have $\bar{g}^{ii}/\bar{g}^{jj} = g^{ii}/g^{jj}$. Note that equation (6.1) are equivalent to equations

$$\partial^2_{ij} \ln|g_{ii}| - \partial_i \ln|g_{kk}| \partial_j \ln|g_{ii}| + \partial_i \ln|g_{jj}| \partial_j \ln|g_{kk}| + \partial_j \ln|g_{ii}| \partial_i \ln|g_{kk}| = 0.$$

With the substitution $g_{ii} = e_i H_i^2$, $e_i = \pm 1$, they coincide with the equations given by Eisenhart (Ref. 14, Appendix 13).

Let us apply the results of Sec. IV to the function

$$\mathcal{H} = \tfrac{1}{2} g^{ii} p_i^2 + V - E, \quad E \in \mathbb{R}.$$

Theorem 6.1: *The HJE*

$$\tfrac{1}{2} g^{ii} p_i^2 + V - E = 0, \tag{6.5}$$

is separable in the orthogonal coordinates (q^i), *for a fixed value* $E \in \mathbb{R}$, *if and only if equations*

$$\frac{1}{g^{hh}} S_{ij}(g^{hh}) = \frac{1}{g^{kk}} S_{ij}(g^{kk}), \quad S_{ij}(V) = \frac{V-E}{g^{hh}} S_{ij}(g^{hh}), \tag{6.6}$$

are satisfied for all indices h, k *and* $i \neq j$.

Proof: Due to Theorem 4.2, a necessary and sufficient condition for the separation of $\mathcal{H}=0$ is that the Levi–Civita conditions be satisfied when restricted to the submanifold $\mathcal{E}_0$, that is for $\mathcal{H}=0$. By (6.5), equation $\mathcal{H}=0$ is equivalent to

$$p_1^2 = -\sum_{k=2}^{n} \frac{g^{kk}}{g^{11}} p_k^2 + \frac{2}{g^{11}}(E - V). \tag{6.7}$$

Thus, by (6.2) and (6.7), we get

$$L_{ij}(\mathcal{H})|_{\mathcal{E}_0} = g^{ii} g^{jj} p_i p_j \left(\frac{1}{2}\sum_{k=2}^{n} S_{ij}(g^{kk}) p_k^2 + S_{ij}(V) - \frac{1}{2} S_{ij}(g^{11}) \sum_{k=2}^{n} \frac{g^{kk}}{g^{11}} p_k^2 + \frac{S_{ij}(g^{11})}{g^{11}}(E - V)\right). \tag{6.8}$$

Functions (6.8) vanish if and only if

$$\frac{1}{2}\sum_{k=2}^{n}\left[S_{ij}(g^{kk})-\frac{g^{kk}}{g^{11}}S_{ij}(g^{11})\right]p_k^2-S_{ij}(E-V)-\frac{S_{ij}(g^{11})}{g^{11}}(E-V)=0. \tag{6.9}$$

Since conditions (6.9) must be satisfied for all values of $p_2,\ldots,p_n$, they are equivalent to

$$S_{ij}(g^{kk})-\frac{g^{kk}}{g^{11}}S_{ij}(g^{11})=0,\quad S_{ij}(E-V)-\frac{E-V}{g^{11}}S_{ij}(g^{11})=0,$$

hence to (6.6). ■

The first equations (6.6) mean that the functions $(1/g^{hh})S_{ij}(g^{hh})$ do not depend on the choice of the index h. This is a necessary condition for the separability of equation (6.5), which has the following equivalent formulation.

Theorem 6.2: *The necessary condition for the separability of the HJE* (6.5)

$$\frac{1}{g^{hh}}S_{ij}(g^{hh})=\frac{1}{g^{kk}}S_{ij}(g^{kk})\quad \textit{for all indices } h,\ k \textit{ and } i\neq j, \tag{6.10}$$

is equivalent to the existence of a function $\sigma\neq 0$ such that the conformal metric $\bar{g}^{ii}=g^{ii}/\sigma$ is a Stäckel metric,

$$\bar{S}_{ij}(\bar{g}^{kk})=0. \tag{6.11}$$

Proof: (i) If such a function σ exists, then by the second equation (6.4) we derive $(1/g^{kk})S_{ij}(g^{kk})=(1/\sigma)S_{ij}(\sigma)$. Hence, (6.10) follows. (ii) Conversely, assume that (6.10) holds. If we choose $\sigma=g^{11}$ then from the second equation (6.4) we get $\bar{S}_{ij}(\bar{g}^{kk})=0$. ■

Remark 6.2: From this proof it follows that condition (6.10) is verified if and only if (6.11) is satisfied with $\sigma=g^{jj}$ for any arbitrary choice of the index j.

This theorem suggests the following.

Definition 6.1: We call *conformal separable coordinates* orthogonal coordinates $\underline{q}=(q^i)$ for which conditions (6.10) or (6.11) hold.

Remark 6.3: A special class of orthogonal conformal separable coordinates is that for which $g^{ii}=c^i\sigma(\underline{q})$, $c^i\in\mathbb{R}$. In this case the components of the orthogonal conformal metric are $c^i=g^{ii}/\sigma$ =constant; hence, they are obviously of the Stäckel type. Up to a rescaling of the coordinates we can reduce to the case $g^{ii}=\pm\sigma$, according to the signature of the metric. Note that in this case the original metric (g^{ii}) is conformally flat. Orthogonal coordinates for which $g^{ii}=g^{jj}$ are called isothermal.

Now we apply Theorem 6.1 to the following three special cases:

$$\begin{aligned}
&V=0,\quad E\neq 0, && \text{non-null geodesics},\\
&V=0,\quad E=0, && \text{null geodesics},\\
&V-E\neq 0, && \text{dynamical trajectories with total energy } E.
\end{aligned}$$

The results for the null geodesics case date back to Stäckel[32] (see also Ref. 20).

Theorem 6.3: *The HJE $\frac{1}{2}g^{ii}p_i^2=E$ with a fixed value $E\neq 0$, is separable in orthogonal coordinates (q^i) if and only if (g^{ii}) is a Stäckel metric, i.e., if and only if it is separable in the ordinary sense for all values of E.*

Proof: For $V=0$ the second equation (6.6) gives $S_{ij}(g^{kk})=0$. ■

Theorem 6.4: *The HJE of the null geodesics*

$$g^{ii}p_i^2=0 \tag{6.12}$$

is separable in the orthogonal coordinates (q^i) if and only if these coordinates are conformal separable.

Proof: For $V=0$ and $E=0$ the second equations (6.6) are trivially satisfied, so that only the first equations characterize the separation. ■

Theorem 6.5: *The HJE*

$$\tfrac{1}{2}g^{ii}p_i^2 + V - E = 0 \quad (V - E \neq 0), \tag{6.13}$$

is separable if and only if the conformal metric

$$\bar{g}^{ii} = \frac{1}{E - V}g^{ii} \tag{6.14}$$

is a Stäckel metric, or equivalently, if and only if for all indices h, k and $i \neq j$,

$$\frac{1}{g^{kk}}S_{ij}(g^{kk}) = \frac{1}{V - E}S_{ij}(V). \tag{6.15}$$

This means that the coordinates are conformal separable, according to Definition 6.1, but the conformal factor σ must be equal to the function $V-E$.

Proof: For $V-E \neq 0$ system (6.6) is equivalent to (6.15). Moreover, let us consider the conformal metric (6.14) and the associated Stäckel operators $\bar{S}_{ij}$. From the second formula (6.4) with $\sigma=E-V$, we get

$$\bar{S}_{ij}(\bar{g}^{kk}) = \frac{1}{E - V}S_{ij}(g^{kk}) - \frac{g^{kk}}{(E - V)^2}S_{ij}(E - V).$$

Thus, $\bar{S}_{ij}(\bar{g}^{kk})=0$ is equivalent to (6.15). ∎

Remark 6.4: The metric $\bar{g}^{ii}=(E-V)^{-1}g^{ii}$ is called the *Jacobi metric* or action metric (see, e.g., Ref. 33 and the references cited therein) of the natural Hamiltonian $H=G+V$ for the fixed value E of the total energy. Then, Theorem 6.5 can be reformulated as follows

Theorem 6.6: *The HJE* (6.13) *is separable if and only if the corresponding Jacobi metric is a Stäckel metric.*

We adapt to the Jacobi metric the considerations about the conformal Hamiltonians stated in Proposition 4.3 and Theorem 4.4.

With a natural Hamiltonian $H=G+V=\frac{1}{2}g^{ij}p_ip_j+V$ and a fixed value of the energy $E\in\mathbb{R}$ we associate two Hamiltonians,

$$H_E = \tfrac{1}{2}g^{ij}p_ip_j + V - E, \quad J_E = \tfrac{1}{2}\tfrac{g^{ij}}{E-V}p_ip_j.$$

The passage from the natural Hamiltonian $H=G+V$ to the geodesic Hamiltonian J_E is called Jacobi transformation[16,28,30] or Maupertuis transformation.[9,33]

Let $\mathbf{X}_H$ be the Hamiltonian vector field generated by H (it coincides with that generated by H_E) and $\mathbf{X}_J$ the Hamiltonian vector field generated by J_E. Adapting to these cases Proposition 4.3, Theorem 4.4, and Remark 4.5, we get the following.

