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Hamiltonian Optics and Generating Families

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BIBLIOPOLIS

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Preface

This monograph grew out of a series of lectures given at the XXVI Summer School of Mathematical Physics, Ravello, September 2001, organized by G.N.F.M. (Gruppo Nazionale di Fisica Matematica) of I.N.d.A.M. (Istituto Nazionale di Alta Matematica, Roma), at the Department of Mathematics of the University of Torino in the academic years 2000/2001 and 2001/2002, and at the Department of Physical Sciences of the University of Napoli, May 2003.

The elements of Symplectic Geometry and Analytical Mechanics on which these lectures are based can be found in the literature of the seventies and eighties of the last century. The bibliography is of course far from complete and refers the reader to some of the important contributions. Here, we introduce only the essential notions of symplectic geometry needed for application to the geometrical theory of the Hamilton-Jacobi equation and to the control theory of static systems. Most of these notions are well known, but the way they are assembled and used is new in many respects.

A fundamental role in the present approach is played by the notion of *generating family* and by two operations: the *composition of generating families* of symplectic relations and the *canonical lift* from objects on manifolds (submanifolds, relations, mappings, vector fields, etc.) to symplectic objects on the corresponding cotangent bundles. Generating families describe special subsets of cotangent bundles which we call *Lagrangian sets.* A Lagrangian set is a Lagrangian submanifold (which may be immersed) if the generating family is a *Morse family.* However, there are physically interesting examples of Lagrangian sets which are not Lagrangian submanifolds. An advantage of considering generating families as fundamental objects is that, while the composition of two symplectic relations may not be a smooth relation, the composition of two generating families is always a smooth function. In other words, the *symplectic creed* as formulated by A. Weinstein in his article *Symplectic geometry* (1981) in the form *everything is a Lagrangian submanifold*, which means that one should try to express objects in symplectic geometry and mechanics in terms of Lagrangian submanifolds, is here replaced by *everything has a generating family.*

The geometrical theory of the Hamilton-Jacobi equation is closely related to Geometrical Optics. The symplectic formulation of Hamiltonian Optics presented here differs from other formulations illustrated in papers and well known reference books cited in the Bibliography and it is, in my opinion, very close to the original ideas of Hamilton. From a geometrical view-point a Hamilton-Jacobi equation is a coisotropic submanifold of a cotangent bundle. A *geometrical solution* is a Lagrangian set described by a generating family and contained in the coisotropic submanifold. There are two fundametal symplectic relations associated with a Hamilton-Jacobi equation, the *characteristic relation* and the *characteristic reduction*. The two corresponding generating families are the *Hamilton principal function* and the *complete solution of the Hamilton-Jacobi equation*, respectively. By composing the latter with its *transpose* we get the former. Since the characteristic relation is a singular Lagrangian submanifold, the Hamilton principal function is necessarily a generating family and not a two-point function as in the classical theory. Cauchy data (or *sources* of systems of rays), mirror and lenses are represented by symplectic relations thus, by generating families. Then the Cauchy problem and the actions of a lens or of a mirror on a system of rays are translated into the composition of generating families.

What is presented here is only a first approach to Geometrical Optics based on the notions of symplectic relation and generating family. We do not cover many important examples of optical phenomena, which can be found in standard reference books (e.g. Synge, Luneburg, Buchdahl) and which probably can be treated within this framework.

Perhaps, the use of generating families and symplectic relations does not yield a revolutionary progress in Hamiltonian Optics, but we are *obliged* to introduce these concepts if, for example, we want to give a global meaning to the Hamilton characteristic function, as shown in Chapters 3 and 4.

Symplectic relations and generating families can play an interesting role also in the control theory of static systems, including thermostatic systems. Chapter 5 is devoted to this matter. Our approach is based on the notion of *control relation* and on an extended version of the *virtual work principle* for constrained systems with non-controlled degrees of freedom (*hidden variables*). Several examples of singular phenomena concerning static systems and thermostatics are illustrated. In particular, it is shown how the *Maxwell rule* follows as a theorem from the extended virtual work principle. Thermostatics of simple and composite systems is here described in the four-dimensional state space, with global coordinates (S, V, P, T), entropy, volume, pressure, absolute temperature, endowed with the natural symplectic structure induced by the first principle of thermodynamics.

An outline of the basic tools of calculus on manifolds needed in our discussion is given in Appendix A. A supplementary note (Appendix B) written in collaboration with Franco Cardin (Dipartimento di Matematica Pura e Applicata, Università di Padova), is devoted to the calculus of global principal Hamilton functions for the eikonal equations on the two-dimensional sphere $\mathbb{S}_2$ and pseudo-sphere $\mathbb{H}_2$.

Acknowledgments

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Chapter 1

Symplectic manifolds and symplectic relations

1.1 Symplectic manifolds

A **symplectic manifold** is a pair $(\mathcal{S}, \omega)$ consisting of an even-dimensional differentiable manifold $\mathcal{S}$ endowed with a **symplectic form** ω i.e., a non-degenerate closed 2-form.

In local coordinates $\underline{x} = (x^{\mathrm{A}})$, $\mathrm{A} = 1, \ldots, m$, $m = \dim(\mathcal{S})$, any 2-form on $\mathcal{S}$ admits the representation

$$\omega = \tfrac{1}{2}\, \omega_{\mathrm{AB}}\, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}},$$

with

$$dx^{\mathrm{A}} \wedge dx^{\mathrm{B}} = dx^{\mathrm{A}} \otimes dx^{\mathrm{B}} - dx^{\mathrm{B}} \otimes dx^{\mathrm{A}}, \qquad \omega_{\mathrm{AB}} = \omega(\partial_{\mathrm{A}}, \partial_{\mathrm{B}}).$$

Here, ∂_{A} denotes the partial derivative $\partial/\partial x^{\mathrm{A}}$ interpreted as a vector field. The components $\omega_{AB}(\underline{x})$ form a skew-symmetric $m \times m$ matrix, $[\omega_{AB}]$, $\omega_{\mathrm{AB}} = -\omega_{\mathrm{BA}}$. A 2-form is **non-degenerate** if $\omega(u, v) = 0$ for all vectors v implies $u = 0$. This is equivalent to

$$\det\,[\omega_{\mathrm{AB}}] \neq 0.$$

This shows that the dimension of a symplectic manifold is even. A 2-form is **closed** if $d\omega = 0$. This is equivalent to $\partial_{\mathrm{A}}\omega_{\mathrm{BC}}\, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}} \wedge dx^{\mathrm{C}} = 0$ i.e., to

$$\partial_{\{\mathrm{A}}\omega_{\mathrm{BC}\}} = 0,$$

where $\{\ldots\}$ denotes the sum over the cyclic permutation of the indices. Let us denote by $[\Omega^{\mathrm{AB}}]$ the inverse matrix of $[\omega_{\mathrm{AB}}]$,

$$\Omega^{\mathrm{AB}}\omega_{\mathrm{CB}} = \delta^{\mathrm{A}}_{\mathrm{C}}.$$

The symplectic form generates two basic operations:

(i) A $\mathbb{R}$-linear mapping from smooth (C^∞) real-valued functions on $\mathcal{S}$ to vector fields on $\mathcal{S}$,

$$\mathcal{F}(\mathcal{S}) \to \mathcal{X}(\mathcal{S}) \colon H \mapsto X_H,$$

defined by

$$\omega(X_H, X) = -\langle X, dH\rangle, \quad \forall X \in \mathcal{X}(\mathcal{S}). \tag{1}$$

The vector field X_H is called the **Hamiltonian vector field** generated by the **Hamiltonian** H.

Here we denote by $\langle v, \theta\rangle$ and $\langle X, \theta\rangle$ the evaluation between tangent vectors v, or vector fields X, and 1-forms θ. In local coordinates,

$$\theta = \theta_{\mathrm{A}}\, dx^{\mathrm{A}}, \quad v = v^{\mathrm{A}}\, \partial_{\mathrm{A}}, \quad X = X^{\mathrm{A}}(x)\, \partial_{\mathrm{A}},$$

so that

$$\langle v, \theta\rangle = v^{\mathrm{A}}\, \theta_{\mathrm{A}}(x), \qquad \langle X, \theta\rangle = X^{\mathrm{A}}\, \theta_{\mathrm{A}}.$$

Equation (1) is equivalent to equation

$$i_{X_H}\omega = -dH,$$

where i_X denotes the **interior product** of a differential p-form with respect to a vector field X. The result is a $p-1$-form defined by

$$(i_X\omega)_{B\ldots C} = X^A\, \omega_{AB\ldots C}.$$

Thus, the coordinate expression of equation (1) is

$$X^A \omega_{AB} = -\partial_B H.$$

By applying the inverse matrix $[\Omega^{AB}]$ we get the explicit definition of the components of X_H,

$$X^A = -\Omega^{AB}\, \partial_B H.$$

(ii) A binary internal operation in the space $\mathcal{F}(\mathcal{S})$ of smooth functions on $\mathcal{S}$, called **Poisson bracket** (PB), defined by

$$\{F, G\} = \omega(X_F, X_G). \tag{2}$$

In local coordinates,

$$\{F, G\} = \Omega^{\mathrm{AB}}\, \partial_{\mathrm{A}} F \partial_{\mathrm{B}} G.$$

The following properties hold:

$$
\begin{array}{lll}
 & \{F,G\} = -\{G,F\} & \text{(anticommutativity)} \\
 & \{aF+bG,H\} = a\{F,H\} + b\{G,H\} & \text{(bilinearity: } a,b \in \mathbb{R}) \\
(3) & \{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0 & \text{(cyclic or Jacobi identity)} \\
 & \{F,GH\} = \{F,G\}\,H + \{F,H\}\,G & \text{(Leibniz rule)} \\
 & \{F,G\} = 0,\ \forall F \quad \Longrightarrow \quad dG = 0 & \text{(regularity).}
\end{array}
$$

The first three properties show that the space $\mathcal{F}(\mathcal{S})$ endowed with the PB is a Lie-algebra. It can be seen that the Jacobi identity is equivalent to $d\omega = 0$.

Two functions are said to be **in involution** if $\{F, G\} = 0$. By the last property (3), if a function is in involution with all functions, then it is constant on the connected components of $\mathcal{S}$. A manifold endowed with a bracket on functions satisfying conditions (3), except the last one, is called **Poisson manifold** (for further information and references see for instance [Weinstein, 1998], [Libermann, Marle, 1987], [Vaisman, 1994]).

A remarkable property, relating the Poisson bracket of functions to the Lie bracket of vector fields, is expressed by formula

$$
[X_F, X_G] = X_{\{F,G\}} \tag{4}
$$

which shows that *the Lie bracket* $[X_F, X_G]$ *of two Hamiltonian vector fields is the Hamiltonian vector field generated by the function* $\{F, G\}$. This means that the mapping $H \mapsto X_H$ is a Lie-algebra homomorphism. An equivalent form of (4) is

$$
i_{[X_F,\, X_G]}\omega = -\, d\{F, G\}. \tag{5}
$$

1.2 Symplectic vector spaces

Several notions and properties of symplectic geometry are related to basic notions of *linear symplectic algebra*, since any tangent space $T_p\mathcal{S}$ of a symplectic manifold is a symplectic vector space. A **symplectic vector space** is a real even-dimensional vector space E_{2n} endowed with a real-valued bilinear, non-degenerate and skew-symmetric form $\omega: E \times E \to \mathbb{R}$,

$$
\begin{cases}
\omega(u,v) = -\,\omega(v,u) \\
\omega(u,v) = 0,\ \forall v \in E \quad \Longrightarrow \quad u = 0.
\end{cases}
$$

With a symplectic vector space (E, ω) we associate its **dual symplectic space** (E^*, Ω). The **dual symplectic form** Ω is defined by

$$
\Omega(u^\flat, v^\flat) = \omega(u, v) \tag{1}
$$

where

$$u^\flat = \omega(u, \cdot) \in E^*, \qquad \langle v, u^\flat \rangle = \omega(u, v). \tag{2}$$

The mapping $\flat: E \to E^*: u \mapsto u^\flat$ is a linear isomorphism, since ω is non-degenerate. If $\omega_{\mathrm{AB}} = \omega(e_{\mathrm{A}}, e_{\mathrm{B}}) \in \mathbb{R}$ are the components of ω in any basis (e_{A}) of E, then the mapping $\flat$ corresponds to the operation of lowering the indices, $(v^\flat)_{\mathrm{A}} = v^{\mathrm{B}}\omega_{\mathrm{BA}}$, and the components Ω^{AB} of Ω are the elements of the inverse matrix of $[\omega_{AB}]$, $\Omega^{\mathrm{AB}}\omega_{\mathrm{CB}} = \delta^{\mathrm{A}}_{\mathrm{C}}$.

With each subspace A of a symplectic vector space E we associate its **polar dual subspace**

$$A^\circ = \{\alpha \in E^* \mid \langle u, \alpha \rangle = 0,\ \forall u \in A\} \subseteq E^* \tag{3}$$

and its **symplectic orthogonal subspace**

$$A^\S = \{v \in E \mid \omega(u, v) = 0,\ \forall u \in A\} \subseteq E. \tag{4}$$

The **dual polar operator** $\circ$ and the **symplectic polar operator** $\S$ satisfy the same formal rules of the **orthogonal operator** $\perp$ in Euclidean spaces:

$$\begin{cases} \dim(A) + \dim(A^\circ) = \dim(E) \\ A^\circ \subset B^\circ \iff B \subset A \\ (A + B)^\circ = A^\circ \cap B^\circ \\ A^\circ + B^\circ = (A \cap B)^\circ \\ A^{\circ\circ} = \iota(A), \end{cases} \tag{5}$$

where $\iota: E \to E^{**}$ is the natural isomorphism defined by $\langle \alpha, \iota(v) \rangle = \langle v, \alpha \rangle$, $\alpha \in E^*$, $v \in E$;

$$\begin{cases} \dim(A) + \dim(A^\S) = \dim(E) \\ A^\S \subset B^\S \iff B \subset A \\ (A + B)^\S = A^\S \cap B^\S \\ A^\S + B^\S = (A \cap B)^\S \\ A^{\S\S} = A. \end{cases} \tag{6}$$

The correspondence between (5) and (6) follows from equation

$$\flat(A^\S) = A^\circ. \tag{7}$$

We observe that

$$A^{\S\circ} = A^{\circ\S}. \tag{8}$$

A subspace $A \subseteq E$ is called

$$\begin{cases} \textbf{isotropic} & & A \subseteq A^{\S} \\ \textbf{coisotropic} & \text{if} & A^{\S} \subseteq A \\ \textbf{Lagrangian} & & A^{\S} = A. \end{cases}$$

By using these formulae it can be proved that:

(i) *If A is isotropic (coisotropic, Lagrangian) then $A^{\S}$ is coisotropic (isotropic, Lagrangian).*

(ii) *If A is isotropic (coisotropic, Lagrangian) then* $\dim(A) \leq \frac{1}{2}\dim(E)$ *($\geq$, $=$, respectively).*

(iii) *A subspace A is isotropic if and only if $\omega(u,v) = 0$ for all $u, v \in A$.*

(iv) *A subspace of dimension 1 (codimension 1) is isotropic (coisotropic).*

(v) *A subspace $A \subseteq E$ is coisotropic (isotropic, Lagrangian) if and only if its polar $A^{\circ} \subseteq E^{*}$ is isotropic (coisotropic, Lagrangian) in the symplectic dual space* (E^{*}, Ω).

(vi) *A subspace $\subseteq E$ contained in (containing) a Lagrangian subspace L is isotropic (coisotropic).*

(vii) *A Lagrangian subspace L of a coisotropic subspace C contains the symplectic polar $C^{\S}$.*

All these "linear" results will be applied in the following analysis of the special submanifolds of a symplectic manifold.

1.3 Special submanifolds

Let K be a submanifold of a symplectic manifold $\mathcal{S}_{2n}$. We define, for each $p \in K$,

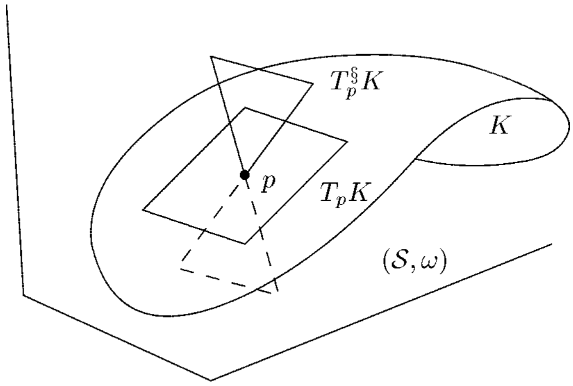
$$T_p^{\S} K = \{v \in T_p\mathcal{S} \mid p \in K,\ \omega(u,v) = 0,\ \forall u \in T_pK\}.$$

We consider the set TK of all vectors tangent to K and

$$T^{\S} K = \cup_{p \in K} T_p^{\S} K.$$

Both TK and $T^{\S}K$ are submanifolds of $T\mathcal{S}$. If $\dim(K) = k$, then

$$\dim(TK) = 2k, \qquad \dim(T^{\S}K) = k + (2n - k) = 2n.$$



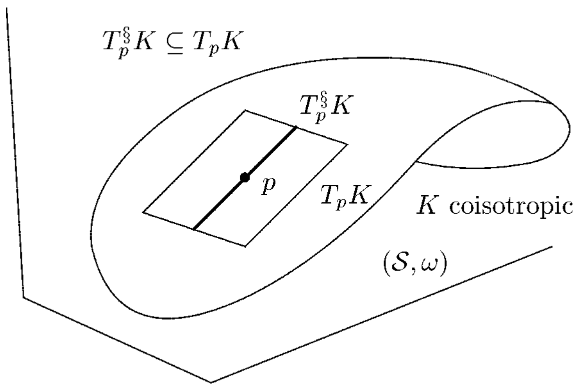
A submanifold K is called

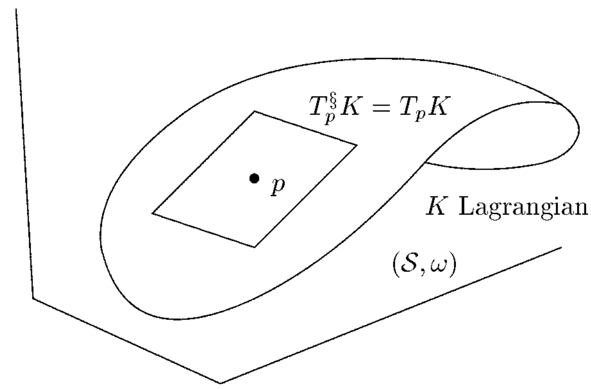
$$\begin{cases} \textbf{isotropic} & & TK \subseteq T^{\S}K, \\ \textbf{coisotropic} & \text{if} & T^{\S}K \subseteq TK, \\ \textbf{Lagrangian} & & T^{\S}K = TK. \end{cases}$$

In these three cases we have respectively,

$$\dim(K) \leq n, \quad \dim(K) \geq n, \quad \dim(K) = n.$$

Notice that a Lagrangian submanifold is simultaneously coisotropic and isotropic, and that it is an isotropic submanifold of maximal dimension or a coisotropic submanifold of minimal dimension.

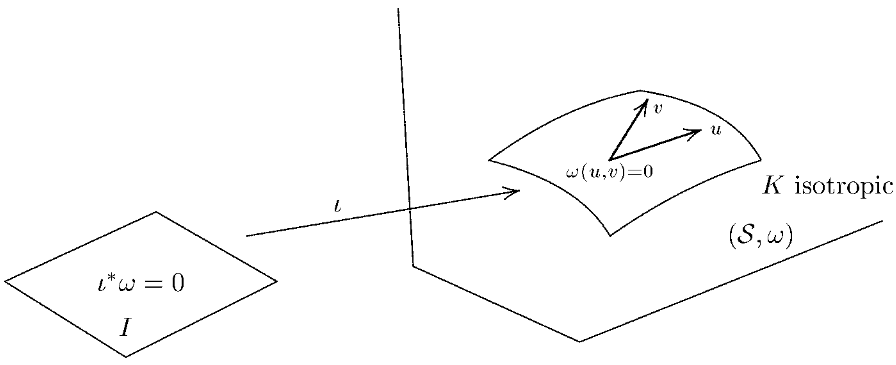




As shown by the following theorems, the isotropy of a submanifold is characterized by means of the symplectic form, while the coisotropy is characterized by means of the Poisson bracket.

Theorem 1. *A submanifold K is isotropic if and only if* (i) $\omega|K = 0$ *i.e.,* $\omega(u, v) = 0\ \forall u, v \in TK$, *or* (ii) $\iota^*\omega = 0$, *where* $\iota: I \to \mathcal{S}$ *is any injection (injective immersion) of a manifold I into $\mathcal{S}$ with image $\iota(I) = K$.* (iii) *If the injection ι is represented by parametric equations $x^{\mathrm{A}} = x^{\mathrm{A}}(\kappa^\alpha)$, then its image is isotropic if and only if*

$$\omega_{\mathrm{AB}} \frac{\partial x^{\mathrm{A}}}{\partial \kappa^\alpha} \frac{\partial x^{\mathrm{B}}}{\partial \kappa^\beta} = 0.$$



This suggests the following extension of the definition of Lagrangian submanifold: a **Lagrangian immersion** is an immersion $\iota: K \to \mathcal{S}$ into a symplectic manifold such that $\dim(K) = \frac{1}{2}\dim(\mathcal{S})$ and $\iota^*\omega = 0$. An **immersed Lagrangian**

submanifold is the image $\Lambda = \iota(K)$ of a Lagrangian immersion. If the immersion is an embedding, then we have a Lagrangian submanifold in the ordinary sense.[1]

Theorem 2. (i) *A submanifold K is coisotropic if and only if $\{F, G\}|K = 0$ for all functions $F, G \in \mathcal{F}(\mathcal{S})$ whose restrictions $F|K$ and $G|K$ to K are constant.* (ii) *If K is defined by independent equations $K^a = 0$, then it is coisotropic if and only if $\{K^a, K^b\}|K = 0$.*

From these characterizations it follows that,

Theorem 3. *A submanifold of dimension 1 is isotropic. A submanifold of codimension 1 is coisotropic.*

Theorem 4. *On a symplectic manifold of dimension $2n$, the maximal number of independent functions in involution is n (i.e., if $n + k$ $(k > 0)$ functions are in involution then they are necessarily dependent).*

1.4 The characteristic foliation of a coisotropic submanifold

If $C_m \subseteq \mathcal{S}_{2n}$ is a coisotropic submanifold, then $T^{\S}C$ is a subbundle of TC, whose fibres have dimension $r = 2n - m$, equal to the codimension of C. In other words, $T^{\S}C$ is a **regular distribution** on C of rank $r = 2n - m$, which we call the **characteristic distribution** of C and denote by Γ_C. Note that it is an isotropic distribution.

A **characteristic vector field** of C is a vector field X on $\mathcal{S}$, tangent to C and such that its image is contained in the characteristic distribution,

$$X(p) \in T_p^{\S}C, \quad \forall\, p \in C.$$

This is equivalent to equation

$$\omega(X, Z) = 0,$$

for all vectors Z tangent to C. The characteristic vector fields form a linear subspace of $\mathcal{X}(\mathcal{S})$.

Theorem 1. *A Hamiltonian vector field X_H is a characteristic vector of a coisotropic submanifold C if and only if H is constant on C.*

Proof. This follows from the identity $\omega(X_H, v) = \langle v, i_{X_H}\omega\rangle = -\langle v, dH\rangle$ for all $v \in TC$. ∎

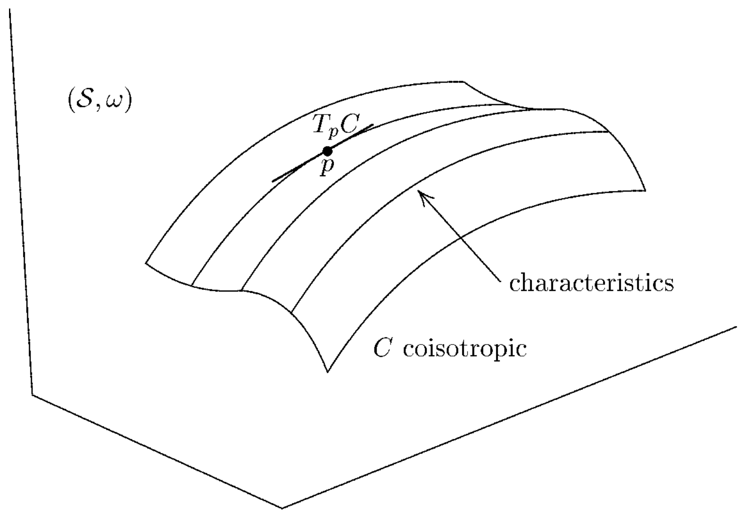
Two fundamental geometrical properties of the coisotropic submanifolds are stated in the following two theorems.

[1] For a detailed discussion and further references on Lagrangian immersions and Lagrangian embeddings, as well as for special submanifolds of symplectic manifolds, see e.g. [Marmo, Morandi, Mukunda, 1990].

Theorem 2. *The characteristic distribution of a coisotropic submanifold is completely integrable.*

This means that for each point $p \in C$ there exists an **integral manifold** of Γ_C i.e., a submanifold of dimension $r = \text{codim}(C)$ containing p and tangent to Γ_C. The integral manifolds of Γ_C are called **characteristics** of C. A maximal connected integral manifold is called a **maximal characteristic** of C. Thus, any coisotropic submanifold admits a **characteristic foliation** made of maximal characteristics.

Proof. Let the submanifold C be (locally) described by $r = 2n - m$ independent equations $C^a = 0$. Because of the coisotropy, the functions C^a are in involution (at least on C), $\{C^a, C^b\}|C = 0$ (Theorem 2, §1.3). It follows that the Hamiltonian vector fields X_a generated by the functions C^a commute, $[X_a, X_b]|C = 0$ (formula (4), §1.1). Since the differentials dC^a are pointwise linearly independent, these vector fields are pointwise independent characteristic vector fields of C (Theorem 1). Thus, they span the characteristic distribution. The corresponding (local) flows φ^a_t commute. Thus, starting from any fixed point $x_0 \in C$, the set of points $x \in C$ such that $x = \phi^1_{t_1} \circ \ldots \circ \varphi^r_{t_r}(x_0)$ defines a submanifold of dimension r which is tangent at each point to the vector fields X_a. Hence, this submanifold is an integral manifold of the characteristic distribution of C. ∎



Theorem 3. *A Lagrangian submanifold Λ contained in a coisotropic submanifold C is the union of characteristics of C.*

This property is known as **absorption principle** [Vinogradov, Kuperschmidt, 1977].

Proof. It is a consequence of a property of linear symplectic algebra: if $L \subseteq K \subseteq E$, where E is a symplectic space, K is a coisotropic subspace and L is Lagrangian, then $K^{\S} \subseteq L^{\S} = L$. ■

Remark 1. (i) The dimension of the characteristics is equal to the codimension of C. The characteristics are isotropic submanifolds. (ii) Any submanifold of codimension 1 is coisotropic; hence, its characteristics are (one-dimensional) curves. (iii) If C is Lagrangian then the maximal characteristics coincide with the connected components of C.

As a complement of Theorem 1 we have

Theorem 4. *A Hamiltonian vector field X_H is tangent to a coisotropic submanifold C if and only if H is constant on the characteristics of C.*

Proof. Apply the identity $\omega(X_H, v) = \langle v, i_{X_H}\omega\rangle = -\langle v, dH\rangle$, $\forall\, v \in T^{\S}C$. ■

Remark 2. In §6.5 we shall apply Theorem 2 for proving Frobenius' theorem concerning the complete integrability of regular distributions. On the other hand, we can use Frobenius' theorem to prove Theorem 1 as follows. We show that the characteristic vector fields form a Lie-subalgebra of $\mathcal{X}(\mathcal{S})$. (i) The Lie bracket of two vector fields tangent to a submanifold C is tangent to C (this is a general property of the Lie bracket of vector fields). (ii) The intrinsic definition of the differential of a 2-form is expressed by the following formula:

$$\begin{aligned} d\omega(X,Y,Z) = d_X\big(\omega(Y,Z)\big) + d_Y\big(\omega(Z,X)\big) + d_Z\big(\omega(X,Y)\big) - \\ \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y). \end{aligned}$$

If ω is closed, X and Y are characteristic vectors and Z is tangent to C, then we get

$$0 = \omega([X,Y],Z)$$

for each vector Z tangent to C. This shows that the Lie bracket of two characteristic vector fields is a characteristic vector field. ■

1.5 Relations

A (binary) **relation** is a subset of the Cartesian product of two sets:

$$R \subseteq B \times A.$$

The sets A and B are called the **domain** and the **codomain** of the relation, respectively. For a relation we also use the notation

$$R\colon A \to B$$

or

$$A \overset{R}{\rightarrow} B.$$

The **transpose relation** or **inverse relation**

$$R^\top \subseteq A \times B$$

is made of the same pairs of R, but in the opposite order. A relation $R \subseteq B \times A$ is **symmetric** if

$$A = B \quad \text{and} \quad R^\top = R.$$

Relations can be composed according to the following rule: if

$$R \subseteq B \times A, \qquad S \subseteq C \times B,$$

then

$$S \circ R = \{(c, a) \in C \times A \mid \exists b \in B,\ (b, a) \in R,\ (c, b) \in S\} \subseteq C \times A.$$

The composition of relations is associative

$$(S \circ Q) \circ R = S \circ (Q \circ R)$$

and satisfies the contravariant transposition rule

$$(S \circ R)^\top = R^\top \circ S^\top.$$

For relations and their composition we use the notation *from the right to the left* in agreement with the standard notation for the composition of mappings. A **mapping** is a relation $R \subseteq B \times A$ characterized by conditions

$$\begin{cases} R^\top \circ B = A, \\ (b, a) \in R,\ (b', a) \in R \quad \Longrightarrow \quad b' = b. \end{cases}$$

We identify a mapping $\rho \colon A \to B$ with its graph $R = \operatorname{graph}(\rho) \subset B \times A$:

$$(b, a) \in R \qquad \Longleftrightarrow \qquad b = \rho(a).$$

The **diagonal** of $A \times A$ is denoted by Δ_A,

$$\Delta_A = \{(a, a') \in A \times A \mid a = a'\}.$$

It behaves as the **identity relation** over the set A; if $R \subseteq B \times A$ then $R \circ \Delta_A = R$ and $\Delta_B \circ R = R$.

In this framework, it is convenient to interpret a subset $S \subseteq A$ as a relation $S \subseteq A \times \{0\}$ where $\{0\}$ is a *singleton*, an arbitrary set made of a single element. If $R \subseteq B \times A$ then $R \circ S$ is the **image of the subset** S by the relation R. In particular $R \circ A \subseteq B$ is the **image** of the relation R and $R^\top \circ B \subseteq A$ is the **inverse image** of R.

1.6 Symplectic relations

A **smooth relation** is a submanifold $R \subseteq \mathcal{S}_2 \times \mathcal{S}_1$ of the product of two differentiable manifolds $\mathcal{S}_1$ and $\mathcal{S}_2$. In general, the composition of two smooth relations is not a smooth relation.

Let $(\mathcal{S}_1, \omega_1)$ and $(\mathcal{S}_2, \omega_2)$ be two symplectic manifolds. Let us consider the 2-form on the product manifold $\mathcal{S}_2 \times \mathcal{S}_1$ defined by

$$\omega_2 \ominus \omega_1 = \mathrm{pr}_2^* \omega_2 - \mathrm{pr}_1^* \omega_1,$$

where

$$\mathrm{pr}_1 \colon \mathcal{S}_2 \times \mathcal{S}_1 \to \mathcal{S}_1, \qquad \mathrm{pr}_2 \colon \mathcal{S}_2 \times \mathcal{S}_1 \to \mathcal{S}_2,$$

are the canonical projections. This is a symplectic form. Thus, $(\mathcal{S}_2 \times \mathcal{S}_1, \omega_2 \ominus \omega_1)$ is a symplectic manifold, also denoted by $(\mathcal{S}_2, \omega_2) \times (\mathcal{S}_1, -\omega_1)$.

A **canonical** or **symplectic relation** is a Lagrangian submanifold of a symplectic manifold $(\mathcal{S}_2 \times \mathcal{S}_1, \omega_2 \ominus \omega_1)$. These Lagrangian submanifolds have been named "canonical relations" by Hörmander [Hörmander, 1971] and "symplectic relations" by Tulczyjew [Sniatycki, Tulczyjew, 1972]. This definition is suggested by the following property:

Theorem 1. ([Sniatycki, Tulczyjew, 1972], [Tulczyjew, 1974, 1977b]) *A diffeomorphism $\varphi \colon \mathcal{S}_1 \to \mathcal{S}_2$ between two symplectic manifolds is a* **symplectomorphism** *i.e., it preserves the symplectic forms,*

$$\varphi^* \omega_2 = \omega_1,$$

if and only if its graph $R \subset \mathcal{S}_2 \times \mathcal{S}_1$ is a Lagrangian submanifold of $(\mathcal{S}_2 \times \mathcal{S}_1, \omega_2 \ominus \omega_1)$.

Proof. The graph R is the image set of the injective mapping

$$\iota = (\varphi, \mathrm{id}_{\mathcal{S}_1}) \colon \mathcal{S}_1 \to \mathcal{S}_2 \times \mathcal{S}_1 \colon a \mapsto \big(\varphi(a), a\big).$$

Since $\mathrm{pr}_1 \circ \iota = \mathrm{id}_{\mathcal{S}_1}$ and $\mathrm{pr}_2 \circ \iota = \varphi$, it follows that

$$\begin{aligned} \iota^*(\omega_2 \ominus \omega_1) &= \iota^*(\mathrm{pr}_2^* \omega_2 - \mathrm{pr}_1^* \omega_1) = \iota^* \mathrm{pr}_2^* \omega_2 - \iota^* \mathrm{pr}_1^* \omega_1 \\ &= (\mathrm{pr}_2 \circ \iota)^* \omega_2 - (\mathrm{pr}_1 \circ \iota)^* \omega_1 = \varphi^* \omega_2 - \omega_1. \end{aligned}$$

Thus, R is isotropic if and only if $\varphi^*\omega_2 = \omega_1$. For a diffeomorphism, $\dim(R) = \frac{1}{2}(\dim(\mathcal{S}_2) + \dim(\mathcal{S}_1))$. ■

When $\mathcal{S}_1 = \mathcal{S}_2$, a symplectomorphism is called **canonical transformation**. Thus, the notion of symplectic relation is an extension of that of canonical transformation.

Remark 1. A Lagrangian submanifold Λ of a symplectic manifold $(\mathcal{S}, \omega)$ can be considered as a symplectic relation, since it is a Lagrangian submanifold of the symplectic manifold $\mathcal{S} \times \{0\}$.

In general symplectic relations do not compose nicely. A sufficient condition is given by the following theorem proved in [Sniatycki, Tulczyjew, 1972].

Theorem 2. *Let R_{21} and R_{32} be symplectic relations from $(\mathcal{S}_1, \omega_1)$ to $(\mathcal{S}_2, \omega_2)$ and from $(\mathcal{S}_2, \omega_2)$ to $(\mathcal{S}_3, \omega_3)$, respectively, such that: (i) $R_{31} = R_{32} \circ R_{21}$ is a submanifold, (ii) for each (p_3, p_2, p_1) such that $(p_3, p_2) \in R_{32}$ and $(p_2, p_1) \in R_{21}$, $T_{(p_3,p_1)}R_{31} = T_{(p_3,p_2)}R_{32} \circ T_{(p_2,p_1)}R_{21}$. Then R_{31} is a symplectic relation.*

Remark 1. Another sufficient condition for the smooth composition of two symplectic relations is illustrated in Remark 1, §1.10. As we shall see below, there are physically interesting examples of non-smooth composition of symplectic relations. Moreover, symplectic relations between cotangent bundles can be represented by functions, called "generating families", in such a way that the composition of symplectic relations is translated into a suitable composition of generating families. This composition rule always yields a smooth function, even if the composed relation is not smooth.