Theorem 6.7: *Assume that equation $H_E=0$ (i.e., $J_E=1$) defines a regular hypersurface of T^*Q. Then* (i) *on this hypersurface the Hamiltonian vector fields $\mathbf{X}_H$ and $\mathbf{X}_J$ are parallel,*

$$(E - V)\mathbf{X}_J = \mathbf{X}_H, \tag{6.16}$$

and outside this surface the difference $(E-V)\mathbf{X}_J-\mathbf{X}_H$ is a vertical vector field. (ii) *On $H_E=0$ the integral curves of the vector fields $\mathbf{X}_H$ and $\mathbf{X}_J$ coincide, up to a reparametrization, and the affine parameters t and $\bar{t}$ of $\mathbf{X}_H$ and $\mathbf{X}_J$, respectively, are related by* $\mathrm{d}\bar{t}=(E-V)\mathrm{d}t$. (iii) *Any first integral F of $\mathbf{X}_J$ is constant along the integral curves of $\mathbf{X}_H$ contained on $H_E=0$.* (iv) *If a complete solution of the geodesic HJE $J_E=h$ is known, then for $h=1$ we get the orbits of the field $\mathbf{X}_H$ on the hypersurface $H_E=0$.*

Proof: By (4.23), we get (6.16). Moreover, due to (4.26), we have that $\mathbf{X}_H-(E-V)\mathbf{X}_J$ is vertical outside the hypersurface $H_E=0$, since it is generated by the function $\Lambda=E-V$ which is constant along the fibers. ∎

Hence, as a corollary of Theorem 6.6, we have the following.

Theorem 6.8: *The orthogonal separation (in the ordinary sense) of the geodesic HJE* $J_E=h$ *is equivalent to the orthogonal separation of the HJE* $H_E=0$ *for a fixed value E of the energy. For* $h=1$ *we get the orbits corresponding to the integral curves of* $\mathbf{X}_H$ *with total energy E.*

Remark 6.5: Equation (6.15) shows that $S_{ij}(V)=0$ if and only if $S_{ij}(g^{kk})=0$. These two conditions characterize the orthogonal Stäckel separation for a natural Hamiltonian. In this case the Jacobi metric is a Stäckel metric for all values of E. From (6.15) it follows also that if the conformal Jacobi metric (6.14) is a Stäckel metric for two distinct values $E_1\neq E_2$ of the energy, then it is a Stäckel metric for all values of E. Indeed, from (6.15) written for $E=E_1$ and $E=E_2$ it follows that

$$\frac{1}{V-E_1}S_{ij}(V)=\frac{1}{V-E_2}S_{ij}(V).$$

Thus, $S_{ij}(V)=0$, so that also $S_{ij}(g^{kk})=0$. As a consequence, we have the following.

Theorem 6.9: *The HJE* (6.13) *is separable for two distinct values of the energy E if and only if it is separable in the ordinary sense.*

Remark 6.6: According to Theorem 6.9, we have that if a natural Hamiltonian $H=G+V$ is not separable in the ordinary sense, then there exists at most one value of the energy E such that $H=E$ is separable.

Remark 6.7: For a natural Hamiltonian in orthogonal coordinates the Lagrangian multipliers λ_i or the function Λ, involved in Theorems 4.1 and 4.3, respectively, which in general are functions on T^*Q, are necessarily constant along the fibers, i.e., they reduce in this case to functions on Q.

We conclude this section with the formulation of Theorems 6.2 and 6.3 in terms of Stäckel matrices. We recall that an orthogonal metric is a Stäckel metric if and only if it is a row of the inverse of a Stäckel matrix $\mathbf{S}=[\varphi_i^{(j)}(q^i)]$. By applying this definition to the general conformal metric $\bar{g}^{ii}=g^{ii}/\sigma$ and to the Jacobi metric (6.14) we get the following.

Theorem 6.10: (i) *Coordinates* (q^i) *are conformal separable if and only if there exists a Stäckel matrix* $\mathbf{S}=[\varphi_i^{(j)}(q^i)]$ *such that*

$$\frac{g^{ii}}{\varphi^i_{(n)}}=\frac{g^{jj}}{\varphi^j_{(n)}}, \tag{6.17}$$

where $[\varphi^i_{(j)}]=\mathbf{S}^{-1}$. (ii) *The Jacobi metric* (6.14) *is a Stäckel metric if and only if there exists a Stäckel matrix* $\mathbf{S}=[\varphi_i^{(j)}(q^i)]$ *such that* (6.17) *holds and moreover,*

$$E-V=\sum_i \varphi_i^{(n)}g^{ii}. \tag{6.18}$$

Proof: We have

$$\exists\sigma\left|\frac{g^{ii}}{\sigma}=\varphi^i_{(n)}\Leftrightarrow\frac{g^{ii}}{\varphi^i_{(n)}}=\frac{g^{jj}}{\varphi^j_{(n)}},\right.$$

$$\frac{g^{ii}}{E-V}=\varphi^i_{(n)}\Leftrightarrow\frac{g^{ii}}{\varphi^i_{(n)}}=\frac{g^{jj}}{\varphi^j_{(n)}}\wedge E-V=\sum\nolimits_i \varphi_i^{(n)}g^{ii}.$$

■

Remark 6.8: Let us denote by M^i_j the cofactor of $\varphi_j^{(i)}$. We have $\det\mathbf{S}=\Sigma_i\varphi_i^{(n)}M_i^n$ and

$$\varphi^i_{(n)}=\frac{M_i^n}{\det\mathbf{S}}.$$

Hence, (6.17) is equivalent to

$$\frac{g^{ii}}{M_i^n} = \cdots = \frac{g^{jj}}{M_j^n}.$$

We observe that in these conditions, only the first $n-1$ columns of the Stäckel matrix are involved, while the last column is involved only in the expression (6.18) of $E-V$. Hence, in the characterization of the null geodesic separation only a rectangular $n\times(n-1)$ Stäckel matrix is involved,

$$\begin{bmatrix} \varphi_1^{(1)} & \dots & \varphi_1^{(n-1)} \\ \vdots & \vdots & \vdots \\ \varphi_n^{(1)} & \dots & \varphi_n^{(n-1)} \end{bmatrix}.$$

VII. THE INTRINSIC CHARACTERIZATION OF THE ORTHOGONAL SEPARATION

Theorems 6.4 and 6.5 show that the separation of variables of the HJE for the null geodesics and for a fixed value of the energy is equivalent to the ordinary complete orthogonal separation of a conformal (contravariant) metric

$$\overline{\mathbf{G}} = \frac{1}{\sigma}\mathbf{G},$$

where $\overline{\mathbf{G}}=(\overline{g}^{ii})$, $\mathbf{G}=(g^{ii})$ and σ is a nowhere vanishing function on Q. In these two cases we have, respectively,

$$\sigma = \text{a suitable function on } Q \quad \text{for the null geodesics},$$

$$\sigma = E - V \quad \text{for the Jacobi metric}.$$

Since the ordinary geodesic separation can be characterized by means of Killing tensors (KT's), in both cases we are led to consider KT's of a conformal metric. A basic well known property is the following.

Proposition 7.1: A symmetric two-tensor $\mathbf{K}$ *is a KT for the conformal metric* $\overline{\mathbf{G}}=(1/\sigma)\mathbf{G}$ *i.e.,*

$$[\overline{\mathbf{G}},\mathbf{K}] = 0 \ (\Leftrightarrow \{P_{\overline{\mathbf{G}}},P_{\mathbf{K}}\} = 0), \tag{7.1}$$

if and only if

$$[\mathbf{G},\mathbf{K}] = -\frac{2}{\sigma}\mathbf{K}\,\nabla\,\sigma\odot\mathbf{G} \quad \left(\Leftrightarrow \{P_{\mathbf{G}},P_{\mathbf{K}}\} = -\frac{2}{\sigma}P_{\mathbf{K}\nabla\sigma}P_{\mathbf{G}}\right). \tag{7.2}$$

Notation: Here we denote by $[\cdot,\cdot]$ the Lie–Schouten bracket of contravariant symmetric tensors and by $\odot$ the symmetric tensor product. If we consider the homogeneous polynomial functions $P_{\mathbf{K}}$ on T^*Q associated with contravariant symmetric tensors $\mathbf{K}=(K^{i\dots j})$ on Q, then this bracket is defined by $P_{[\mathbf{K}_1,\mathbf{K}_2]}=\{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\}$ and the symmetric product by $P_{\mathbf{K}_1\odot\mathbf{K}_2}=P_{\mathbf{K}_1}P_{\mathbf{K}_2}$. We say that $\mathbf{K}_1$ and $\mathbf{K}_2$ are in involution if $[\mathbf{K}_1,\mathbf{K}_2]=0$. We denote by $\mathbf{KX}$ the image of a vector field $\mathbf{X}$ by $\mathbf{K}$ interpreted as a (1,1) tensor.