1.7 Linear symplectic relations

A **linear relation** $R \subseteq B \times A$ is a linear subspace of the direct sum $B \oplus A$ of two vector spaces A and B. The direct sum is the Cartesian product endowed with the natural structure of vector space. This definition is suggested by the fact that *a mapping $f: A \to B$ is linear if and only if its graph R is a linear subspace of $B \oplus A$.* It can be shown that *the composition of two linear relations is a linear relation.* Vector spaces (as objects) and linear relations (as morphisms) form a category [Benenti, Tulczyjew, 1979]. Let (A, α) and (B, β) be symplectic vector spaces. On the direct sum $B \oplus A$ a bi-linear skew-symmetric form $\beta \ominus \alpha$ is defined by

$$(1) \qquad (\beta \ominus \alpha)\big((b, a), (b', a')\big) = \beta(b, b') - \alpha(a, a').$$

This form is non-singular, thus it is a symplectic form. With a linear relation $R \subseteq B \times A$ we associate its **symplectic dual relation** $R^\S \subseteq B \times A$, where § is the symplectic polar operator with respect to $\beta \ominus \alpha$,

$$(2) \qquad (b, a) \in R^\S \quad \Longleftrightarrow \quad \forall\, (b', a') \in R,\ \beta(b, b') - \alpha(a, a') = 0.$$

According to the terminology used for special subspaces, a linear relation $R \subseteq B \times A$ between symplectic vector spaces is called **Lagrangian** if $R^{\S} = R$, **isotropic** if $R \subseteq R^{\S}$, **coisotropic** if $R^{\S} \subseteq R$. A Lagrangian linear relation is also called **symplectic** or **canonical**. Note that if R is a linear symplectic relation, then $R^{\top}$ is symplectic. Symplectic spaces and symplectic linear relations form a category [Benenti, Tulczyjew, 1981].

Remark 1. Many properties concerning with symplectic relations have a "linear" background. Indeed, if $R \subset \mathcal{S}_2 \times \mathcal{S}_1$ is a smooth relation between symplectic manifolds, then it is a symplectic relation if and only if for each pair of points $(p_2, p_1) \in R$ the tangent subspace $T_{(p_2,p_1)}R \subset T_{p_2}\mathcal{S}_2 \oplus T_{p_1}\mathcal{S}_1$ is a symplectic (linear) relation (this follows from the fact that a submanifold Λ of a symplectic manifold $\mathcal{S}$ is Lagrangian if and only if at each point $p \in \Lambda$, $T_p\Lambda$ is a Lagrangian subspace of $T_p\mathcal{S}$).

A basic property of linear relations between symplectic vector spaces is the following **functorial rule**,

Theorem 1. *If $S \circ R$ is the composition of two linear relations between symplectic vector spaces, then*

$$\boxed{(S \circ R)^{\S} = S^{\S} \circ R^{\S}} \tag{3}$$

A proof of this formula, taken from [Benenti, 1988], is given in §6.7. It follows from general properties of the linear relations.

By using this formula we can easily prove the following two fundamental statements:

Theorem 2. *The composition of two linear symplectic relations is a linear symplectic relation.*

Proof. If $R^{\S} = R$ and $S^{\S} = S$, then $(S \circ R)^{\S} = S^{\S} \circ R^{\S} = S \circ R$. ∎

Theorem 3. *The image $R \circ K$ of an isotropic [coisotropic, Lagrangian] subspace K by a linear symplectic relation R is an isotropic [coisotropic, Lagrangian] subspace.*

Proof. If K is an isotropic subspace of A, $K \subseteq K^{\S}$, and R is symplectic, then by (1) and the inclusion property of the composition of relations, we find $(R \circ K)^{\S} = R \circ K^{\S} \supseteq R \circ K$. This shows that the image $R \circ K$ is isotropic. Similarly, if K is coisotropic, $K^{\S} \subseteq K$, it follows that also $R \circ K$ is coisotropic. ∎

1.8 Symplectic reductions

A **reduction** is a special case of relation between differentiable manifolds. It is a relation $R \subseteq \mathcal{S}_0 \times \mathcal{S}$ which is the graph of a surjective submersion $\rho\colon C \to \mathcal{S}_0$ from a submanifold $C \subseteq \mathcal{S}$ onto $\mathcal{S}_0$. The transpose $R^{\top}$ of a reduction is called

coreduction. Special cases of reductions are the surjective submersions, the diffeomorphisms, the transpose of the embeddings of submanifolds. Reductions and symplectic reductions are morphisms of categories [Benenti, 1983]. The notion of symplectic reduction plays a fundamental role in the global symplectic formulation of the Cauchy problem for a first-order partial differential equation (§1.10) and of the Jacobi theorem (§3.6).

In order to establish the basic properties of the symplectic reductions we need to consider them within the category of linear relations. A **linear symplectic reduction** is a Lagrangian subspace $R \subseteq B \times A$ which is the graph of a linear surjective mapping from a subspace $K \subseteq A$ onto B.

Theorem 1. *If $R \subseteq B \times A$ is a linear symplectic reduction then* (i) $K = R^\top \circ B$ *is a coisotropic subspace,* (ii) $R^\top \circ \{0\} = K^\S$.

Proof. By the functorial rule (3), §1.7, we have $K^\S = (R^\top \circ B)^\S = (R^\S)^\top \circ B = R^\top \circ B^\S = R^\top \circ \{0\} \subset K$. ∎

Then we can prove

Theorem 2. *Let $R \subseteq \mathcal{S}_0 \times \mathcal{S}$ be a symplectic reduction. Then:* (i) *the inverse image $R^\top \circ N \subseteq \mathcal{S}$ of a coisotropic (isotropic, Lagrangian) submanifold $N \subseteq \mathcal{S}_0$ is a coisotropic (isotropic, Lagrangian) submanifold;* (ii) *the inverse image $C = R^\top \circ \mathcal{S}_0$ of R is a coisotropic submanifold;* (iii) *the inverse image $R^\top \circ \{p_0\}$ of a point $p_0 \in \mathcal{S}_0$ is an integral manifold of the characteristic distribution of C.*

Proof. The inverse image of a submanifold by a submersion is a submanifold. Being a reduction R the graph of a (surjective) submersion $\rho\colon C \to \mathcal{S}_0$, the inverse image of any submanifold $N \subseteq \mathcal{S}_0$ (in particular of a point) is a submanifold of $C = R^\top \circ \mathcal{S}_0$. Let $(p_0, p) \in R$. Then $T_{(p_0,p)}R \subset T_{p_0}\mathcal{S}_0 \times T_p\mathcal{S}$ is a linear symplectic relation i.e., a Lagrangian subspace,

$$(T_{(p_0,p)}R)^\S = T_{(p_0,p)}R.$$

Because of the definition of submersion, this linear relation is the graph of a surjective linear mapping. Thus, $T_{(p_0,p)}R$ is a linear symplectic reduction, with inverse image T_pC.

$$(T_{(p_0,p)}R)^\top \circ T_{p_0}\mathcal{S}_0 = T_pC.$$

Due to Theorem 3, §1.7, items (i) and (iii) are proved (note that $\mathcal{S}_0$ is coisotropic). Moreover,

$$(T_{(p_0,p)}R)^\top \circ \{0\} = T_p^\S C, \qquad 0 \in T_{p_0}\mathcal{S}_0.$$

Let us consider the fibre

$$I_{p_0} = R^\top \circ \{p_0\}.$$

We have

$$\begin{aligned} T_pI_{p_0} &= \{v \in T_pC \mid T\rho(v) = 0\} = \{v \in T_pC \mid T_{(p,p_0)}R \circ \{v\} = 0\} \\ &= (T_{(p_0,p)}R)^\top \circ \{0\} = T_p^\S C. \end{aligned}$$

This shows that the tangent space of a fibre at a point p coincides with the tangent space of the characteristic containing that point. This proves item (iii). ∎

The operation considered in the preceding theorem is called **coreduction** or **counter-reduction** of a submanifold. As we have seen, this operation preserves the submanifold structure and the symplectic kind. Instead, the **reduction of a submanifold**, $N \subseteq \mathcal{S} \mapsto N_0 = R \circ N \subseteq \mathcal{S}_0$, is more delicate and for preserving the submanifold structure for the image N_0 it requires some special assumptions on the intersection $C \cap N$.

Definition 1. Two submanifolds N and C of a manifold $\mathcal{S}$ are said to have **clean intersection** [Bott, 1954] [Weinstein, 1973, 1977] if

(i) $C \cap N \neq \emptyset$ is a submanifold and
(ii) $T_x C \cap T_x N = T_x(C \cap N)$ for every $x \in C \cap N$.

Remark 1. The inclusion $T(C \cap N) \subseteq TC \cap TN$ always holds for two submanifolds N and C whose intersection $C \cap N$ is a submanifold. Indeed, if $v = [\gamma] \in T(N \cap C)$ is a vector represented by a curve γ on $N \cap C$, then γ is a curve on N and on C simultaneously, so that $v \in TN$ and $v \in TC$.

Remark 2. The case $N \subset C$ is a special case of clean intersection. For the characterization of the clean intersection in terms of equations of submanifolds see Appendix A.4.

As above, in what follows $R \subseteq \mathcal{S}_0 \times \mathcal{S}$ is a symplectic reduction, $C = R^\top \circ \mathcal{S}_0$ is its inverse image, $\rho \colon C \to \mathcal{S}_0$ is the associated submersion and $\rho_N \colon N \cap C \to \mathcal{S}_0$ the restriction to $N \cap C$.

Theorem 3. (i) *If N and C have clean intersection and* $\dim\left(T_x^\S C \cap T_x N\right)$ *does not depend on $x \in N \cap C$ then each $x \in N \cap C$ admits a neighborhood U such that $R \circ U$ is a submanifold of $\mathcal{S}_0$.* (ii) *If N is of special type (i.e., coisotropic, isotropic, Lagrangian) then $R \circ U$ is of the same type.*

Remark 3. The condition $\dim\left(T_x^\S C \cap T_x N\right) =$ constant in this statement can be replaced by $\dim\left(T_x^\S C + T_x N\right) =$ constant, due to the Grassmann formula,

$$\dim\left(T_x^\S C \cap T_x N\right) + \dim\left(T_x^\S C + T_x N\right) = \dim\left(T_x^\S C\right) + \dim\left(T_x N\right).$$

Proof. The subspace $T_x^\S C$ is the kernel of the linear mapping $T_x\rho \colon T_x C \to T_{\rho(x)}\mathcal{S}_0$. Then,

$$\begin{aligned}\dim\left(\mathrm{Ker}(T_x\rho_N)\right) &= \dim\left(T_x^\S C \cap T_x(N \cap C)\right) = \dim\left(T_x^\S C \cap T_x C \cap T_x N\right)\\ &= \dim\left(T_x^\S C \cap T_x N\right).\end{aligned}$$

It follows that ρ_N has constant rank i.e., it is a subimmersion. Thus, for each $x \in N \cap C$ there exists a neighborhood U such that $\rho_N(U) = R \circ U$ is a submanifold (cf.

[Dieudonné], 16.8.8 and [Libermann, Marle, 1987] p.344). Assume for simplicity that $U = N \cap C$ and $V = R \circ U$. We have

$$\begin{aligned} T(R \circ N) &= T\big(R \circ (N \cap C)\big) = T\big(\rho_N(N \cap C)\big) = T\big(\rho(N \cap C)\big) \\ &= T\rho\big(T(N \cap C)\big) = TR \circ \big(T(N \cap C)\big) = TR \circ (TN \cap TC) \\ &= TR \circ TN. \end{aligned}$$

We have used the property $T(\rho(A)) = T\rho(TA)$ which holds for any submersion $\rho\colon C \to \mathcal{S}_0$ and any submanifold $A \subseteq C$ such that $\rho(A)$ is a submanifold, and the fact that TC is the inverse image of TR. For $x \in N \cap C$ and $y = \rho(x)$ we have $T_y V = T_{(y,x)} R \circ T_x N$. Since $T_{(y,x)} R$ is a linear symplectic relation, it follows that $T_y V$ is a subspace of the same type of $T_x N$ (Theorem 3, §1.7). ∎

If we apply Theorem 3 to the case of a Lagrangian submanifold $N = \Lambda$ then we get the following fundamental result [Weinstein, 1977]:

Theorem 4. *Let $R \subseteq \mathcal{S}_0 \times \mathcal{S}$ be a symplectic reduction and let $\Lambda \subset \mathcal{S}$ be a Lagrangian submanifold. If Λ and $C = R^\top \circ \mathcal{S}_0$ have clean intersection, then $\Lambda_0 = R \circ \Lambda$ is an immersed Lagrangian submanifold of $\mathcal{S}_0$.*

Proof. If $N = \Lambda$ is Lagrangian then the clean intersection implies the second assumption in item (i) of Proposition 1 (see Remark 3). Indeed we have

$$TN + T^\S C = T^\S K + T^\S C = (TK \cap TC)^\S = T^\S (N \cap C),$$

and $T^\S_x (K \cap C)$ has a constant dimension, equal to the codimension of $K \cap C$. ∎

A special case of clean intersection is the transverse intersection:

Definition 2. Two submanifolds N and C of a manifold $\mathcal{S}$ are said to be **tranverse** or to have **transverse intersection** if

$$(1) \qquad T_x C + T_x N = T_x \mathcal{S} \qquad \forall\, x \in C \cap N \neq \emptyset.$$

It can be proved that (a prof can be found in the books [Lang, 1972] and [Abraham, Robbin, 1967]; see also [Libermann, Marle, 1987], Appendix 1).

Theorem 5. *If N and C are transverse submanifolds of $\mathcal{S}$, of dimensions n, c and s, respectively, then:* (i) *$N \cap C$ is an $(n + c - s)$-dimensional submanifold of $\mathcal{S}$ (as well as of N or of C);* (ii) *$T_x C + T_x N = T_x \mathcal{S}$, $\forall\, x \in C \cap N$.*

This means that *transverse intersection implies clean intersection.*

Remark 4. If $N = \Lambda$ is Lagrangian and C is coisotropic in the symplectic manifold $\mathcal{S}$, then the tranverse intersection (1) is equivalent to

$$(2) \qquad T^\S_x C \cap T_x \Lambda = 0, \quad \forall\, x \in \Lambda \cap C \neq \emptyset.$$

This means that Λ is not tangent to the characteristics of C. The dimension of $C \cap \Lambda$ is $n + c - 2n = c - n$, where $c = \dim(C)$ and $2n = \dim(\mathcal{S})$. If $k = 2n - c$ is the codimension of C, then $\dim(C \cap \Lambda) = n - k$. This is in agreement with Theorem 4: the dimension of $\Lambda \cap C$, which is transversal to the characteristics, is equal to the dimension of the reduced submanifold $R \circ \Lambda$, which must be Lagrangian. On the other hand, the dimension of the reduced symplectic manifold is $2(n-k)$.

Remark 5. If the intersection of Λ and $C = R^\top \circ \mathcal{S}_0$ is not clean, then $\Lambda_0 = R \circ \Lambda$ may not be a submanifold. In this case we say that it is a **Lagrangian set** (cf. §2.5; physically interesting examples will be given in Chapters 4 and 5).

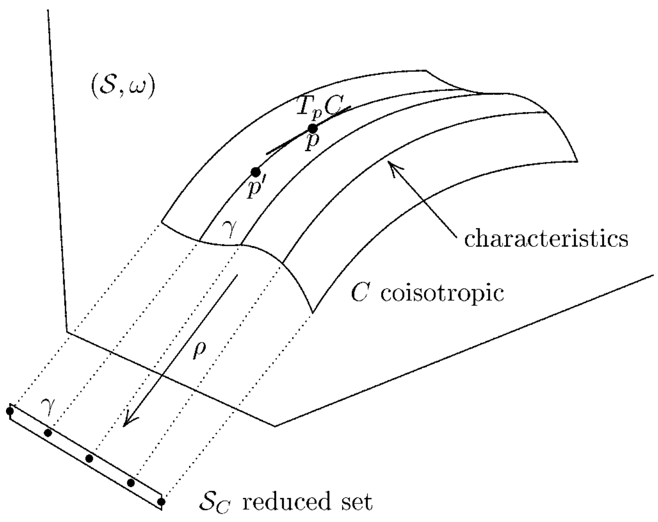
1.9 Symplectic relations generated by a coisotropic submanifold

Let $C_m \subseteq \mathcal{S}_{2n}$ be a coisotropic submanifold. Let us denote by $\mathcal{S}_C$ the **reduced set** of $\mathcal{S}$ by C i.e., the set of the maximal connected characteristics of C. A coisotropic submanifold $C \subseteq \mathcal{S}$ generates two relations [Tulczyjew, 1975]: (i) the **characteristic relation**

$$D_C \subseteq \mathcal{S} \times \mathcal{S},$$

made of pairs of points (p, p') belonging to a same characteristic of C,

$$(1) \qquad (p, p') \in D_C \iff \exists\, \gamma \in \mathcal{S}_C \mid p, p' \in \gamma;$$



(ii) the **characteristic reduction**

$$R_C \subseteq \mathcal{S}_C \times \mathcal{S},$$

defined by

$$(\gamma, p) \in R_C \iff p \in \gamma. \tag{2}$$

It follows that

$$D_C = R_C^\top \circ R_C. \tag{3}$$

According to its definition, D_C is a relation from C to C, since it involves points of C only. However, it is convenient to consider D_C as a relation in $\mathcal{S}$. Indeed, by using local coordinates adapted to the characteristics (whose existence is due to the local Frobenius theorem) it can be proved that it is (locally) a $2n$-dimensional submanifold of $\mathcal{S} \times \mathcal{S}$ and moreover, that it is a Lagrangian submanifold with respect to the symplectic form $\omega \ominus \omega$ [Benenti, Tulczyjew, 1980].

In general, the reduced set $\mathcal{S}_C$ is not a differentiable manifold, so that R_C is not a smooth relation. However,

Theorem 1. *If the reduced set $\mathcal{S}_C$ has a differentiable structure such that the canonical projection*

$$\rho : C \to \mathcal{S}_C : p \mapsto \gamma \mid p \in \gamma$$

is a surjective submersion, then: (i) *There is a unique* **reduced symplectic form** ω_C *such that* ([Lichnerowicz, 1975], [Weinstein, 1977])

$$\omega | C = \rho^*(\omega_C). \tag{4}$$

(ii) *With respect to this symplectic form $R_C \subset \mathcal{S}_C \times \mathcal{S}$ is a symplectic relation.*
(iii) $D_C = R_C^\top \circ R_C$ *is a symplectic relation.*

We call $(\mathcal{S}_C, \omega_C)$ the **reduced symplectic manifold** and R_C the **symplectic reduction** associated with C. Note that

$$\dim(\mathcal{S}_C) = \dim(C) - \operatorname{codim}(C) = 2\dim(C) - \dim(\mathcal{S}) = 2(m-n). \tag{5}$$

Proof. (i) The reduced symplectic form is defined by equation

$$\omega_C(v, v') = \omega(w, w'), \quad \forall\, (w, w') \in TC \times_C TC \ s.t.\ T\rho(w) = v,\ T\rho(w') = w. \tag{6}$$

By definition of pull-back this is equivalent to (4). Definition (6) does not depend on the choice of the vectors (w, w'). Consider a point $p \in C$ and $w, w', \bar{w}, \bar{w}' \in T_pC$ with $T\rho(w) = v = T\rho(\bar{w})$, $T\rho(w') = v' = T\rho(\bar{w}')$. Then the vectors $w - \bar{w}$ and $w' - \bar{w}'$ projects onto the zero vector; this means that they are tangent to the characteristic at p. Hence, $w-\bar{w}$, $w'-\bar{w}' \in T_p^{\S}C$. Since this subspace is isotropic, $\omega(w-\bar{w}, w'-\bar{w}') = 0$ and $\omega(w,w') = \omega(\bar{w},\bar{w}')$. This proves the independence of the definition (6) from the choice of the vectors (w,w') at a fixed point p. Consider the (local) flow $\phi_t\colon C \to C$ of a Hamiltonian characteristic vector field (generated by a Hamiltonian constant on C). For all admissible $t \in \mathbb{R}$ we have $p_t = \phi_t(p) \in C$ and $\rho \circ \phi_t = \rho$. Then the vectors $w_t = T\phi_t(w) \in T_{p_t}C$ and $w'_t = T\phi_t(w') \in T_{p_t}C$ still project onto the vectors (v,v'). Since ϕ_t is symplectic, $\omega(w,w') = \omega(w_t,w'_t)$. Moreover, we observe that any two points p and p' on the same characteristic can be joined by a finite number of integral curves of characteristic Hamiltonian vector fields. This proves the independence of the definition (6) from the choice of the point p on a fixed characteristic. Hence, definition (6) is well posed. The fact that ω_C is nondegenerate is a consequence of the fact that $T_p\rho$ is everywhere surjective (by definition of submersion). The fact that ω_C is closed follows from (4). (ii) The relation R_C is the image of the topological immersion $\iota\colon C \to \mathcal{S}_C \times \mathcal{S}\colon p \mapsto \big(\rho(p), p\big)$. It follows that

$$\begin{aligned}\iota^*(\omega_C \ominus \omega) &= \iota^*(\mathrm{pr}^*_{\mathcal{S}_C}\omega_C - \mathrm{pr}^*_{\mathcal{S}}\omega) \\ &= (\mathrm{pr}_{\mathcal{S}_C} \circ \iota)^*\omega_C - (\mathrm{pr}_{\mathcal{S}} \circ \iota)^*\omega \\ &= \rho^*\omega_C - \omega|C = 0.\end{aligned}$$

This shows that R_C is isotropic. On the other hand, $\dim(\mathcal{S}_C) = 2(m-n)$, $\dim(\mathcal{S}_C \times \mathcal{S}) = 2m$, and $\dim(R_C) = \dim\operatorname{graph}(\rho) = \dim(C) = m$. This shows that R_C is Lagrangian. (iii) From an atlas on $\mathcal{S}$ of charts adapted to the submersion ρ we can construct an atlas of charts adapted to D_C. Thus, D_C is a submanifold of dimension equal to $2n = \dim(\mathcal{S})$. Any vector tangent to D_C, interpreted as an equivalence class of curves on D_C, is a class of pairs of curves on C which are pointwise projected by ρ on a same curve of $\mathcal{S}_C$. This implies that a vector w tangent to D_C at a point (p,p') is a pair of vectors $(v,v') \in T_{(p,p')}C \subset T_{(p,p')}\mathcal{S}$ whose images by $T\rho$ coincide, $T\rho(v) = T\rho(v')$. Since the projections of $w = (v,v')$ onto the first and second factor $\mathcal{S}$ are $v \in T_pC \subset T_p\mathcal{S}$ and $v' \in T_{p'}C \subset T_{p'}\mathcal{S}$ respectively, it follows that for two vectors w and $\bar{w}$ at the same point (p,p'),

$$\omega \ominus \omega(w,\bar{w}) = \omega(v,\bar{v}) - \omega(v',\bar{v}').$$

By definition of ω_C, this last difference is equal to

$$\omega_C(T\rho(v), T\rho(\bar{v})) - \omega_C(T\rho(v') - T\rho(\bar{v}')).$$

But these two terms are equal, since $T\rho(v') = T\rho(v)$ and $T\rho(\bar{v}') = T\rho(\bar{v})$. Thus, $\omega \ominus \omega(w,\bar{w}) = 0$ and D_C is isotropic. ■

Example 1. Let $\mathcal{S} = \mathbb{R}^4 = (x,y,u,v)$, $\omega = dx \wedge du + dy \wedge dv$, $C = \mathbb{S}_3$ the unit sphere $x^2 + y^2 + u^2 + v^2 = 1$. Then: (i) the characteristics of C are the maximal circles, (ii) the reduced set $\mathcal{S}_C$ is the unit sphere $\mathbb{S}_2$, (iii) the surjective submersion

$\rho: \mathbb{S}_3 \to \mathbb{S}_2$ is the Hopf fibration, (iv) the reduced symplectic form ω_C is the area element of $\mathbb{S}_2$ ([Weinstein, 1977], [Benenti, 1988], p.139, [Libermann, Marle, 1987], §14.14).

Remark 1. The reduced Poisson bracket associated with the reduced symplectic form can be directly defined as follows. Let $f: \mathcal{S}_C \to \mathbb{R}$ be a smooth function on the reduced manifold. Let us denote by $f': C \to \mathbb{R}$ its extension to C constant on the characteristics, so that

$$f' = f \circ \rho.$$

Furthermore, let us denote by f'' any (local) extension of f' in a neighborhood of C, so that $f''|C = f'$. Then the reduced Poisson bracket is completely determined by equation

$$(\{f, g\}_C)' = \{f'', g''\}|C.$$

This definition is well posed because of the following properties, whose proof is straightforward.

Proposition 1. *Let C be a coisotropic submanifold of a symplectic manifold $\mathcal{S}$ and let (F_1, F_2) and (G_1, G_2) be two pairs of functions on $\mathcal{S}$ such that $F_1|C = F_2|C$, $G_1|C = G_2|C$. Then $\{F_1, G_1\}|C = \{F_2, G_2\}|C$.*

Proposition 2. *Let C be a coisotropic submanifold of a symplectic manifold $\mathcal{S}$. If F and G are two functions on $\mathcal{S}$ constant on the characteristics of C, then their Poisson bracket $\{F, G\}$ is constant on the characteristics.*

1.10 The symplectic background of the Cauchy problem

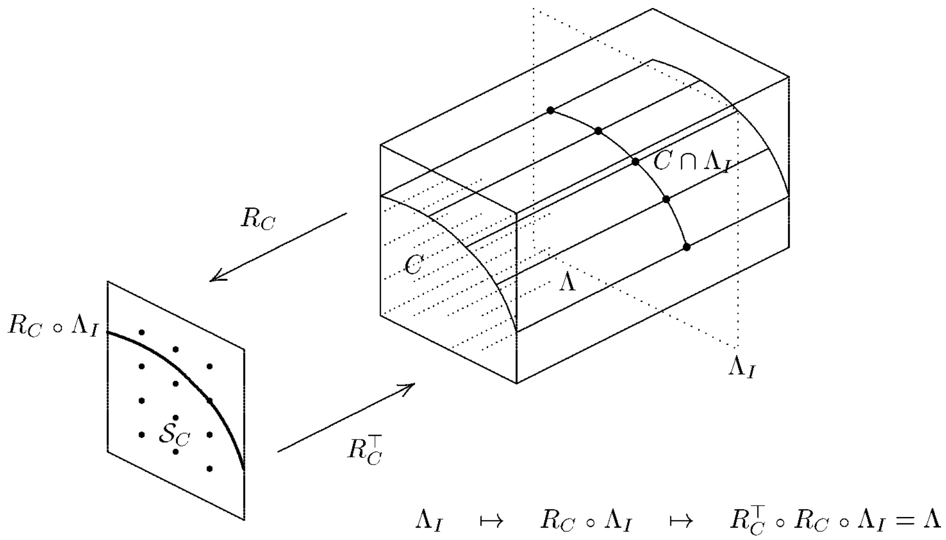
An important application of the results of the preceding section is the following

Theorem 1. *Let $\Lambda_I \subset \mathcal{S}$ be a Lagrangian submanifold having clean intersection with a coisotropic submanifold $C \subseteq \mathcal{S}$. Then there exists a unique connected immersed Lagrangian submanifold Λ contained in C and containing $C \cap \Lambda_I$. This Lagrangian submanifold is defined by the composition formula*

$$\boxed{\Lambda = D_C \circ \Lambda_I}$$

thus, it is the union of the maximal characteristics of C intersecting Λ_I.

Proof. By Theorem 3, §1.8, $R_C \circ \Lambda_I$ is an immersed Lagrangian submanifold of $\mathcal{S}_C$ and by Theorem 2, $\Lambda = R_C^\top \circ R_C \circ \Lambda_I$ is a Lagrangian submanifold of $\mathcal{S}$. The uniqueness follows from the absorption principle. ∎



$$\Lambda_I \quad \mapsto \quad R_C \circ \Lambda_I \quad \mapsto \quad R_C^\top \circ R_C \circ \Lambda_I = \Lambda$$

As we shall see, when applied to cotangent bundles, this statement can be interpreted as a **symplectic background of the Cauchy problem**: C will represent a first-order PDE, Λ_I (or $C \cap \Lambda_I$) the **initial** or **boundary conditions** and Λ the corresponding solution. However, the case in which the intersection $C \cap \Lambda_I$ is not clean occurs in many interesting applications of this theory. In this case, $R_C \circ \Lambda_I$ and $\Lambda = D_C \circ \Lambda_I$ may not be Lagrangian submanifolds.

Remark 1. A remarkable application of the above considerations concerns the composition of symplectic relations (cf. [Weinstein, 1977]). Let $R_1 \subset (\mathcal{S}_1 \times \mathcal{S}_0, \omega_1 \ominus \omega_0)$ and $R_2 \subset (\mathcal{S}_2 \times \mathcal{S}_1, \omega_2 \ominus \omega_1)$ be smooth symplectic relations; then the composed relation can be interpreted as a reduced set,

$$R_2 \circ R_1 = R_C \circ (R_2 \times R_1),$$

where $R_2 \times R_2 \subset \mathcal{S} = \mathcal{S}_2 \times \mathcal{S}_1 \times \mathcal{S}_1 \times \mathcal{S}_0$ is interpreted as a Lagrangian submanifold with respect to the symplectic form

$$\omega = \omega_2 \ominus \omega_1 \oplus \omega_1 \ominus \omega_0,$$

and R_C is the reduction relation generated by the coisotropic submanifold

$$C = \mathcal{S}_2 \times \Delta_{\mathcal{S}_1} \times \mathcal{S}_0,$$

where $\Delta_{\mathcal{S}_1} \subset \mathcal{S}_1 \times \mathcal{S}_1$ is the diagonal. In this case the reduced symplectic manifold $\mathcal{S}_C$ is just $(\mathcal{S}_2 \times \mathcal{S}_0, \omega_2 \ominus \omega_0)$. It follows that if C and $R_2 \times R_1$ have clean (or transverse) intersection, then the composite relation $R_2 \circ R_1$ is a smooth (possibly immersed) symplectic relation.

Chapter 2

Symplectic relations on cotangent bundles

2.1 Cotangent bundles

A tangent **covector** on a manifold $\mathcal{Q}$ is a linear mapping

$$f: T_q\mathcal{Q} \to \mathbb{R}: v \mapsto \langle v, f \rangle.$$

We denote by

$$T_q^*\mathcal{Q}$$

the **cotangent space** at the point q i.e., the dual space of the tangent space $T_q\mathcal{Q}$.

The **cotangent bundle** $T^*\mathcal{Q}$ of a manifold $\mathcal{Q}_n$ is the set of all covectors i.e., the union of all cotangent spaces. It is a $2n$-dimensional manifold. We denote by

$$\pi_{\mathcal{Q}}: T^*\mathcal{Q} \to \mathcal{Q}$$

the **cotangent fibration** of $\mathcal{Q}$, which maps a covector $f \in T^*\mathcal{Q}$ to the point $q \in \mathcal{Q}$ where it is applied. We denote by

$$(q^i, p_i)$$

the **canonical coordinates** on $T^*\mathcal{Q}$ corresponding to coordinates $\underline{q} = (q^i)$ $(i = 1, \ldots, n)$ on $\mathcal{Q}$. They are defined as follows: if f is a covector at a point q in the domain of the coordinates, then $q^i(f)$ are the coordinates of q and $p_i(f)$ are the components of the covector in these coordinates, such that for all $v \in T_q\mathcal{Q}$,

$$\langle v, f \rangle = \dot{q}^i\, p_i = p_i\, \delta q^i.$$

There are two mechanical interpretations of a cotangent bundle.

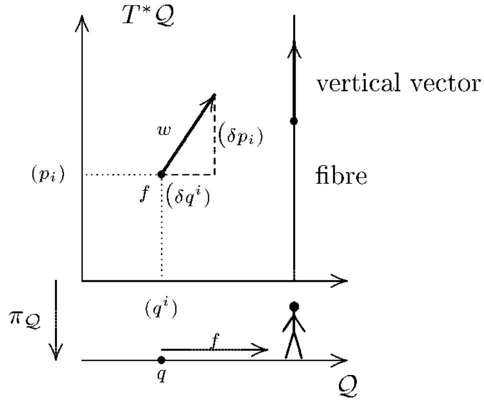
(1) If Q is a configuration manifold, then a covector $f \in T^*Q$ represents a **force** and the evaluation $\langle v, f\rangle$ the **virtual work** of the force corresponding to the virtual displacement v (or the **virtual power** if v is interpreted as a **virtual velocity**).

(2) If Q is a configuration manifold, then a covector $p \in T^*Q$ represents an **impulse** and the cotangent bundle T^*Q the **phase space** of the mechanical system.

It is useful to consider the vectors tangent to a cotangent bundle: they form the manifold $T(T^*Q) = TT^*Q$. Natural canonical coordinates (q^i, p_i) on T^*Q generate coordinates

$$(q^i, p_i, \dot{q}^i, \dot{p}_i) = (q^i, p_i, \delta q^i, \delta p_i)$$

on TT^*Q, where $(\dot{q}^i, \dot{p}_i) = (\delta q^i, \delta p_i)$ are the components of the tangent vectors $w \in TT^*Q$.



A vector w tangent to T^*Q is called **vertical** if is tangent to a fibre. A vector is vertical if and only if $T\pi_Q(w) = 0$. The vertical vectors are characterized by equations $\delta q^i = 0$.

2.2 The canonical symplectic structure of a cotangent bundle

With each vector w tangent to a cotangent bundle T^*Q we can associate "in a natural way" a real number. Let $w \in T_p(T^*Q)$. The point p where w is attached is an element of T^*Q i.e., a covector. As a covector, p is attached at a point $q \in Q$. Let u be the image of the vector w by the tangent mapping $T\pi_Q: TT^*Q \to TQ$ of the cotangent fibration $\pi_Q: T^*Q \to Q$. It is a vector $u \in T_qQ$ attached at the same point q of p. Thus, the evaluation $\langle u, p\rangle$ makes sense. This is just the number associated with w. If we consider the mapping defined in this way,

$$\theta_Q: TT^*Q \to \mathbb{R}: w \mapsto \langle u, p\rangle, \tag{1}$$

we can see that it is linear over the fibres of $T(T^*Q)$, so that it can be interpreted as a 1-form over T^*Q, and instead of (1) we can write

$$\langle w, \theta_Q\rangle = \langle u, p\rangle. \tag{2}$$

This is the **fundamental form** of a cotangent bundle, also called the **Liouville 1-form**. By the process illustrated above we have

$$u = T\pi_Q(w), \qquad p = \tau_{T^*Q}(w)$$

where

$$\tau_{T^*Q}: T(T^*Q) \to T^*Q$$

is the tangent fibration over T^*Q. It follows that the formal definition of the Liouville form is

$$\boxed{\langle w, \theta_Q\rangle = \langle T\pi_Q(w), \tau_{T^*Q}(w)\rangle} \tag{3}$$

Its representation in **natural canonical coordinates** (q^i, p_i) corresponding to coordinates (q^i) of Q is

$$\boxed{\theta_Q = p_i\, dq^i} \tag{4}$$

Indeed, if $w = (\dot{q}^i, \dot{p}_i)$, then $u = (\dot{q}^i)$ and (2) reads $\langle w, \theta_Q\rangle = p_i\dot{q}^i$ (†). On the other hand, any 1-form θ over T^*Q can be locally written $\theta = \theta_i\, dq^i + \theta^i\, dp_i$, so that $\langle w, \theta\rangle = \theta_i\, \dot{q}^i + \theta^i\, \dot{p}_i$. In order to get (†) we must have $\theta^i = 0$ and $\theta_i = p_i$.

The differential of the fundamental 1-form is the **canonical symplectic form** on T^*Q,

$$\boxed{\omega_Q = d\theta_Q} \tag{5}$$

In coordinates,

$$\boxed{\omega_Q = dp_i \wedge dq^i} \tag{6}$$

The corresponding expression of the **canonical Poisson bracket** is

$$\boxed{\{F, G\} = \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}} \tag{7}$$

and the first-order equations corresponding to a Hamiltonian vector field X_H are the **Hamilton equations**

$$\boxed{\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}} \tag{8}$$

2.3 One-forms as sections of cotangent bundles

A **1-form** (or **linear differential form**) on a manifold $\mathcal{Q}$ can be interpreted in three equivalent ways: (i) as a mapping

(1) $$\sigma: T\mathcal{Q} \to \mathbb{R}: v \mapsto \langle v, \sigma \rangle,$$

which is linear when restricted to each tangent space $T_q\mathcal{Q}$, $q \in \mathcal{Q}$; (ii) as a **section** of the cotangent bundle, that is as a mapping

(2) $$\sigma: \mathcal{Q} \to T^*\mathcal{Q}: q \mapsto \sigma(q),$$

such that $\sigma(q)$ is a covector in $T^*_q\mathcal{Q}$ (in this interpretation, we can say that a 1-form is a **field of covectors**; (iii) as an object locally expressed as a linear combination of the differentials of coordinates

(3) $$\sigma = \sigma_i(\underline{q})\, dq^i,$$

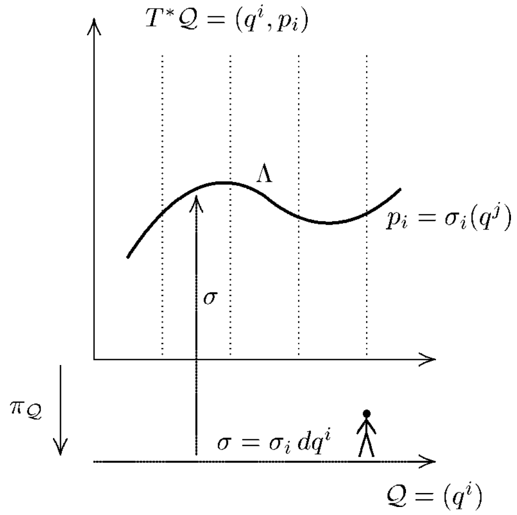
where σ_i are the **components** of σ. The link between (1) and (3) is given by

$$\langle v, \sigma \rangle = \dot{q}^i\, \sigma_i.$$

The link between (2) and (3) is given by

$$p_i = \sigma_i(\underline{q}).$$

The image of a 1-form



There is an important link between 1-forms and Lagrangian submanifolds:

Theorem 1. *The image $\sigma(Q) \subset T^*Q$ of a 1-form (interpreted as a section) is a Lagrangian submanifold if and only if σ is closed, $d\sigma = 0$.*

Proof. The image $\Lambda = \sigma(Q)$ is a submanifold of dimension $n = \frac{1}{2}\dim(T^*Q)$. If we consider the canonical symplectic form restricted to Λ, then we get

$$\omega_Q|\Lambda = d\sigma_i \wedge dq^i = \partial_j\sigma_i\, dq^j \wedge dq^i.$$

It follows that $\omega_Q|\Lambda = 0$ (isotropy condition) if and only if $\partial_j\sigma_i = \partial_i\sigma_j$, that is $d\sigma = 0$. ∎

Remark 1. Let $G\colon Q \to \mathbb{R}$ be a differentiable function. Its differential dG is an exact, thus closed, 1-form. Then its image

$$\Lambda = dG(Q),$$

which is locally described by the n equations

$$\boxed{p_i = \frac{\partial G}{\partial q^i}} \tag{4}$$

is a Lagrangian submanifold. This is the case when the closed 1-form σ is exact: $\sigma = dG$. Then G is said to be a **global generating function** of Λ. Of course any other function $G +$ constant is a generating function.

2.4 Lagrangian singularities and caustics

Let $\Lambda \subset T^*Q$ be a Lagrangian submanifold of a cotangent bundle. A **regular point** $p \in \Lambda$ is a point where Λ is **transversal to the fibres** i.e., where the tangent space $T_p\Lambda$ is complementary to the space V_p of the vertical vectors at p:

$$\begin{cases} T_p\Lambda \cap V_p = 0 \\ T_p\Lambda + V_p = T_p(T^*Q). \end{cases} \tag{1}$$

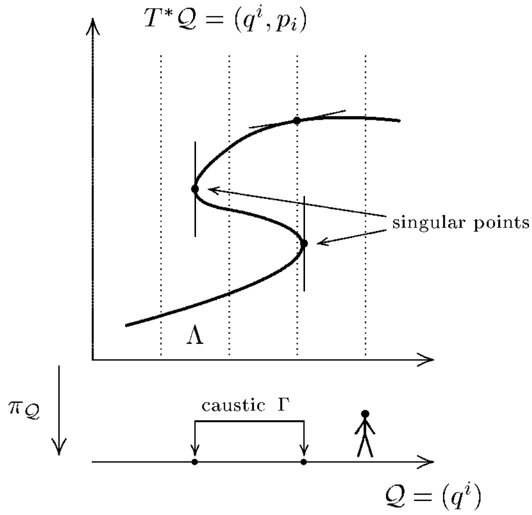
We remark that these two conditions, which express the complementarity of the two subspaces $T_p\Lambda$ and V_p, are in fact equivalent, since the subspaces $T_p\Lambda$ and V_p are both Lagrangian subspaces of $T_p(T^*Q)$,

$$\begin{aligned} T_p\Lambda \cap V_p = 0 \quad &\Leftrightarrow \quad (T_p\Lambda \cap V_p)^\S = T_p(T^*Q) \\ &\Leftrightarrow \quad (T_p\Lambda)^\S + (V_p)^\S = T_p(T^*Q) \\ &\Leftrightarrow \quad T_p\Lambda + V_p = T_p(T^*Q). \end{aligned}$$

A non-regular point is called **singular**. A singular point of a Lagrangian submanifold is also called **Lagrangian singularity** or **catastrophe**. The set $\Gamma(\Lambda) \subset Q$

of the points of Q on which are based all the singular points is called the **caustic** of Λ.

Remark 1. (i) A point p is regular if and only if the tangent space $T_p\Lambda$ does not contain vertical vectors except the zero vector. A point is regular if and only if the restriction $\pi\colon \Lambda \to Q$ of the cotangent fibration $\pi_Q\colon T^*Q \to Q$ to Λ is a submersion at p.



To "measure" the degree of singularity we introduce the **rank** of a point $p \in \Lambda$: it is the dimension of the projection onto T_qQ, $q = \pi_Q(p)$, of the tangent space $T_p\Lambda$:

$$\operatorname{rank}(p) = \dim\big(T\pi_Q(T_p\Lambda)\big). \tag{2}$$

A point p is regular if and only if $\operatorname{rank}(p) = n = \dim(Q)$. A Lagrangian submanifold is called **regular** if all its points are regular.

As for any submanifold, a Lagrangian submanifold can be represented, at least locally, by *parametric equations* or by *implicit equations*. For a Lagrangian submanifold there is however a third local representation, by means of **generating families**. This will be examined in the next section. In this section we consider the particular case of a representation by means of **generating functions**. In the following discussion we shall assume that the Lagrangian submanifold under consideration is a C^∞ submanifold.

2.4.1 Parametric equations. A system of $2n$ parametric equations in n parameters (λ^k),

$$\begin{cases} q^i = q^i(\lambda^k), \\ p_i = p_i(\lambda^k), \end{cases} \tag{1}$$

represents a *local immersion* $\iota: \Lambda \to T^*Q$ of a n-dimensional manifold Λ, with coordinates (λ^k), if and only if

(2) $$\operatorname{rank}\left[\frac{\partial q^i}{\partial \lambda^k} \,\middle|\, \frac{\partial p_i}{\partial \lambda^k}\right] = n \qquad (\max).$$

Indeed, the tangent mapping $T\iota$ is represented by equations

(3) $$\begin{cases} \dot{q}^i = \dfrac{\partial q^i}{\partial \lambda^k}\,\dot{\lambda}^k, \\ \dot{p}_i = \dfrac{\partial p_i}{\partial \lambda^k}\,\dot{\lambda}^k, \end{cases}$$

and this is a linear injective mapping at each point if and only if the rank of the matrix (2) is maximal ($\dot{q}^i = 0$ and $\dot{p}_i = 0$ must imply $\dot{\lambda}^k = 0$). The submanifold Λ is Lagrangian if and only if

(4) $$\frac{\partial q^i}{\partial \lambda^k}\frac{\partial p_i}{\partial \lambda^j} - \frac{\partial q^i}{\partial \lambda^j}\frac{\partial p_i}{\partial \lambda^k} = 0.$$

Indeed, these equations are equivalent to the isotropy condition $\iota^* d\theta_Q = 0$. The left hand sides of equations (4) are called **Lagrangian brackets**. In this representation, the rank of a point is given by

(5) $$\operatorname{rank}(p) = \operatorname{rank}\left[\frac{\partial q^i}{\partial \lambda^k}\right]_p,$$

where the evaluation at the point p of the matrix means its evaluation at those values of the parameters λ^k corresponding to the point p. Indeed, the tangent mapping $T\pi: T\Lambda \to TQ$ is described by equations $\dot{q}^i = \dfrac{\partial q^i}{\partial \lambda^k}\,\dot{\lambda}^k$. Hence,

(6) $$p \text{ regular} \quad \Longleftrightarrow \quad \det\left[\frac{\partial q^i}{\partial \lambda^k}\right]_p \neq 0.$$

It follows that the set of the singular points is described by equation

(7) $$\det\left[\frac{\partial q^i}{\partial \lambda^k}\right] = 0.$$

Note that this is a single equation on the parameters λ^k. Hence, it is an equation on Λ.

2.4.2 Implicit equations. A submanifold $\Lambda \subset T^*Q$ of codimension n (thus, of dimension n) can be represented (at least locally) by n independent equations

$$\Lambda^i(\underline{q}, \underline{p}) = 0. \tag{8}$$

This means that

$$\text{rank}\left[\frac{\partial \Lambda^i}{\partial q^k} \,\middle|\, \frac{\partial \Lambda^i}{\partial p_k}\right] = n \qquad \text{(max)} \tag{9}$$

at each point of Λ i.e., for each set of values of the coordinates satisfying equations (8). The submanifold Λ is Lagrangian if and only if it is coisotropic i.e., if and only if

$$\{\Lambda^i, \Lambda^j\}|\Lambda = 0. \tag{10}$$

In this representation,

$$\text{rank}(p) = \text{rank}\left[\frac{\partial \Lambda^i}{\partial p_k}\right]_p . \tag{11}$$

Indeed, the tangent subbundle $T\Lambda \subset TT^*Q$ is described by equations

$$\frac{\partial \Lambda^i}{\partial q^k}\, \dot{q}^k + \frac{\partial \Lambda^i}{\partial p_k}\, \dot{p}_k = 0,$$

so that at any point p the dimension of the space of the vertical vectors, for which $\dot{q}^i = 0$, is given by the corank of the matrix (11) at that point; this is the codimension of the space $T_p\pi(T_p\Lambda)$. Hence,

$$p \text{ regular} \iff \det\left[\frac{\partial \Lambda^i}{\partial p_k}\right]_p \neq 0. \tag{12}$$

and the caustic $\Gamma \subseteq Q$ of Λ is described by equation

$$\det\left[\frac{\partial \Lambda^i}{\partial p_k}\right]_p = 0. \tag{13}$$

together with equations (8).

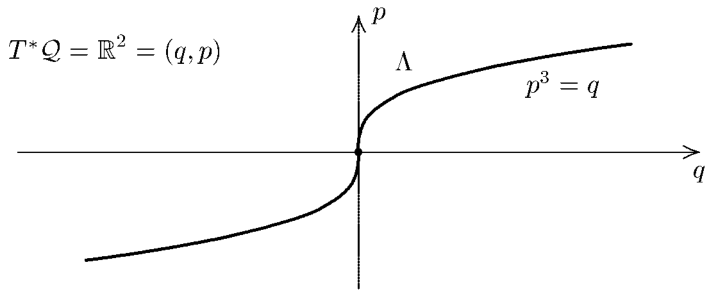
2.4.3 Generating functions. In a neighborhood of a regular point p a Lagrangian submanifold can be described by equations of the kind

$$p_i = \frac{\partial G}{\partial q^i}, \tag{14}$$

where $G(\underline{q})$ is a function in a neighborhood of the point $q = \pi_Q(p)$. Indeed, due to condition (12), equations (8) are locally solvable with respect to p_i, $p_i = \sigma_i(q^j)$. Being Λ Lagrangian, the form $\sigma = \sigma_i\, dq^i$ is closed, thus locally exact.

However, there are cases in which a representation of the kind (14) holds also in a neighborhood of a singular point. A simple example is the following.

Example 1. $\mathcal{Q} = \mathbb{R} = (q)$, $T^*\mathcal{Q} = \mathbb{R}^2 = (q,p)$, Λ the curve $q = p^3$. All points of Λ are regular except $(q,p) = (0,0)$. Since $p = q^{\frac{1}{3}}$, this Lagrangian submanifold is the image of the 1-form $\sigma = q^{\frac{1}{3}}\, dq$; thus, it is generated by the function $G(q) = \frac{3}{4}\, q^{\frac{4}{3}}$. We remark, however, that this function is only C^1 (it does not admits the second derivative at the point $q = 0$). In spite of this, as we shall see in the next section, this Lagrangian submanifold admits a C^∞ "generating family".



2.5 Generating families

A way for extending the notion of generating function is the following.

Definition 1. We call **generating family** on a manifold $\mathcal{Q}$ a smooth function $G\colon \mathcal{Q} \times U \to \mathbb{R}$, where U is a **supplementary manifold.**[1] A generating family

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[1] A more general definition of generating family is illustrated in Remark 3 below: it is a function $G\colon \mathcal{Z} \to \mathbb{R}$ on a manifold $\mathcal{Z}$ endowed with a surjective submersion $\zeta\colon \mathcal{Z} \to \mathcal{Q}$. However, in all the examples of application we shall encounter in these lectures, $\mathcal{Z}$ is a Cartesian product $\mathcal{Q} \times U$ and $\zeta\colon \mathcal{Q} \times U \to \mathcal{Q}$ is the natural projection. For a general survey on the notion of "generating family", or "generating function", and its applications to Analytical Mechanics see [Tulczyjew, 1974], [Weinstein, 1977], [Tulczyjew, 1977b], [Arnold, 1986], [Libermann, Marle, 1987]. The definition given here differs from that given for instance in [Arnold, 1986] and in [Chaperon, 1995], where the term "generating family" is introduced for what here (or in [Weinstein, 1977]) is called "Morse family". In [Chaperon, 1995], §1.3, the term "phase function" is used for a function on $\mathcal{Q} \times E$, where E is a finite-dimensional vector space. As remarked in [Arnold, 1986], the notion of generating family already appears in the works of Jacobi and Lie and in [Whittaker,

generates a **Lagrangian set** $\Lambda \subset T^*Q$, locally defined by equations

$$(1) \qquad \boxed{\begin{aligned} p_i &= \frac{\partial G}{\partial q^i} \\ 0 &= \frac{\partial G}{\partial u^\alpha} \end{aligned}}$$

where $G = G(\underline{q}, \underline{u})$ is a local representative of G in coordinates (q^i, u^α) of $Q \times U$; $i = 1, \dots, n$; $\alpha = 1, \dots, r$; $n = \dim(Q)$, $r = \dim(U)$. We call $\underline{u} = (u^\alpha)$ **supplementary coordinates** or **supplementary (extra) variables**. A point $p \in T^*Q$ belongs to Λ if and only if its coordinates $(\underline{q}, \underline{p})$ satisfy equations (1) for some values of the supplementary coordinates. The **critical set** of a generating family is the subset $\Xi \subseteq Q \times U$ of the stationary points of G *along the fibres*

$$\Xi = \{(q, u) \in Q \times U \mid d_u G(q) = 0\}.$$

This set is locally described by equations

$$(2) \qquad \frac{\partial G}{\partial u^\alpha} = 0.$$

Definition 2. A generating family is called **Morse family** if the $r \times (n + r)$ matrix

$$[\mathbf{G}_{qu} \,|\, \mathbf{G}_{uu}] = \left[\frac{\partial^2 G}{\partial u^\alpha \partial q^i} \middle| \frac{\partial^2 G}{\partial u^\alpha \partial u^\beta} \right]$$

has maximal rank at each point of the critical set,

$$(3) \qquad \operatorname{rank} \left[\frac{\partial^2 G}{\partial u^\alpha \partial q^i} \middle| \frac{\partial^2 G}{\partial u^\alpha \partial u^\beta} \right]_\Xi = r.$$

Theorem 1. *If G is a Morse family then the Lagrangian set Λ described by equations* (1) *is an immersed Lagrangian submanifold of T^*Q.*

Proof. (i) To prove that Λ is a n-dimensional submanifold of T^*Q we consider the set $\Lambda' \subset T^*Q \times U$ defined by the $n + r$ equations

$$(\dagger) \qquad F_i \doteq p_i - \frac{\partial G}{\partial q^i} = 0, \qquad F_\alpha \doteq \frac{\partial S}{\partial u^\alpha} = 0.$$

We remark that: (i) the canonical projection of Λ' to $Q \times U$ is the critical set,

$$\xi = (\pi_Q \times \mathrm{id}_U)(\Lambda');$$

1927]. The expression *generating family* or *Morse family* is suggested by the fact that G is regarded as a family of functions parametrized by the points $q \in Q$.

(ii) the canonical projection of Λ' to T^*Q is the set Λ,

$$\Lambda = \mathrm{pr}_{T^*Q}(\Lambda').$$

The matrix
$$\begin{bmatrix} \partial^j F_i & \partial_j F_i & \partial_\beta F_i \\ \partial^j F_\alpha & \partial_j F_\alpha & \partial_\beta F_\alpha \end{bmatrix} = \begin{bmatrix} \delta_i^j & \partial_j F_i & \partial_\beta F_i \\ 0 & \partial_j \partial_\alpha G & \partial_{\beta\alpha} G \end{bmatrix},$$
where $\partial_\alpha = \partial/\partial u^\alpha$, has maximal rank at the points of Λ', since the submatrix

$$[\,\partial_j \partial_\alpha G \quad | \quad \partial_{\beta\alpha} G\,]$$

has maximal rank, due to condition (3). This proves that equations (†) are independent, so that Λ' is a submanifold of codimension $n + r$ in a manifold of dimension $2n + r$ thus, a submanifold of dimension n. The set Λ is the projection on T^*Q of Λ, since it is defined by the same equation of Λ', leaving out the coordinates (u^α). The Cartesian projection $T^*Q \times U \to T^*Q$ preserves the submanifold structure if Λ' is transversal to its fibres. This happens if Λ' has no tangent vectors (except the zero vectors) which are vertical in this projection. Let $T\Lambda'$ be the set of the vectors tangent to Λ'. It is described by equations

$$(\ddagger) \qquad \begin{cases} \dot{p}_i - \partial_{ij} G\, \dot{q}^j - \partial_i \partial_\alpha G\, \dot{u}^\alpha = 0, \\ \partial_i \partial_\alpha G\, \dot{q}^i + \partial_{\alpha\beta} G\, \dot{u}^\beta = 0. \end{cases}$$

together with equations (†). A vertical vector is defined by $\dot{q}^i = \dot{p}_i = 0$. From (‡) it follows that
$$\begin{cases} \partial_i \partial_\alpha G\, \dot{u}^\alpha = 0, \\ \partial_{\alpha\beta} G\, \dot{u}^\beta = 0. \end{cases}$$
This is a linear homogenous system in $(\dot{u}^\alpha)$. The matrix of the coefficients is again the matrix (3) of maximal rank. It follows that $\dot{u}^\alpha = 0$ i.e., that the only vertical vectors tangent to Λ' are the zero vectors. This proves the assertion. (ii) It remains to prove that Λ is isotropic (since it has dimension n, it is Lagrangian). By equations (1) we get

$$\omega|\Lambda = (dp_i \wedge dq^i)|\Lambda = \partial_j \partial_i G\, dq^j \wedge dq^i + \partial_\alpha \partial_i g\, du^\alpha \wedge dq^i = 0,$$

since $\partial_{ij} G$ is symmetric in the indices, and $\partial_i \partial_\alpha G = 0$ on Λ, because of (1). ■

Remark 1. Equations (1) are equivalent to the *"differential equation"*

$$(4) \qquad \boxed{p_i\, dq^i = dG}$$

or to the *"variational equation"*

$$(4') \qquad \boxed{p_i\, \delta q^i = \delta G}$$

Theorem 2. *If G is a Morse family, then the caustic $\Gamma \subseteq \mathcal{Q}$ of Λ is described by equations*

$$\boxed{\det\left[\frac{\partial^2 G}{\partial u^\alpha \partial u^\beta}\right] = 0, \quad \frac{\partial G}{\partial u^\alpha} = 0} \tag{5}$$

A point $q \in \mathcal{Q}$ belongs to Γ if and only if its coordinates satisfy equations (5) for some values of the supplementary variables.[2]

Remark 2. Equations (5) are equivalent to the single equation

$$\boxed{\det\left[\frac{\partial^2 G}{\partial u^\alpha \partial u^\beta}\right]_\Xi = 0} \tag{6}$$

Remark 3. Global definition of generating family. The above-given definitions and statements have a coordinate-independent meaning. It is convenient to introduce the notion of generating family within a more general setting. Let $\zeta\colon \mathcal{Z} \to \mathcal{Q}$ be a surjective submersion (in most of the applications ζ is a trivial fibration and $\mathcal{Z} = \mathcal{Q} \times U$) and let $V(\zeta) \subset T\mathcal{Z}$ be the subbundle of the **vertical vectors**,

$$V(\zeta) = \{v \in T\mathcal{Z} \mid T\zeta(v) = 0\}.$$

Let $C = V^\circ(\zeta)$ be the **conormal bundle** of $V(\zeta)$ i.e., the set of the covectors annihilating the vertical vectors,

$$C = V^\circ(\zeta) = \{f \in T^*_z\mathcal{Z} \mid \langle v, f\rangle = 0, \forall v \in V_z(\zeta)\}. \tag{7}$$

[2] Proof. A point of Λ is regular if at that point $\dot q^i = 0 \quad \Rightarrow \quad \dot p_i = 0$. If we put $\dot q^i = 0$ in equations (‡) we get, in matrix notation

$$\begin{cases} \dot{\mathbf{p}} - \mathbf{G}_{qu}\,\dot{\mathbf{u}} = 0, \\ \mathbf{G}_{uu}\,\dot{\mathbf{u}} = 0. \end{cases} \tag{$\star$}$$

(i) Assume $\det \mathbf{G}_{uu} = 0$. Then $\mathbf{K} = \mathrm{Ker}\mathbf{G}_{uu} \neq 0$. Suppose that $\mathbf{G}_{qu}(\mathbf{K}) = 0$. This means that $\mathbf{K} \neq 0$ is in the kernel of both $\mathbf{G}_{uu}$ and $\mathbf{G}_{qu}$, so that $\mathbf{K}$ is sent to the zero vector also by $[\mathbf{G}_{qu}, \mathbf{G}_{uu}]$. But this matrix has zero kernel, since it has maximal rank: absurd. Hence, $\mathbf{G}_{qu}(\mathbf{K}) \neq 0$. This means that there exists a vector $\dot{\mathbf{u}} \in \mathbf{K}$ whose image $\dot{\mathbf{p}} = \mathbf{G}_{qu}\,\dot{\mathbf{u}}$ is not zero. Thus, equations ($\star$) admits a non-zero solution and the point is singular. (ii) If $\det \mathbf{G}_{uu} \neq 0$, then from the second part of equations ($\star$) we get $\dot{\mathbf{u}} = 0$, and from the first part $\dot{\mathbf{p}} = 0$. In this case the point is regular. Hence, the points of the caustic are such that their coordinates (q^i) satisfy equation $\det[\partial_{\alpha\beta}G] = 0$ for some values of the supplementary coordinates (u^α). ■

Let us consider the graph of ζ as a relation $R_\zeta \subset \mathcal{Q} \times \mathcal{Z}$. Its "canonical lift" (cf. §2.9) is a symplectic reductions $\widehat{R}_\zeta \subset T^*\mathcal{Q} \times T^*\mathcal{Z}$, whose inverse image is just $C = V^\circ(\zeta)$.[3]

Let $\Lambda_G = dG(\mathcal{Z})$ be the regular Lagrangian submanifold generated by a smooth function $G: \mathcal{Z} \to \mathbb{R}$. Then the reduced set

$$\Lambda = \widehat{R}_\zeta \circ \Lambda_G \tag{8}$$

is the Lagrangian set generated by the generating family G. It can be shown that this set is locally described by equations (1), being $(\underline{q}, \underline{u})$ local coordinates on $\mathcal{Z}$ adapted to the fibration ζ. We say that G is a **Morse family** if C and Λ_G have *transverse intersection* (cf. [Weinstein, 1977]). This means that

$$T_p\Lambda_G + T_pC = T_p(T^*\mathcal{Z}), \qquad \forall\, p \in \Lambda_G \cap C, \tag{9}$$

or equivalently,

$$T_p\Lambda_G \cap T_p^\S C = 0, \qquad \forall\, p \in \Lambda_G \cap C. \tag{10}$$

The transversality is locally expressed by condition (3).[4]

A fundamental fact is that *any Lagrangian submanifold of a cotangent bundle is locally generated by Morse families*:

Theorem 3 (Maslov-Hörmander theorem). *If $\Lambda \subset T^*\mathcal{Q}$ is a Lagrangian submanifold, then for each $p \in \Lambda$ there exists a Morse family generating Λ in a neighborhood of p.*

---

[3] Cf. [Weinstein, 1977]. See also the proof in the following footnote.

[4] Proof. Let us consider coordinates (q^i, u^α) on $\mathcal{Z}$ adapted to the submersion ζ. Let $(q^i, u^\alpha; p_i, r_\alpha)$ be the canonical coordinates on $T^*\mathcal{Z}$. The vertical vectors are then characterized by equations $\dot{q}^i = 0$. Since $\langle v, f\rangle = p_i\,\dot{q}^i + r_\alpha\,\dot{u}^\alpha$, the condition expressed in the definition (7) is equivalent to $r_\alpha\,\dot{u}^\alpha = 0$ for all $(\dot{u}^\alpha) \in \mathbb{R}^r$. Thus, the conormal bundle $C = V^\circ$ is described by equation $r_\alpha = 0$ (since $\{r_\alpha, r_\beta\} = 0$, this shows that C is coisotropic). Due to Theorem 1, §1.4, the characteristic distribution of C is spanned by the Hamiltonian vector fields generated by the Hamiltonians $H_\alpha(\underline{p}, \underline{r}) = r_\alpha$. It follows that the characteristic distribution $T^\S C$ is described by equations

$$r_\alpha = 0, \quad \dot{q}^i = 0, \quad \dot{p}_i = 0, \quad \dot{u}^\alpha = \lambda^\alpha, \quad \dot{r}_\alpha = 0.$$

On the other hand, Λ_G is described by equations

$$p_i = \frac{\partial G}{\partial q^i}, \quad r_\alpha = \frac{\partial G}{\partial u^\alpha},$$

Proof. [5] Let us consider a Lagrangian immersion

$$q^i = q^i(\lambda^k), \qquad p_i = p_i(\lambda^k) \tag{11}$$

(equation (1) of §2.4.1) and the $2n \times n$ matrix with maximal rank

$$\begin{bmatrix} Q^i_k \\ \\ P_{jk} \end{bmatrix} = \begin{bmatrix} \frac{\partial q^i}{\partial \lambda^k} \\ \\ \frac{\partial p_j}{\partial \lambda^k} \end{bmatrix}. \tag{12}$$

Since the immersion is Lagrangian, equation (4) of §2.4.1 hold i.e.,

$$Q^i_k P_{ij} - Q^i_j P_{ik} = 0.$$

Under this condition, it can be proved (see §6.8) that, up to a reordering of the coordinates (q^i), the matrix (12) admits a regular $n \times n$ submatrix of the kind

$$\begin{bmatrix} Q^a_k \\ \\ P_{\alpha k} \end{bmatrix} = \begin{bmatrix} \frac{\partial q^a}{\partial \lambda^k} \\ \\ \frac{\partial p_\alpha}{\partial \lambda^k} \end{bmatrix}, \qquad a = 1, \dots, m, \qquad \alpha = m+1, \dots, n,$$

so that $T\Lambda_G$ is described by equations

$$\begin{cases} p_i - \dfrac{\partial G}{\partial q^i} = 0, \quad r_\alpha - \dfrac{\partial G}{\partial u^\alpha} = 0, \\ \dot{p}_i - \dfrac{\partial^2 G}{\partial q^i \partial q^j}\,\dot{q}^j - \dfrac{\partial^2 G}{\partial q^i \partial u^\beta}\,\dot{u}^\beta = 0, \\ \dot{r}_\alpha - \dfrac{\partial^2 G}{\partial u^\alpha \partial q^j}\,\dot{q}^j - \dfrac{\partial^2 G}{\partial u^\alpha \partial u^\beta}\,\dot{u}^\beta = 0. \end{cases}$$

It follows that $C \cap \Lambda_G$ is described by equations

$$p_i = \frac{\partial G}{\partial q^i}, \quad 0 = \frac{\partial G}{\partial u^\alpha},$$

which are just the equations (1) of the Lagrangian set Λ generated by G, while the intersection $T\Lambda_G \cap T^{\S}C$ is described by equations

$$\frac{\partial G}{\partial u^\alpha} = 0, \qquad \frac{\partial^2 G}{\partial q^i \partial u^\beta}\,\dot{u}^\beta = 0, \qquad \frac{\partial^2 G}{\partial u^\alpha \partial u^\beta}\,\dot{u}^\beta = 0.$$

This system admits the unique solution $\dot{u}^\alpha = 0$ if and only if the rank of the involved matrix is maximal, condition (3). ■

[5] See also [Libermann, Marle, 1987], [Weinstein, 1977]. The proof given here is taken from [Benenti, 1988].

so that the subsystem of (11),

$$q^a = q^a(\lambda^k), \qquad p_\alpha = p_\alpha(\lambda^k),$$

can be solved (locally) with respect to (λ^k): $\lambda^k = \lambda^k(q^a, p_\alpha)$. This means that we can take (q^a, p_α) as parameters of the immersion, so that a Lagrangian submanifold Λ can be always represented by local immersions of the kind[6]

$$q^\alpha = q^\alpha(q^b, p_\beta), \qquad p_a = p_a(q^b, p_\beta). \tag{13}$$

The 1-form

$$\theta = p_a\, dq^a - q^\alpha\, dp_\alpha$$

is such that $d\theta = \omega = dp_i \wedge dq^i$ (the canonical symplectic form). Its pull-back to Λ is closed (since Λ is Lagrangian), thus locally exact. It follows that there exists a function $F(q^a, p_\alpha)$ such that (13) are equivalent to

$$p_a = \frac{\partial F}{\partial q^a}, \qquad q^\alpha = -\frac{\partial F}{\partial p_\alpha}. \tag{14}$$

Let us consider the function

$$G(q^i; p_\alpha) = F(q^a, p_\alpha) + p_\alpha\, q^\alpha.$$

This is a Morse family on (q^i) with supplementary variables (p_α). Indeed, in the matrix

$$\left[\frac{\partial^2 G}{\partial p_\alpha \partial p_\beta} \;\middle|\; \frac{\partial^2 G}{\partial p_\alpha \partial q^i}\right]$$

we find the regular square matrix

$$\left[\frac{\partial^2 F}{\partial p_\alpha \partial q^\beta}\right] = [\delta^\alpha_\beta].$$

The equations of the Lagrangian set generated by this Morse family are

$$\begin{cases} p_i = \dfrac{\partial G}{\partial q^i} \quad \to \quad \begin{cases} p_a = \dfrac{\partial G}{\partial q^a} = \dfrac{\partial F}{\partial q^a} \\ p_\alpha = \dfrac{\partial G}{\partial q^\alpha} = p_\alpha \end{cases} \\ 0 = \dfrac{\partial G}{\partial p_\alpha} = q^\alpha + \dfrac{\partial F}{\partial p_\alpha}. \end{cases}$$

[6] The parameters (q^a, p_α) are local coordinates on the Lagrangian submanifold Λ. They are called **canonical coordinates** of Λ. For more information about the existence and the use of a **canonical atlas** of a Lagrangian submanifold see [Mishchenko, Shatalov, Sternin, 1978].

These equations coincide with equations (14) of Λ. ■
Remark 4. Two generating families (or Morse families) are **equivalent** if they generate the same Lagrangian set (or the same Lagrangian submanifold). Two generating families differing by an additive constant are obviously equivalent.[7]
Remark 5. In some cases it is possible to "lower" the dimension of the supplementary manifold (i.e., the number of the supplementary variables) of a given generating family G and get an equivalent **reduced generating family**. This depends on the critical set of G. For instance, if the critical set is the image of a section $\xi: \mathcal{Q} \to \mathcal{Z}$ of ζ, then Λ is generated (in the ordinary sense) by the function $G \circ \xi: \mathcal{Q} \to \mathbb{R}$. This is of course an extreme case. In general, we can remove some of the (u^α) when, by solving equations (2), they can be expressed as functions of the coordinates and of the remaining ones. Note that <u>all</u> the supplementary variables can be removed, and get an ordinary generating function, if and only if

$$\text{(14)} \qquad \det \left[\frac{\partial^2 G}{\partial u^\alpha \partial u^\beta} \right]_\Xi \neq 0,$$

and this is the case of a regular Lagrangian submanifold (no caustic), in accordance with Theorem 2 and Remark 2.
Example 1. The Lagrangian submanifold of Example 1, §2.4, is generated by the global C^∞ Morse family $G(q; u) = uq - \frac{1}{4}\, u^4$.
Remark 6. In considering generating families which are Morse families, as it is commonly done in the literature, we lose most of the interest and of the power of this concept. Indeed, as we shall see in the following, there are physical and mathematical interesting examples related to generating families which are not, or cannot be, even up to equivalences, Morse families. For instance, there are *systems of rays* or sets of *equilibrium states* of static systems which are Lagrangian sets, and not Lagrangian submanifolds (cf. Chapters 4 and 5); there are global Hamilton principal functions which are not Morse families (cf. Appendix B; see also [Cardin, 2002], where such an example arises in the construction of global solutions of the Cauchy problem for a t-dependent Hamilton-Jacobi equation).