Proof of Proposition 7.1: The equivalence of (7.1) and (7.2) is proved by the following calculation:

$$[\overline{\mathbf{G}},\mathbf{K}] = \left[\frac{1}{\sigma}\odot\mathbf{G},\mathbf{K}\right] = \frac{1}{\sigma}\odot[\mathbf{G},\mathbf{K}] + \left[\frac{1}{\sigma},\mathbf{K}\right]\odot\mathbf{G}$$

$$= \frac{1}{\sigma}[\mathbf{G},\mathbf{K}] - 2\mathbf{K}\,\nabla\,\frac{1}{\sigma}\odot\mathbf{G} = \frac{1}{\sigma}\left([\mathbf{G},\mathbf{K}] + \frac{2}{\sigma}\mathbf{K}\,\nabla\,\sigma\odot\mathbf{G}\right). \qquad \blacksquare$$

A symmetric two-tensor $\mathbf{K}$ is a conformal Killing tensor (CKT) if there exists a vector field $\mathbf{C}$ such that

$$[\mathbf{G},\mathbf{K}] = 2\mathbf{C}\odot\mathbf{G} \ (\Leftrightarrow\{P_{\mathbf{G}},P_{\mathbf{K}}\} = 2P_{\mathbf{C}}P_{\mathbf{G}}). \tag{7.3}$$

We say that $\mathbf{K}$ is a CKT of *gradient type* if there exists a function U such that $\mathbf{C}=\nabla U$. We say that $\mathbf{K}$ is a CKT of *self-gradient type* if $\mathbf{C}=\mathbf{K}\nabla U$.

We remark that in Proposition 7.1 the tensor $\mathbf{K}$ is a CKT of self-gradient type with respect to the metric $\mathbf{G}$ with $U=-\ln|\sigma|$.

Remark 7.1: The eigenvectors of $\mathbf{K}$ in Proposition 7.1 are the same with respect both metrics $\bar{\mathbf{G}}$ and $\mathbf{G}$. If $\bar{\rho}^i$ are the eigenvalues of $\mathbf{K}$ with respect to $\bar{\mathbf{G}}$, then the eigenvalues with respect to $\mathbf{G}$ are

$$\rho^i = \frac{\bar{\rho}^i}{\sigma}. \tag{7.4}$$

If the eigenvalues are simple with respect to $\bar{\mathbf{G}}$, then they are also simple with respect to $\mathbf{G}$.

Remark 7.2: Tensors of the kind $f\mathbf{G}$ are at the same time CKT's of gradient type (with $\mathbf{C}=\nabla f$, i.e., $U=f$) and of self-gradient type (with $\mathbf{C}=f\mathbf{G}\nabla \ln f$, i.e., $U=\ln f$).

Definition 7.1: Two CKT's $\mathbf{K}$ and $\mathbf{K}'$ are said to be *equivalent* if $\mathbf{K}'=\mathbf{K}+f\mathbf{G}$ for some function f.

Equivalent CKT's have the same eigenvectors. We shall be interested in equivalence classes of this kind. In any equivalence class there exists a trace-free representative, so that only trace-free CKT's are considered by some authors.[20,29,34]

As shown by the following proposition, in some special case a CKT $\mathbf{K}$ is equivalent to a tensor $\mathbf{K}'$ of self-gradient type (hence, a KT of a conformal metric).

Proposition 7.2: (i) *A CKT* $\mathbf{K}$ *which is diagonalized in orthogonal coordinates is equivalent to a CKT* $\mathbf{K}'$ *of self-gradient type.* (ii) *For any given orthogonal coordinate system there exists a function U such that any CKT* $\mathbf{K}$ *diagonalized in these coordinates is equivalent to a CKT* $\mathbf{K}'$ *of self-gradient type such that* $[\mathbf{G},\mathbf{K}']=2\mathbf{K}'\nabla U\odot\mathbf{G}$, *i.e., to a KT of the conformal metric* $\bar{\mathbf{G}}=e^U\mathbf{G}$. (iii) *The n functions* $U_k=-\ln|g^{kk}|$ *satisfy item (ii).*

Proof: If $g^{ij}=0$ and $K^{ij}=0$ for $i\neq j$, then $K^{ii}=\rho^i g^{ii}$ and Eq. (7.3) is equivalent to

$$\partial_i\rho^j = (\rho^i-\rho^j)\partial_i \ln g^{jj} + \partial_i\rho^i, \quad C_i = \partial_i\rho^i. \tag{7.5}$$

Let us take the tensor $\mathbf{K}'=\mathbf{K}-\rho^n\mathbf{G}$ with eigenvalues $\rho'^i=\rho^i-\rho^n$. By using (7.5) we get

$$\partial_i\rho'^j = (\rho'^i-\rho'^j)\partial_i \ln|g^{jj}| - \rho'^i\partial_i \ln|g^{nn}|.$$

This shows that $\mathbf{K}'$ is a CKT with $C_i'=-\rho'^i\partial_i\ln|g^{nn}|$, hence of self-gradient type with $U=-\ln|g^{nn}|$ and a KT for the conformal metric $\mathbf{G}/g^{nn}$. We remark that U does not depend on $\mathbf{K}$ but only on the given coordinates. ■

In the following two sections we give intrinsic versions of Theorems 6.4 and 6.5, respectively, for the case considered in Theorem 6.3 the intrinsic characterizations are just that of the ordinary orthogonal separation.[4,18] We shall use the following.

Definition 7.2: A (conformal) Killing two-tensor with simple eigenvalues and normal eigenvectors is called *characteristic (conformal) Killing tensor*.

A. The orthogonal separation of the null geodesics

A first characterization is related to the existence of a single CKT.

Theorem 7.1: *The HJE* (6.12) *for the null geodesics is separable in orthogonal coordinates if and only if on Q there exists a characteristic CKT* $\mathbf{K}$.

Proof: According to the intrinsic characterization of the orthogonal separation of a geodesic Hamiltonian,[4,18] a metric $\bar{\mathbf{G}}$ is orthogonally separable if and only if it admits a KT $\mathbf{K}$, $[\bar{\mathbf{G}},\mathbf{K}]=0$, with simple eigenvalues and normal eigenvectors. Since $\bar{\mathbf{G}}=\mathbf{G}/\sigma$, due to Proposition 7.1, this is equivalent to the existence of a characteristic CKT satisfying equation (7.2). Proposition 7.2 shows that this is equivalent to the existence of a characteristic CKT without any other condition. ■

Remark 7.3: Theorem 7.1 was first stated by Kalnins and Miller (Ref. 20, Theorem 1, Sec. II with a different proof, not involving the use of self-gradient CKT's.

A second characterization is related to n CKT's.

Theorem 7.2: *The HJE* (6.12) *for the null geodesics is separable in orthogonal coordinates if and only if on Q there exist n CKT's* $(\mathbf{K}_i)=(\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_n)$, (i) *pointwise independent*, (ii) *with common eigenvectors,* (iii) *in involution*.

Proof: By Theorem 8.8 of Ref. 7 the common eigenvectors are normal. There exists orthogonal coordinate systems in which all the tensors are diagonalized. Then, the pointwise independence implies the existence of a linear combination (with constant coefficients) $\mathbf{K}=c^i\mathbf{K}_i$ with simple eigenvalues. This is a conformal characteristic tensor. Then we apply Theorem 7.1. Conversely, since the separation of (6.12) is equivalent to the ordinary separation of a conformal metric $\overline{\mathbf{G}}=\mathbf{G}/\sigma$, there exist n KT's $\mathbf{K}_i$ for the conformal metric satisfying (i), (ii), (iii). Due to Proposition 7.1, these tensors are CKT's for $\mathbf{G}$. ■

Remark 7.4: In the intrinsic characterization of the ordinary orthogonal separation in terms of n independent KT's (in involution), the metric $\mathbf{G}$ may be one of them. On the contrary, in Theorem 7.2 none CKT's $\mathbf{K}_i$ can be the metric. Indeed, if one of the $\mathbf{K}_i$ is the metric, then condition (iii) implies that all $\mathbf{K}_i$ are KT's and we reduce to the ordinary orthogonal separation. In other words, the metric cannot belong to the linear space generated by the $\mathbf{K}_i$ (by linear combinations with constant coefficients). However,

Proposition 7.3: Given, n CKT's $\mathbf{K}_i$ *with common normal eigenvectors, there exist a linear combination with constant coefficients and a function f such that* $c^i\mathbf{K}_i=f\mathbf{G}$.

Proof: We apply the second part of Proposition 7.2. Then, there are equivalent CKT's of self-gradient type $\mathbf{K}_i'=\mathbf{K}_i+f_i\mathbf{G}$ with the same function U. They are KT's of the metric $e^U\mathbf{G}=\overline{\mathbf{G}}$ with common normal eigenvectors. Thus, there exists a linear combination with constant coefficients such that $c^i\mathbf{K}_i'=\overline{\mathbf{G}}$. It follows that $c^i\mathbf{K}_i=-c^if_i\mathbf{G}+e^U\mathbf{G}=f\mathbf{G}$ with $f=e^U-c^if_i$. ■

A third characterization of the separability for (6.12) involves $n-1$ CKT's.

Theorem 7.3: *The HJE* (6.12) *for the null geodesics is separable in orthogonal coordinates if and only if on Q there exist* $n-1$ *CKT's* $(\mathbf{K}_\alpha)=(\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{n-1})$ *with* (i) *common normal eigenvectors (i.e., all simultaneously diagonalizable in orthogonal coordinates) and such that* (ii) $\mathbf{G},\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{n-1}$ *are pointwise independent*.

Proof: Due to the pointwise independence of the tensors, there exists a linear combination with constant coefficients having distinct eigenvalues, i.e., which is a characteristic CKT and by Theorem 7.1 we have the separation of variables for (6.12) Conversely, if (6.12) is separable, then the conformal metric $\overline{\mathbf{G}}$ is separable and there exists $n-1$ tensors $(\mathbf{K}_\alpha)$ which are (a) KT's with respect to $\overline{\mathbf{G}}$, (b) with common normal eigenvectors, and such that (c) $(\overline{\mathbf{G}},\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{n-1})$ are pointwise independent. Hence, $\mathbf{K}_\alpha$ satisfy (i), (ii), and (iii). ■

This theorem is a slightly modified version of Theorem 2, Sec. II of Ref. 20. In general, a set of tensors $(\mathbf{K}_\alpha)$ satisfying the hypotheses of Theorem 7.3 may not be in involution. However,

Proposition 7.4: The tensors $(\mathbf{K}_\alpha)$ *in Theorem 7.3 are equivalent to CKT's in involution*.

Proof: First of all we remark that also the tensors $\mathbf{K}_\alpha+f_\alpha\mathbf{G}$ satisfy the hypotheses of Theorem 7.3, for any choice of the $n-1$ nonzero functions f_α. By using equations (7.5), we see that two CKT's $\mathbf{K}_\alpha$, $\mathbf{K}_\beta$ (diagonalized in orthogonal coordinates) are in involution if and only if for all indices i,

$$\frac{C_{\alpha i}}{\rho^i_\alpha}=\frac{C_{\beta i}}{\rho^i_\beta}\quad(\alpha,\beta\in\{1,\ldots,n-1\}),\tag{7.6}$$

where ρ^i_α are the eigenvalues of $\mathbf{K}_\alpha$ and $C_{\alpha i}$ are the covariant components of the vector fields $\mathbf{C}_\alpha$ satisfying $[\mathbf{G},\mathbf{K}_\alpha]=2\mathbf{C}_\alpha\odot\mathbf{G}$. Condition (7.6) is not preserved by replacing the tensors by equivalent ones. Moreover, by Proposition 7.2 (ii), $\mathbf{K}_\alpha$ are equivalent to CKT's $\mathbf{K}_\alpha'$ of self-gradient type with the same function U. By Proposition 7.1, $\mathbf{K}'_\alpha$ are KT's of the conformal metric $e^U\mathbf{G}$, having common normal eigenvectors. Hence, they are in involution. ■

There is an alternative formulation of Theorem 7.3, still involving $n-1$ CKT's, due to Kalnins and Miller (Ref. 20, Theorem 4, Sec. II):

Theorem 7.4: *The HJE* (6.12) *for the null geodesics is separable in orthogonal coordinates if and only if on Q there exist* $n-1$ *CKT's* $(\mathbf{K}_\alpha)=(\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{n-1})$ (i) *with common eigenvectors,* (ii) *in involution and such that* (iii) $\mathbf{G},\mathbf{K}_1,\mathbf{K}_2,\ldots,\mathbf{K}_{n-1}$ *are pointwise independent.*

We give here a proof which is based on the following general characterization of the integrability of frames, which is an extension of that given in Ref. 7, Theorem 8, Sec. VIII.

Theorem 7.5: *Let* $(\mathbf{X}_i)$ *a frame on Q. Let* $(\mathbf{K}_a)$ *be n contravariant symmetric two-tensors* (i) *pointwise independent,* (ii) *simultaneously diagonalized in the frame* $(\mathbf{X}_i)$ *and such that* (iii) *for each* $a\neq b$ *there exists a vector field* $\mathbf{C}_{ab}$ *and a symmetric two tensor* $\mathbf{M}_{ab}$ *diagonalized in the frame* $(\mathbf{X}_i)$ *such that*

$$\{P(\mathbf{K}_a),P(\mathbf{K}_b)\} = 2P(\mathbf{C}_{ab})P(\mathbf{M}_{ab}). \tag{7.7}$$

Then, the frame is integrable and all the two-tensors are simultaneously diagonalized in a same coordinate system.

We recall (cf. Ref. 7) that (I) a frame is called *integrable* if for each index i the distribution Δ_i spanned by the vectors $(\mathbf{X}_1,\ldots,\mathbf{X}_{i-1},\mathbf{X}_{i+1},\ldots,\mathbf{X}_n)$ is completely integrable; (II) a frame is integrable if and only if the distributions Δ_{ij} spanned by pairs of vectors $(\mathbf{X}_i,\mathbf{X}_j)$ are completely integrable; (III) a frame is integrable if and only if there exist local coordinates (q^i) such that $\mathbf{X}_i=f_i\,\partial/\partial q^i$, where f_i are nowhere vanishing functions.

Proof: Let us set $[\mathbf{X}_i,\mathbf{X}_j]=\Omega^h_{ij}\mathbf{X}_h$, $\Omega^h_{ij}=-\Omega^h_{ji}$, $P(\mathbf{X}_i)=x_i$, so that $\{x_i,f\}=\langle\mathbf{X}_i,\mathrm{d}f\rangle$, $\{x_i,x_j\}=P([\mathbf{X}_i,\mathbf{X}_j])=\Omega^h_{ij}x_h$. Assumptions (ii) and (iii) mean that $\mathbf{K}_a=K^i_a\mathbf{X}_i\odot\mathbf{X}_i$, $\mathbf{M}=M^i\mathbf{X}_i\odot\mathbf{X}_i$. By recalling the calculation of Ref. 7, Sec. VIII, we have

$$\{P(\mathbf{K}_a),P(\mathbf{K}_b)\} = 2\big(2K^i_aK^h_b\Omega^j_{ih} + (K^i_a\langle\mathbf{X}_i,\mathrm{d}K^k_b\rangle - K^i_b\langle\mathbf{X}_i,\mathrm{d}K^k_a\rangle)\,\delta^h_k\delta^j_k\big)x_ix_hx_j.$$

Being $2P(\mathbf{C}_{ab})P(\mathbf{M}_{ab})=2C^k_{ab}x_kM^l_{ab}x_l^2=2C^k_{ab}M^l_{ab}x_kx_l^2$, from Eq. (7.7) it follows that

$$\big(2K^i_aK^h_b\Omega^j_{ih} + (K^i_a\langle\mathbf{X}_i,\mathrm{d}K^k_b\rangle - K^i_b\langle\mathbf{X}_i,\mathrm{d}K^k_a\rangle - C^i_{ab}M^k_{ab})\,\delta^h_k\delta^j_k\big)x_ix_hx_j = 0.$$

This is a homogeneous polynomial equation which must be identically satisfied for all values of the variables (p_k), i.e., for all values of the variables (x_i), since $x_i=P(\mathbf{X}_i)=X^k_ip_k$, and $\det[X^k_i]\neq 0$. Thus, all coefficients vanish. In particular, the coefficient of $x_1x_2x_3$ (as well as for all possible choices of three distinct indices) gives rise to equation

$$K^1_aK^2_b\Omega^3_{12} + K^1_aK^3_b\Omega^2_{13} + K^2_aK^3_b\Omega^1_{23} + K^2_aK^1_b\Omega^3_{21} + K^3_aK^1_b\Omega^2_{31} + K^3_aK^2_b\Omega^1_{32} = 0.$$

From now on the proof is the same of Theorem 8.8 of Ref. 7. ■

Proof of Theorem 7.4: The tensors $(\mathbf{K}_a)=(\mathbf{G},\mathbf{K}_\alpha)$ fulfill the assumptions of Theorem 7.5. In particular, Eqs. (7.7) become

$$\{P(\mathbf{K}_\alpha),P(\mathbf{K}_\beta)\} = 0, \quad \{P(\mathbf{K}_\alpha),P(\mathbf{G})\} = 2P(\mathbf{C}_\alpha)P(\mathbf{G}).$$

Hence, the common eigenvectors are normal. ■

A final important remark is that Theorems 7.2 and 7.4 can be derived from more general statements.