2.6 Generating families of symplectic relations

Let $\zeta: \mathcal{Q}_2 \times \mathcal{Q}_1 \times U \to \mathcal{Q}_2 \times \mathcal{Q}_1$ be a trivial fibration. Let $G: \mathcal{Q}_1 \times \mathcal{Q}_2 \times U \to \mathbb{R}$ be a smooth function, locally represented by functions

$$G(\underline{q}_2, \underline{q}_1; \underline{u})$$

[7] For a general theory of the equivalence of Morse families, and related references, see [Arnold, 1986], [Libermann, Marle, 1987], [Viterbo, 1992], [Théret, 1999].

in the coordinates of the factor manifolds. By setting $Q = Q_1 \times Q_2$ and applying the definitions and theorems of the preceding section, it follows that if G is a Morse family, then it generates a Lagrangian submanifold R' of $T^*Q \simeq T^*Q_2 \times T^*Q_1$ with respect to the canonical symplectic form $\omega_{Q_2} \oplus \omega_{Q_1}$. This is not a symplectic relation. In order to get a symplectic relation we apply the symplectic transformation

$$\iota: T^*Q_2 \times T^*Q_1 \to T^*Q_2 \times T^*Q_1: (p_2, p_1) \mapsto (p_2, -p_1)$$

so that

$$\iota^*(\omega_{Q_2} \ominus \omega_{Q_1}) = \omega_{Q_2} \oplus \omega_{Q_1}.$$

Thus we get a Lagrangian submanifold $R = \iota(R')$ with respect to the symplectic form

$$\omega_{Q_2} \ominus \omega_{Q_1} = d\underline{p}_2 \wedge d\underline{q}_2 - d\underline{p}_1 \wedge d\underline{q}_1$$

i.e., a symplectic relation R from T^*Q_1 to T^*Q_2. Then we say that G is a **generating family** of the symplectic relation R. This relation is locally described by equations

$$(1) \qquad \boxed{\begin{aligned} \underline{p}_1 &= -\frac{\partial G}{\partial \underline{q}_1} \\ \underline{p}_2 &= \frac{\partial G}{\partial \underline{q}_2} \\ 0 &= \frac{\partial G}{\partial \underline{u}} \end{aligned}}$$

which are equivalent to the single "differential equation"

$$(2) \qquad \boxed{\underline{p}_2 d\underline{q}_2 - \underline{p}_1 d\underline{q}_1 = dG}$$

We denote a generating family of a symplectic relation by

$$G(Q_2 \times Q_1; U).$$

Theorem 1. *If the symplectic relation $R \subset T^*Q_2 \times T^*Q_1$ is generated by $G(Q_2 \times Q_1; U)$, then the inverse relation $R^\top \subset T^*Q_1 \times T^*Q_2$ is generated by $G^\top(Q_1 \times Q_2; U)$ defined by*

$$(3) \qquad G^\top(q_1, q_2; u) = -G(q_2, q_1; u), \qquad (q_1, q_2) \in Q_1 \times Q_2.$$

Proof. If R is described by equations (1) then $R^\top$ is described by similar equations

$$(4) \qquad \underline{p}_2 = -\frac{\partial G^\top}{\partial \underline{q}_2}, \quad \underline{p}_1 = \frac{\partial G^\top}{\partial \underline{q}_1}, \quad 0 = \frac{\partial G^\top}{\partial \underline{u}}.$$

The relations R and $R^\top$ must be described by equivalent equations, since these two relations differ only by the order in the pairs. Equations (1) and (4) coincide if $S^\top$ is defined as in (3). ∎

It follows that

Theorem 2. *A symplectic relation $D \subset T^*Q \times T^*Q$ generated by $G(Q \times Q; U)$ is symmetric,*

$$D^\top = D,$$

if and only if G is skew-symmetric, up to a constant, on the pairs (q, q') belonging to the critical set,

$$G(q, q', u) = -G(q', q, u), \quad (q, q', u) \in \Xi.$$

Proof. If D is symmetric then $G(q, q', u) = -G(q', q, u) + c$, where c is a constant. If we replace G by $\bar{G} = G - \frac{c}{2}$, then $\bar{G}(q, q'; u) + \bar{G}(q', q; u) = G(q, q'; u) + G(q', q; u) - c = 0$. ∎

2.7 The composition of generating families

Theorem 1. *If the symplectic relations*

$$R_2 \subset T^*Q_2 \times T^*Q_1, \qquad R_1 \subset T^*Q_1 \times T^*Q_0$$

are generated by the families

$$G_2(Q_2 \times Q_1; U_2), \qquad G_1(Q_1 \times Q_0; U_1),$$

then the relation $R_2 \circ R_1 \subset T^*Q_2 \times T^*Q_0$ is generated by the family

$$G_{21}(Q_2 \times Q_0; Q_1 \times U_2 \times U_1)$$

defined by[8]

$$(1) \qquad \boxed{G_{21}(q_2, q_0; q_1, u_2, u_1) = G_2(q_2, q_1; u_2) + G_1(q_1, q_0; u_1)}$$

[8] The composition rule of the "generating forms" of linear symplectic relations has been introduced in [Lawruk, Sniatycki, Tulczyjew, 1975] and [Benenti, Tulczyjew, 1981] (see also [Benenti, 1983c]) and extended to generating families of symplectic relations in [Benenti, 1988].

We shall denote this composition rule by

$$G_{21} = G_2 \oplus G_1.$$

Remark 1. In the generating family G_{21} the manifold $\mathcal{Q}_1$ plays the role of supplementary manifold, together with U_1 and U_2. The composed generating family G_{21} may not be a Morse family, but in any case it generates the relation $R_2 \circ R_1$, which may not be a smooth relation. In other words, while smooth relations do not always compose *nicely*, the composed generating family is always a smooth function.

Proof. The two relations are respectively described by equations

$$(2) \qquad R_1 : \begin{cases} \underline{p}_0 = -\dfrac{\partial G_1}{\partial \underline{q}_0} \\ \underline{p}_1 = \dfrac{\partial G_1}{\partial \underline{q}_1} \\ 0 = \dfrac{\partial G_1}{\partial \underline{u}_1} \end{cases} \qquad R_2 : \begin{cases} \underline{p}_1 = -\dfrac{\partial G_2}{\partial \underline{q}_1} \\ \underline{p}_2 = \dfrac{\partial G_2}{\partial \underline{q}_2} \\ 0 = \dfrac{\partial G_2}{\partial \underline{u}_2}. \end{cases}$$

In composing the two relations the two sets of coordinates p_1 coincide. Thus the relation $R_2 \circ R_1$ is described by equations

$$(3) \qquad \begin{cases} \underline{p}_0 = -\dfrac{\partial G_1}{\partial \underline{q}_0} \\ \underline{p}_2 = \dfrac{\partial G_2}{\partial \underline{q}_2} \end{cases} \qquad \begin{cases} 0 = \dfrac{\partial G_2}{\partial \underline{q}_1} + \dfrac{\partial G_1}{\partial \underline{q}_1} \\ 0 = \dfrac{\partial G_2}{\partial \underline{u}_2} \\ 0 = \dfrac{\partial G_1}{\partial \underline{u}_1}. \end{cases}$$

These equations are equivalent to the single equation

$$(4) \qquad \boxed{\underline{p}_2\, d\underline{q}_2 - \underline{p}_0\, d\underline{q}_0 = d(G_2 + G_1)}$$

This proves the composition rule (1). ∎

Equation (4) is an equivalent form of the composition rule (1) of the generating families.

2.8 The canonical lift of submanifolds

There is an operation, which we call **canonical lift** or **canonical prolongation** and denote by $\widehat{\ }$, which transforms "geometrical objects" on a manifold $\mathcal{Q}$ (vector fields, transformations, submanifolds, etc.) into "symplectic objects" on the corresponding cotangent bundle T^*Q. This operation plays an important role in the theory of symplectic relations and in its applications. The basic definition, from which the canonical prolongation of all other classes of objects are derived, is the canonical lift of a submanifold.

Definition 1. The **canonical lift of a submanifold** $\Sigma \subseteq Q$ is the set $\widehat{\Sigma} = T^\circ\Sigma \subset T^*Q$ of the covectors annihilating the vectors tangent to Σ,

$$\text{(1)} \qquad \boxed{p \in \widehat{\Sigma} \quad \Longleftrightarrow \quad \begin{cases} p \in T_q^*Q, & q \in \Sigma \\ \langle v, p\rangle = 0, & \forall v \in T_q\Sigma \end{cases}}$$

Remark 1. The set $\widehat{\Sigma}$ has the following mechanical interpretation: if Σ is a **smooth constraint** imposed on the configuration manifold Q of a holonomic system, then $\widehat{\Sigma}$ is the set of the **reaction forces**, whose virtual work is zero. If we interpret these forces as vectors (by means of a metric tensor) then $\widehat{\Sigma}$ is the set of all vectors orthogonal to Σ. Special remarkable cases are

$$\begin{cases} \Sigma = q \in Q \quad \text{(a point of } Q) \quad \longmapsto \quad \widehat{q} = T_q^*Q \quad \text{(the fibre over } q) \\ \Sigma = Q \quad \longmapsto \quad \widehat{Q} = Q \quad \text{(interpreted as the zero-section of } T^*Q) \end{cases}$$

The **zero-section** of T^*Q is the set of all zero covectors; thus it is identified with Q itself.

Definition 2. Let $\Sigma \subseteq Q$ be a submanifold and $F\colon \Sigma \to \mathbb{R}$ a smooth function. The **canonical lift of a submanifold with function** is the set $(\widehat{\Sigma, F}) \subset T^*Q$ defined by

$$\text{(2)} \qquad \boxed{p \in (\widehat{\Sigma, F}) \quad \Longleftrightarrow \quad \begin{cases} p \in T_q^*Q, & q \in \Sigma \\ \langle v, p\rangle = \langle v, dF\rangle, & \forall v \in T_q\Sigma \end{cases}}$$

This is the set the covectors p whose evaluation with any vector v tangent to Σ is equal to the derivative of the function F with respect to v. In the above definition F can be a function on the whole Q or on an open neighborhood of Σ. Indeed, only the restriction of F to Σ is involved.

Remark 2. The second line of definition (2) is equivalent to

$$\text{(3)} \qquad p - d_qF \in \widehat{\Sigma}.$$

Note that

$$\text{(4)} \qquad (\Sigma, \text{const.})\widehat{\ } = \widehat{\Sigma}.$$

If F represents a potential energy, then $\widehat{(\Sigma, F)}$ is the set of **equilibrium states**.

Theorem 1. *The canonical lifts* $\widehat{\Sigma}$ *and* $\widehat{(\Sigma, F)}$ *are Lagrangian submanifolds.*[9]

This can be proved in a direct way[10] or by using Morse families, as shown by the following

Theorem 2. *Let* Σ *be defined by independent equations*

$$\Sigma_\alpha(q) = 0, \qquad \alpha = 1, \ldots, r, \tag{5}$$

then, $\widehat{(\Sigma, F)}$ *is generated by the Morse family* $G\colon \mathcal{Q} \times U \to \mathbb{R}$, $\quad U = \mathbb{R}^r$, *defined by*

$$\boxed{G(q; u^\alpha) = u^\alpha\, \Sigma_\alpha(q) + F(q)} \tag{6}$$

Proof. The critical set Ξ is determined by equations $\partial G/\partial u^\alpha = \Sigma_\alpha = 0$, thus it coincides with $\Sigma \times U$. The maximal rank condition is fulfilled:

$$\operatorname{rank}\left[\frac{\partial^2 G}{\partial u^\alpha \partial q^i}\,\middle|\,\frac{\partial^2 G}{\partial u^\alpha \partial u^\beta}\right]_\Xi = \operatorname{rank}\left[\frac{\partial \Sigma_\alpha}{\partial q^i}\,\middle|\,0\right]_\Xi = \operatorname{rank}\left[\frac{\partial \Sigma_\alpha}{\partial q^i}\right]_\Xi = r.$$

Thus, G is a Morse family and generates a Lagrangian submanifold Λ by equations

$$\begin{cases} p_i = \dfrac{\partial G}{\partial q^i} = u^\alpha\, \partial_i \Sigma_\alpha + \partial_i F \\ 0 = \dfrac{\partial G}{\partial u^\alpha} = \Sigma_\alpha. \end{cases}$$

The vectors v tangent to Σ are characterized by equations

$$\frac{\partial \Sigma_\alpha}{\partial q^i}\, \dot{q}^i = 0,$$

thus,

$$\langle v, p\rangle = \dot{q}^i\, p_i = \dot{q}^i\, (u^\alpha\, \partial_i \Sigma_\alpha + \partial_i F) = \dot{q}^i\, \partial_i F = \langle v, dF\rangle$$

---

[9] Lagrangian submanifolds of this kind have been introduced by Tulczyjew [Tulczyjew, 1977b]. The definition of the canonical lift $\widehat{\Sigma}$ can be extended to any subset $\Sigma \subset \mathcal{Q}$, by a suitable definition of the tangent $T\Sigma$ of a subset given in [Tulczyjew, 1989].

[10] Let (q^i) be local coordinates on $\mathcal{Q}$ adapted to Σ. This means that Σ is locally described by equations $q^\alpha = 0$ $(\alpha = 1, \ldots, r)$, $r = \operatorname{codim}(\Sigma)$. Then, $T\Sigma$ is described by equations $q^\alpha = 0$, $\dot{q}^\alpha = 0$, and the condition $\langle v, p\rangle = \langle v, dF\rangle$, $\forall v \in T_q\Sigma$ becomes $\dot{q}^a\, (p_a - \partial_a F) = 0$, $\forall (\dot{q}^a) \in \mathbb{R}^{n-r}$ (where $a = r+1, \ldots, n$). Thus, $\Lambda = \widehat{(\Sigma, F)}$ is a submanifold of dimension n described by the n equations $q^\alpha = 0$, $p_a = \partial_a F$. The isotropy follows from Remark 5.

for all $p \in \Lambda$. This shows that $\Lambda = \widehat{\Sigma}$. ■

Remark 3. The supplementary variables (u^α) in definition (5) play the role of **Lagrangian multipliers**. We say that $\widehat{(\Sigma, F)}$ is the **Lagrangian submanifold generated by the function F on the constraint** Σ.

Remark 4. The Lagrangian submanifold $\Lambda = \widehat{(\Sigma, F)}$ projects onto Σ,

$$\pi_Q\big(\widehat{(\Sigma, F)}\big) = \Sigma$$

and the restriction $\pi \colon \Lambda \to \Sigma$ of π_Q to $\Lambda = \widehat{(\Sigma, F)}$ is a surjective submersion. Hence, all points have constant rank equal to $\dim \Sigma$ (if everything is of class C^2) and the caustic is Σ, unless Σ is an open subset of Q; in this case, Λ is regular.

Remark 5. The canonical lift $\Lambda = \widehat{(\Sigma, F)}$ is a Lagrangian submanifold of a special kind, which we call **exact**. Indeed, from definition (2) it follows that for all $w \in T_p\Lambda$, $\langle w, \theta_Q \rangle = \langle v, p \rangle = \langle v, dF \rangle = \langle w, \pi^* dF \rangle$, where $v = T\pi(w)$. This means that

$$\boxed{\theta_Q | \widehat{(\Sigma, F)} = d\pi^* F} \tag{7}$$

This shows that the pull-back of the Liouville 1-form to Λ is exact. For $F = 0$ (or constant) we get in particular

$$\boxed{\theta_Q | \widehat{\Sigma} = 0} \tag{8}$$

See §6.6 for further details.

2.9 The canonical lift of relations

We can extend to relations the definition of canonical lift of submanifolds.

Definition 1. The **canonical lift of a smooth relation** $R \subseteq Q_2 \times Q_1$ is the symplectic relation $\widehat{R} \subset T^*Q_2 \times T^*Q_1$ defined by

$$\boxed{(p_2, p_1) \in \widehat{R} \iff \begin{cases} (p_2, p_1) \in T^*_{q_2} Q_2 \times T^*_{q_1} Q_1, \quad (q_2, q_1) \in R \\ \langle v_2, p_2 \rangle = \langle v_1, p_1 \rangle, \quad \forall (v_2, v_1) \in T_{(q_2, q_1)} R \end{cases}} \tag{1}$$

Remark 2. Note that this is <u>not</u> the "true" canonical lift $\widehat{R}$ of R as a submanifold, that is the set

$$\widehat{R} = \big\{\bar{p} \in T^*_{(q_2, q_1)}(Q_2 \times Q_1) \mid (q_2, q_1) \in R, \ \langle \bar{v}, \bar{p} \rangle = 0, \ \forall \bar{v} \in T_{(q_2, q_1)} R\big\}$$

which is a Lagrangian submanifold of $\widehat{R} \subset T^*(Q_2 \times Q_1)$ with respect to the canonical symplectic form $d\theta_{Q_2 \times Q_1}$. Indeed, in order to get a symplectic relation from T^*Q_1 to T^*Q_2 we use the natural identification $T^*(Q_2 \times Q_1) \simeq T^*Q_2 \times T^*Q_1$ and the symplectomorphism $\iota: T^*(Q_1 \times Q) \to T^*Q_1 \times T^*Q$ considered in §2.6. Then we find definition (1). We use the same symbol $\widehat{R}$ for simplicity, since there is no danger of confusion. Indeed, if we consider a submanifold $\Sigma \subseteq Q$ as a zero-relation $\Sigma \subseteq Q \times 0$, then the canonical lift of Σ as a relation, defined in this section, is just the symplectic zero-relation in $T^*Q \times 0$ associated with the Lagrangian submanifold $\widehat{R} \subset T^*Q$ defined in the preceding section.

In a similar way we can introduce the canonical lift of a relation $R \subseteq Q_2 \times Q_1$ associated with a function $F: R \to \mathbb{R}$ or $F: Q_2 \times Q_1 \to \mathbb{R}$. It is the smooth symplectic relation defined by

$$(2) \quad \boxed{\begin{array}{l} (p_2, p_1) \in \widehat{(R, F)} \iff \\ \qquad \begin{cases} (p_2, p_1) \in T_{(q_2, q_1)}(Q_2 \times Q_1), \quad (q_2, q_1) \in R \\ \langle v_2, p_2 \rangle - \langle v_1, p_1 \rangle = \langle (v_2, v_1), dF \rangle, \quad \forall (v_2, v_1) \in T_{(q_2, q_1)} R \end{cases} \end{array}}$$

We say that this is the **symplectic relation generated by the function** F **over the relation** R. From this definition it follows that (cf. (7) of the preceding section)

$$(3) \quad \boxed{\theta_{Q_2} \ominus \theta_{Q_1} | \widehat{(R, F)} = \pi^* dF}$$

where $\pi: \widehat{(R, F)} \to R$ is the surjective submersion associated with $\pi_{Q_2} \times \pi_{Q_1}$. Hence, in particular,

$$(4) \quad \boxed{\theta_{Q_2} \ominus \theta_{Q_1} | \widehat{R} = 0}$$

Chapter 3

The geometry of the Hamilton-Jacobi equation

3.1 The Hamilton-Jacobi equation

A **Hamilton-Jacobi equation** is a first-order PDE of the kind

$$C\left(q^i, \frac{\partial G}{\partial q^i}\right) = 0, \tag{1}$$

in the n variables $\underline{q} = (q^i)$ which involves only the partial derivatives of the unknown function $G(\underline{q})$, where $C(\underline{q}, \underline{p})$ is a smooth function in the $2n$ variables $(\underline{q}, \underline{p}) = (q^i, p_i)$.

This kind of equation has the following geometrical interpretation. Let us consider a regular function $C: T^*\mathcal{Q} \to \mathbb{R}$ on the cotangent bundle of a manifold $\mathcal{Q}$ with local coordinates $\underline{q}$. "Regular" means that $dC \neq 0$ at all points such that $C(\underline{q}, \underline{p}) = 0$. Thus, equation

$$C(\underline{q}, \underline{p}) = 0 \tag{2}$$

defines a submanifold $C \subset T^*\mathcal{Q}$ of codimension 1 (a hypersurface). Let us consider the submanifold $\Lambda \subset T^*\mathcal{Q}$ described by equations

$$p_i = \frac{\partial G}{\partial q^i}. \tag{3}$$

This is a Lagrangian submanifold, generated by the function $G(\underline{q})$ on the manifold $\mathcal{Q}$. It follows that G is a solution of the Hamilton-Jacobi equation (1) if and only if $\Lambda \subset C$. Hence, the Hamilton-Jacobi equation can be interpreted as a submanifold C of the cotangent bundle $T^*\mathcal{Q}$ and a solution can be interpreted as a Lagrangian submanifold Λ contained in C. This submanifold is regular, since it is the image of the differential of a function.

We can extend these classical concepts (i) by calling **Hamilton-Jacobi equation** any submanifold (of any dimension) of a cotangent bundle T^*Q and (ii), by considering as a solution any Lagrangian submanifold $\Lambda \subset C$. We call it a **smooth geometrical solution**: this will include the cases in which Λ has singular points.[1] A Hamilton-Jacobi equation is then (at least) locally represented by a system of independent equations

$$C_{\mathsf{a}}(\underline{q}, \underline{p}) = 0 \qquad (\mathsf{a} = 1, \ldots, k). \tag{4}$$

This means that the differentials dC_{a} are independent at all points satisfying equations (4). Furthermore, a *classical* solution $G(\underline{q})$ is replaced by a generating family $G(\underline{q}, \underline{u})$ satisfying equations

$$\boxed{\begin{aligned} & C_{\mathsf{a}}\left(\underline{q}, \frac{\partial G}{\partial \underline{q}}\right) = 0 \\ & \frac{\partial G}{\partial u^\alpha} = 0 \end{aligned}} \qquad \begin{aligned} \mathsf{a} &= 1, \ldots, k \\ \alpha &= 1, \ldots, r \end{aligned} \tag{5}$$

for some values of the variables $\underline{u} = (u^\alpha)$. If this solution is a Morse family, then it generates a smooth geometrical solution $\Lambda \subset C$ described by equations

$$\Lambda : \begin{cases} p_i = \dfrac{\partial G}{\partial q^i} \\ \dfrac{\partial G}{\partial u^\alpha} = 0. \end{cases} \tag{6}$$

If G is not a Morse family, then the subset Λ of T^*Q described by equations (6) may not be a Lagrangian submanifold. In any case, it is a Lagrangian set (according to the definition given in §2.5). So, by **geometrical solution** we mean any Lagrangian set $\Lambda \subset C$ generated by a generating family.

[1] The common geometrical interpretation of the "Hamilton-Jacobi equation" is a hypersurface (i.e., a submanifold of codimension 1) of a cotangent bundle or of a contact manifold (cf. [Vinogradov, Kupershmidt, 1977] and [Arnold, 1980]). In this case it is represented by a single partial differential equation and it is always integrable in the sense given below. The fundamental elements of the geometrical theory of the Hamilton-Jacobi equation, interpreted as a submanifold $C \subset T^*Q_n$ of any dimension $k < n$, are given in the short note [Tulczyjew, 1975]. Some of these elements have been developed in [Benenti, Tulczyjew, 1980] and [Benenti, 1983a,b] and in further papers cited below.

About the existence of solutions, we can consider the following definition: a Hamilton-Jacobi equation $C \subset T^*\mathcal{Q}$ is **integrable** if for each $p \in C$ there exists a Lagrangian submanifold $\Lambda \subset C$ containing p (in other words, we require the existence of a smooth geometrical solution at each point of C).

Theorem 1. *A Hamilton-Jacobi equation $C \subset T^*\mathcal{Q}$ is integrable if and only if it is a coisotropic submanifold.*

Proof. (i) Assume that C is integrable. From $T_p\Lambda \subset T_pC$ it follows that $T_p^\S C \subset T_p^\S \Lambda = T_p\Lambda$, since Λ is Lagrangian. Thus, $T_p^\S C \subset T_pC$, and C is coisotropic. (ii) Assume that C is coisotropic. Take a point $p \in C$. There always exists a neighborhood $C_p \subset C$ of p, such that the reduced set $\mathcal{S}/C_p$ (here, $\mathcal{S} = T^*\mathcal{Q}$) is a manifold, thus a symplectic manifold and a symplectic reduction $R = R_{C_p}$ is defined from $\mathcal{S}$ to this manifold. Consider the reduced point $\gamma = R \circ \{p\}$. We observe that for each point of a symplectic manifold there always exists a Lagrangian submanifold containing that point.[2] Consider a Lagrangian submanifold Λ_γ containing γ. The inverse image $\Lambda = R^\top \circ \Lambda_\gamma$ is a Lagrangian submanifold (Theorem 2, §1.8) containing p. ∎

Definition 1. Since we are interested in dealing with the integrable cases only, by **Hamilton-Jacobi equation** we mean a coisotropic submanifold of a cotangent bundle: $C \subset T^*\mathcal{Q}$.

This means that the systems of equations (4) we are going to consider are such that the left hand sides are in involution, at least at the points of C,

$$\{C_\mathsf{a}, C_\mathsf{b}\}|C = 0. \tag{7}$$

One of the main problems related to a Hamilton-Jacobi equation is how to generate a (possibly unique) maximal solution from suitable *initial conditions* (**Cauchy problem**). We shall give a geometrical construction of such a solution, by using the composition rule of symplectic relations, then we shall transform this geometrical construction into an analytical method. The terminology we shall use is taken from Geometrical Optics, since one of the most important examples of Hamilton-Jacobi equation is the **eikonal equation**,

$$g^{ij}(\underline{q})\, p_i\, p_j - n^2(\underline{q}) = 0, \tag{8}$$

determined by a positive-definite contravariant metric tensor $\mathbf{G} = (g^{ij})$ on a (Riemannian) manifold $\mathcal{Q}$ and by a function $n \colon \mathcal{Q} \to \mathbb{R}$. A special but fundamental case is that of an *isotropic medium*, where $\mathcal{Q} = \mathbb{R}^3$ is the Euclidean three-space and n is the **refraction index**,

$$n = \frac{c}{v}, \tag{9}$$

[2] This follows from the existence of local canonical coordinates (Darboux theorem). Indeed, if $\omega = dy_i \wedge dx^i$, where (x^i, y_i) are canonical coordinates such that $x^i(p) = 0$, then equations $x^i = 0$ define a Lagrangian submanifold containing p.

where v is the velocity of the light in the medium. A homogeneous medium is characterized by $n =$ constant, the vacuum by $n = 1$.[3]

Other basic examples of Hamilton-Jacobi equations C are the following:

(i) The Hamilton-Jacobi equation of a holonomic time-independent and conservative dynamical system, for a fixed value of the total energy $E \in \mathbb{R}$,

$$\tfrac{1}{2}\, g^{ij}(\underline{q})\, p_i\, p_j + V(\underline{q}) - E = 0. \tag{10}$$

The reduced symplectic manifold R_C is the *manifold of the orbits* of total energy E [Souriau, 1970] (see Remark 5, §3.2, for an example).

(ii) The Hamilton-Jacobi equation of a holonomic time-independent conservative system

$$\tfrac{1}{2}\, g^{ij}(\underline{q})\, p_i\, p_j + V(\underline{q}) + p_0 = 0, \tag{11}$$

in the cotangent bundle of the extended configuration manifold $\mathbb{R} \times \mathcal{Q}$, where $\mathbb{R} = (t) = (q^0)$ is the time-axis. This way of considering classical dynamics is called **homogeneous formalism**: time is considered as a Lagrangian coordinate. It can be extended to time-dependent holonomic systems,

$$\tfrac{1}{2}\, g^{ij}(t, \underline{q})\, p_i\, p_j + V(t, \underline{q}) + p_0 = 0. \tag{12}$$

(iii) The Hamilton-Jacobi equation associated with a vector field $X = (X^i)$ on a manifold $\mathcal{Q}$,

$$X^i p_i = 0, \tag{13}$$

whose solutions are the first integrals of X (see Remark 4, §3.2, and Remark 2, §3.3). In order to avoid singularities, it is convenient to consider its extension to the cotangent bundle of $\mathbb{R} \times \mathcal{Q}$,

$$X^i p_i + p_0 = 0. \tag{14}$$

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[3] In the gravitational lensing theory the effective refraction index is

$$n(\mathbf{x}) = 1 - 2U(\mathbf{x})/c^2,$$

where $U(\mathbf{x})$ is the Newtonian potential of the mass distribution $\rho(\mathbf{x})$,

$$U(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'.$$

See e.g. the article by N. Straumann in [Straumann, Jetzer, Kaplan, 1998].

(iv) The Hamilton-Jacobi equation associated with a completely integrable distribution,

$$X^i_\alpha p_i = 0, \tag{15}$$

where $X_\alpha = (X^i_\alpha)$ are $r \leq n$ independent vector fields spanning the distribution. In this case the coisotropic submanifold C has codimension $r \geq 1$ (see §6.5).

3.2 Characteristics and rays

Due to Theorem 1, §1.4, if a coisotropic submanifold C is represented by $k \leq n$ independent equations $C_a = 0$, then the functions C_a generate characteristic vector fields $\mathbf{X}_a$. The corresponding Hamilton equations are (notation: $\partial^i = \frac{\partial}{\partial p_i}$, $\partial_i = \frac{\partial}{\partial q^i}$)

$$\begin{cases} \dot{q}^i = \partial^i C_a \\ \dot{p}_i = -\partial_i C_a. \end{cases} \tag{1}$$

These vectors are pointwise independent since the rank of the $2n \times k$ matrix

$$\left[\partial^i C_a \,|\, \partial_i C_a\right] \tag{2}$$

is maximal. By linear combinations $\mathbf{X} = \lambda^a \, \mathbf{X}_a$ these fields span the characteristic distribution. Thus, the differential system associated with a characteristic vector field is of the kind

$$\begin{cases} \dot{q}^i = \lambda^a \, \partial^i C_a \\ \dot{p}_i = -\lambda^a \, \partial_i C_a, \end{cases} \tag{3}$$

where λ^a are arbitrary functions.

Definition 1. The **rays** of a Hamilton-Jacobi equation $C \subset T^*Q$ are the projections onto the configuration manifold Q of the characteristics of C.

Remark 1. The characteristics of C projects onto (immersed) submanifolds of Q of dimension equal to the codimension of C if the characteristics are transversal to the fibres,[4]

$$T^{\S}C \cap V(T^*Q) = 0. \tag{4}$$

[4] The converse is not true in general. Let us consider for instance the Lagrangian submanifold of Example 1 in §2.4.3. It is a coisotropic submanifold with only one characteristic, the submanifold itself, which is not transversal to the fibre at the origin, while it projects onto a submanifold, the q-axis.

In this case we say that the Hamilton-Jacobi equation C is **regular**. This condition is fulfilled if the rank of the $n \times k$ matrix $[\partial^i C_{\mathsf{a}}]$ is maximal,

$$\operatorname{rank}[\partial^i C_{\mathsf{a}}] = k. \tag{5}$$

Indeed, the vertical vectors are characterized by equations $\dot{q}^i = 0$. A vertical vector in $T^{\S}C$ is the zero vector if equations $\lambda^{\mathsf{a}}\, \partial^i C_{\mathsf{a}} = 0$ imply $\lambda^{\mathsf{a}} = 0$ thus, $\dot{p}_i = 0$. This happens if the matrix (5) has maximal rank.

Remark 2. In the case $\operatorname{codim}(C) = 1$, the characteristics are transversal to the fibres if $\operatorname{rank}[\partial^i C] = 1$. Then the rays are (one-dimensional) curves.

Remark 3. In the case of the eikonal equation, we have $C = |p|^2 - n^2(\underline{q})$ so that $\partial^i C = g^{ij}\, p_j$. It follows that the transversality condition is satisfied for all $(p_i) \neq 0 \in \mathbb{R}^n$ thus, for all $p \in C$, since $n \neq 0$.

Theorem 1. *For the homogenous (or vacuum) eikonal equation, $g^{ij} p_i p_j = 1$, the rays are oriented geodesics of the Riemannian manifold $(\mathcal{Q}, g^{ij})$. For the eikonal equation, $g^{ij} p_i p_j = n^2$ with $n \neq 0$, the rays are the oriented geodesics of the* **Jacobi metric**

$$\bar{g}^{ij} = \frac{1}{n^2}\, g^{ij}. \tag{6}$$

Proof. The integral curves on $T^*\mathcal{Q}$ of the Hamiltonian dynamical system generated by $H = \frac{1}{2}\, g^{ij}\, p_i p_j$ project onto the integral curves (on $\mathcal{Q}$) of the Lagrange equations associated with $L = \frac{1}{2}\, g_{ij}\, \dot{q}^i\, \dot{q}^j$. These integral curves describe motions with constant scalar velocity on geodesic trajectories. ∎

Remark 4. (i) The characteristics of the Hamilton-Jacobi equation (13) of §3.1, $X^i p_i = 0$, associated with a vector field X on $\mathcal{Q}$, are the unparametrized integral curves, starting from points $(\underline{q}, \underline{p})$ satisfying this equation, of the Hamiltonian vector field $\widehat{X}$ on $T^*\mathcal{Q}$ generated by the Hamiltonian $P_X = X^i p_i$ (cf. §6.4). The rays are the unparametrized integral curves of X i.e., the orbits of X. (ii) The rays of the Hamilton-Jacobi equation (14), §3.1, are parametrized integral curves of X: two integral curves describing the same unparametrized curve differs by the initial point. (iii) The rays of the Hamilton-Jacobi equation (15), §3.1, $X^i_\alpha p_i = 0$, associated with an integrable distribution, are the integral manifolds of the distribution (see §6.5 for details).

Remark 5. An interesting example of Hamilton-Jacobi equation with fixed value of the energy of the kind (10), §3.1, is that of the Kepler motions in the Euclidean space $\mathbb{R}^n$. The Hamiltonian is (for the notation see §4.1)

$$H(\mathbf{x}, \mathbf{p}) = \tfrac{1}{2}\, |\mathbf{p}|^2 - \frac{1}{|\mathbf{x}|}.$$

For each value of the constant E, equation $H = E$ defines a coisotropic submanifold C_E of $T^*\mathbb{R}^n$. For each dimension n, we have three distinct cases, corresponding to $E > 0$ (hyperbolic case), $E = 0$ (parabolic case) and $E < 0$ (elliptic case). The manifolds of the oriented orbits i.e., the reduced manifolds R_{C_E}, have been determined in [Moser, 1970] (for $n = 2, 3$) and in [Pham Mau Quan, 1980]. For example, for $n = 2$ these manifolds are: $\mathbb{H}_2$ (hyperbolic case), $\mathbb{S}_1 \times \mathbb{R}$ (parabolic case) and $\mathbb{S}_2$ (elliptic case). Further investigations on the corresponding reduced symplectic structures are needed. For an accurate treatment of all the aspects of the Kepler problem see the recent book [Cordani, 2003].

3.3 Systems of rays and wave fronts

Definition 1. A **system of rays** associated with a Hamilton-Jacobi equation $C \subseteq T^*Q$ is the set of the projections on the configuration manifold Q of the characteristics contained in a geometrical solution $\Lambda \subseteq C$.

If C is regular (§3.2), then all characteristics project onto submanifolds of dimension equal to the codimension of C; so, if Λ is a smooth solution (i.e., a Lagrangian submanifold) then the corresponding system of rays is made of a set of these submanifolds, with possible points of intersection. In all other cases a system of rays may be a complicated family of subsets of Q.

Let us consider for simplicity the case of a smooth geometrical solution. Assume that Λ is the image of a closed one-form φ on an open domain $U \subset Q$, and that $\varphi \neq 0$ everywhere. Then on the domain U two regular and integrable distributions are defined.

The first distribution $\Delta_W \subset TU$ has the 1-form φ as a characteristic form i.e., it is made of the vectors annihilated by φ. Since the 1-form is closed, this distribution is completely integrable with integral manifolds of codimension **1**. These integral manifolds are called **wave fronts** of the solution Λ. If $\varphi = dG$, then the wave fronts are described by the equations $G =$ constant.

The second distribution $\Delta_R \subset TU$ is the projection onto TU of the characteristic distribution $T^{\S}C$ restricted to Λ. By the *absorption principle* this restriction is well defined: $T^{\S}_{\Lambda}C \subset T\Lambda$. If we assume that on Λ the characteristic distribution is transversal to the fibres (as we have seen, also this condition is satisfied by the eikonal equation), then the distribution R is completely integrable and its integral manifolds form a system of rays (whose dimension coincide with the codimension of the Hamilton-Jacobi equation C). Note that Δ_W and Δ_R have a complementary rank.

The distribution Δ_R is spanned by the projections on Q of the characteristic vector fields $\mathbf{X}_{\mathsf{a}}$ restricted to Λ. The dynamical systems corresponding to these projected vector fields are the first set of the Hamilton equations

$$\dot{q}^i = \partial^i C_{\mathsf{a}}(\underline{q}, \underline{p}), \tag{1}$$

where in the right hand sides the momenta $\underline{p}$ are replaced by their expressions in terms of the coordinates $\underline{q}$, defined by the components of the 1-form φ:

$$p_i = \varphi_i(\underline{q}) \qquad (\varphi = \varphi_i \, dq^i). \tag{2}$$

Remark 1. In the case of the eikonal equation equations (1) become

$$\dot{q}^i = 2\, g^{ij}\, p_j, \tag{3}$$

and on a regular solution Λ generated by G,

$$\dot{q}^i = 2\, g^{ij}\, \partial_j G. \tag{4}$$

This shows that the gradient of the generating function G is a vector field spanning the distribution Δ_R. Since G is constant on the wave fronts, we have that *a regular solution of the eikonal equation generates a system of geodesics* (the rays) *orthogonal to a system of hypersurfaces* (the wave fronts). In fact, this is an equivalence: any orthogonally integrable system of geodesics corresponds to a regular solution of the eikonal equation. This was one of the leading idea of Hamilton's *Theory of systems of rays* [Hamilton, 1828]. Since the wave fronts are orthogonal to a system of geodesics, they are **geodesically parallel** i.e., the ray segments between two given wave fronts have constant length.

Remark 2. The wave fronts of the Hamilton-Jacobi equations (13) or (14), §3.1, $X^i p_i = 0$, are defined by equations $G =$ const., where G is any first integral of the vector field X. Since G is constant along the integral curves, in this case any wave front is made of rays. The same property holds for the Hamilton-Jacobi equation (15) associated with a completely integrable distribution.

Remark 3. The above description of the wave fronts and rays fails in a neighborhood of a singular point i.e., when rays and wave fronts approach the caustic of Λ. An even more complicated situation is that arising from a non-smooth geometrical solution i.e., from a Lagrangian set generated by a solution G (with supplementary variables) which is not a Morse family. In this case wave fronts and caustics are not defined.

Remark 4. The vectorial form of the (vacuum) eikonal equation in a Euclidean affine space $\mathcal{Q} = \mathbb{R}^n$ is (cf. §4.3)

$$|\mathbf{p}|^2 = 1. \tag{5}$$

A system of parallel (oriented) rays is represented by a unit (constant) vector $\mathbf{u}$. The generating function of the corresponding Lagrangian submanifold is

$$G(\mathbf{x}) = \mathbf{x} \cdot \mathbf{u}. \tag{6}$$

The wave fronts are the $n-1$-planes orthogonal to $\mathbf{u}$.

Remark 5. In a Euclidean space, the system of rays originated by a fixed point $\mathbf{x}_0$ is generated by the function

$$G_1(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0| \tag{7}$$

(this *distance function*, as a generating family, will be examined in detail in §4.1) or by the Morse family

$$G_2(\mathbf{x}; \mathbf{a}) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{a}, \qquad \mathbf{a} \in \mathbb{S}_{n-1}, \tag{8}$$

with supplementary manifold $\mathbb{S}_{n-1}$. The generating function G_1 yields *outgoing rays* only, since

$$\mathbf{p} = \frac{\partial G_1}{\partial \mathbf{x}} = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}. \tag{9}$$

Note that it is not differentiable for $\mathbf{x} = \mathbf{x}_0$, so that the Lagrangian submanifold described by equation (9) is not defined over the point $\mathbf{x}_0$. The generating family G_2 is globally defined and differentiable. The corresponding equations

$$\begin{cases} 0 = \dfrac{\partial G_2}{\partial \mathbf{a}} \\ \mathbf{p} = \dfrac{\partial G_2}{\partial \mathbf{x}} \end{cases}$$

are equivalent to (cf. Remark 4, §4.3)

$$\begin{cases} \mathbf{x} - \mathbf{x}_0 \parallel \mathbf{a} & (\parallel = \text{parallel to}) \\ \mathbf{p} = \mathbf{a}. \end{cases}$$

Thus, G_2 gives outgoing as well as incoming rays.

3.4 The characteristic functions

The characteristic relation $D_C \subset T^*Q \times T^*Q$ defined by a coisotropic submanifold $C \subset T^*Q$ is a symmetric symplectic relation between cotangent bundles. Thus, it is generated (at least locally) by generating families on the product manifold $Q \times Q$.

Definition 1. A **Hamilton principal function** (or a **characteristic function**) of a Hamilton-Jacobi equation (i.e., of a coisotropic submanifold) $C \subset T^*Q$ is a generating family $S(Q \times Q; A)$ of the characteristic relations $D_C \subset T^*Q \times T^*Q$.

As we shall see, a global Hamilton characteristic function (if it exists) can be used for computing (i) all solutions of the Hamilton-Jacobi equation and (ii) the system of rays associated with a solution. If $S(\underline{q}, \underline{q}'; \underline{a})$ is a local representative of a Hamilton principal function (where $\underline{q} = (q^i)$ and $\underline{q}' = (q'^i)$ are local coordinates on Q and $\underline{a} = (a^\alpha)$ local coordinates on A) then D_C is locally described by equations

$$(1) \qquad \begin{cases} p'_i = -\dfrac{\partial S}{\partial q'^i} \\ p_i = \dfrac{\partial S}{\partial q^i} \\ 0 = \dfrac{\partial S}{\partial a^\alpha}. \end{cases}$$

Remark 1. If p_0 is a point of C, then the set $D_C \circ \{p_0\}$ is the maximal characteristic containing (or passing through) p_0. If in equations (1) the coordinates $(\underline{q}', \underline{p}')$ are just the coordinates of p_0, then these equations describe (locally) this characteristic and consequently the corresponding ray. A system of rays corresponding to a solution of the Hamilton-Jacobi equation C can be computed in this way.

Theorem 1. *If the coisotropic submanifold C is not a section of the cotangent bundle T^*Q, then the characteristic relation D_C is singular over the diagonal of $Q \times Q$.*

Proof. Assume that D_C is regular at a point $(q, q) \in \Delta_Q$. Hence, it is locally generated by a function $S(\underline{q}, \underline{q}')$ which is skew-symmetric (Theorem 2, §2.6). In this case equations (1) reduce to

$$\begin{cases} p'_i = -\dfrac{\partial S}{\partial q'^i} \\ p_i = \dfrac{\partial S}{\partial q^i} \end{cases}$$

and show that for $q = q'$ we have $p = p'$. This means that we have a unique covector $p \in T_q^*Q \cap C$. This is in contradiction with the assumption that C is a section. ∎

Remark 2. If C is a section, then it is a Lagrangian submanifold. If it is connected, then we have only one characteristic (the manifold C itself) and the characteristic relation D_C is defined by

$$(p, p') \in D_C \iff p, p' \in C.$$

If $G\colon Q \to \mathbb{R}$ is a global generating function of C, then D_C is generated by the global (skew-symmetric) generating function

$$S(q, q') = G(q) - G(q').$$

Remark 3. The eikonal equation and the Hamilton-Jacobi equation of classical mechanics are not sections, since at each point q the intersection $T_q^*Q \cap C$ is diffeomorphic to a sphere. Hence, the theorem above shows that for these equations *a global Hamilton characteristic function is necessarily a generating family.* This is a novelty with respect to the classical Hamilton-Jacobi theory, where S is a function of pairs of points of Q, locally represented by a function $S(\underline{q}_1, \underline{q}_0)$ of $2n$ coordinates.[5] Moreover, in the classical theory the Hamilton principal function is defined as an **action integral**. This is due to the following general property.

Theorem 2. *If we exclude the singular points and assume that the remaining part of D_C is an exact Lagrangian submanifold* (§6.6), *then a potential function of D_C is the integral*

$$I(p_1, p_0) = \int_{c[p_0, p_1]} \theta_Q, \tag{2}$$

where $c[p_0, p_1]$ is any path with extremal points (p_0, p_1) and contained in the characteristic containing these two points.

Proof. From the definition of potential function of a Lagrangian submanifold it follows that a potential function of the Lagrangian submanifold $D_C \subset (T^*Q \times$

[5] The Hamilton principal function S has been introduced by Hamilton as a function depending on $2n+2$ variables $(\underline{q}_1, \underline{q}_0, t_1, t_0)$, where $\underline{q}_0$ are the initial values of coordinates (at the time t_0) of a holonomic system, and $\underline{q}_1$ are their final values (at the time t_1). This function satisfy the Hamilton-Jacobi equations

$$\begin{cases} \dfrac{\partial S}{\partial t_1} + H\Big(\underline{q}_1, \dfrac{\partial S}{\partial \underline{q}_1}, t_1\Big) = 0 \\ \dfrac{\partial S}{\partial t_0} + H\Big(\underline{q}_0, -\dfrac{\partial S}{\partial \underline{q}_0}, t_0\Big) = 0, \end{cases}$$

where $H = H(\underline{q}, \underline{p}, t)$ is the (time-dependent) Hamiltonian function of the mechanical system. We have used here the classical notation adopted by Levi-Civita ([Levi-Civita, Amaldi, 1927], Ch. XI, n. 27). In the homogenous formalism of Hamiltonian dynamics, time is considered as a coordinate and the n-dimensional configuration manifold Q of the system is replaced by the $n+1$-dimensional **extended configuration manifold** $\mathbb{R} \times Q$. The notion of Hamilton principal function as a generating function of the characteristic relation of a coisotropic submanifold or as a generating function of the symplectic relation $D_t \subset T^*Q \times T^*Q$, between the initial values (at $t_0 = 0$) and the final values (at $t = t_1$) of the coordinates in the motions generated by the Hamiltonian H, has been introduced by Tulczyjew [Tulczyjew, 1975, 1977b]. In Hamiltonian Optics other "characteristic functions" are considered. See e.g. [Synge, 1962], Chapter II, [Luneburg, 1964], p.100, [Buchdahl, 1970], p.8. What we are considering here is just a generalization of the so-called **point-characteristic function**.

$T^*Q, d\theta_Q \ominus d\theta_Q)$ is given by the integral

$$I(p_1, p_0) = \int_c \theta_Q \ominus \theta_Q,$$

taken along any path c over D_C joining a fixed pair $(\bar{p}_1, \bar{p}_0)$ with the pair (p_1, p_0). This path can be represented by two curves $c_1(t)$ and $c_0(t)$ on C and defined on the real closed interval $[0,1]$ such that $c_i(0) = \bar{p}_i$, $c_i(1) = p_i$ $(i = 0,1)$ and $\big(c_1(t), c_0(t)\big) \in D_C$. Hence,

$$I(p_1, p_0) = \int_{c_1} \theta_Q - \int_{c_0} \theta_Q = \int_{c_1'} \theta_Q - \int_{c_0'} \theta_Q$$

where $c' = (c_1', c_0')$ is another path having the same property. By choosing $c_0' = c_0$ we see that

$$\int_{c_1} \theta_Q = \int_{c_1'} \theta_Q.$$

This means that θ_Q is exact for the chosen paths from $\bar{p}_1$ to p_1, as well as for those from $\bar{p}_0$ to p_0. Moreover, for each $t \in [0,1]$ the two points $c_1(t)$ and $c_0(t)$ are the end points of a curve $\gamma_t(s)$ defined for $s \in [0,1]$ with image on a characteristic. Since the characteristic are isotropic, θ_Q is exact on all γ_1. It follows that

$$A = \int_{c_1} \theta_Q + \int_{\gamma_1} \theta_Q - \int_{c_0} \theta_Q$$

is a number depending only on the fixed end points $(\bar{p}_1, \bar{p}_0)$. Thus,

$$I(p_1, p_0) = \int_{c_1} \theta_Q - \int_{c_0} \theta_Q = A - \int_{\gamma_1} \theta_Q.$$

Since A depends only on the fixed points $(\bar{p}_1, \bar{p}_0)$, and the path γ_1 goes from p_1 to p_0, the integral (2) is a potential function. ∎

As a consequence of this theorem we have another link with the classical theory.

Theorem 3. *For the eikonal equation $g^{ij}p_ip_j = 1$ the characteristic relation D_C outside the diagonal is locally generated by the distance function*

$$d(q_0, q_1) = \int_{t_0}^{t_1} \sqrt{g_{ij}\dot{q}^i\dot{q}^j}\, dt$$

where the integral is taken along the geodesic $q(t)$ such that $q(t_0) = q_0$, $q(t_1) = q_1$.

Proof. Since the rays are the geodesics, because of the preceding theorem a local potential function is given by

$$I(p_1, p_0) = \int_c p_i dq^i = \int_{t_0}^{t_1} p_i \dot{q}^i\, dt = 2 \int_{t_0}^{t_1} g^{ij} p_i p_j\, dt,$$

where c is a characteristic from p_0 to p_1 and the integrals are taken along a geodesic $q^i(t)$ such that $\dot{q}^i = 2\, g^{ij} p_j$ (cf. (1) of §3.4). The kinetic energy of the motion $q^i(t)$ is $K = \frac{1}{2}\, v^2 = \frac{1}{2}\, g_{ij}\dot{q}^i\dot{q}^j = 2\, g^{ij} p_i p_j = 2$, so that the scalar constant velocity is $v = \frac{ds}{dt} = 2$. This means that the Euclidean distance is such that $ds = 2dt$. Hence the last integral above is just the integral of ds. This shows that the characteristic function projects onto the generating function given by the distance. ∎

The Hamilton principal function can be derived, at least locally, from another *characteristic function* associated with the Hamilton-Jacobi equation.

Definition 2. A **complete solution** or a **complete integral** of a Hamilton-Jacobi $C \subseteq T^*\mathcal{Q}$ is a smooth function $W\colon \mathcal{Q} \times A \to \mathbb{R}$, where A is a manifold, such that

(i) for each $a \in A$ the function $W_a\colon \mathcal{Q} \to \mathbb{R}$, $W_a(q) = W(q, a)$, is a generating function of a Lagrangian submanifold Λ_a contained in C (thus, a regular solution of the Hamilton-Jacobi equation);

(ii) the set $\{\Lambda_a,\ a \in A\}$ is a Lagrangian foliation of C (this means that for each $p \in C$ there exists a unique $a \in A$ such that $p \in \Lambda_a$);

(iii) the canonical projection $\pi\colon C \to A\colon p \mapsto a$ such that $p \in \Lambda_a$ is differentiable.

Remark 4. From this definition it follows that

$$\dim(A) = \dim(\mathcal{Q}) - \operatorname{codim}(C). \tag{3}$$

As we shall see below (§3.6) the canonical projection π (which is obviously surjective) is a submersion.

Remark 5. We can extend this definition by considering generating families $W_a(q; v)$ parametrized by $a \in A$ and defining a Lagrangian foliation of C with singular points. In the classical Hamilton-Jacobi theory, W is locally represented by a function of the coordinates $\underline{q}$ of Q and a set of **constants of integration** (a^α), which represent a point $a \in A$. No extra variables v are present, since in the classical theory only ordinary generating functions are considered.[6]

[6] According to Levi-Civita, [Levi-Civita, Amaldi, 1927], Ch. X, n. 38, the letter W is used for denoting a complete solution of the time-independent **reduced Hamilton-Jacobi equation** $H - E = 0$, for any fixed value of the energy E. It is a function of the n Lagrangian coordinates $\underline{q}$ and of n constant parameters $\underline{\pi} = (\pi^i)$, satisfying the **completeness condition**

$$\det\left[\frac{\partial^2 W}{\partial \underline{q}\, \partial \underline{\pi}}\right] \neq 0$$

Actually, since the energy E becomes a function of these constants, for a fixed value of E, they are not all independent. So, they can be expressed as functions of $n-1$ independent parameters $\underline{a}$ satisfying the completeness condition (3). This is in accordance with Definition 2, being in this case $m = n - 1$ since $\operatorname{codim}(C) = 1$. For a time-

As shown by Theorem 6 below, by means of a global complete solution (if it exists) we can easily construct a global Hamilton principal function. For proving this theorem we need the following global version of the classical **theorem of Jacobi**.

Theorem 4. (i) *A complete solution* $W: Q \times A \to \mathbb{R}$ *of* C *generates a symplectic coreduction* $R^\top \subset T^*Q \times B$ *such that* $R^\top \circ B = C$, *where* B *is an open submanifold of* T^*A. (ii) *The characteristics of* C *are the connected components of the images* $R^\top \circ \{b\}$ *of the points* $b \in B$. (iii) *The reduction* R *is* **isomorphic** *to the reduction relation* R_C *i.e., there exists a symplectomorphism* $\varphi: T^*Q/C \to B$ *such that* $R = \varphi \circ R_C$.[7]

Its local version is

Theorem 5. *A function* $W(\underline{q}, \underline{a})$ *of* n *coordinates* $\underline{q} = (q^i)$ *on* Q *and of* m *parameters* $\underline{a} = (a^\alpha)$ *is a local representative of a complete integral of* C *if and only if:* (i) $m = n - k = \dim(Q) - \mathrm{codim}(C)$; (ii) *for each values of these parameters it satisfies the differential equations*

$$C_{\mathsf{a}}\left(\underline{q}, \frac{\partial W}{\partial \underline{q}}\right) = 0$$

where $C_{\mathsf{a}}(\underline{q}, \underline{p}) = 0$ *are local independent equations of* C; (iii) *the* $n \times m$ *matrix* $[\partial^2 W/\partial \underline{q} \partial \underline{a}]$ *has maximal rank,*

$$\text{rank}\left[\frac{\partial^2 W}{\partial q^i \partial a^\alpha}\right] = m. \tag{4}$$

In this case the reduction relation $R \subset T^*A \times T^*Q$ *is locally described by equations*

$$p_i = \frac{\partial W}{\partial q^i}, \qquad b_\alpha = -\frac{\partial W}{\partial a^\alpha}, \tag{5}$$

where $\underline{a} = (a^\alpha)$ *are interpreted as local coordinates on* A *and* $(\underline{a}, \underline{b}) = (a^\alpha, b_\alpha)$ *as the corresponding canonical coordinates on* T^*A.

The proofs of these theorems are given in §3.6.

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independent Hamiltonian H one can think of a complete solution of the Hamilton-Jacobi equation $\partial V/\partial t + H(\underline{q}, \partial V/\partial \underline{q}) = 0$ of the form $V = -Et + W$. Then, this equation reduces to $H - E = 0$. In this reduction procedure, Jacobi considered W as depending on E and on further $n - 1$ constant parameters $\underline{a}$.

[7] This is an improved simplified version of the **global Jacobi theorem** treated in [Tulczyjew, 1975], [Benenti, Tulczyjew, 1980, 1982a, 1982b], [Benenti, 1983a], [Benenti, 1988]. See [Libermann, Marle, 1987] for a general review.

Theorem 6. *If $W(\underline{q},\underline{a})$ is a complete integral of C, then the generating family $S(Q \times Q; A)$ defined by*

$$S(q,q';a) = W(q,a) - W(q',a) \tag{6}$$

is a Hamilton principal function.

Proof. Since $D_C = R_C^\top \circ R_C$, it follows from the last item of Theorem **1** that

$$D_C = R^\top \circ R.$$

Thus, if $W^\top$ is the generating function of the reduction R, then

$$W^\top(a,q) = -W(q,a)$$

and by composing, in the order, $W^\top$ with W we get a generating family of D_C. ∎

Remark 6. Note that in the generating family S the manifold A plays the role of supplementary manifold. Note that S is skew-symmetric, in accordance with the symmetry of D_C and Theorem 2, §2.6. The Hamilton principal function S defined by (6) is a Morse family. Indeed, due to (4), the matrix

$$\left[\frac{\partial^2 S}{\partial q^i \partial a^\alpha} \,\middle|\, \frac{\partial^2 S}{\partial q^{i'} \partial a^\alpha} \,\middle|\, \frac{\partial^2 S}{\partial a^\beta \partial a^\alpha} \right] =$$

$$= \left[\frac{\partial^2 W(\underline{q},\underline{a})}{\partial q^i \partial a^\alpha} \,\middle|\, -\frac{\partial^2 W(\underline{q}',\underline{a})}{\partial q^{i'} \partial a^\alpha} \,\middle|\, \frac{\partial^2 W(\underline{q},\underline{a})}{\partial a^\beta \partial a^\alpha} - \frac{\partial^2 W(\underline{q}',\underline{a})}{\partial a^\beta \partial a^\alpha} \right]$$

has maximal rank m everywhere.

Remark 7. From item (iii) of Theorem 4 it follows that a necessary condition for the existence of a global complete integral is that the reduced symplectic manifold T^*Q/C be symplectomorphic to a cotangent bundle (or at least to an open subset of a cotangent bundle). If $W(\underline{q},\underline{a})$ is a local representative of a complete solution, then equations (5) generate an open subrelation of $R^\top$. In general, by the integration the Hamilton-Jacobi equation we can find only **local complete solutions**, which generate Lagrangian foliations on open subsets of C. Then the composition formula (6) generates local principal functions.

Remark 8. There are cases in which a global principal function S exists, while a global complete solution does not. An example is the eikonal equation on the sphere $\mathbb{S}_2 \subset \mathbb{R}^3$, for which the reduced manifold is $\mathbb{S}_2$ (indeed, any oriented geodesic on the unit sphere $\mathbb{S}_2 \subset \mathbb{R}^3$ is represented by a unit vector orthogonal to the plane on which it lies). In these cases a global principal function may be determined by other methods. For $\mathbb{S}_2$ see Appendix B.

Remark 9. All the above definitions and results concerning a complete solution can be extended to the case of a generating family: $W(Q \times A; V)$, where V is a supplementary manifold, with coordinates (v^a). Then W and S depend on these extra variables and equations

$$0 = \frac{\partial S}{\partial v^a}, \qquad 0 = \frac{\partial W}{\partial v^a}, \tag{7}$$

must be added to systems (1) and (5), respectively.
Example 1. Let us consider for the eikonal equation of the Euclidean plane $Q = \mathbb{E}_2 = \mathbb{R}^2$,

$$C(x, y, p_x, p_y) \doteq p_x^2 + p_y^2 - 1 = 0. \tag{8}$$

In §4.3 it will be proved that: (i) *The reduced symplectic manifold* $(T^*Q)/C$ *is symplectomorphic to* $T^*\mathbb{S}_1$. (ii) *A global complete integral is*

$$W: \mathbb{R}^2 \times \mathbb{S}_1 \to \mathbb{R}, \qquad W(\mathbf{x}, \mathbf{a}) = \mathbf{a} \cdot \mathbf{x}. \tag{9}$$

(iii) *The characteristic relation* D_C *is generated by the Hamilton principal function*

$$S: \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{S}_1 \to \mathbb{R}, \qquad S(\mathbf{x}, \mathbf{x}', \mathbf{a}) = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a}. \tag{10}$$

Note that R_C is a regular Lagrangian submanifold which admits a global ordinary generating function (without supplementary variables), while D_C is singular over the diagonal $\Delta_Q \subset Q \times Q$. Out of the diagonal, D_C is made of two branches, which are regular Lagrangian submanifolds generated by the functions

$$S(\mathbf{x}, \mathbf{x}') = \pm |\mathbf{x} - \mathbf{x}'|. \tag{11}$$

Note that these functions are not differentiable for $\mathbf{x} = \mathbf{x}'$ i.e., over the diagonal. All these results have a natural extension to the space $\mathbb{R}^n$. The unit circle $\mathbb{S}_1$ is replaced by the unit sphere $\mathbb{S}_{n-1}$ (cf. §4.3). Note that for $n = 2$, instead of the generating families W and S defined in (9) and (10), one can use the equivalent generating families

$$W: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \qquad W(\mathbf{x}, \theta) = x \cos\theta + y \sin\theta, \tag{12}$$

and

$$S: \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, \qquad S(\mathbf{x}, \mathbf{x}', \theta) = (x - x') \cos\theta + (y - y') \sin\theta. \tag{13}$$

3.5 Sources, mirrors, lenses

Let $\Sigma \subset Q$ be a submanifold. Let us compose its canonical lift $\widehat{\Sigma}$ with the characteristic relation D_C,

$$\Lambda = D_C \circ \widehat{\Sigma}. \tag{1}$$

The set Λ is the union of the (maximal) characteristics of C intersecting $\widehat{\Sigma}$. If Λ is a Lagrangian submanifold, then it is a smooth geometrical solution of the Hamilton-Jacobi equation C. As we have seen in §1.8, a sufficient condition is the clean (or transversal) intersection of $C \cap \widehat{\Sigma}$ (see Remark 2 below). In this case the submanifold Σ behaves as a **source** of the system of rays represented by Λ and the characteristic relation D_C as a **propagator**. The composition formula (1) leads to the following

Theorem 1. *Let S be a Hamilton principal function of C and let G_Σ be a generating family of $\widehat{\Sigma}$, then the composed family $S \oplus G_\Sigma$ is a solution of the Hamilton-Jacobi equation generating the geometrical solution $\Lambda = D_C \circ \widehat{\Sigma}$.*

Remark 1. A generating family G_Σ is given by formula (5) of §2.8 (with $F = 0$). Note that $S \oplus G_\Sigma$ is anyway a smooth solution of the Hamilton-Jacobi equation, while Λ may not be a smooth geometrical solution. If it is a Morse family, then Λ is an immersed Lagrangian submanifold. Similar results hold if, instead of the pure canonical lift $\widehat{\Sigma}$, we consider the canonical lift with a function $(\widehat{\Sigma, F})$ and the set

$$\Lambda = D_C \circ (\widehat{\Sigma, F}). \tag{2}$$

In this case the objects (Σ, F) behave as **initial data** [Cardin, 1989, 2002].

Remark 2. Let $\underline{q} = (q^\alpha, q^a)$ be coordinates adapted to Σ, so that Σ is locally described by equations $q^\alpha = 0$. It follows that $(\widehat{\Sigma, F})$ is described by equations $q^\alpha = 0$ and $p_a - \partial_a F = 0$ (cf. footnote 10, §2.8). Assume that C is defined by independent equations $C^A(\underline{q}, \underline{p}) = 0$. Then the intersection of $C \cap (\widehat{\Sigma, F})$ is clean if the matrix (the symbol $\boxed{}$ denotes a square submatrix)

$$(\star) \qquad \begin{bmatrix} \boxed{\delta^\alpha_\beta} & -\partial_\beta \partial_a F & \partial_\beta C^A \\ 0 & \boxed{-\partial_b \partial_a F} & \partial_b C^A \\ \boxed{0} & 0 & \partial^\alpha C^A \\ 0 & \boxed{\delta^a_b} & \partial^b C^A \end{bmatrix}$$

has constant rank in a neighborhood of $C \cap (\widehat{\Sigma, F})$ (Remark 4, appendix A.4). Since only the restriction of F to Σ is relevant, the coordinates q^α can be chosen such that F does not depend on them, so that the above matrix becomes

$$(\star\star)\qquad \begin{bmatrix} \boxed{\delta^\alpha_\beta} & 0 & \partial_\beta C^A \\ 0 & \boxed{-\partial_b\partial_a F} & \partial_b C^A \\ \boxed{0} & 0 & \partial^\beta C^A \\ 0 & \boxed{\delta^a_b} & \partial^b C^A \end{bmatrix}.$$

These matrices must be computed for $C = 0$ (or $C^A = 0$) and $p_i = \partial_i F$. In the case of the canonical lift of Σ (with $F = 0$ or constant) we have

$$(\star\star\star)\qquad \begin{bmatrix} \boxed{\delta^\alpha_\beta} & 0 & \partial_\beta C^A \\ 0 & \boxed{0} & \partial_b C^A \\ \boxed{0} & 0 & \partial^\beta C^A \\ 0 & \boxed{\delta^a_b} & \partial^b C^A \end{bmatrix}.$$

In the case of the eikonal equation we have $C \doteq g^{ij}p_ip_j - n^2$ and the last columns in the above matrices reduce to a single colum,

$$(\circ)\qquad \begin{bmatrix} \boxed{\delta^\alpha_\beta} & 0 & \partial_\beta C \\ 0 & \boxed{-\partial_b\partial_a F} & \partial_b C \\ \boxed{0} & 0 & 2g^{\beta i}p_i \\ 0 & \boxed{\delta^a_b} & 2g^{bi}p_i \end{bmatrix}.$$

For $F = 0$ on $\widehat{\Sigma}$ we have $p_a = 0$ and the matrix becomes

$$(\circ\circ)\qquad \begin{bmatrix} \boxed{\delta^\alpha_\beta} & 0 & \partial_\beta C \\ 0 & \boxed{0} & \partial_b C \\ \boxed{0} & 0 & 2g^{\beta\gamma}p_\gamma \\ 0 & \boxed{\delta^a_b} & 2g^{b\gamma}p_\gamma \end{bmatrix}.$$

Since at least one $v^\beta \doteq g^{\beta\gamma}p_\gamma \neq 0$ (due to the eikonal equation, otherwise all $p_i = 0$), this last matrix has maximal rank. This shows that for the eikonal equation, C and $\widehat{\Sigma}$ have clean intersection.

Let $\Lambda_I \subset C$ be a geometrical solution and let $\Sigma \subset Q$ be a submanifold. Let us consider the diagonal relation

$$\Delta_\Sigma = \{(q, q') \in Q \times Q \mid q = q' \in \Sigma\} \subset Q \times Q$$

and compose its canonical lift $\widehat{\Delta}_\Sigma$ with the characteristic relation,

$$\Lambda_O = D_C \circ \widehat{\Delta}_\Sigma \circ \Lambda_I. \tag{3}$$

In this case, as it is shown below, the submanifold Σ represents a **mirror**: it transforms an **input** or **incident system of rays** Λ_I into an **output** or **reflected system** Λ_O.

Definition 1. We call **mirror relation** associated with a submanifold $\Sigma \subset Q$ the canonical lift

$$M_\Sigma = \widehat{\Delta}_\Sigma \subset T^*Q \times T^*Q$$

defined by

$$\boxed{(p, p') \in M_\Sigma = \widehat{\Delta}_\Sigma = \iff \begin{cases} p, p' \in T_q^*Q, & q \in \Sigma \\ p - p' \in T_\Sigma^\circ = \widehat{\Sigma} \end{cases}} \tag{4}$$

This definition follows from the general Definition 3 in §2.8, of canonical lift of smooth relation. Indeed, by applying formula (7) of §2.8 to the present case we have

$$\widehat{\Delta}_\Sigma = \{(p, p') \in T_{(q,q')}(Q \times Q),\ (q, q') \in \Delta_\Sigma,\ \langle v, p\rangle = \langle v', p'\rangle\ \forall (v, v') \in T_{(q,q')}\Delta_\Sigma\}.$$

However, a tangent vector $(v, v') \in T_{(q,q')}\Delta_\Sigma$ is an equivalence class of a curve γ on Δ_Σ and such a curve is necessarily of the form $t \mapsto \gamma(t) = \big(q(t), q(t)\big)$. Thus, the pair $(v, v') \in T_{(q,q')}\Delta_\Sigma$ is represented by a unique vector $v \in T_q\Sigma$. So that

$$\widehat{\Delta}_\Sigma = \{(p, p') \in T_{(q,q')}(Q \times Q),\ q = q' \in \Sigma,\ \langle v, p - p'\rangle = 0\ \forall v \in T_q\Sigma\}$$

and we get the definition (4). Note that the second condition in (4) means that $\langle v, p - p'\rangle = 0$, $\forall v \in T_q\Sigma$.

The output set Λ_O given by the composition (3) may not be a submanifold (even if Λ_I is a submanifold), so that it represents a rather complicated system of rays. But in any case,

Theorem 2. *The composition, in the order, of a generating family of the incident system of rays Λ_I, of a generating family of M_Σ and of a Hamilton principal function S of C gives a generating family of the reflected system of rays Λ_O.*

Remark 3. Similar results hold if we consider the canonical lift with function $(\widehat{\Delta_\Sigma, F})$ and the set

$$\Lambda_O = D_C \circ (\widehat{\Delta_\Sigma, F}) \circ \Lambda_I. \tag{5}$$

Then the pair (Σ, F) represents an **ideal lens**.

Definition 2. We call **lens relation** corresponding to the pair (Σ, F) the canonical lift with function

$$L_{(\Sigma,F)} = (\widehat{\Delta_\Sigma, F}) \subset T^*Q \times T^*Q$$

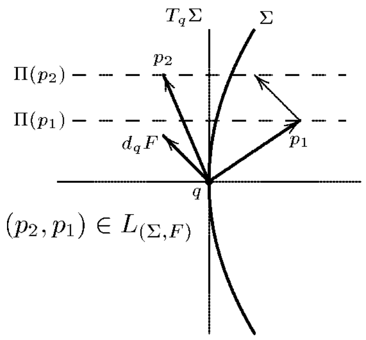
defined by

$$(6) \qquad \boxed{(p,p') \in L_{(\Sigma,F)} = (\widehat{\Delta_\Sigma, F}) \iff \begin{cases} p, p' \in T_q^*Q, \quad q \in \Sigma, \\ \langle v, p - p' - d_qF \rangle = 0, \quad \forall v \in T_q\Sigma \end{cases}}$$

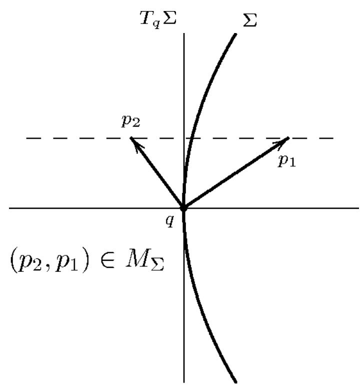
Remark 4. Note that

$$(7) \qquad L_{(\Sigma,c)} = M_\Sigma, \qquad (\widehat{\Sigma, F}) = L_{(\Sigma,F)} \circ \widehat{\Sigma}, \qquad L_{(\Sigma,-F)} = L^\top_{(\Sigma,F)}.$$

Remark 5. All the above definitions and theorems find a clear interpretation in the case of the n-dimensional Euclidean space, $Q = \mathbb{E}_n = \mathbb{R}^n$. If Σ is a regular r-dimensional surface, and covectors are interpreted as vectors, then a pair (p_2, p_1) based on $q \in \Sigma$ belongs to the relation $L_{(\Sigma,F)}$ if and only if the *"vector"* $p_2 - (p_1 + d_qF)$ is orthogonal to the tangent plane $T_q\Sigma$. It follows that all p_2 in relation with a fixed p_1 belong to the $n-r$-dimensional plane $\Pi(p_2)$ orthogonal to $T_q\Sigma$ determined by the *"end point"* of $p_1 + d_qF$. Moreover, if $\Pi(p_1)$ is the plane passing through p_1 and orthogonal to $T_q\Sigma$, then all pairs (p_2, p_1) whose end points are on these two planes belong to the relation. If $r = n - 1$, then these planes become straight lines. For $F = 0$ (or constant) we get the mirror relation.