Definition 7.2: We say that two symmetric two-tensors $\mathbf{K}_1$ and $\mathbf{K}_2$ on a Riemannian manifold are in conformal involution if there exists a vector field $\mathbf{C}_{12}$ such that

$$[\mathbf{K}_1,\mathbf{K}_2] = 2\mathbf{C}_{12}\odot\mathbf{G} \quad (\Leftrightarrow \{P_{\mathbf{K}_1},P_{\mathbf{K}_2}\} = 2P_{\mathbf{C}_{12}}P_{\mathbf{G}}). \tag{7.8}$$

Theorem 7.6: *The HJE* (6.12) *for the null geodesics is separable in orthogonal coordinates if and only if on Q there exist n CKT's* $(\mathbf{K}_i)=(\mathbf{K}_0,\mathbf{K}_1,\ldots,\mathbf{K}_{n-1})$ (i) *pointwise independent,* (ii) *with common eigenvectors and* (iii) *in conformal involution.*

Proof: The common eigenvectors are normal, since equation (7.8) is a particular case of (7.7). Item (i) implies the existence of a linear combination $\mathbf{K}=c^i\mathbf{K}_i$ with simple eigenvalues. Then we apply Theorem 7.1. Conversely, due to Theorem 7.2, the separation implies the existence of independent CKT's $\mathbf{K}_i$ satisfying (7.8) with $\mathbf{C}_{ij}=0$. ■

Remark 7.5: Theorem 7.6 is in perfect analogy with the intrinsic characterization of the ordinary orthogonal separation in terms of n independent KT's in involution: it is enough to cancel the word "conformal." This shows that the notion of conformal involution is a natural and useful extension of the ordinary involution.

Proposition 7.5: All CKT's diagonalized in orthogonal coordinates (q^i) *are in conformal involution.*

Proof: According to Proposition 7.2, two tensors $\mathbf{K}_1$, $\mathbf{K}_2$ diagonalized in (q^i) are equivalent to two KT's $\mathbf{K}_1'=\mathbf{K}_1-\rho_1^n\mathbf{G}$, $\mathbf{K}_2'=\mathbf{K}_2-\rho_2^n\mathbf{G}$ of the conformal metric $\overline{\mathbf{G}}=\mathbf{G}/g^{nn}$. Two simultaneously diagonalized KT's are in involution (Ref. 4, Sec. II). Hence,

$$[\mathbf{K}_1,\mathbf{K}_2]=[\mathbf{K}_1'+\rho_1^n\mathbf{G},\mathbf{K}_2'+\rho_2^n\mathbf{G}]=2(\mathbf{K}_1\nabla\rho_2^n-\mathbf{K}_2\nabla\rho_1^n)\odot\mathbf{G}.$$

■

Remark 7.6: As a consequence of this proposition, for two CKT's simultaneously diagonalized in orthogonal coordinates equation (7.7), $[\mathbf{K}_1,\mathbf{K}_2]=2\mathbf{C}_{12}\odot\mathbf{M}_{12}$, implies $\mathbf{M}_{12}=\mathbf{G}$, thus the conformal involution (7.8). In other words, in Theorem 7.6 by replacing the conformal involution conditions (iii), $[\mathbf{K}_i,\mathbf{K}_j]=2\mathbf{C}_{ij}\odot\mathbf{G}$ with $[\mathbf{K}_i,\mathbf{K}_j]=2\mathbf{C}_{ij}\odot\mathbf{M}_{ij}$ we do not get an extension of the theorem.

Remark 7.7: The CKT's $\mathbf{K}_i$ of Theorem 7.6 generate an n-dimensional space $\mathcal{K}$ of CKT's in conformal involution which are simultaneously diagonalized in orthogonal coordinates. We call such a space a *conformal Killing–Stäckel space* (CKS space). The existence of such a space is necessary and sufficient for the orthogonal separation of the null geodesic HJE. However, since properties (i), (ii), and (iii) in this theorem are invariant with respect to the equivalence transformations $\mathbf{K}_i\to\mathbf{K}_i'=\mathbf{K}_i+f_i\mathbf{G}$, there are infinitely many CKS-spaces $\mathcal{K}'$ associated with $\mathcal{K}$, corresponding to any choice of the functions f_i, having the same properties and diagonalized in the same coordinates. We remark that (I) each CKS space contain a tensor of the kind $f\mathbf{G}$ (i.e., a symmetric tensor with n coinciding eigenvalues). (II) There exists a CKS space which contains the metric tensor $\mathbf{G}$. Property (I) follows from Proposition 7.3. To prove (II), starting from the given $\mathbf{K}_i$, according to Proposition 7.3, we can find a linear combination such that $c^i\mathbf{K}_i=f\mathbf{G}$. Thus, if $c^0\neq 0$, we replace $\mathbf{K}_0$ by the equivalent tensor $\mathbf{K}_0'=\mathbf{K}_0+[(1-f)/c^0]\mathbf{G}$. Then the CKS space generated by $(\mathbf{K}_0',\mathbf{K}_1,\ldots,\mathbf{K}_{n-1})$ contains the metric $\mathbf{G}=c^0\mathbf{K}_0'+c^1\mathbf{K}_1+\cdots+c^{n-1}\mathbf{K}_{n-1}$. A consequence of these remarks is that we can reformulate Theorem 7.6 assuming that the metric tensor $\mathbf{G}$ is one of the $\mathbf{K}_i$. This shows that Theorem 7.4 follows from Theorem 7.6.

B. The orthogonal separation for $E-V\neq 0$

Theorem 7.7: *The HJE* (6.13) *for a fixed value E of the energy and for* $E-V\neq 0$ *is separable in orthogonal coordinates if and only if on Q there exists a characteristic CKT* $\mathbf{K}$ *such that*

$$[\mathbf{G},\mathbf{K}]=\frac{2}{E-V}\mathbf{K}\nabla V\odot\mathbf{G} \tag{7.9}$$

or, equivalently, if and only if there exist a function f and a characteristic CKT $\mathbf{K}'$ *such that*

$$[\mathbf{G},\mathbf{K}']=\frac{2}{E-V}(\mathbf{K}'\nabla V+\nabla f)\odot\mathbf{G}. \tag{7.10}$$

Proof: The proof of the first part of this statement follows the same pattern of that of Theorem 7.1, with $\sigma=E-V$. Moreover, if we find a characteristic CKT $\mathbf{K}'$ satisfying (7.10), then the equivalent tensor $\mathbf{K}=\mathbf{K}'-[f/(E-V)]\mathbf{G}$ satisfies condition (7.9). ■

Theorem 7.8: *The* HJE *(6.13) for a fixed value E of the energy and for* $E-V\neq 0$ *is separable* *in orthogonal coordinates if and only if on Q there exist n CKT's* $(\mathbf{K}_i)=(\mathbf{K}_0,\mathbf{K}_1,\ldots,\mathbf{K}_{n-1})$ *(i) pointwise independent, (ii) with common eigenvectors, (iii) in conformal involution and such that*

$$[\mathbf{G},\mathbf{K}_i] = \frac{2}{E-V}(\mathbf{K}_i \nabla V + \nabla f_i)\odot\mathbf{G} \tag{7.11}$$

with suitable functions f_i.

Proof: Due to Theorem 7.5 and item (iii) the common eigenvectors are normal. Items (i) and (ii) imply the existence of a CKT with simple eigenvalues satisfying (7.10). Then we apply Theorem 7.7. Conversely, if (6.13) is separable, then the Jacobi metric $\overline{\mathbf{G}}=\mathbf{G}/(E-V)$ is separable. This means that there exists n KT's $\mathbf{K}_i$ for $\overline{\mathbf{G}}$, pointwise independent, with common eigenvectors, in involution, hence in conformal involution. Recalling Proposition 7.1, we have

$$[\mathbf{G},\mathbf{K}_i] = \tfrac{2}{E-V}\mathbf{K}_i \nabla V\odot\mathbf{G}.$$

This is a particular case of (7.11). ■

Remark 7.8: This theorem shows that, in other words, the orthogonal separation of the Jacobi metric is equivalent to the existence of a CKS space satisfying the additional condition (7.11). We observe that we can always modify the basis $(\mathbf{K}_i)$ in order to include the metric tensor $\mathbf{G}$. Due to Proposition 7.2, there exist a function f and n real numbers c^i not all equal to zero, such that $f\mathbf{G}=\Sigma_i c^i\mathbf{K}_i$. Up to a reordering of the tensors, we can suppose that $c^0\neq 0$. Then, $(\mathbf{G},\mathbf{K}_\alpha) =(\mathbf{G},\mathbf{K}_1,\ldots,\mathbf{K}_{n-1})$ satisfy items (ii), (iii) and are pointwise independent,

$$\det\begin{bmatrix} g^{ii} \\ K^{ii}_\alpha\end{bmatrix} = \frac{1}{f}\det\begin{bmatrix} c^jK^{ii}_j \\ K^{ii}_\alpha\end{bmatrix} = \sum_{\beta=1}^{n-1}\frac{c^\beta}{f}\det\begin{bmatrix} K^{ii}_\beta \\ K^{ii}_\alpha\end{bmatrix} + \frac{c^0}{f}\det\begin{bmatrix} K^{ii}_0 \\ K^{ii}_\alpha\end{bmatrix} = \frac{c^0}{f}\det[K^{ii}_j] \neq 0.$$

VIII. SEPARATED EQUATIONS

Summarizing the results of Sec. VII A, we have five intrinsic characterizations of the orthogonal separation of the null geodesic HJE: Theorem 7.1 (involving a single characteristic CKT), Theorem 7.2 (involving n CKT's in involution), Theorem 7.3 (involving $n-1$ simultaneously diagonalized CKT's), Theorem 7.4 (involving $n-1$ CKT's in involution), and Theorem 7.6 (involving n CKT's in conformal involution). We show how, for each one of these characterizations, we can reduce the HJE to separated ordinary differential equations. This reduction involves the use of Stäckel matrices. As shown in Ref. 2, we can state the following.