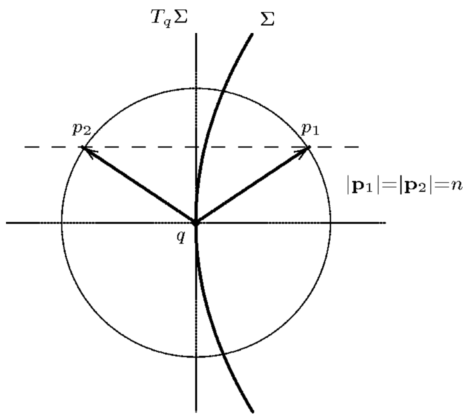


Lens relation



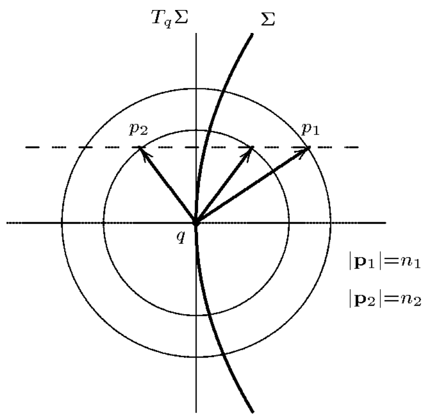
Mirror relation

If the covectors have a prescribed constant length $n = v/c$ (due for instance to an eikonal equation), then only the vectors belonging to the sphere of radius n are involved in the relation. Thus, in the case of hypersurface of codimension 1, the mirror relation yields the usual reflection law.



Reflection law

If the covectors have a different length in the two half-spaces separated by the boundary Σ, then the mirror relation M_Σ yields the refraction law.



Refraction law

Actually, with an **incident vector** $\mathbf{p}_1$ this relation associates two **refracted vectors** $\mathbf{p}_2$. Only one of them (and only a half-line determined by it) has a physical meaning.

This is the case of two different Hamilton-Jacobi equations C_1 and C_2 in T^*Q (for instance, two eikonal equations corresponding to two different media, with refraction index n_1 and n_2 respectively, separated by a surface Σ). If we consider these two media coexisting in the whole space, then we have to deal with two characteristic relations D_{C_1} and D_{C_2} and with a surface Σ. Let Λ_1 be a solution of C_1 (i.e., a system of rays in the first medium), possibly generated by a source Σ_1, so that $\Lambda_1 = D_{C_1} \circ \widehat{\Sigma}_1$. If Σ is a surface separating the two media, then the composition

$$\Lambda_2 = D_{C_2} \circ M_\Sigma \circ \Lambda_1 \tag{8}$$

gives a solution of C_2 representing the refracted system of rays.

3.6 The theorem of Jacobi

When a coisotropic submanifold C of a symplectic manifold $(\mathcal{S}, \omega)$ (in particular, of a cotangent bundle T^*Q) is given, then we are faced by the problem of finding its characteristics. Indeed, if the characteristics are known then we can solve the Cauchy problem i.e., we can construct geometrical solutions (Lagrangian submanifolds) $\Lambda \subset C$ starting from initial data (cf. §1.10). The geometrical background of the problem of finding the characteristics is given by the following theorems.

Theorem 1. *Let C be a coisotropic submanifold of a symplectic manifold $(\mathcal{S}, \omega)$. Assume that we know a symplectic reduction $R \subset \mathcal{S}_0 \times \mathcal{S}$ from $(\mathcal{S}, \omega)$ onto a symplectic manifold $(\mathcal{S}_0, \omega_0)$, whose inverse image is C,*

$$C = R^\top \circ \mathcal{S}_0.$$

Then the characteristics of C are the connected components of the fibres of the reduction R,

$$R^\top \circ \{p_0\}, \qquad p_0 \in \mathcal{S}_0.$$

We call **fibres** of a reduction the inverse images of points.

Proof. We recall item (iii) of Theorem 2 of §1.8. Since we assume that the inverse images of points are connected, they coincide with the characteristics. ∎

Theorem 2. *Assume that a symplectic reduction $R \subset \mathcal{S}_0 \times \mathcal{S}$ has connected fibres. Then the symplectic reduction R_C associated with the coisotropic submanifold $C = R^T \circ \mathcal{S}_0$ is* **isomorphic** *to R i.e., the composition*

$$\Phi = R_C \circ R^\top \subset \mathcal{S}/C \to \mathcal{S}_0$$

is the graph of a symplectomorphism $\varphi \colon \mathcal{S}_0 \to \mathcal{S}/C$ from $(\mathcal{S}_0, \omega_0)$ to the reduced symplectic manifold $(\mathcal{S}/C, \omega/C)$.

Proof. It is clear that Φ is a one-to-one smooth relation. In order to prove that it is the graph of a symplectomorphism we have to prove that Φ is a Lagrangian submanifold. For this we consider the relation $R_C \times R$ from $(\mathcal{S}, \omega) \times (\mathcal{S}, -\omega)$ to $(\mathcal{S}/C, \omega/C) \times (\mathcal{S}_0, -\omega_0)$. We observe that it is a symplectic reduction with inverse image $C \times C$ and that Φ is the image by R_C of the characteristic relation $D_C \subset C \times C$. Since D_C is a Lagrangian submanifold having clean intersection with $C \times C$ (since it is contained in $C \times C$), it follows that Φ is a Lagrangian submanifold (Remark 1 and Theorem 3, §1.8). ■

These two theorems prove the second part, items (ii) and (iii), of Theorem 1, §3.4. The further geometrical foundation of the theorem of Jacobi (expressed by item (i) of Theorem 1, §3.4) is strictly related to the cotangent bundle structure: *if the symplectic manifold $\mathcal{S}$ is a cotangent bundle $T^*\mathcal{Q}$, then a reduction R with inverse image C is determined by any Lagrangian foliation of C, $\{\Lambda_a,\ a \in A\}$ described by a parametrized set of generating functions $W_a \colon \mathcal{Q} \to \mathbb{R}$* (or generating families $W_a \colon \mathcal{Q} \times U \to \mathbb{R}$). This is the concept of complete solution of the Hamilton-Jacobi (Definition 2, §3.4). Indeed,

Theorem 3. *A smooth function $W \colon \mathcal{Q} \times A \to \mathbb{R} \colon (q, a) \mapsto W_a(q)$ is a complete solution of the Hamilton-Jacobi equation $C \subset T^*\mathcal{Q}$ if and only if it generates a symplectic coreduction $R^\top \subset T^*\mathcal{Q} \times B$ such that $C = R^\top \circ B$, where B is an open submanifold of T^*A.*

Proof. (i) Assume that W is a complete solution according to Definition 2, §3.4. Let $(\underline{q}, \underline{a}) = (q^i, a^\alpha)$ be local coordinates of $\mathcal{Q} \times A$. Let $(\underline{q}, \underline{p}; \underline{a}, \underline{b}) = (q^i, p_i; a^\alpha, b_\alpha)$ be the corresponding canonical coordinates on $T^*\mathcal{Q} \times T^*A$. Let us consider the symplectic relation $R \subset T^*\mathcal{Q} \times T^*A$ generated by the function W and described by equations

$$
\begin{cases}
p_i = \dfrac{\partial W(\underline{q}, \underline{a})}{\partial q^i} \\[2ex]
b_\alpha = -\dfrac{\partial W(\underline{q}, \underline{a})}{\partial a^\alpha}.
\end{cases} \tag{1}
$$

Since the Lagrangian submanifolds Λ_a described by the first equations (1) form a foliation parametrized by $\underline{a}$, these equations are solvable with respect to the variables $\underline{a}$. Thus,

$$
\operatorname{rank}\left[\frac{\partial^2 W}{\partial \underline{q}\, \partial \underline{a}}\right] = m = \dim(A) \tag{2}
$$

and by the implicit function theorem we get smooth functions

$$
a^\alpha = a^\alpha(\underline{q}, \underline{p}) \tag{3}
$$

representing the canonical projection $\pi \colon C \to A$. By inserting these function into the second equations (1) we get functions

$$
b_\alpha = b_\alpha(\underline{q}, \underline{p}). \tag{4}
$$

Equations (3-4) are equivalent to equations (1). This shows that R is the graph of a smooth mapping $\rho: C \to T^*A$. By a formal derivation of equations (1) we get equations

$$(5) \qquad \begin{cases} \dot{p}_i = \dfrac{\partial^2 W}{\partial q^i \partial q^j}\, \dot{q}^j + \dfrac{\partial^2 W}{\partial q^i \partial a^\alpha}\, \dot{a}^\alpha \\ \dot{b}_\alpha = \dfrac{\partial^2 W}{\partial b_\alpha \partial q^j}\, \dot{q}^j + \dfrac{\partial^2 W}{\partial b_\alpha \partial a^\beta}\, \dot{a}^\beta, \end{cases}$$

which represent the tangent mapping $T\rho$. If we assign any arbitrary value to $(\dot{a}^\alpha, \dot{b}_\alpha)$, due to the maximality condition (2) the second equations (5) admits a solution $\dot{q}^i$ and due to the first equations (5) we get values for $\dot{p}_i$. This shows that ρ is a submersion thus, that R is a reduction. The image set of a surjective submersion is an open submanifold [Dieudonné, 16.7.5]. This shows that R is a symplectic reduction onto an open subset of T^*A. (ii) Conversely, assume that W is the generating function of a symplectic coreduction $R^\top \subset T^*Q \times B$, $B \subseteq T^*A$, with B open and $C = R^\top \circ B$. Then the first equations (1) describe Lagrangian submanifolds $\Lambda_a = R^\top \circ (T^*A \cap B)$ which are contained in C. Since the fibres $T^*_a A \cap B$ form a Lagrangian foliation, then the Lagrangian submanifolds Λ_a form a foliation. The canonical projection π is a submersion, since it is the composition of two submersion, $\pi = \pi_A \circ \rho$, where ρ is the submersion associated with the reduction R. ∎

Remark 1. About the requirement (iii) in Definition 2, §3.4 of complete solution we observe that there are cases in which a differentiable function $W: Q \times A \to \mathbb{R}$ generates a Lagrangian foliation of C such that the canonical projection π is not differentiable. An example is the following: $Q = \mathbb{R}$, $C = T^*Q = \mathbb{R}^2$, $A = \mathbb{R}$, $W(q,a) = a^3 q$. The Lagrangian submanifold Λ_a is described by equation $p = a^3$ and the mapping π is described by $a = p^{\frac{1}{3}}$. This mapping is not differentiable for $p = 0$ $(a = 0)$. Hence, W is not a complete integral. Instead, the function $W(q,a) = aq$ is a complete integral and defines the same foliation.

Chapter 4

Hamiltonian optics in Euclidean spaces

4.1 The distance function

Let $\mathcal{Q} = \mathbb{R}^n = \{\mathbf{x}\} = \{(x^i)\}$ be the Euclidean n-space. We can identify the tangent bundle $T\mathcal{Q} = \{(\mathbf{x}, \mathbf{p})\}$ with the cotangent bundle $T^*\mathcal{Q} = \{(x^i, p_i)\}$. Notation: $\mathbf{u} \cdot \mathbf{v} = \sum_i u^i v^i$ is the scalar product of two vectors and $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$; for $n = 3$, $\mathbf{u} \times \mathbf{v}$ is the cross product of two vectors.

Let $U \subset \mathcal{Q}$ be a regular and orientable r-dimensional surface (locally) described by parametric equations $\mathbf{x} = \mathbf{u}(u^\alpha)$, $\alpha = 1, \ldots, r$. Let us consider the **distance function** $\Phi \colon \mathcal{Q} \times U \to \mathbb{R}$

(1)
$$\boxed{\Phi(\mathbf{x}, \mathbf{u}) = |\mathbf{x} - \mathbf{u}|}$$

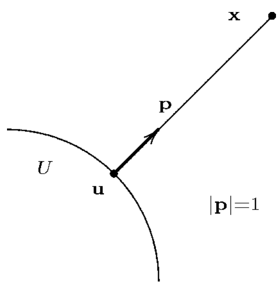
as a generating family on $\mathcal{Q}$ with supplementary manifold U (and supplementary coordinates (u^α)).

Theorem 1. *The distance function is a differentiable Morse family for* $\mathbf{x} \neq \mathbf{u}$. *It generates the Lagrangian submanifold* Λ_U *defined by*

(2)
$$(\mathbf{x}, \mathbf{p}) \in \Lambda_U \quad \Longleftrightarrow \quad \begin{cases} \mathbf{p} = \dfrac{\mathbf{x} - \mathbf{u}}{|\mathbf{x} - \mathbf{u}|}, & \mathbf{u} \in U, \quad \mathbf{u} \neq \mathbf{x}, \\ \mathbf{p} \perp U, \end{cases}$$

and contained in the 1-codimensional coisotropic submanifold $C \subset T^*\mathcal{Q}$ *defined by equation*

(3)
$$|\mathbf{p}|^2 = \textstyle\sum_i p_i^2 = 1.$$



Proof. The equations of the Lagrangian set Λ_U generated by Φ are

$$(4)\qquad \begin{cases} \mathbf{p} = \dfrac{\partial\Phi}{\partial\mathbf{x}} = \dfrac{\mathbf{x}-\mathbf{u}}{|\mathbf{x}-\mathbf{u}|}, \\ 0 = \dfrac{\partial\Phi}{\partial u^\alpha} = \dfrac{1}{|\mathbf{x}-\mathbf{u}|}\,\partial_\alpha(\mathbf{x}-\mathbf{u})\cdot(\mathbf{x}-\mathbf{u}) = -\,\mathbf{e}_\alpha\cdot\mathbf{p}, \end{cases}$$

where

$$(5)\qquad \mathbf{e}_\alpha = \partial_\alpha\mathbf{u} \qquad \left(\partial_\alpha = \frac{\partial}{\partial u^\alpha}\right)$$

are the coordinate vectors tangent to U. Equations (4) are equivalent to (2). From the first equation (4) it follows that $|\mathbf{p}| = \mathbf{1}$. From the second equation (4) it follows that the critical set Ξ is made of pairs of vectors $(\mathbf{x}, \mathbf{u})$ such that $\mathbf{x}-\mathbf{u}$ is perpendicular to U at the point $\mathbf{u}$. Let us set

$$(6)\qquad \begin{cases} A_{\alpha\beta} = \mathbf{e}_\alpha\cdot\mathbf{e}_\beta, \\ \partial_\alpha\mathbf{e}_\beta = \Gamma^\gamma_{\alpha\beta}\,\mathbf{e}_\gamma + \mathbf{B}_{\alpha\beta}, \end{cases}$$

where: $A_{\alpha\beta}$ are the components of the first fundamental form of the surface U, $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols, and $\mathbf{B}_{\alpha\beta}$ are vector fields orthogonal to U, representing the external curvature of the surface. Note that if U is of codimension 1, then $\mathbf{B}_{\alpha\beta} = B_{\alpha\beta}\,\mathbf{n}$, where $\mathbf{n} =$ is a unit vector field orthogonal to U. Thus, $B_{\alpha\beta}$ are the components of the second fundamental form of the hypersurface U. It follows that

$$(7)\qquad \begin{aligned} \frac{\partial\mathbf{p}}{\partial u^\alpha} = \frac{\partial^2\Phi}{\partial\mathbf{x}\partial u^\alpha} &= -\,\frac{\partial_\alpha\mathbf{u}}{|\mathbf{x}-\mathbf{u}|} - \tfrac{1}{2}\,\frac{\mathbf{x}-\mathbf{u}}{|\mathbf{x}-\mathbf{u}|^3}\,2\,(\mathbf{x}-\mathbf{u})\cdot\partial_\alpha(\mathbf{x}-\mathbf{u}) \\ &= \frac{(\mathbf{p}\cdot\mathbf{e}_\alpha)\,\mathbf{p}-\mathbf{e}_\alpha}{|\mathbf{x}-\mathbf{u}|}, \end{aligned}$$

and

$$(8)\qquad \begin{aligned}\frac{\partial^2\Phi}{\partial u^\alpha \partial u^\beta} &= -\,\partial_\alpha(\mathbf{e}_\beta \cdot \mathbf{p}) = -\,\partial_\alpha \mathbf{p}\cdot \mathbf{e}_\beta - \mathbf{p}\cdot(\Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma + B_{\alpha\beta}\,\mathbf{n}) \\ &= \frac{A_{\alpha\beta} - (\mathbf{p}\cdot\mathbf{e}_\alpha)(\mathbf{p}\cdot\mathbf{e}_\beta)}{|\mathbf{x}-\mathbf{u}|} - \mathbf{p}\cdot(\Gamma^\gamma_{\alpha\beta}\mathbf{e}_\gamma + B_{\alpha\beta}\,\mathbf{n}).\end{aligned}$$

On the critical set Ξ we have $\mathbf{p}\cdot\mathbf{e}_\alpha = 0$, so that

$$(9)\qquad \begin{cases} \dfrac{\partial^2\Phi}{\partial \mathbf{x}\partial u^\alpha} = \dfrac{\partial \mathbf{p}}{\partial u^\alpha} = -\,\dfrac{\mathbf{e}_\alpha}{|\mathbf{x}-\mathbf{u}|}, \\ \dfrac{\partial^2\Phi}{\partial u^\alpha \partial u^\beta} = \dfrac{A_{\alpha\beta}}{|\mathbf{x}-\mathbf{u}|} - \mathbf{B}_{\alpha\beta}\cdot\mathbf{p}. \end{cases}$$

Since the vectors $\mathbf{e}_\alpha$ are independent, the first sub-matrix of the matrix

$$\left[\frac{\partial^2\Phi}{\partial u^\alpha \partial x^i} \;\middle|\; \frac{\partial\Phi}{\partial u^\alpha \partial u^\beta}\right]_\Xi$$

has maximal rank. This proves that the distance function is a Morse family. ■

Remark 1. The distance function has no derivative for $\mathbf{x} = \mathbf{u} \in U$. Thus, the Lagrangian submanifold Λ_U is not defined over the points $\mathbf{x}$ of U.

Remark 2. The Lagrangian submanifold Λ_U is contained in the coisotropic submanifold C defined by equation (3). This is the eikonal equation of the Euclidean plane. The rays are oriented straight lines (Theorem 1, §3.2). The system of rays corresponding to Λ_U is the set of outgoing straight lines perpendicular to U. Here, the submanifold U behaves as a source of a system of rays, in accordance with the theory developed in §3.5 (where a source has been denoted by Σ).

Remark 3. The caustic Γ_U of Λ_U is described by equations (Theorem 2, §2.5)

$$(10)\qquad \begin{cases} \det[\partial_{\alpha\beta}\Phi] = 0, \\ \partial_\alpha\Phi = 0. \end{cases}$$

Due to $(4)_2$ and to the second equation (9), these equations are equivalent to

$$(11)\qquad \begin{cases} \det\left[\mathbf{p}\cdot\mathbf{B}_{\alpha\beta} - \dfrac{1}{|\mathbf{x}-\mathbf{u}|}\,A_{\alpha\beta}\right] = 0, \\ \mathbf{p}\cdot\mathbf{e}_\alpha = 0. \end{cases}$$

Remark 4. Let us consider the particular case of an oriented surface U in the 3-space: $n = 3$, $r = 2$. The first equation (11) is equivalent to

$$(12)\qquad \det\left[B_{\alpha\beta} - \frac{1}{(\mathbf{x}-\mathbf{u})\cdot\mathbf{n}}\,A_{\alpha\beta}\right] = 0,$$

and the second one to $\mathbf{x}-\mathbf{u} \perp U$. Since the characteristic equation of the main curvatures of a surface is

$$\det\,[B_{\alpha\beta} - \lambda\, A_{\alpha\beta}] = 0,$$

we find

$$\lambda = \frac{1}{(\mathbf{x}-\mathbf{u}) \cdot \mathbf{n}} = \frac{1}{|\mathbf{x}-\mathbf{u}|}.$$

This shows that

Theorem 2. *The caustic Γ_U of the Lagrangian manifold Λ_U generated by the distance function $\Phi(\mathbf{x},\mathbf{u}) = |\mathbf{x}-\mathbf{u}|$ is the set of the centers of curvature of the surface U.*

As a consequence,

Theorem 3. *The only sources U which generate systems of rays without caustics are the plane surfaces.*

Remark 5. Let us consider the following parametric representation of Λ_U in the three parameters (u^α, μ):

$$\begin{cases} \mathbf{p} = \mathbf{n}(u^\alpha) = \dfrac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}, \\ \mathbf{x} = \mathbf{u} + \mu\, \mathbf{p}, \qquad \mu \in \mathbb{R}. \end{cases} \tag{13}$$

The determinant of the second order derivatives of $\mathbf{x}$ is equal to

$$\begin{aligned} \mathbf{p} \cdot (\mathbf{e}_1 + \mu\, \partial_1 \mathbf{p}) \times (\mathbf{e}_2 + \mu\, \partial_2 \mathbf{p}) &= \mu^2\, \mathbf{p} \cdot \partial_1 \mathbf{p} \times \partial_2 \mathbf{p} + \mathbf{p} \cdot \mathbf{e}_1 \times \mathbf{e}_2 \\ &= \mu^2\, \mathbf{p} \cdot \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{x}-\mathbf{u}|^2} + \mathbf{p} \cdot \mathbf{e}_1 \times \mathbf{e}_2 \\ &= \left(1 + \frac{\mu^2}{|\mathbf{x}-\mathbf{u}|^2}\right) \mathbf{n} \cdot \mathbf{e}_1 \times \mathbf{e}_2 \neq 0. \end{aligned}$$

Hence, the representation (13) is an immersion.

Example 1. The above results can be adapted to the case of a curve U in the Euclidean plane $\mathbb{R}^2 = (x,y)$. Let us prove that *if U is described by parametric equations $x = x(t)$, $y = y(t)$, then the caustic Γ_U is described by parametric equations*

$$\begin{cases} x = x(t) - \dfrac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\, \dot{y}, \\ y = y(t) + \dfrac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}\, \dot{x}. \end{cases} \tag{14}$$

Let us consider the Morse family

$$G(\mathbf{x}; \mathbf{a}, t) = \mathbf{a} \cdot \big(\mathbf{x} - \mathbf{x}(t)\big) \tag{15}$$

with supplementary variables $\mathbf{a} \in \mathbb{S}_1$ and $t \in \mathbb{R}$. By setting

$$\mathbf{a} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

we observe that this Morse family is equivalent to

$$G(x, y; \theta, t) = \big(x - x(t)\big) \cos\theta - \big(y - y(t)\big) \sin\theta, \tag{16}$$

with supplementary variables $\theta, t \in \mathbb{R}$. The corresponding Lagrangian submanifold is then described by equations

$$\begin{cases} \dfrac{\partial G}{\partial \theta} \equiv \big(y - y(t)\big) \cos\theta - \big(x - x(t)\big) \sin\theta = 0, \\ \dfrac{\partial G}{\partial t} \equiv \dot{x} \cos\theta + \dot{y} \sin\theta = 0, \end{cases} \tag{17}$$

and

$$\begin{cases} p_x = \dfrac{\partial G}{\partial x} = \cos\theta, \\ p_y = \dfrac{\partial G}{\partial y} = \sin\theta. \end{cases} \tag{18}$$

Equations (17) describe the critical set. The vectorial expression of equations (17) and (18) is

$$\mathbf{p} = \mathbf{a}, \qquad \big(\mathbf{x} - \mathbf{u}(t)\big) \times \mathbf{p} = 0, \qquad \dot{\mathbf{u}}(t) \cdot \mathbf{p} = 0.$$

The last equation means that $\mathbf{p} \perp U$. The second equation means that $\mathbf{x} - \mathbf{u}(t)$ is parallel to $\mathbf{p}$. Since $|\mathbf{p}|^2 = 1$, this equation becomes equivalent to

$$\mathbf{p} = \pm \frac{\mathbf{x} - \mathbf{u}(t)}{|\mathbf{x} - \mathbf{u}(t)|}.$$

Thus, the Lagrangian submanifold generated by this function has two connected components. By choosing the + sign we find the equations (2) of the Lagrangian submanifold Λ_U generated by the distance function. The caustic of this Lagrangian submanifold is described by equation

$$\det \begin{bmatrix} \dfrac{\partial^2 G}{\partial\theta\partial\theta} & \dfrac{\partial^2 G}{\partial\theta\partial t} \\ \dfrac{\partial^2 G}{\partial t\partial\theta} & \dfrac{\partial^2 G}{\partial t\partial t} \end{bmatrix} = 0 \tag{19}$$

together with the equations (17). From (19) we get equation

$$\left[\big(x - x(t)\big)\,\cos\theta + \big(y - y(t)\big)\,\sin\theta\right](\ddot{x}\,\cos\theta + \ddot{y}\,\sin\theta) - (\dot{x}\,\sin\theta - \dot{y}\,\cos\theta)^2 = 0.$$

By combining this equation with the first equation (17) we obtain the linear system

$$\begin{cases} \xi\cos\theta + \eta\sin\theta = X, \\ \xi\sin\theta - \eta\cos\theta = 0, \end{cases}$$

where

$$X = \frac{(\dot{x}\sin\theta - \dot{y}\cos\theta)^2}{\ddot{x}\sin\theta + \ddot{y}\sin\theta}, \quad \xi = x - x(t), \quad \eta = y - y(t).$$

It follows that

$$\xi = -\begin{vmatrix} X & \sin\theta \\ 0 & -\cos\theta \end{vmatrix}, \quad \eta = -\begin{vmatrix} \cos\theta & X \\ \sin\theta & 0 \end{vmatrix}.$$

From the second equation (17) we get

$$\cos\theta = \rho\,\dot{y}, \qquad \sin\theta = -\rho\,\dot{x},$$

with

$$1 = \rho^2\,(\dot{x}^2 + \dot{y}^2).$$

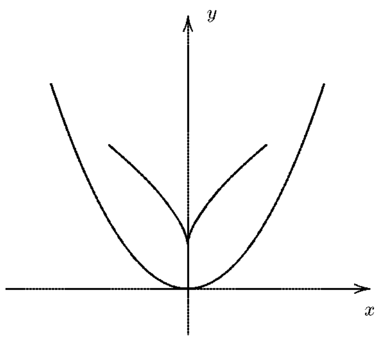
Thus,

$$\xi = -\begin{vmatrix} \rho\dfrac{(\dot{x}^2+\dot{y}^2)^2}{\ddot{x}\dot{y} - \ddot{y}\dot{x}} & -\rho\dot{x} \\ 0 & -\rho\dot{y} \end{vmatrix} = \rho^2\,\frac{(\dot{x}^2+\dot{y}^2)^2}{\ddot{x}\dot{y} - \ddot{y}\dot{x}}\,\dot{y} = \frac{\dot{x}^2+\dot{y}^2}{\ddot{x}\dot{y} - \ddot{y}\dot{x}}\,\dot{y}.$$

This proves the first equation (14). The second equation is proved in a similar way.

Example 2. For the parabola $y = \frac{1}{2}\,x^2$, by setting $x = t$, equations (14) yields the following parametric equations of the caustic:

$$x = -\,t^3, \qquad y = 1 + \tfrac{3}{2}\,t^2.$$



The caustic U' of a curve U in the plane i.e., the set of the centers of curvature of U according to Theorem 2, is tangent to all lines orthogonal to U.

Remark 6. Instead of the distance function we can consider the function

$$\Phi'(\mathbf{x},\mathbf{u}) = \tfrac{1}{2}\,|\mathbf{x}-\mathbf{u}|^2.$$

This is the Euclidean version of the **world function** introduced by Synge for a generic Riemannian manifold [Synge, 1960]. By considering this function as a generating family we write equations

$$\begin{cases} \mathbf{p} = \dfrac{\partial \Phi'}{\partial \mathbf{x}} = \mathbf{x}-\mathbf{u}, \\ 0 = \dfrac{\partial \Phi'}{\partial u^\alpha} = -\,\mathbf{p}\cdot\mathbf{e}_\alpha. \end{cases}$$

From the first equation it follows that

$$\frac{\partial \mathbf{p}}{\partial u^\alpha} = -\,\mathbf{e}_\alpha.$$

This shows that Φ' is a Morse family. This Morse family is now everywhere differentiable. The corresponding Lagrangian submanifold is defined by

$$\Lambda'_U \ni (\mathbf{x},\mathbf{p}) \qquad \Longleftrightarrow \qquad \begin{cases} \mathbf{p} = \mathbf{x}-\mathbf{u},\ \mathbf{u}\in U, \\ \mathbf{p}\perp U. \end{cases}$$

Moreover,

$$\frac{\partial^2 \Phi'}{\partial u^\beta \partial u^\alpha} = -\,\partial_\beta \mathbf{p}\cdot\mathbf{e}_\alpha - \mathbf{p}\cdot\partial_\beta\cdot\mathbf{e}_\alpha = A_{\alpha\beta} - \mathbf{p}\cdot\big(\Gamma^\gamma_{\beta\alpha}\,\mathbf{e}_\gamma + B_{\beta\alpha}\,\mathbf{n}\big).$$

Hence, under the condition $\partial_\alpha \Phi' = 0$, which is also in this case equivalent to $\mathbf{p}\cdot\mathbf{e}_\alpha = 0$, we find

$$\partial_\beta\partial_\alpha \Phi = A_{\beta\alpha} - \mathbf{n}\cdot(\mathbf{x}-\mathbf{u})\,B_{\beta\alpha},$$

and the equation of the caustic is identical to (12). Thus, $\Gamma'_U = \Gamma_U$. Note that the Lagrangian submanifold Λ'_U is not contained in the submanifold C of equation $\mathbf{p}^2 = 1$.

4.2 From wave optics to geometrical optics

Any component $u(\mathbf{x},t)$ of an electromagnetic potential is a solution of the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} - \Delta U = 0, \tag{1}$$

where Δ is the Laplace-Beltrami operator. In Cartesian coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

1 Let us consider the special class of **spherical solutions**,

$$u = u(r,t), \qquad r = |\mathbf{x}|.$$

Since $\Delta u = \mathrm{div}(\mathrm{grad}(u))$, for such a function we have

$$\Delta u = \frac{1}{r}\frac{\partial^2}{\partial r^2}(ru),$$

and from (1) it follows that

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} = 0, \qquad v = ru. \tag{2}$$

The general solution of this last equation is

$$v = f(r+t) + g(r-t), \tag{3}$$

where f and g are arbitrary (smooth) functions. Solutions of the kind $f(r+t)$ and $g(r-t)$ are called **incoming waves** and **outgoing waves**, respectively. Thus, the general spherical solution of the wave equation is

$$u = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}. \tag{4}$$

Such a function represents the **electromagnetic radiation generated by a point source** (the origin of the coordinates).

2 Among the spherical solutions let us consider **oscillatory outgoing solutions** of the kind

$$u_\omega(r,t) = \frac{c}{r}\, e^{i\omega(r-t)}, \qquad \omega \in \mathbb{R}_+, \;\; c \in \mathbb{R}. \tag{5}$$

If the source is located at the point $\mathbf{u}$, then the corresponding solution of this kind is

$$u_{(\omega,\mathbf{u})} = c(\mathbf{u})\, e^{-i\omega t}\, \frac{e^{i\omega|\mathbf{x}-\mathbf{u}|}}{|\mathbf{x}-\mathbf{u}|}. \tag{6}$$

The factor

$$I = c(\mathbf{u})\,\frac{e^{i\omega|\mathbf{x}-\mathbf{u}|}}{|\mathbf{x}-\mathbf{u}|} \tag{7}$$

is called the **intensity** of the radiation.

3 Let us consider the radiation generated by a surface U made of pointwise sources. The resulting intensity at any point $\mathbf{x}$ is given by the integral

$$\boxed{I_\omega(\mathbf{x}) = \int_U c(\mathbf{u})\,\frac{e^{i\omega|\mathbf{x}-\mathbf{u}|}}{|\mathbf{x}-\mathbf{u}|}\,d\mathbf{u}} \tag{8}$$

This is a surface integral of the kind

$$I_\omega = \int_U a(\mathbf{u})\,e^{i\omega\,\Phi(\mathbf{u})}\,d\mathbf{u}, \tag{9}$$

called **oscillatory integral**. The function $\Phi(\mathbf{u})$ is called the **phase function**. In the present case

$$\Phi(\mathbf{u}) = |\mathbf{x}-\mathbf{u}|, \qquad a(\mathbf{u}) = \frac{c(\mathbf{u})}{|\mathbf{x}-\mathbf{u}|}. \tag{10}$$

About this integral we have two fundamental **theorems on the stationary phase** (see e.g. [Guillemin, Sternberg, 1977], Chapter I, for proofs and references).

Theorem 1. *If $\mathbf{x}$ is such that $d_{\mathbf{u}}\Phi \neq 0$ at all points of U, then for all $m \in \mathbb{N}$*

$$I_\omega(\mathbf{x}) = O(\omega^{-m}). \tag{11}$$

The meaning of this theorem is that for $\omega \to \infty$, the radiations of all sources interfere in such a way that the total intensity is "negligible" at any point $\mathbf{x}$.

Theorem 2. *If $\mathbf{x}$ is such that $d_{\mathbf{u}}\Phi = 0$ at a finite number of points $\mathbf{u}_* \in U$, then the following asymptotic formula holds*

$$\boxed{I_\omega(\mathbf{x}) = \left(\frac{2\pi}{\omega}\right)^{\frac{n}{2}} \sum\nolimits_* a(\mathbf{u}_*)\,e^{i\omega\Phi(\mathbf{u}_*)}\,\frac{e^{i\frac{\pi}{4}\operatorname{sign}\mathbf{H}_*}}{\sqrt{\det\mathbf{H}_*}}\,\left(1 + O(\omega^{-1})\right)} \tag{12}$$

where $n = \dim(U)$ and $\mathbf{H}$ is the Hessian matrix of Φ and

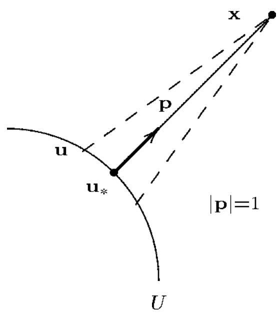
$$\operatorname{sign}(\mathbf{H}) = \#(\text{positive eigenvalues}) - \#(\text{negative eigenvalues}). \tag{13}$$

A first consequence of this theorem is that the non negligible contribution to the intensity I_ω comes only from those points $\mathbf{u}_*$ where $d_{\mathbf{u}}\Phi = 0$, that is

$$\frac{\partial \Phi}{\partial \mathbf{u}} = 0. \tag{14}$$

Since in our case $\Phi(\mathbf{u}) = |\mathbf{x} - \mathbf{u}|$, for any chosen $\mathbf{x}$, the points $\mathbf{u}_*$ satisfying this condition are the points where the line from $\mathbf{u}_*$ to $\mathbf{x}$ (i.e., the vector $\mathbf{x} - \mathbf{u}_*$) is orthogonal to U, and this holds for any other point $\mathbf{x}'$ on this line. This means that for an observer located at any point $\mathbf{x}$ on the line perpendicular to U at the point $\mathbf{u}_*$ only the radiation emitted by the source $\mathbf{u}_*$ is detected. This line is called the **ray** issued from $\mathbf{u}_*$. It is parallel to the vector

$$\mathbf{p} = \frac{\partial \Phi}{\partial \mathbf{x}}. \tag{15}$$



Furthermore, from formula (12) we observe that the intensity $I_\omega(\mathbf{x})$ is unbounded at the points $\mathbf{x}$ where $\det \mathbf{H}_* = 0$ i.e.,

$$\det \left[\frac{\partial^2 \Phi}{\partial u^\alpha \partial u^\beta} \right]_* = 0, \tag{16}$$

where (u^α) are local parameters of U. These points form the **caustic**. Equations (14), (15) and (16) are just the equations of the Lagrangian submanifold Λ_U and of the corresponding caustic generated by the distance function Φ, as it has been shown in the preceding section.

4.3 The eikonal equation and the global Hamilton principal function

Let $C \subset T^*\mathbb{R}^n$ be the coisotropic submanifold of codimension 1 defined by equation

$$\textstyle\sum_i p_i^2 = |\mathbf{p}|^2 = 1. \tag{1}$$

This is the **eikonal equation** for the homogeneous (empty) Euclidean n-space. The characteristics of C can be found by integrating the Hamilton equations generated by the Hamiltonian $H = \sum_i p_i^2$,

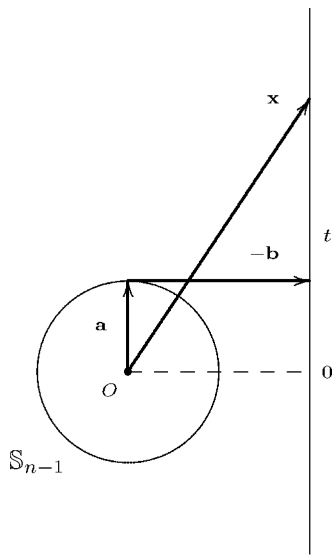
$$\begin{cases} \dot{x}^i = \lambda\, p_i \\ \dot{p}_i = 0, \end{cases}$$

where λ is any function. By choosing $\lambda = 1$, we find that: (i) the characteristics are the straight lines described by parametric equations

$$\begin{cases} \mathbf{x} = t\,\mathbf{a} - \mathbf{b} \\ \mathbf{p} = \mathbf{a} \end{cases} \qquad \mathbf{a} \in \mathbb{S}_{n-1}, \quad \mathbf{b} \in \mathbb{R}^n, \tag{2}$$

since $\mathbf{p} \in \mathbb{S}_{n-1}$ (the unit sphere) is equivalent to $\mathbf{p} \in C$; (ii) the rays are oriented straight lines in $\mathbb{R}^n$.

It is convenient to choose $\mathbf{b}$ orthogonal to $\mathbf{a}$ i.e., $\mathbf{b}$ tangent to the sphere $\mathbb{S}_{n-1}$ at the "point" $\mathbf{a}$. In this way, through equations (2), any characteristic of C is determined by a pair of vectors $(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} \in \mathbb{S}_{n-1}$ and $\mathbf{b}$ is a vector tangent to the sphere and orthogonal to $\mathbf{a}$. This defines a one-to-one mapping from the set $\mathcal{S}_C$ of the characteristics to the tangent bundle $T\mathbb{S}_{n-1}$, which is identified with the cotangent bundle $T^*\mathbb{S}_{n-1}$. The minus sign in front of $\mathbf{b}$ is chosen in order to get a symplectomorphism between the reduced symplectic manifold $\mathcal{S}_C$ and the cotangent bundle $T^*A = T^*\mathbb{S}_{n-1}$ (see below).



It follows from (2) that two pairs $(\mathbf{x}, \mathbf{p})$ and $(\mathbf{x}', \mathbf{p}')$, representing two points of $T^*\mathcal{Q}$, belong to a same characteristic if and only if

$$(3) \qquad \begin{cases} \mathbf{p} = \mathbf{p}' = \mathbf{a} \in \mathbb{S}_{n-1} \\ \mathbf{x} - \mathbf{x}' \parallel \mathbf{p} \end{cases} \qquad (\parallel = \text{parallel to}).$$

Thus, the characteristic relation D_C is defined by these conditions.

Remark 1. (i) With each pair $(\mathbf{a}, \mathbf{x}) \in A \times \mathcal{Q} = \mathbb{S}_{n-1} \times \mathbb{R}^n$ we associate a unique element $\big((\mathbf{a}, \mathbf{b}), (\mathbf{x}, \mathbf{p})\big) \in T^*A \times T^*\mathcal{Q}$ belonging to the reduction relation R_C. Indeed, as it will be proved below, R_C is a regular Lagrangian submanifold of $T^*(A \times \mathcal{Q})$. (ii) With each pair $(\mathbf{x}, \mathbf{x}') \in \mathbb{R}^n \times \mathbb{R}^n = \mathcal{Q} \times \mathcal{Q}$ such that $\mathbf{x} \neq \mathbf{x}'$ we can associate two elements of D_C differing by the sign, $\big((\mathbf{x}, \mathbf{p}), (\mathbf{x}', \mathbf{p}')\big)$ and $\big((\mathbf{x}, -\mathbf{p}), (\mathbf{x}', -\mathbf{p}')\big)$. This means that D_C is two-folded over the points of $\mathcal{Q} \times \mathcal{Q}$ out of the diagonal, while over the diagonal (i.e., for $\mathbf{x}' = \mathbf{x}$) it is made of pairs $\big((\mathbf{x}, -\mathbf{p}), (\mathbf{x}, -\mathbf{p}')\big)$ such that $\mathbf{p} = \mathbf{p}' \in \mathbb{S}_{n-1}$. Indeed, as it will be proved below, D_C is a regular (and two-folded) Lagrangian submanifold of $T^*(\mathcal{Q} \times \mathcal{Q})$ out of the diagonal, while it is singular over the diagonal (§3.4).

For the construction and the analysis of the characteristic relation D_C we can follow another way: to look for a complete solution of C and use it for constructing a Hamilton principal function.

Theorem 1. *A global complete solution of the eikonal equation* (1) $|\mathbf{p}|^2 = 1$ *is the function on* $\mathcal{Q} \times \mathbb{S}_{n-1}$ *defined by*

$$(4) \qquad \boxed{W(\mathbf{x}, \mathbf{a}) = \mathbf{a} \cdot \mathbf{x}, \qquad \mathbf{a} \in \mathbb{S}_{n-1}}$$

Proof. The partial differential equation associated with the eikonal equation (1) is

$$\textstyle\sum_i \left(\frac{\partial W}{\partial x^i}\right)^2 = 1.$$

It is integrable by separation of variables. A solution is

$$(5) \qquad W = \sum_i a_i \, x^i$$

with integration constants such that

$$\sum_i a_i^2 = 1.$$

This means $\mathbf{a} = (a_i) \in \mathbb{S}_{n-1}$. The function (5) is a complete solution since for each $\mathbf{p} \in C$ there is a unique Lagrangian submanifold $\Lambda_{\mathbf{a}}$, generated by the functions

$W_{\mathbf{a}}(\mathbf{x})$ containing $\mathbf{p}$. Indeed, the vectorial equation of Λ_a is $\mathbf{p} = \mathbf{a}$. Moreover, the mapping $\pi: C \to A$ is a submersion. ∎

Remark 2. Each Lagrangian submanifold $\Lambda_{\mathbf{a}}$ corresponds to a system of **parallel rays** or **plane waves** (Example 1, §3.3).

Remark 3. The function W generates the transpose $R^\top$ of a symplectic reduction $R \subset T^*A \times T^*Q$, $A = \mathbb{S}_{n-1}$, whose inverse image in C. As a consequence, (i) the reduced set $\mathcal{S}_C$ is symplectomorphic to the cotangent bundle $T^*A = T^*\mathbb{S}_{n-1}$, (ii) the equations of $R^\top \subset T^*Q \times T^*A$ are

$$\text{(6)} \qquad \begin{cases} \mathbf{b} = -\dfrac{\partial W}{\partial \mathbf{a}} = -\big(\mathbf{x} - \mathbf{x} \cdot \mathbf{a}\,\mathbf{a}\big) = -P_{\mathbf{a}}(\mathbf{x}), \\ \mathbf{p} = \dfrac{\partial W}{\partial \mathbf{x}} = \mathbf{a}. \end{cases}$$

Remark 4. In the first equation (6) $P_{\mathbf{a}}$ denotes the projection operator onto the plane orthogonal to $\mathbf{a}$,

$$\text{(7)} \qquad P_{\mathbf{a}}(\mathbf{x}) = (\mathbf{I} - \mathbf{a} \otimes \mathbf{a})(\mathbf{x}) \qquad (\mathbf{I} = \text{identity}).$$

Here, we have used the following general property: assume that a hypersurface A (of codimension 1) in $\mathbb{R}^n$ is (locally) described by a vector function $\mathbf{a}(a^\alpha)$ depending on $n - \mathbf{1}$ parameters (surface coordinates) in such a way that the vectors tangent to A

$$\text{(8)} \qquad \mathbf{e}_\alpha = \frac{\partial \mathbf{a}}{\partial a^\alpha} = \partial_\alpha \mathbf{a}$$

are pointwise independent. Let $f(\mathbf{a})$ be any function on A. This function is locally represented by a function $f(a^\alpha)$ of the surface coordinates. The partial derivatives

$$\text{(9)} \qquad b_\alpha = \frac{\partial f}{\partial a^\alpha}$$

are the covariant components of a vector $\mathbf{b} = b^\alpha\, \mathbf{e}_\alpha$ tangent to the surface, being $b_\alpha = A_{\alpha\beta} b^\beta$ and $A_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ the components of the first fundamental form of the surface (see §4.1). Since $b_\alpha = \mathbf{b} \cdot \mathbf{e}_\alpha$, we can write (9) in the vectorial form

$$\text{(10)} \qquad \mathbf{b} = \frac{\partial f}{\partial \mathbf{a}}.$$

Let $f(\mathbf{x})$ be any (local) extension of $f(\mathbf{a})$ in a neighborhood of the surface. Its gradient

$$\nabla f = \frac{\partial f}{\partial \mathbf{x}}$$

is not (in general) a vector tangent to the surface. However, its tangent component coincides with the vector (10). It follows that

$$\frac{\partial f}{\partial \mathbf{a}} = P_{\mathbf{n}}\big(\nabla f(\mathbf{a})\big), \tag{11}$$

where $\mathbf{n}$ is a unit vector orthogonal to the surface at the point $\mathbf{a}$, and

$$P_{\mathbf{n}} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} \tag{12}$$

is the projection operator onto the $(n-1)$-dimensional plane orthogonal to $\mathbf{n}$ (and tangent to the surface). Note that for $A = \mathbb{S}_{n-1}$ we have $\mathbf{n} = \mathbf{a}$. Note that equations (6) coincide with equations (2) for $t = \mathbf{x} \cdot \mathbf{a}$.

Theorem 2. *The generating family $S(Q \times Q; \mathbb{S}_{n-1})$, with supplementary manifold $A = \mathbb{S}_{n-1}$, defined by*

$$\boxed{S(\mathbf{x}, \mathbf{x}'; \mathbf{a}) = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a}, \quad \mathbf{a} \in \mathbb{S}_{n-1}} \tag{13}$$

is a global Hamilton principal function of the eikonal equation on the Euclidean space $Q = \mathbb{R}^n$.

Proof. The co-reduction relation $R^\top$ is generated by $W(\mathbf{x}, \mathbf{a})$ (Remark 3). The reduction relation R is generated by $W^\top(\mathbf{a}, \mathbf{x}) = -W(\mathbf{x}, \mathbf{a})$. In accordance with Theorem 3 of §3.4 and formula (4), by composing these generating families we get the generating family (13) of the characteristic relation $D_C = R^\top \circ R$. ■

Remark 5. The equations of D_C generated by S are

$$\begin{cases} \mathbf{p}' = -\dfrac{\partial S}{\partial \mathbf{x}'} = \mathbf{a}, \\ \mathbf{p} = \dfrac{\partial S}{\partial \mathbf{x}} = \mathbf{a}, \\ 0 = \dfrac{\partial S}{\partial \mathbf{a}} = P_{\mathbf{a}}(\mathbf{x} - \mathbf{x}'). \end{cases} \tag{14}$$

These equations coincide with equations (3).

Remark 6. The reduction relation R_C is a regular Lagrangian submanifold, since it is generated by an ordinary generating function W (without extra variables). On the contrary, the characteristic relation (Remark 1, §3.4) is singular over the diagonal, so that in the neighborhood of the diagonal, is generated by a generating family.

Theorem 3. *The generating family $S(\mathbf{x}, \mathbf{x}'; \mathbf{a}) = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a}$ is a Morse family and the caustic of D_C is the diagonal of $Q \times Q$.*

Proof. Let us consider a parametric representation $\mathbf{a}(u^\alpha)$ of the sphere in the $n-1$ parameters (u^α). The vectors $\mathbf{e}_\alpha = \partial_\alpha \mathbf{a}$ are independent and tangent to the sphere. Since

$$\partial_\alpha S = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{e}_\alpha, \qquad \partial_\alpha = \partial/\partial u^\alpha,$$

the critical set Ξ is given by the pair of vectors such that $\mathbf{x}-\mathbf{x}' \perp \mathbb{S}_{n-1}$. Moreover,

$$\partial_\alpha \partial_\beta S = (\mathbf{x} - \mathbf{x}') \cdot \partial_\beta \mathbf{e}_\alpha = (\mathbf{x} - \mathbf{x}') \cdot (\Gamma^\gamma_{\beta\alpha}\, \mathbf{e}_\gamma + B_{\beta\alpha}\, \mathbf{a})$$

and

$$\begin{cases} \partial_i \partial_\alpha S = e^i_\alpha, & \partial_i = \partial/\partial x^i, \\ \partial_{i'} \partial_\alpha S = e^i_\alpha, & \partial_{i'} = \partial/\partial x'^i, \end{cases}$$

where e^i_α are the Cartesian components of the vector $\mathbf{e}_\alpha$. Then the matrix

$$\left[\partial_\alpha \partial_i S \,\middle|\, \partial_\alpha \partial_{i'} S \,\middle|\, \partial_\alpha \partial_\beta S \right]$$

has maximal rank everywhere, since the submatrix $[\partial_\alpha \partial_i S] = [e^i_\alpha]$ has maximal rank, being the vectors $(\mathbf{e}_\alpha)$ independent. Hence, S is a Morse family. On the critical set,

$$\partial_\alpha \partial_\beta S = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a}\, B_{\beta\alpha}.$$

Since $\mathbf{x}-\mathbf{x}'$ is parallel to $\mathbf{a}$ and on the sphere $\det[B_{\beta\alpha}] \neq 0$, we have $\det\left[\partial_\alpha \partial_\beta S\right] = 0$ if and only if $\mathbf{x} - \mathbf{x}' = 0$. ∎

Theorem 4. *Outside the diagonal of $Q \times Q$ the characteristic relation D_C is the union of two disjoint regular symplectic relations generated by the functions*

(15)
$$\boxed{S_\pm(\mathbf{x}, \mathbf{x}') = \pm |\mathbf{x} - \mathbf{x}'|}$$

Proof. The symplectic relation generated by S_+ is represented by equations

(16)
$$\begin{cases} \mathbf{p}' = -\dfrac{\partial S_+}{\partial \mathbf{x}'} = \dfrac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}, \\ \mathbf{p} = \dfrac{\partial S_+}{\partial \mathbf{x}} = \dfrac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x} - \mathbf{x}'|}. \end{cases}$$

The requirements (3) are fulfilled. With S_- we get the opposite pair $(-\mathbf{p}, -\mathbf{p}')$. ∎

Remark 7. This theorem is in accordance with the general Theorem 3 of §3.4. In accordance with Theorem 2, §2.6 (cf. Remark 6, §3.4), the global Hamilton principal function (13) is skew-symmetric in $(\mathbf{x}, \mathbf{x}')$. Instead, the generating functions (15) $S_\pm$ are symmetric. This is not a contradiction, since these functions are "non-global" Hamilton principal functions: each one of them generates only a branch of the symplectic relation D_C.

Example 1. Generating families of systems of rays generated by a surface in $\mathbb{R}^n$. Let $U \subset \mathcal{Q} = \mathbb{R}^n$ be a r-dimensional regular surface described by independent $m = n - r$ equations

$$U_a(\mathbf{x}) = 0. \tag{17}$$

Then the canonical lift $\widehat{U}$ is generated by the Morse family (cf. (6), §2.8)

$$G_U(\mathbf{x}; \boldsymbol{\lambda}) = \lambda^a\, U_a(\mathbf{x}), \qquad \boldsymbol{\lambda} = (\lambda^a) \in \mathbb{R}^{n-r}. \tag{18}$$

In accordance with what we have seen in §4.1 about the distance function, the system of rays issued from U is represented by the Morse family

$$\boxed{G_1(\mathbf{x}; \mathbf{u}) = |\mathbf{x} - \mathbf{u}|, \quad \mathbf{u} \in U} \tag{19}$$

In accordance with Theorem 1 of §3.5, the system of outgoing and incoming rays is described by the generating family

$$\boxed{G_2(\mathbf{x}; \mathbf{x}', \mathbf{a}, \boldsymbol{\lambda}) = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a} + \lambda^a\, U_a(\mathbf{x}')} \tag{20}$$

with supplementary variables $\mathbf{x}' \in \mathbb{R}^n$, $\mathbf{a} \in \mathbb{S}_{n-1}$, $(\lambda^a) \in \mathbb{R}^{n-r}$. This follows from the composition of G_U (18) with the Hamilton principal function (13). It is remarkable the fact that this is always a Morse family, whatever U (cf. Remark 2, §3.5).

Example 2. Generating families of systems of rays generated by a point in $\mathbb{R}^n$. If the surface U of Example 1 reduces to a point $\mathbf{x}_0$, then $\widehat{U} = \widehat{\mathbf{x}_0}$ is the fibre over this point and the generating family (18) becomes

$$G_{\mathbf{x}_0}(\mathbf{x}; \boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot (\mathbf{x} - \mathbf{x}_0). \tag{21}$$

Then the generating family (19) becomes

$$\boxed{G_1(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|} \tag{22}$$

On the other hand, the generating family (20) becomes

$$G_2(\mathbf{x}; \mathbf{x}', \mathbf{a}, \boldsymbol{\lambda}) = (\mathbf{x} - \mathbf{x}') \cdot \mathbf{a} + \boldsymbol{\lambda} \cdot (\mathbf{x}' - \mathbf{x}_0), \quad \mathbf{a} \in \mathbb{S}_{n-1}, \;\; \boldsymbol{\lambda} \in \mathbb{R}^n.$$

However, one of the equations of the critical set of G_2 is

$$0 = \frac{\partial G_1}{\partial \mathbf{x}'} = \boldsymbol{\lambda} - \mathbf{a}.$$

Thus, the generating family G_2 is reducible to (we use the same symbol)

$$\boxed{G_2(\mathbf{x}; \mathbf{a}) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{a}, \quad \mathbf{a} \in \mathbb{S}_{n-1}} \tag{23}$$

Note that *the generating family* (22) *describe only outgoing rays, while the family* (23) *describes both incoming and outgoing rays.*

Example 3. Generating family of a reflected system of rays. Let V be a regular surface described by $m = n - r$ independent equations $V_a(\mathbf{x}) = 0$. Then the diagonal relation $\Delta_V \subset \mathbb{R}^n \times \mathbb{R}^n$ is represented by equations

$$(\mathbf{x}, \mathbf{x}') \in \Delta_V \quad \Longleftrightarrow \quad V_a(\mathbf{x}) = 0, \quad \mathbf{x} - \mathbf{x}' = 0.$$

It follows from Theorem 2, §2.8 that the canonical lift $\widehat{\Delta}_V$ is generated by the Morse family

$$G_M(\mathbf{x}, \mathbf{x}'; \boldsymbol{\mu}, \boldsymbol{\lambda}) = \boldsymbol{\mu} \cdot (\mathbf{x} - \mathbf{x}') + \lambda^a V_a(\mathbf{x}'), \quad \boldsymbol{\mu} \in \mathbb{R}^n, \ \boldsymbol{\lambda} = (\lambda^a) \in \mathbb{R}^m. \tag{24}$$

This is the mirror relation associated with V (cf. Definition 1, §3.5). Let $G_I(\mathbf{x}; \mathbf{u})$ be the generating family of a system of rays Λ_I. Then, in accordance with Theorem 2 of §3.5, the system of rays reflected by V is described by the generating family

$$G_O(\mathbf{x}; \mathbf{x}', \mathbf{x}'', \mathbf{a}, \mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}') + \boldsymbol{\mu} \cdot (\mathbf{x}' - \mathbf{x}'') + \lambda^a V_a(\mathbf{x}'') + G_I(\mathbf{x}''; \mathbf{u}).$$

However, one of the equations of the critical set of this family is

$$0 = \frac{\partial G_O}{\partial \boldsymbol{\mu}} = \mathbf{x}' - \mathbf{x}''.$$

Then this generating family is reducible to

$$\boxed{G_O(\mathbf{x}; \mathbf{x}', \mathbf{a}, \mathbf{u}, \boldsymbol{\lambda}) = \mathbf{a} \cdot (\mathbf{x} - \mathbf{x}') + \lambda^a V_a(\mathbf{x}') + G_I(\mathbf{x}'; \mathbf{u})} \tag{25}$$

with supplementary variables $\mathbf{x}' \in \mathbb{R}^n$, $\mathbf{a} \in \mathbb{S}_{n-1}$, $\boldsymbol{\lambda} = (\lambda^a) \in \mathbb{R}^m$ and $\mathbf{u}$.

Remark 8. If we exclude the points $\mathbf{x}, \mathbf{x}' \in V$, then the mirror relation is generated by

$$\boxed{G_M^\circ(\mathbf{x}, \mathbf{x}'; \mathbf{v}) = |\mathbf{x} - \mathbf{v}| + |\mathbf{v} - \mathbf{x}'|, \quad \mathbf{v} \in V} \tag{26}$$

Indeed, the equations associated with this generating family are

$$
(27)\qquad \begin{cases} \mathbf{p} = \dfrac{\partial G}{\partial \mathbf{x}} = \dfrac{\mathbf{x}-\mathbf{v}}{|\mathbf{x}-\mathbf{v}|}, \qquad \mathbf{p}' = -\dfrac{\partial G}{\partial \mathbf{x}'} = -\dfrac{\mathbf{x}'-\mathbf{v}}{|\mathbf{x}'-\mathbf{v}|}, \\ 0 = \dfrac{\partial G}{\partial \mathbf{v}} = P_{\mathbf{n}}\left(\dfrac{\mathbf{v}-\mathbf{x}}{|\mathbf{v}-\mathbf{x}|} + \dfrac{\mathbf{v}-\mathbf{x}'}{|\mathbf{v}-\mathbf{x}'|}\right). \end{cases}
$$

Due to the first two equations, the last equation (describing the critical set) is equivalent to $P_{\mathbf{n}}(\mathbf{p}'-\mathbf{p}) = 0$. This means that $\mathbf{p}-\mathbf{p}'$ is orthogonal to the surface V (Remark 4). This is in accordance with the reflection law, formula (4) of §3.5. Note that *the generating function G_M° (26) is the total "optical" length from $\mathbf{x}$ to $\mathbf{x}'$ through the point $\mathbf{v} \in V$ and that equations (27) express the principle that in the reflection by the mirror V this length is stationary* [Hamilton, 1828].

Example 4. Generating family of a system of rays generated by a source U and reflected by a mirror V. It is described by the generating family (25), where G_I is of the form (20). There is however a simpler description: *the system of rays generated by U and reflected by V is described by the generating family*

$$
(28)\qquad \boxed{G(\mathbf{x};\mathbf{u},\mathbf{v}) = |\mathbf{x}-\mathbf{v}| + |\mathbf{v}-\mathbf{u}|, \qquad \mathbf{u} \in U,\ \mathbf{v} \in V}
$$

This can be shown by composing the generating function (19) $|\mathbf{x}'-\mathbf{u}|$ with the generating function (26) and observing that on the critical set of the generating family so obtained,

$$
|\mathbf{x}-\mathbf{v}| + |\mathbf{x}'-\mathbf{v}| + |\mathbf{x}'-\mathbf{u}|,
$$

we have (take the partial derivative w.r. to $\mathbf{x}'$)

$$
\frac{\mathbf{x}'-\mathbf{v}}{|\mathbf{x}'-\mathbf{v}|} + \frac{\mathbf{x}'-\mathbf{u}}{|\mathbf{x}'-\mathbf{u}|} = 0.
$$

This means $\mathbf{x}' = k(\mathbf{v}-\mathbf{u})$, so that $|\mathbf{x}'-\mathbf{v}| + |\mathbf{x}'-\mathbf{u}| = |\mathbf{v}-\mathbf{u}|$.

Remark 9. The conclusion of Example 4 can be extended as follows: *the system of rays generated by a source U and reflected by a sequence of mirrors $V_1, \ldots, V_m$ is described by the generating family*

$$
(29)\qquad \begin{gathered} G(\mathbf{x};\mathbf{u},\mathbf{v}_1,\ldots,\mathbf{v}_m) = |\mathbf{x}-\mathbf{v}_m| + |\mathbf{v}_m-\mathbf{v}_{m-1}| + \ldots + |\mathbf{v}_1-\mathbf{u}|, \\ \mathbf{v}_i \in V_i, \quad \mathbf{u} \in U. \end{gathered}
$$

Example 5. Ideal lens. Let V be a regular surface in $\mathbb{R}^n$ representing an ideal lens. The lens relation $(\widehat{\Delta_V, F})$ (§3.5) is generated by the Morse family (cf. Example 3)

$$
(30)\qquad \boxed{\begin{gathered} G_L(\mathbf{x},\mathbf{x}';\boldsymbol{\mu},\boldsymbol{\lambda}) = \boldsymbol{\mu}\cdot(\mathbf{x}-\mathbf{x}') + \lambda^a\, V_a(\mathbf{x}') + F(\mathbf{x}') \\ \boldsymbol{\mu} \in \mathbb{R}^n,\ (\lambda^a) \in \mathbb{R}^{n-r} \end{gathered}}
$$

The lens relation is also generated by

$$G'_L(\mathbf{x},\mathbf{x}';\mathbf{v}) = |\mathbf{x}-\mathbf{v}| + |\mathbf{x}'-\mathbf{v}| + F(\mathbf{v}), \quad \mathbf{v}\in V \tag{31}$$

In this second representation the points $\mathbf{x},\mathbf{x}' \in V$ are excluded. Indeed, the equations corresponding to this generating family are similar to equations (27); the only difference occurs in the last equation of the critical set, which now reads

$$0 = \frac{\partial G}{\partial \mathbf{v}} = P_{\mathbf{n}}\big(\mathbf{p}' - \mathbf{p} + \nabla F(\mathbf{v})\big), \quad \mathbf{n}\perp V. \tag{32}$$

This means that the vector $\mathbf{p}'-\mathbf{p}+\nabla F(\mathbf{v})$ is orthogonal to V. This is in accordance with formula (6) of §3.5.

Example 6. Concave and convex ideal lenses. In $\mathbb{R}^2$ let us consider the y-axis as an ideal lens V. Let us look for a function $F(y)$ on V such that the corresponding lens relation maps a system of incident rays Λ_I parallel to (and oriented as) the x-axis to a system of rays Λ_O focused at a given point $(f,0)$ on the x-axis. The input system of rays is described by the Morse family $G_I(\mathbf{x}) = x$. For the lens relation let us consider the generating family (30), written in the form

$$G_L(x,y,x',y';\mu_x,\mu_y,\lambda) = \mu_x(x-x') + \mu_y(y-y') + \lambda x' + F(y'). \tag{33}$$

Let us consider the Hamilton principal function (13) written in the form

$$S(x,y,x',y';\theta) = (x-x')\cos\theta + (y-y')\sin\theta, \tag{34}$$

where $\mathbf{a} = [\cos\theta, \sin\theta]$. Then, by composing these three generating families, we get the generating family of the output system of rays,

$$\begin{aligned} G_O(x,y;x',y',x'',y'',\theta,\mu_x,\mu_y,\lambda) &= (x-x')\cos\theta + (y-y')\sin\theta + \\ &+ \mu_x(x'-x'') + \mu_y(y'-y'') + \lambda\, x'' + F(y'') + x''. \end{aligned} \tag{35}$$

This family is reducible. Indeed, among the equations of the critical set we find

$$0 = \frac{\partial G_O}{\partial \mu_x} = x' - x'', \qquad 0 = \frac{\partial G_O}{\partial \mu_y} = y' - y'',$$

i.e., $x' = x''$, $y' = y''$. Then the reduced family is (we use the same symbol of (35) for simplicity)

$$G_O(x,y;x',y',\theta,\nu,v) = (x-x')\cos\theta + (y-v)\sin\theta + \nu\, x' + F(v), \tag{36}$$

where $\nu = \lambda + 1$ and $v = y' = y''$. A further reduction can be performed. Indeed, one of the equations of the critical set of this last family is

$$0 = \frac{\partial G_O}{\partial \nu} = x'.$$

Thus, the final reduced generating family of the output system of rays is

$$\boxed{G_O(x, y; \theta, v) = x\,\cos\theta + (y - v)\,\sin\theta + F(v)} \tag{37}$$

The corresponding Lagrangian set is described by equations (we write $G_O = G$ for simplicity)

$$\begin{cases} 0 = \dfrac{\partial G}{\partial \theta} = -\,x\,\sin\theta + (y - v)\,\cos\theta, \\ 0 = \dfrac{\partial G}{\partial v} = -\sin\theta + F'(v), \end{cases} \tag{38}$$

$$p_x = \frac{\partial G}{\partial x} = \cos\theta, \qquad p_y = \frac{\partial G}{\partial y} = \sin\theta. \tag{39}$$

Equations (38) describe the critical set. The output system Λ_O is then completely described by equations

$$\begin{cases} p_x = \cos\theta, \\ p_y = \sin\theta, \\ p_y = F'(v), \\ x\,p_y = (y - v)\,p_x. \end{cases} \tag{40}$$

It is two-folded. Since for the input system $p_x = 1 > 0$, the physically interesting part of Λ_O is that for which

$$p_x = \cos\theta \geq 0.$$

The generating family (37) is a Morse family (thus, Λ_O is a Lagrangian submanifold) for $p_x = \cos\theta \neq 0$ or for $\cos\theta = 0$ and $F''(v) \neq 0$. This follows from the analysis of the matrix

$$\begin{bmatrix} \dfrac{\partial^2 G}{\partial\theta\partial\theta} & \dfrac{\partial^2 G}{\partial\theta\partial v} & \dfrac{\partial^2 G}{\partial\theta\partial x} & \dfrac{\partial^2 G}{\partial\theta\partial y} \\ \dfrac{\partial^2 G}{\partial v\partial\theta} & \dfrac{\partial^2 G}{\partial v\partial v} & \dfrac{\partial^2 G}{\partial v\partial x} & \dfrac{\partial^2 G}{\partial v\partial y} \end{bmatrix} = \tag{41}$$

$$= \begin{bmatrix} -\,x\,\cos\theta - (y - v)\,\sin\theta & -\cos\theta & -\sin\theta & \cos\theta \\ & & & \\ -\cos\theta & F''(v) & 0 & 0 \end{bmatrix}.$$

For $\cos\theta \neq 0$ it has maximal rank $(= 2)$. For $\cos\theta = 0$ it reduces to

$$\begin{bmatrix} \mp(y-v) & 0 & \mp 1 & 0 \\ 0 & F''(v) & 0 & 0 \end{bmatrix},$$

and the rank is maximal only for $F''(v) \neq 0$. Let us compute the caustic of the output system of rays. The first equation describing the caustic is obtained by equating to zero the determinant of the first square sub-matrix (41),

$$\det \begin{bmatrix} \dfrac{\partial^2 G}{\partial\theta\partial\theta} & \dfrac{\partial^2 G}{\partial\theta\partial v} \\ \dfrac{\partial^2 G}{\partial v\partial\theta} & \dfrac{\partial^2 G}{\partial v\partial v} \end{bmatrix} = F''(v)\left[x\,\cos\theta + (y-v)\,\sin\theta\right] + \cos^2\theta = 0.$$

The remaining equations are just the equations (38) of the critical set. For $F''(v) \neq 0$ we get the linear system in (x, y)

$$\begin{cases} x\,\cos\theta + (y-v)\,\sin\theta = -\dfrac{\cos^2\theta}{F''(v)}, \\ x\,sin\theta - (y-v)\,\cos\theta = 0, \end{cases}$$

whose solution is

$$x = -\frac{\cos^3\theta}{F''(v)}, \qquad y - v = -\frac{\sin\theta\,\cos^2\theta}{F''(v)}. \tag{42}$$

Since $F'(v) = \sin\theta$, we get the parametric equations (in the parameter v) of the caustic,

$$\begin{cases} x = -\dfrac{\left(1 - F'^2(v)\right)^{\frac{3}{2}}}{F''(v)}, \\ y = v - \dfrac{F'(v)\left(1 - F'^2(v)\right)}{F''(v)}. \end{cases} \tag{43}$$

The caustic is given by the single point $(f, 0)$ if and only if

$$\begin{cases} \cos^3\theta = -f\,F''(v), \\ v = \dfrac{\cos^2\theta\,\sin\theta}{F''(v)}. \end{cases}$$

By the first equation, $F''(v) = -\frac{1}{f}\cos^3\theta$, the second equation gives

$$v = -f\,\frac{\sin\theta}{\cos\theta} = -\frac{f\,F'}{\sqrt{1 - F'^2}} \tag{44}$$

if we consider $p_x = \cos\theta > 0$. It follows that

$$\frac{v^2}{f^2} = \frac{F'^2}{1 - f'^2}, \qquad F'^2 = \frac{v^2}{f^2 + v^2}.$$

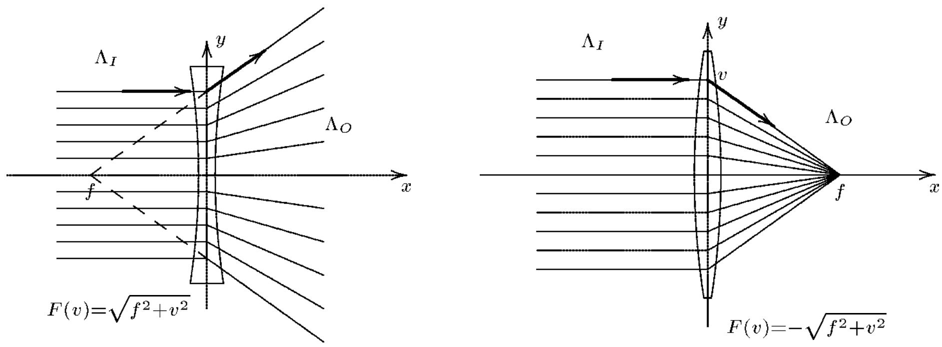
Thus, from

$$F'(v) = \pm\frac{v}{\sqrt{f^2 + v^2}}, \tag{45}$$

we get, up to an inessential additive constant,

$$\boxed{F(v) = \pm\sqrt{f^2 + v^2}} \tag{46}$$

These are the functions on the $(y = v)$-axis which represent an ideal lens. Due to (44) and (45), the function $F(v) = \sqrt{f^2 + v^2}$ corresponds to the case $f < 0$ thus, to a concave lens. The function $F(v) = -\sqrt{f^2 + v^2}$ corresponds to the case $f > 0$ thus, to a convex lens. Note that in these two cases we have $p_y = F'(v) > 0$ and $p_y < 0$, respectively.



Example 7. All the preceding discussion can be extended to the more realistic case of a lens V in $\mathbb{R}^3$ of equation $x = 0$ (coinciding with the (y, z)-plane). Then the generating family (37) of the output system is replaced by

$$G_O(\mathbf{x}; \mathbf{v}, \mathbf{a}) = (\mathbf{x} - \mathbf{v}) \cdot \mathbf{a} + F(\mathbf{v}), \tag{47}$$

where $\mathbf{v}$ is a vector on the (y, z)-plane representing a point of the lens. For an axially symmetric lens $F(\mathbf{v}) = F(v)$, where $v = |\mathbf{v}|$ is the distance from the x-axis.