Lemma 8.1: Let $(F_i)=(F_1,\ldots,F_n)$ *be n independent functions of the form* $F_i=\varphi^j_{(i)}p_j^2$. *They are in involution if and only if the matrix* $[\varphi^j_{(i)}]$ *is the inverse of a Stäckel matrix* $\mathbf{S}=[\varphi^{(j)}_i]$.

Proof: We prove this statement in a direct way, without any reference with the known links between Stäckel matrices and the orthogonal separation. The condition

$$\{F_i,F_h\} = 2(\varphi^k_{(i)}\partial_k\varphi^j_{(h)} - \varphi^k_{(h)}\partial_k\varphi^j_{(i)})p_kp_j^2 = 0$$

is equivalent to equations

$$\varphi^k_{(i)}\partial_k\varphi^j_{(h)} = \varphi^k_{(h)}\partial_k\varphi^j_{(i)}, \quad k \text{ n.s.} \tag{8.1}$$

(i) Multiplying by $\varphi^{(i)}_l$ and summing over i, we get the equivalent system $\delta^k_l\partial_k\varphi^{(h)}_j =-\varphi^k_{(h)}\Sigma_i\varphi^j_{(i)}\partial_k\varphi^{(i)}_l$. For $k\neq l$, $\varphi^k_{(h)}\Sigma_i\varphi^j_{(i)}\partial_k\varphi^{(i)}_l=0$. For any fixed index k there always exists an index h such that $\varphi^k_{(h)}\neq 0$. It follows that $\Sigma_i\varphi^j_{(i)}\partial_k\varphi^{(i)}_l=0$. And this is equivalent to $\partial_k\varphi^{(i)}_l=0$, for $k\neq l$. (ii) Conversely, let $[\varphi^{(j)}_i]$ be a Stäckel matrix. By applying ∂_k to equation $\varphi^i_{(h)}\varphi^{(l)}_i=\delta^l_h$, we get $\Sigma_i(\varphi^{(l)}_i\partial_k\varphi^i_{(h)})+\varphi^k_{(h)}\partial_k\varphi^{(l)}_k=0$. Let us multiply by $\varphi^j_{(l)}$ and sum over the index l; we get $\partial_k\varphi^j_{(h)} -\varphi^k_{(h)}\Sigma_l\varphi^{(l)}_k\partial_k\varphi^j_{(l)}=0$. If we multiply by $\varphi^k_{(i)}$ without summing over k, then we find $\varphi^k_{(i)}\partial_k\varphi^j_{(h)} =\varphi^k_{(i)}\varphi^k_{(h)}\Sigma_l\varphi^{(l)}_k\partial_k\varphi^j_{(l)}$. This shows that $\varphi^k_{(i)}\partial_k\varphi^j_{(h)}$ is symmetric with respect to the indices (i,h). Thus, (8.1) is proved. ■

Case of Theorem 7.2: Let $(\mathbf{K}_i)$ be n CKT's satisfying the conditions of this theorem. The functions $F_i = K_i^{jj} p_j^2$ fulfill Lemma 8.1. Hence, $\varphi^j_{(i)} \doteq K_i^{jj}$ form the inverse $\mathbf{S}^{-1}$ of a Stäckel matrix. Moreover, since they are CKT's, see (7.3), we have $\{P_{\mathbf{G}}, F_i\} = 2P_{\mathbf{C}_i} P_{\mathbf{G}}$, where $P_{\mathbf{G}} = g^{ii} p_i^2$. This shows that F_i *are null first integrals* of the null geodesics [cf. (4.27)]. Since F_i are independent and in involution, equations

$$F_i(\underline{q}, \underline{p}) \doteq K_i^{jj} p_j^2 = c_i \tag{8.2}$$

describe a Lagrangian foliation on an open subset of T^*Q which is compatible with the submanifold of equation $g^{ii} p_i^2 = 0$ (see Sec. II). This foliation is the geometrical counterpart of an extended complete solution of the null HJE. This complete solution is separable. Indeed, by solving equations (8.2), $\varphi^j_{(i)} p_j^2 = c_i$, we get the separated equations

$$p_j^2 = \varphi_j^{(i)} c_i. \tag{8.3}$$

Case of Theorem 7.3: Let $(\mathbf{K}_\alpha) = (\mathbf{K}_1, \ldots, \mathbf{K}_{n-1})$ be $n-1$ CKT's satisfying the conditions of this theorem. They are diagonalized in orthogonal coordinates. Let us consider the diagonalized tensors $\mathbf{K}'_\alpha = \mathbf{K}_\alpha - \rho_\alpha^n \mathbf{G}$, where ρ_α^n are the last eigenvalues of $\mathbf{K}_\alpha$. Since $\rho_\alpha^n = K_\alpha^{nn} / g^{nn}$, the diagonal components of $\mathbf{K}'_\alpha$ are

$$K_\alpha^{jj} - K_\alpha^{nn} \frac{g^{jj}}{g^{nn}}. \tag{8.4}$$

By recalling the proof of Proposition 7.2, all $\mathbf{K}'_\alpha$ are KT's of the conformal metric $\mathbf{G}/g^{nn}$, simultaneously diagonalized, hence in involution. As a consequence, the $n-1$ functions

$$F_\alpha \doteq \left(K_\alpha^{jj} - K_\alpha^{nn} \frac{g^{jj}}{g^{nn}} \right) p_j^2$$

are null first integrals in involution of the null geodesics. Moreover, the function

$$F_n \doteq \frac{g^{jj}}{g^{nn}} p_j^2$$

is a further null first integral in involution. These n first integrals in involution are independent because of item (ii) of Theorem 7.3. Thus, due to Lemma 8.1, the functions

$$\varphi^j_{(\alpha)} \doteq K_\alpha^{jj} - K_\alpha^{nn} \frac{g^{jj}}{g^{nn}}, \quad \varphi^j_{(n)} \doteq \frac{g^{jj}}{g^{nn}},$$

form the inverse of a Stäckel matrix. It follows that equations

$$F_\alpha = \varphi^j_{(\alpha)} p_j^2 = c_\alpha, \quad F_n = \varphi^j_{(n)} p_j^2 = 0, \tag{8.5}$$

define a Lagrangian foliation of the submanifold $g^{ii} p_i^2 = 0$, which is the geometrical counterpart of an internal complete solution of the null HJE. This complete solution is separable. Indeed, by solving equations (8.5), $\varphi^j_{(i)} p_j^2 = c_i$, with $c_n = 0$, we get separated equations of the kind (8.3), but with $n-1$ constant parameters (c_α),

$$p_j^2 = \varphi_j^{(\alpha)} c_\alpha. \tag{8.6}$$

This result is in agreement with Remark 6.8.

Case of Theorem 7.4: The procedure is the same as for the case of Theorem 7.3.

Case of Theorem 7.6: Let $(\mathbf{K}_i)$ be n CKT's satisfying the conditions of this theorem. By recalling Remark 7.7, we can always find a linear (constant coefficients) combination such that $a^i \mathbf{K}_i = f\mathbf{G}$. When the constants a^i and the function f are determined, assuming (up to a reordering) that $a^0 \neq 0$, we can replace $\mathbf{K}_0$ by $\mathbf{G}$, and we are in the case of Theorem 7.3.

Case of Theorem 7.1: We point out that Theorem 7.1 is convenient for characterizing the separation, since it involves only a single CKT. However, in order to get separated equations (involving a Stäckel matrix) we need to know $n-1$ CKT's. Let ρ^i be the eigenvalues of the given characteristic CKT $\mathbf{K}$. According to Proposition 7.2, the tensor $\widetilde{\mathbf{K}}=\mathbf{K}-\rho^n\mathbf{G}$ is a characteristic KT for the conformal metric $\overline{\mathbf{G}}=\mathbf{G}/g^{nn}$ (instead of the last one n, we can choose any other index). As it is well known, any characteristic KT generates a n-space of KT's simultaneously diagonalized in orthogonal coordinates, whose eigenvalues $\overline{\rho}^i$ with respect to the metric $\overline{\mathbf{G}}$ satisfy the Killing–Eisenhart equations

$$\partial_i\overline{\rho}^j = (\overline{\rho}^i - \overline{\rho}^j)\partial_i \ln \overline{g}^{jj} \tag{8.7}$$

which form a complete integrable system. Since $\overline{\rho}^j=(\rho^j-\rho^n)g^{nn}$, we observe that for the given tensor $\widetilde{\mathbf{K}}$ we have $\overline{\rho}^n=0$. We observe that, following Kalnins and Miller,[20] if we set $\mu^i=\rho^i-\rho^n$, then from (8.7) we obtain equations

$$\partial_i\mu^j = (\mu^i - \mu^j)\partial_i \ln g^{jj} - \mu^i\partial_i \ln g^{nn},$$

which summarize Eqs. (2.8) of Ref. 20. Let us take n independent solutions $\overline{\rho}_i^j$ of system (8.7) with $\overline{\rho}_n^j=1$ for all j. The corresponding tensors $\mathbf{K}_i$ of components $K_i^{jj}=\overline{\rho}_i^j g^{jj}/g^{nn}$ are independent KT's for $\overline{\mathbf{G}}$ such that $\mathbf{K}_n=\overline{\mathbf{G}}$. This means that

$$\varphi_{(i)}^j = K_i^{jj} = \overline{\rho}_i^j g^{jj}/g^{nn}$$

form the inverse of a Stäckel matrix and equations $\varphi_{(i)}^j p_j^2=c_i$ are equivalent to the separated equations $p_j^2=\varphi_j^{(i)}c_i$. This gives an extended separated solution. For $c_n=0$ we get the null geodesics.

Finally, let us consider the case of $E-V\neq 0$.

Case of Theorem 7.8: Let $(\mathbf{K}_i)=(\mathbf{K}_0,\ldots,\mathbf{K}_{n-1})$ be n CKT's satisfying the conditions of this theorem. By recalling the proof of Theorem 7.7, if we perform the equivalence transformation $\widetilde{\mathbf{K}}_i=\mathbf{K}_i-[f_i/(E-V)]\mathbf{G}$ we get n KT's of the Jacobi metric $\overline{\mathbf{G}}=\mathbf{G}/(E-V)$, characterizing its orthogonal separation. Then the functions $\varphi_{(i)}^j=K_i^{jj}-f_i/(E-V)$ form the inverse of a Stäckel matrix. By solving equations $\varphi_{(i)}^j p_j^2=c_i$, we get the separated equations

$$p_j^2 = \varphi_j^{(i)}c_i \tag{8.8}$$

thus, a complete separated solution of the HJE,

$$(E-V)^{-1}g^{ii}p_i^2 = 2h. \tag{8.9}$$

The separated solution following from (8.8) is an extended separated solution of the HJE $\frac{1}{2}g^{ii}p_i^2 =E-V$ with the fixed value E of the energy. By substituting in (8.9) the expressions of p_j given by (8.8), we get an equation of the kind $h=h(c_i)$. It follows that for $h=1$ we get equation $h(c_i)=1$. When the constants c_i satisfy this equation we get an internal separated solution of the HJE for the given value E of the energy.

Case of Theorem 7.7: If we have a characteristic CKT tensor $\mathbf{K}'$ satisfying (7.10), then $\mathbf{K} =\mathbf{K}'-[f/(E-V)]\mathbf{G}$ is characteristic KT of the Jacobi metric $\overline{\mathbf{G}}=\mathbf{G}/(E-V)$. System (8.7) with $\overline{g}^{jj}=g^{jj}/(E-V)$ is completely integrable and provides n independent solutions $\overline{\rho}_i^j$ with $\overline{\rho}_n^j=1$ for all j. With such a solution we define the inverse of a Stäckel matrix by setting $\varphi_{(i)}^j=\overline{\rho}_i^j g^{jj}/(E-V)$. Then, by solving equations $\varphi_{(i)}^j p_j^2=c_i$ we get separated equations which define an extended separated solution. By setting $c_n=1$ we get an internal separated solution. We remark that in both cases the Stäckel matrices depend on the value E.

IX. THE TWO-DIMENSIONAL CASE

A two-dimensional Riemannian manifold is always conformally flat. The link between the conformal separation in two dimensions, the analytical functions and the CKT's is examined in Ref. 28, and used for generalizing a result of Ref. 25. We show here how some known results follow from the general theory developed in the preceding sections.

We can write the most general 2×2 Stäckel matrix in two variables in the form

$$\mathbf{S}=\begin{bmatrix}\phi_1(q^1) & \psi_1(q^1)\\ \phi_2(q^2) & \psi_2(q^2)\end{bmatrix},\quad \phi_1\psi_2-\phi_2\psi_1\neq 0. \tag{9.1}$$

The inverse matrix is

$$\mathbf{S}^{-1}=\frac{1}{\phi_1\psi_2-\phi_2\psi_1}\begin{bmatrix}\psi_2 & -\psi_1\\ -\phi_2 & \phi_1\end{bmatrix}. \tag{9.2}$$

The components $\bar{g}^{ii}$ of a separable orthogonal metric $\overline{\mathbf{G}}$ and of the associated KT $\mathbf{K}$ are given by the second and the first line of $\mathbf{S}^{-1}$,

$$[\bar{g}^{11},\bar{g}^{22}]=\frac{1}{\phi_1\psi_2-\psi_1\phi_2}[-\phi_2,\phi_1],\quad [K^{11},K^{22}]=\frac{1}{\phi_1\psi_2-\psi_1\phi_2}[\psi_2,-\psi_1].$$

Then Theorem 6.10 implies the following.

Theorem 9.1: (i) *The HJE of the null geodesics* (6.12) *is separable in the orthogonal coordinates* (q^1,q^2) *if and only if there exist two nowhere vanishing functions* $\xi_1(q^1)$ *and* $\xi_2(q^2)$ *such that*

$$\frac{g^{11}}{g^{22}}=\frac{\xi_1}{\xi_2}. \tag{9.3}$$

(ii) *The HJE* (6.13), *for a fixed value of* E *and for* $E-V\neq 0$, *is separable in the orthogonal coordinates* (q^1,q^2) *if and only if there exist four functions* $(\xi_1(q^1),\xi_2(q^2),\psi_1(q^1),\psi_2(q^2))$, *with*

$$\frac{\psi_2}{\xi_1}+\frac{\psi_1}{\xi_2}\neq 0, \tag{9.4}$$

such that (9.3) *holds and moreover,*

$$E-V=\psi_1 g^{11}+\psi_2 g^{22}. \tag{9.5}$$

Proof: From (6.17) and (9.2) it follows that there exist functions $\phi_1(q^1)$ and $\phi_2(q^2)$ such that $g^{11}/g^{22}=-\phi_2/\phi_1$. The functions ξ_i of the statement are then given by $\xi_1=1/\phi_1$ and $\xi_2=-1/\phi_2$. Formula (9.5) follows from (6.18) and (9.1). Condition (9.4) is the regularity condition of the Stäckel matrix (9.1). ∎

From Theorem 9.1 it follows that on a two-dimensional Riemannian manifold, orthogonal coordinates (q^1,q^2) are conformal separable coordinates if and only if the ratio g^{11}/g^{22} has the form (9.3), which is equivalent to say that g^{11}/g^{22} is a product of two functions depending only on q^1 and q^2, respectively. In Remark 6.3 we have seen that coordinates satisfying $g^{ii}=c^i\sigma(\underline{q})$ $(c^i\in\mathbb{R})$ are conformal separable. The following theorem shows that in fact any two-dimensional conformal separable system is of this kind.

Theorem 9.2: *On a two-dimensional manifold an orthogonal coordinate system is conformal separable if and only if, up to a rescaling,*

$$g^{11}=\pm g^{22}. \tag{9.6}$$

Proof: According to Remark 6.3, if (9.6) holds, then the coordinates are conformal separable. Conversely, assume that (9.3) holds. Then $g_{11}=\rho/\xi_1$, $g_{22}=\rho/\xi_2$, and

$$ds^2 = g_{11}(dq^1)^2 + g_{22}(dq^2)^2 = \rho\left(\frac{(dq^1)^2}{\xi_1} + \frac{(dq^2)^2}{\xi_2}\right).$$

In the coordinates $(\tilde{q}^i)$ defined by the rescaling $d\tilde{q}^i = |\xi_i|^{-1/2}\, dq^i$ we have $ds^2 = \rho(e_1(d\tilde{q}^1)^2 + e_2(d\tilde{q}^2)^2)$, where $e_i = \text{sign}(\xi_i)$. It follows that $\tilde{g}_{ii} = \rho e_i$. ∎

Remark 9.1: If (9.6) holds, from (9.3) and (9.5) it follows that $\xi_2 = \pm\xi_1 = \text{constant}$ and $E - V = (\psi_1 \pm \psi_2) g^{11}$. Then the Stäckel matrix (9.1) and its inverse (9.2) have the form

$$\mathbf{S} = \begin{bmatrix} c & \psi_1(q^1) \\ \mp c & \psi_2(q^2) \end{bmatrix}, \quad \psi_2 \pm \psi_1 \neq 0, \quad \mathbf{S}^{-1} = \frac{1}{c(\psi_2 \pm \psi_1)} \begin{bmatrix} \psi_2 & -\psi_1 \\ \pm c & c \end{bmatrix}.$$

Let us consider the particular case of the Euclidean plane $\mathbb{E}_2$, with Cartesian coordinates (x,y). We recall the following (see, e.g., Ref. 28).

Proposition 9.1: *If $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, is a non constant analytic function, then in the open domain where $\nabla u \neq 0$ the real and the imaginary parts define conformal separable coordinates $q^1 = u(x,y)$ and $q^2 = v(x,y)$ such that $g^{11} = g^{22}$.*

Proof: From the Cauchy–Riemann conditions

$$u_x = v_y, \quad u_y = -v_x \tag{9.7}$$

(the suffixes denote the partial derivatives) it follows that

$$g^{12} = u_x v_x + u_y v_y = 0, \quad g^{11} = (u_x)^2 + (u_y)^2 = (v_y)^2 + (v_x)^2 = g^{22}. \tag{9.8}$$

Then we apply Remark 9.1. The coordinate transformation is singular at those points where the partial derivatives (9.7) vanish, since

$$\det\begin{bmatrix} u_x & v_x \\ u_y & v_y \end{bmatrix} = u_x v_y - v_x u_y = (u_x)^2 + (u_y)^2 .$$ ∎

By applying Theorem 9.2, we prove the converse of Proposition 9.1.

Proposition 9.2: *Up to a rescaling, every conformal separable system of the Euclidean plane is generated by a nonconstant analytic function*.

*Proof:*According to Theorem 9.2, we can rescale a conformal separable coordinate system in order to have $g^{11} = g^{22}$ and such that the corresponding coordinate transformation satisfies (9.8). The solutions of (9.8) are

$$\begin{aligned} u_x &= v_y, & u_x &= -v_y, \\ u_y &= -v_x, & u_y &= v_x, \end{aligned}$$

which are the Cauchy–Riemann conditions for $f = u(x,y) + iv(x,y)$ or $\tilde{f} = v(x,y) + iu(x,y)$. Hence the coordinates are generated by an analytic function. ∎

Remark 9.2: The real and imaginary part of a given analytic function are both harmonic functions on $\mathbb{R}^2$, i.e., solutions of the Laplace equation in the plane $\Delta u = 0$. Conversely, each harmonic function $u(x,y)$ can be chosen as real part of an analytic function. The corresponding imaginary part $v(x,y)$ is determined up to an additive constant.

Remark 9.3: It is possible to associate with every harmonic function a class of potentials, depending on two real parameters a, b, which are separable for a single value of the energy. The conformal separable coordinates and the suitable value of E depend on (a,b). Let $u(x,y)$ be a harmonic function. Then, the functions $\tilde{u} = u + ax + by$, $a, b \in \mathbb{R}$ are harmonic. According to Remark 9.2, we construct a coordinate transformation

$$q^1 = q^1(x,y) = \tilde{u}, \quad q^2 = q^2(x,y) = \tilde{v},$$

with $\tilde{v}$ such that $\tilde{u} + i\tilde{v}$ is analytic. For these coordinates we have

$$g^{11} = g^{22} = \left(\frac{\partial \tilde{u}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{u}}{\partial y}\right)^2.$$

Hence, (q^1, q^2) are conformal separable coordinates compatible with a natural Hamiltonian $H = G + V$ for a fixed value of the energy E if and only if

$$E - V = (\psi_1(q^1) + \psi_2(q^2))g^{11}. \tag{9.9}$$

In particular, the case $E - V = g^{11}$ is satisfied by choosing $E = a^2 + b^2$ and

$$V(x,y) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 2a\frac{\partial u}{\partial x} + 2b\frac{\partial u}{\partial y}. \tag{9.10}$$

In the following example we consider a dynamical system with a scalar potential depending on a single parameter a, which is a special case of the potential (9.10) obtained by considering $b = 0$ in the preceding discussion.

Example 9.1: In $\mathbb{E}_2$ let us consider the potential

$$V(x,y) = -\frac{2ax + 1}{x^2 + y^2},$$

where (x, y) are Cartesian coordinates and a is a real parameter. By examining the separability condition

$$\mathrm{d}(\mathbf{K}\,\mathrm{d}V) = 0, \tag{9.11}$$

where $\mathbf{K}$ is a generic KT of the Euclidean plane (see Ref. 6 for the details of this technique), we find that, for $a \neq 0$, (9.11) is satisfied only for $\mathbf{K} = \mathbf{G}$ (the metric tensor). Thus, for $a \neq 0$, V is not separable in $\mathbb{E}_2$. However, for any value of $a \neq 0$ there is a suitable value of the energy E, such that the HJE $G + V - E = 0$ is separable in a conformal coordinate system depending on a. Let us consider

$$q^1 = \log\sqrt{x^2 + y^2} + ax = \log \varrho + a\varrho \cos \vartheta, \quad q^2 = \arctan\left(\frac{y}{x}\right) + k\pi + ay = \vartheta + a\varrho \sin \theta.$$

With respect to these coordinates we have

$$g^{11}(q^1, q^2) = g^{22}(q^1, q^2) = \frac{x^2}{(x^2 + y^2)^2} + a^2 + \frac{2ax}{x^2 + y^2} + \frac{y^2}{(x^2 + y^2)^2} = \frac{2ax + 1}{x^2 + y^2} + a^2.$$

Thus, (q^1, q^2) are conformally separable. Moreover, since for $E = a^2$ we get $E - V = g^{11}$, which is of the form (9.9), we have the separation of variables for the fixed value of the energy $E = a^2$. Now we solve the HJE and the corresponding dynamical system. We construct the Stäckel matrix $\mathbf{S}$ associated with (q^1, q^2). By applying to this special case (9.3), (9.4), and (9.9), we have

$$\xi_1 = 1 = \xi_2, \quad \psi_1 = 1, \quad \psi_2 = 0,$$

so that the Stäckel matrix and its inverse are

$$\mathbf{S} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{S}^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

A basis of the conformal Killing–Stäckel space is

$$\mathbf{G} = \frac{\partial}{\partial q^1} \otimes \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} \otimes \frac{\partial}{\partial q^2}, \quad \mathbf{K} = \frac{\partial}{\partial q^2} \otimes \frac{\partial}{\partial q^2}.$$

With respect to the new coordinates, the natural Hamiltonian

$$\mathcal{H} = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 - \frac{2ax+1}{x^2+y^2} - a^2$$

becomes the geodesic Hamiltonian $J_E = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$. The quadratic first integral is $H_K = \frac{1}{2}p_2^2$. The separated equations are given by the system

$$\begin{matrix} J_E = h, \\ H_K = c, \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 = h, \\ \frac{1}{2}p_2^2 = c, \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} p_1^2 = 2(h-c), \\ p_2^2 = 2c. \end{matrix}$$

The constants have to fulfill the conditions $0 \leqslant c \leqslant h$. The integral curves of the Hamilton equations are

$$p_1 = \pm\sqrt{2(h-c)}, \quad p_2 = \pm\sqrt{2c}, \quad q^1 = \pm\sqrt{2(h-c)}t + q_0^1, \quad q^2 = \pm\sqrt{2c}t + q_0^2,$$

and their orbits are described by the equation

$$q^1 - q_0^1 = \pm\sqrt{\frac{h-c}{c}}(q^2 - q_0^2).$$

For $h=1$, we have the orbits of $\mathcal{H}=0$, i.e., of the natural Hamiltonian with potential V and energy $E=a^2$, parametrized by $c \in (0,1)$. In Cartesian coordinates the orbits are given by

$$\log\sqrt{x^2+y^2} + ax = \pm\sqrt{\frac{1-c}{c}}\left[\arctan\left(\frac{y}{x}\right) + k\pi + ay\right] + d$$

and in polar coordinates (ϱ, ϑ) by

$$\log \varrho + a\varrho \cos \vartheta = \pm\sqrt{\frac{1-c}{c}}(\vartheta + a\varrho \sin \theta) + d,$$

where $d = q_0^1 \pm \sqrt{[1-c/c]}q_0^2$ is a constant depending on the initial point.

Remark 9.4: For $n > 2$ it is no longer possible to relate conformal separable coordinates with analytic functions, as for the case $n=2$. However, it can be seen that the orthogonal coordinate systems which allow the R-separation of the Laplace equation in n-dimensional manifolds with constant curvature, obtained by different methods,[8,10,17,19,27] and known in the Euclidean three-space as confocal cyclides[12,13] are all conformal separable coordinates according to our Definition 6.1. This fact exhibits the deep relation between the R-separation and the separation of the HJE with a fixed value of the energy developed in this paper. Indeed, both conformal separable and R-separable coordinates are characterized by CKT's (see, e.g., Ref. 20). A further analysis of this link is in progress.

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