Example 8. Let us extend Example 6 by considering, instead of a parallel system of rays, any arbitrary input system of rays generated by a function $G_I(\mathbf{x}) = G_I(x, y)$. By replacing the last term x'' in (35) with $G_I(x'', y'')$, it follows that the generating family of the output system of rays is given by (37) with the additional term

$$I(v) = G_I(0, v).$$

This means that in all formulae following (37) the term $F(v)$ is replaced by the term $F(v)+I(v)$. In particular the parametric equations of the caustic (43) become

$$(48)\qquad \begin{cases} x = -\dfrac{\left(1-(F'+I')^2(v)\right)^{\frac{3}{2}}}{F''(v)+I''(v)}, \\ y = v - \dfrac{(F'(v)+I'(v))\left(1-(F'+I')^2(v)\right)}{F''(v)+I''(v)}. \end{cases}$$

Assume for instance that the input system of rays is emitted by a point $\mathbf{x}_0 = (x_0, y_0)$. Then G_I is given by (22),

$$G_I = |\mathbf{x} - \mathbf{x}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

and

$$(49)\qquad \begin{aligned} I(v) &= \sqrt{x_0^2 + (v-y_0)^2}, \\ I'(v) &= \frac{v-y_0}{I(v)}, \\ I''(v) &= \frac{I^2(v) - (v-y_0)^2}{I^3(v)}. \end{aligned}$$

For a convex lens,

$$(50)\qquad F'(v) = -\frac{v}{\sqrt{f^2+v^2}}, \qquad F''(v) = -\frac{f^2}{(f^2+v^2)^{\frac{3}{2}}}.$$

The analysis of equations (48) for this case shows that for $x_0 = \pm f$ and for $y_0 \to 0$ we have $x \to \infty$.

4.4 The eikonal equation in a space of constant negative curvature

In the space $T^*\mathbb{R}^n = \{(\mathbf{x}, \mathbf{p})\} = \{(x^a, p_a)\}$ we consider the modified eikonal equation C

$$(1)\qquad |\mathbf{p}|^2 + (\mathbf{p}\cdot\mathbf{x})^2 = 1, \qquad \sum_a p_a^2 + \Big(\sum_a x^a p_a\Big)^2 = 1.$$

This can be interpreted as the eikonal equation associated with the modified contravariant metric tensor

$$\mathbf{H} = \mathbf{G} + \mathbf{x} \otimes \mathbf{x}, \quad H^{ab} = \delta^{ab} + x^a x^b, \tag{2}$$

where $\mathbf{G} = [\delta^{ab}]$ is the natural metric tensor.
Theorem 1. *The eikonal equation* (1) *admits a global complete solution*

$$W \colon Q \times A = \mathbb{R}^n \times \mathbb{S}_{n-1} \to \mathbb{R}$$

defined by

$$W(\mathbf{x}, \mathbf{a}) = \ln\left(\mathbf{a} \cdot \mathbf{x} + \sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}\right) = \operatorname{arcsinh}(\mathbf{a} \cdot \mathbf{x}), \qquad \mathbf{a} \in \mathbb{S}_{n-1}. \tag{3}$$

Proof. The vector

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \frac{\partial W}{\partial \mathbf{x}} = \frac{\mathbf{a}}{\sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}} \tag{4}$$

satisfies equation (1) whatever $\mathbf{a}$. From this equation we derive

$$\mathbf{p} \cdot \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{a}}{\sqrt{1 + (\mathbf{x} \cdot \mathbf{a})^2}},$$

thus,

$$(\mathbf{p} \cdot \mathbf{x})^2 = \frac{(\mathbf{x} \cdot \mathbf{a})^2}{1 + (\mathbf{x} \cdot \mathbf{a})^2} < 1$$

and

$$(\mathbf{x} \cdot \mathbf{a})^2 = \frac{(\mathbf{p} \cdot \mathbf{x})^2}{1 - (\mathbf{p} \cdot \mathbf{x})^2}, \qquad 1 + (\mathbf{x} \cdot \mathbf{a})^2 = \frac{1}{1 - (\mathbf{p} \cdot \mathbf{x})^2}.$$

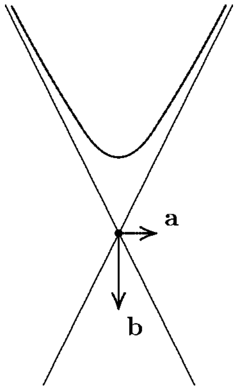
Then the mapping $\pi \colon C \to A \colon \mathbf{p} \mapsto \mathbf{a}$ is given by

$$\mathbf{a} = \frac{\mathbf{p}}{\sqrt{1 - (\mathbf{p} \cdot \mathbf{x})^2}}. \tag{5}$$

This is a submersion. ∎
Remark 1. The symplectic reduction $R \subset T^*\mathbb{S}_{n-1} \times T^*\mathbb{R}^n$ corresponding to this complete solution is described by equation (4) together with equation

$$\mathbf{b} = -\frac{\partial W}{\partial \mathbf{a}} = -\frac{1}{\sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}} P_{\mathbf{a}}(\mathbf{x}) = \frac{1}{\sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}} (\mathbf{x} \cdot \mathbf{a}\, \mathbf{a} - \mathbf{x}). \tag{6}$$

Note that $\mathbf{b} \cdot \mathbf{a} = 0$. From this formula it follows that the ray determined by the pair of orthogonal vectors $(\mathbf{a}, \mathbf{b})$ is the hyperbola on the plane $(\mathbf{a}, \mathbf{b})$ with center at the origin $\mathbf{x} = 0$, asymptotes determined by the vectors $\mathbf{b} \pm \mathbf{a}$ and vertex at the point $\mathbf{x} = -\mathbf{b}$ (cf. Remark 6 below).



The vector $\mathbf{v} = (v^a)$ defined by $v^a = H^{ab}\, p_a$ is tangent to the ray. It follows from (2) and (4) that

$$\mathbf{v} = \mathbf{p} + \mathbf{x} \cdot \mathbf{p}\, \mathbf{x} = \frac{\mathbf{a} + \mathbf{a} \cdot \mathbf{x}\, \mathbf{x}}{\sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}}. \tag{7}$$

From Theorem 1, see also Remark 6, §3.4, we derive

Theorem 2. *The function*

$$\begin{aligned} S(\mathbf{x}, \mathbf{x}'; \mathbf{a}) &= W(\mathbf{x}, \mathbf{a}) - W(\mathbf{x}', \mathbf{a}) = \ln \frac{\mathbf{a} \cdot \mathbf{x} + \sqrt{1 + (\mathbf{a} \cdot \mathbf{x})^2}}{\mathbf{a} \cdot \mathbf{x}' + \sqrt{1 + (\mathbf{a} \cdot \mathbf{x}')^2}} \\ &= \mathit{arcsinh}(\mathbf{x} \cdot \mathbf{a}) - \mathit{arcsinh}(\mathbf{x}' \cdot \mathbf{a}), \qquad \mathbf{a} \in \mathbb{S}_{n-1} \end{aligned} \tag{8}$$

is a global Hamilton principal function of the eikonal equation (1). *It is a Morse family.*

Remark 2. The elements of the inverse matrix $[h_{ab}]$ of $[H^{ab}]$ are

$$h_{ab} = \frac{1}{1 + r^2} \left(\delta_{ab} - x^a\, x^b\right), \qquad r^2 = \mathbf{x} \cdot \mathbf{x} = \textstyle\sum_a (x^a)^2. \tag{9}$$

These are the covariant components of the modified metric tensor (2). In this new metric the scalar product of two vectors $\mathbf{u} = (u^a)$ and $\mathbf{v} = (v^a)$ is given by

$$\mathbf{h}(\mathbf{u}, \mathbf{v}) = h_{ab}\, u^a v^b = \frac{1}{1 + r^2}\, (\mathbf{u} \cdot \mathbf{v} - \mathbf{x} \cdot \mathbf{u}\, \mathbf{x} \cdot \mathbf{v}). \tag{10}$$

Remark 3. The modified metric is invariant under (Euclidean) rotations around the origin $\mathbf{x} = 0$. Thus, the origin is a distinguished point.

Theorem 3. *The metric* $\mathbf{H}$ *has constant negative curvature.*

Proof. Let us consider $\mathbb{R}^n \times \mathbb{R}$ referred to the canonical basis

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots \quad \mathbf{c}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_{n+1} = \mathbf{t} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and endowed with the Minkowskian metric

$$\mathbf{m}(\mathbf{u}, \mathbf{v}) = \textstyle\sum_{a=1}^n u^a v^a - u^{n+1} v^{n+1}.$$

Let us consider the set $\mathbb{H}_n$ of the unit time-like vectors $\mathbf{q}$, $\mathbf{m}(\mathbf{q}, \mathbf{q}) = -\mathbf{1}$, pointing to the "future" i.e., such that $\mathbf{m}(\mathbf{q}, \mathbf{t}) < 0$. It is known that $\mathbb{H}_n$ is a proper Riemannian manifold with constant negative curvature (cf. for instance [Wolf, 1984]). By taking $(x^a) = (x^1, \ldots, x^n)$ as parameters, this is the hyperboloid described by the parametric equation

$$\mathbf{q} = x^a\, \mathbf{c}_a + \sqrt{1 + \textstyle\sum_a (x^a)^2}\; \mathbf{t}.$$

The corresponding tangent frame $(\mathbf{e}_a)$ is then defined by

$$\mathbf{e}_a = \partial_a \mathbf{q} = \mathbf{c}_a + \tfrac{x^a}{z}\, \mathbf{t},$$

being

$$z = \sqrt{1 + \textstyle\sum_a (x^a)^2} = \sqrt{1 + r^2}.$$

It follows that the induced metric tensor (the first fundamental form) of $\mathbb{H}_n$ is

$$g_{ab} = \mathbf{m}(\mathbf{e}_a, \mathbf{e}_b) = \mathbf{m}(\mathbf{c}_a, \mathbf{c}_b) - \frac{x^a\, x^b}{z^2} = \delta_{ab} - \frac{1}{1 + r^2}\, x^a x^b.$$

This is the metric (9). ∎

Note that $\mathbf{q}$ is a unit vector orthogonal to $\mathbb{H}_n$ and that $\partial_b \mathbf{e}_a \cdot \mathbf{q} = \mathbf{B}_{ba} \cdot \mathbf{q} = -g_{ab}$.

For the sake of simplicity in the following we consider the case $n = 2$. All results can be easily extended to any dimension n.

Remark 4. In $\mathbb{R}^2 = (x, y)$ we have

$$(11) \quad [H^{ab}] = \begin{bmatrix} 1 + x^2 & xy \\ xy & 1 + y^2 \end{bmatrix}, \qquad [h_{ab}] = \frac{1}{1 + x^2 + y^2} \begin{bmatrix} 1 + y^2 & -xy \\ -xy & 1 + x^2 \end{bmatrix}$$

and

$$\mathbf{h}(\mathbf{u},\mathbf{v}) = \frac{1}{1+r^2}\Big(\mathbf{u}\cdot\mathbf{v} + (\mathbf{x}\times\mathbf{u})\cdot(\mathbf{x}\times\mathbf{v})\Big) \tag{12}$$

where $\times$ is the standard cross product of vectors in $\mathbb{R}^3 = (x, y, z)$.

Remark 5. Let us consider for instance $\mathbf{a} = \mathbf{c}_1$ (the first vector of the canonical basis of $\mathbb{R}^2$). In this case equation (6) becomes

$$\mathbf{b} = \frac{x\,\mathbf{c}_1 - \mathbf{x}}{\sqrt{1+x^2}} = -\frac{y\,\mathbf{c}_2}{\sqrt{1+x^2}}.$$

By setting $\mathbf{b} = b\,\mathbf{c}_2$ we get equation

$$b\sqrt{1+x^2} = -y. \tag{13}$$

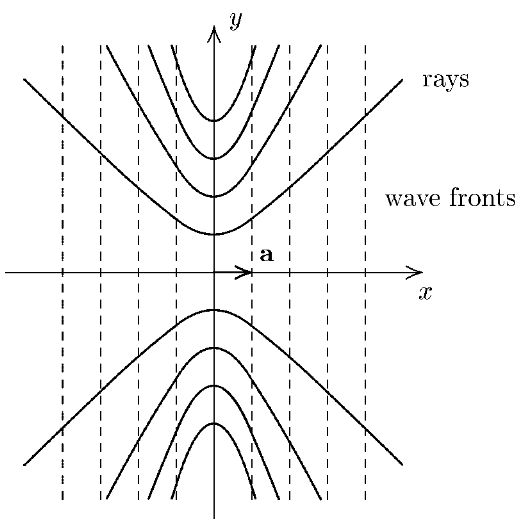
For $b \neq 0$ it follows that

$$\frac{y^2}{b^2} - x^2 = 1.$$

This is the equation of the system of rays associated with the Lagrangian submanifold $\Lambda_{\mathbf{c}_1}$ generated by the function

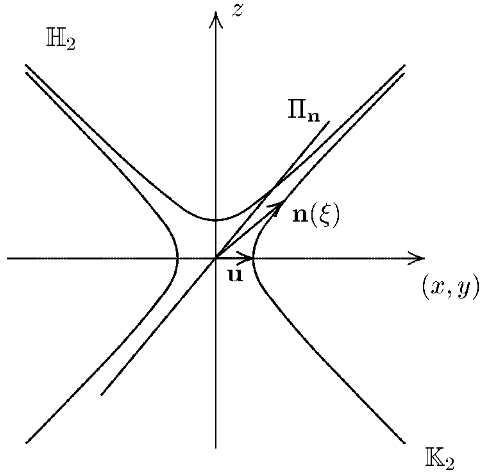
$$G(\mathbf{x}) = \ln\big(x + \sqrt{1+x^2}\big).$$

Equation (13) describes a family of hyperbolas centered at the origin of $\mathbb{R}^2$ and vertices the points $(0, \pm b)$. For $b = 0$ equation (13) reduce to $y = 0$: the x-axis is a ray. The corresponding wave fronts, described by equations $G = \text{const.}$ i.e., $y = \text{const.}$, are the straight lines parallel to the y-axis.

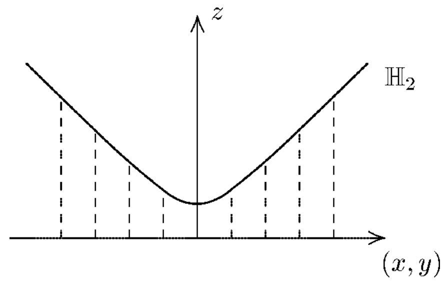


Remark 6. Let $\mathbf{n}$ be a unit space-like vector in the Minkowski three-space. $\mathbf{m}(\mathbf{n}, \mathbf{n}) = 1$. These vectors form the one-folded rotational hyperboloid which we denote by $\mathbb{K}_2$ (it is diffeomorphic to a cylinder). Let $\Pi_{\mathbf{n}}$ be the 2-plane passing through the origin and orthogonal to $\mathbf{n}$, described by equation $\mathbf{m}(\mathbf{q}, \mathbf{n}) = 0$. It can be shown that:

(i) The geodesics of $\mathbb{H}_2$ are the intersections of $\mathbb{H}_2$ with the planes $\Pi_{\mathbf{n}}$ of this kind.

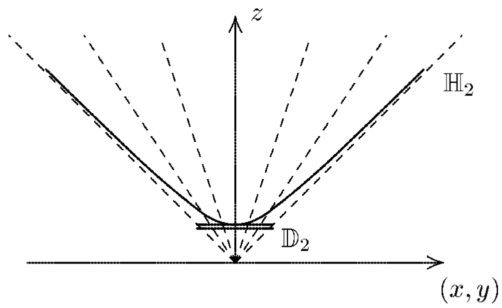


(ii) The geodesics project onto the hyperbolas of the 2-plane (x, y) described in the preceding remarks; hence, the metric properties of the plane (x, y) endowed with the metric H^{ab} can be deduced by those of $\mathbb{H}_2$ by means of the Cartesian projection $(x, y, z) \mapsto (x, y)$ (cf. [Petersen, 1998]).



The geodesics in the plane (x, y) are the projections of the intersections of $\mathbb{H}_2$ with the planes in $\mathbb{R}^3$ passing through the origin.

We recall that $\mathbb{H}_2$ can be also reduced to the Lobachevskij disk $\mathbb{D}_2$ by means of the stereographic projection from the origin to the plane $z = \mathbf{1}$.



(iii) The systems of rays of the kind described in Remark 6 are obtained by considering the unit vectors

$$\mathbf{n}(\xi, \mathbf{u}) = \cosh\xi\, \mathbf{u} + \sinh\xi\, \mathbf{c}_3,$$

where $\xi \in \mathbb{R}$ is a parameter and $\mathbf{u}$ is a unit vector orthogonal to $\mathbf{a}$ in the plane (x, y). The space-like unit vectors $\mathbf{p}$ associated with this family of geodesics (parametrized by ξ) form a section $\Lambda_{\mathbf{u}}$ of $T^*\mathbb{H}_2$. If we take $\mathbf{u} = -\mathbf{c}_2$, then with respect to the frame (13) the components of these covectors are

$$p_y = 0, \qquad p_x = \frac{1}{z}\ \cosh\xi = \frac{1}{\sqrt{1+x^2}}.$$

Since

$$\int \frac{1}{\sqrt{1+x^2}} = \ln\left(x + \sqrt{1+x^2}\right) = \operatorname{arcsinh}(x),$$

the set $\Lambda_{\mathbf{u}}$ is a Lagrangian submanifold generated by the function

$$G(x, y) = \ln\left(x + \sqrt{1+x^2}\right) = \operatorname{arcsinh}(x).$$

Since $x = \mathbf{x} \cdot \mathbf{c}_1$, by replacing $\mathbf{c}_1$ by any unit vector $\mathbf{u}$ we get the complete solution (3). This is an example of a complete solution of a Hamilton-Jacobi equation which is obtained by means of a geometrical process and not by separation of variables. Indeed, the Hamilton-Jacobi equation (1) is integrable by separation of variables in the polar coordinates (ρ, θ), with θ ignorable, since the metric of $\mathbb{H}_2$ is invariant under rotations around the z-axis. Other systems of separable coordinates are known, which are associated with pairs of rotations around time-like vectors. However, these complete separated solutions are not defined on the whole plane (for the general theory of separation of variables in spaces with constant curvature see [Kalnins, 1986]; for the separability in $\mathbb{H}_2$ see [Kalnins, Miller, Pogosyan, 1997] and [Kalnins, Miller, Hakobyan, Pogosyan, 1999]).

In Appendix B it is shown that the eikonal equation on the hyperboloid $\mathbb{H}_2$ admits another global principal Hamilton function S, which does not come from a complete integral W and which is not a Morse family.

Chapter 5

Control of static systems

5.1 Control relations

If a differentiable manifold $\mathcal{Q}$ represents the configuration space of a holonomic mechanical system, then any tangent vector $v \in T\mathcal{Q}$ represents a **virtual velocity** or a **virtual displacement**, while any covector $f \in T^*\mathcal{Q}$ represents a **force**. The evaluation

$$\langle v, f \rangle$$

represents the **virtual power** or the **virtual work** produced by the force f in the virtual velocity (or displacement) v. Let us denote by $(q^i, \delta q^i)$ the coordinates on $T\mathcal{Q}$ associated with coordinates (q^i) on $\mathcal{Q}$. This means that the symbols δq^i represent the components of the tangent vectors, so that the coordinate expression of the virtual work is

$$\langle v, f \rangle = f_i \, \delta q^i, \tag{1}$$

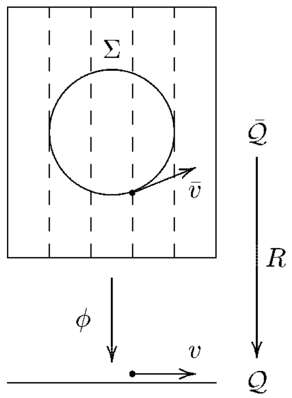
being (f_i) the components of the force f and δq^i the components of the vector v.

In the following discussion $\mathcal{Q}$ will represent a static mechanical system, on which we act by means of "external devices", imposing very slow changes of configuration, in order to avoid inertial or "irreversible" effects. The manifold $\mathcal{Q}$ is related to a "bigger" static system represented by a configuration manifold $\bar{\mathcal{Q}}$, in accordance with the following definition.

Definition 1. A **control relation** is a relation $R \subset \mathcal{Q} \times \bar{\mathcal{Q}}$ of the form

$$\boxed{R = \text{graph}(\phi) \cap (\mathcal{Q} \times \Sigma)} \tag{2}$$

where $\Sigma \subset \bar{\mathcal{Q}}$ is a submanifold, called **constraint**, and $\phi\colon \bar{\mathcal{Q}} \to \mathcal{Q}$ is a fibration or a surjective submersion.



Hence, a control relation is the graph of the restriction to a constraint Σ of a surjective submersion ϕ. We call Q the **control manifold** and $\bar{Q}$ the **extended** or **complete configuration manifold**. Note that R may be a non-smooth relation.
Definition 2. Two virtual displacements $v \in TQ$ and $\bar{v} \in T\bar{Q}$ are called **compatible** with respect to the control relation R if $\bar{v} \in T\Sigma$ (i.e., $\bar{v}$ is *compatible* with the constraint Σ) and $T\phi(\bar{v}) = v$.
Remark 1. The fibration ϕ represents the existence of "hidden" or "internal" variables of the system $\bar{Q}$, which are not controlled and which may assume any value belonging to the constraint Σ. Control relations arise in control problems of static mechanical systems, in catastrophe theory, in thermodynamics.

There is a useful equivalent definition of control relation:
Theorem 1. *Definition* (2) *is equivalent to*

$$\boxed{R = \Phi \circ \Delta_\Sigma} \tag{3}$$

where $\Phi = \mathrm{graph}(\phi)$*, and* $\Delta_\Sigma \subset \bar{Q} \times \bar{Q}$ *is the diagonal of* $\Sigma \times \Sigma$ *i.e., the identity relation on* Σ*.*

Proof. Definition (2) is equivalent to

$$R = \{(q, \bar{q}) \in Q \times \bar{Q} \mid q = \phi(\bar{q}),\ \ \bar{q} \in \Sigma\}.$$

Definition (3) is equivalent to

$$\begin{aligned} R = \Phi \circ \Delta_\Sigma &= \{(q, \bar{q}) \in Q \times \bar{Q} \mid \exists\, \bar{q}' \in \bar{Q} \text{ such that } q = \phi(\bar{q}'),\ \ (\bar{q}, \bar{q}') \in \Delta_\Sigma\} \\ &= \{(q, \bar{q}) \in Q \times \bar{Q} \mid \exists\, \bar{q}' \in \bar{Q} \text{ s.t. } q = \phi(\bar{q}'),\ \ \bar{q} = \bar{q}' \in \Sigma\} \\ &= \{(q, \bar{q}) \in Q \times \bar{Q} \mid q = \phi(\bar{q}),\ \ \bar{q} \in \Sigma\}. \qquad \blacksquare \end{aligned}$$

Definition 3. The **canonical lift of a control relation** $R = \Phi \circ \Delta_\Sigma$ is the composition

$$\boxed{\widehat{R} = \widehat{\Phi} \circ \widehat{\Delta}_\Sigma \subseteq T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}}} \tag{4}$$

of the canonical lifts of Δ_Σ and of Φ.

Theorem 2. *Definition* (4) *is equivalent to*

$$\begin{aligned}\widehat{R} = \{(f, \bar{f}) \in T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}} \mid \pi_\mathcal{Q} \times \pi_{\bar{\mathcal{Q}}}(f, \bar{f}) = (q, \bar{q}) \in R,\\ \langle T\phi(\bar{v}), f\rangle = \langle \bar{v}, \bar{f}\rangle, \ \forall\, \bar{v} \in T_{\bar{q}}\Sigma\}.\end{aligned} \tag{5}$$

Proof. By definition of canonical lift of a smooth relation we have:

$$\begin{aligned}\widehat{\Phi} =&\{(f, \bar{f}) \in T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}} \mid \pi_\mathcal{Q} \times \pi_{\bar{\mathcal{Q}}}(f, \bar{f}) = (q, \bar{q}) \in \Phi,\\ &\langle v, f\rangle = \langle \bar{v}, \bar{f}\rangle, \ \forall\, (v, \bar{v}) \in T_{(q,\bar{q})}\Phi\}\\ =&\{(f, \bar{f}) \in T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}} \mid \pi_\mathcal{Q} \times \pi_{\bar{\mathcal{Q}}}(f, \bar{f}) = (q, \bar{q}) \in \Phi,\\ &\langle T\phi(\bar{v}), f\rangle = \langle \bar{v}, \bar{f}\rangle, \ \forall\, \bar{v} \in T_{\bar{q}}\Sigma\}\end{aligned}$$

and

$$\begin{aligned}\widehat{\Delta}_\Sigma =&\{(\bar{f}, \bar{f}') \in T^*\bar{\mathcal{Q}} \times T^*\bar{\mathcal{Q}} \mid \pi_{\bar{\mathcal{Q}}}(\bar{f}) = \pi_{\bar{\mathcal{Q}}}(\bar{f}') = \bar{q} \in \Sigma,\\ &\langle \bar{v}, \bar{f} - \bar{f}'\rangle = 0, \ \forall\, \bar{v} \in T_{\bar{q}}\Sigma\}.\end{aligned}$$

By applying the composition rule of relations we get

$$\begin{aligned}\widehat{\Phi} \circ \widehat{\Delta}_\Sigma &= \{(f, \bar{f}) \in T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}} \mid \exists\, \bar{f}' \in T^*\bar{\mathcal{Q}} \ \text{ s.t.}\\ &\quad (f, \bar{f}') \in \widehat{\Phi}, \ \ (\bar{f}', \bar{f}) \in \widehat{\Delta}_\Sigma\}\\ &= \{(f, \bar{f}) \in T^*\mathcal{Q} \times T^*\bar{\mathcal{Q}} \mid \exists\, \bar{f}' \in T^*\bar{\mathcal{Q}}, \ \text{ s.t.}\\ &\quad f \in T^*_q\mathcal{Q}, \ \bar{f}' \in T^*_{\bar{q}}\bar{\mathcal{Q}}, \ q = \phi(\bar{q}), \ \ \bar{q} \in \Sigma,\\ &\quad \langle T\phi(\bar{v}), f\rangle = \langle \bar{v}, \bar{f}'\rangle, \ \ \forall\, \bar{v} \in T_{\bar{q}}\bar{\mathcal{Q}}, \ \ \bar{q} = \pi_{\bar{\mathcal{Q}}}(\bar{f}),\\ &\quad \langle \bar{v}, \bar{f} - \bar{f}'\rangle = 0, \ \forall\, \bar{v} \in T_{\bar{q}}\Sigma\}.\end{aligned} \tag{6}$$

From these last conditions it follows that $(q, \bar{q}) \in R$ and $\langle T\phi(\bar{v}), f\rangle = \langle \bar{v}, \bar{f}\rangle$ for all $\bar{v} \in T_{\bar{q}}\Sigma$. This shows that $\widehat{\Phi} \circ \widehat{\Delta}_\Sigma \subseteq \widehat{R}$ as defined in (5). Conversely, if $(f, \bar{f}) \in \widehat{R}$, then the last conditions (6) are satisfied for $\bar{f}' = \bar{f}$. Thus, $\widehat{\Phi} \circ \widehat{\Delta}_\Sigma \supseteq \widehat{R}$. ∎

Remark 2. Formula (5) shows that a pair of forces $(f, \bar{f})$ belongs to the relation $\widehat{R}$ if and only if these two covectors are based at a pair $(q, \bar{q})$ of points belonging to the relation R and such that

$$\langle T\phi(\bar{v}), f\rangle = \langle \bar{v}, \bar{f}\rangle, \tag{7}$$

for any virtual displacement $\bar{v}$ tangent to the constraint Σ. This means that $\langle v, f\rangle = \langle \bar{v}, \bar{f}\rangle$ for two compatible virtual displacements $(v, \bar{v})$.

We use the above-given definitions and theorems for stating the following two *axioms*:

$\boxed{1}$ *The system with configuration manifold $\bar{Q}$ remains in static equilibrium under forces $\bar{f} \in T^*\bar{Q}$ belonging to a Lagrangian submanifold $\bar{\mathcal{E}} \subset T^*\bar{Q}$ generated by a function $\bar{V}\colon \bar{Q} \to \mathbb{R}$, called the* **extended potential energy***:*

$$\boxed{\bar{\mathcal{E}} = d\bar{V}(\bar{Q})} \tag{8}$$

One could consider the general case of a generating family $\bar{V}\colon \bar{Q}\times\bar{U} \to \mathbb{R}$. However, in all examples illustrated below the potential energy $\bar{V}$ will be an ordinary generating function defined on $\bar{Q}$, without supplementary variables.

$\boxed{2}$ *The system with configuration manifold $\bar{Q}$ remains in static equilibrium under the action of an "external mechanism", represented by the control relation R, only with forces f belonging to a certain* **set of equilibrium states** *$\mathcal{E} \subseteq T^*Q$, also called the* **constitutive set** *of the system, defined by*

$$\boxed{\mathcal{E} = \widehat{R} \circ \bar{\mathcal{E}}} \tag{9}$$

Remark 3. Definition (9) means that

$$f \in \mathcal{E} \quad \Longleftrightarrow \quad \exists\, \bar{f} \ \text{ such that } \ (f, \bar{f}) \in \widehat{R}, \ \ \bar{f} \in \bar{\mathcal{E}}. \tag{10}$$

Then (10) and (7) show that $f \in \mathcal{E}$ if and only if for all compatible virtual displacements $(v, \bar{v})$ we have

$$\boxed{\langle v, f\rangle = \langle \bar{v}, d\bar{V}\rangle} \tag{11}$$

We analyze the local coordinate representations of the above concepts by using generating families.

Let ϕ be (locally) represented by equations $q^i = \phi^i(\bar{q}^\alpha)$ and Σ by independent equations $\Sigma_a(\bar{q}^\alpha) = 0$. Thus, in accordance with definition (2), the control relation R is described by equations

$$\begin{cases} q^i - \phi^i(\bar{q}^\alpha) = 0 \\ \Sigma_a(\bar{q}^\alpha) = 0. \end{cases} \tag{12}$$

Theorem 3. *If the control relation R is locally described by equations* (12)*, then its canonical lift $\widehat{R}$ is locally described by the generating family*

$$\boxed{G_R(q^i, \bar{q}^\alpha; \lambda_i, \mu^a) = \lambda_i \left(q^i - \phi^i(\bar{q}^\alpha)\right) + \mu^a\, \Sigma_a(\bar{q}^\alpha)} \tag{13}$$

with Lagrangian multipliers (λ_i, μ^a).

Proof. The generating families of the canonical lifts $\widehat{\Phi}$ and $\widehat{\Delta}_\Sigma$ are, respectively,

$$\begin{cases} G_\Phi(q^i, \bar{q}^\alpha; \lambda_i) = \lambda_i \left(q^i - \phi^i(\bar{q}^\alpha)\right) \\ G_\Sigma(\bar{q}_0^\alpha, \bar{q}^\alpha; \mu^a, \nu_\alpha) = \mu^a \, \Sigma_a(\bar{q}^\alpha) + \nu_\alpha(\bar{q}_0^\alpha - \bar{q}^\alpha). \end{cases}$$

Here, $(\lambda_i, \mu^a, \nu_\alpha)$ are supplementary variables. By the composition rule of the generating families we get the generating family G_R of $\widehat{\Phi} \circ \widehat{\Delta}_\Sigma$,

$$G_R(q^i, \bar{q}^\alpha; \bar{q}_0^\alpha, \lambda_i, \mu^a, \nu_\alpha) = \lambda_i \left(q^i - \phi^i(\bar{q}_0^\alpha)\right) + \mu^a \, \Sigma_a(\bar{q}_0^\alpha) + \nu_\alpha(\bar{q}_0^\alpha - \bar{q}^\alpha),$$

with supplementary variables $(\bar{q}_o^\alpha, \lambda_i, \mu^a, \nu_\alpha)$. Since $\widehat{R}$ is then described by equation

$$f_i \, dq^i - \bar{f}_\alpha \, d\bar{q}^\alpha = dG_R, \tag{14}$$

the vanishing of the coefficients of $d\nu_\alpha$ implies $\bar{q}_0^\alpha = \bar{q}^\alpha$. Thus, the generating family is reducible to (13). ∎

Remark 4. Equation (14) with G_R defined by formula (13) is equivalent to equations obtained by equating to zero the coefficients of $(dq^i, d\bar{q}^\alpha, d\lambda_i, d\mu^a)$,

$$\begin{cases} f_i = \lambda_i \\ \bar{f}_\alpha = \lambda_i \, \dfrac{\partial \phi^i}{\partial \bar{q}^\alpha} - \mu^a \, \dfrac{\partial \Sigma_a}{\partial \bar{q}^\alpha} \\ q^i = \phi^i(\bar{q}^\alpha) \\ \Sigma_a(\bar{q}^\alpha) = 0. \end{cases}$$

By eliminating the Lagrangian multipliers λ_i we get equations

$$\begin{cases} \bar{f}_\alpha = f_i \, \dfrac{\partial \phi^i}{\partial \bar{q}^\alpha} - \mu^a \, \dfrac{\partial \Sigma_a}{\partial \bar{q}^\alpha} \\ q^i = \phi^i(\bar{q}^\alpha) \\ \Sigma_a(\bar{q}^\alpha) = 0. \end{cases} \tag{15}$$

These are the equations describing $\widehat{R}$. The last two equations are the equations of R (fibration and constraint, respectively). The first equation is in accordance with (6). Indeed, if $\bar{v} = (\delta \bar{q}^\alpha)$ and $v = (\delta q^i)$, then

$$\begin{cases} \bar{v} \in T\Sigma \quad \Longleftrightarrow \quad \dfrac{\partial \Sigma_a}{\partial \bar{q}^\alpha} \, \delta \bar{q}^\alpha = 0 \\ v = T\phi(\bar{v}) \quad \Longleftrightarrow \quad \delta q^i = \dfrac{\partial \phi^i}{\partial \bar{q}^\alpha} \, \delta \bar{q}^\alpha. \end{cases} \tag{16}$$

By applying the composition rule of generating families, we prove

Theorem 4. *The constitutive set* $\mathcal{E} = \widehat{R} \circ \bar{\mathcal{E}}$ *is the Lagrangian set in* $T^*\mathcal{Q}$ *(possibly a Lagrangian submanifold) generated by the composite generating family*

$$V = G_R \oplus \bar{V}. \tag{17}$$

It is described by equation

$$\boxed{f_i\, dq^i = d(G_R + \bar{V})} \tag{18}$$

equivalent to the system of equations

$$\boxed{\begin{array}{lll} f_i = \dfrac{\partial G_R}{\partial q^i} & & \\ 0 = \dfrac{\partial G_R}{\partial \bar{q}^\alpha} + \dfrac{\partial \bar{V}}{\partial \bar{q}^\alpha} & & \\ 0 = \dfrac{\partial G_R}{\partial \lambda_i} & \Longleftrightarrow & q^i = \phi^i(\bar{q}^\alpha) \\ 0 = \dfrac{\partial G_R}{\partial \mu^a} & \Longleftrightarrow & \Sigma_a(\bar{q}^\alpha) = 0 \end{array}} \tag{19}$$

If the potential energy $\bar{V}$ depends on supplementary variables $\underline{u}$, then to this system we add equation $0 = \partial \bar{V}/\partial \underline{u}$.

Remark 5. With any smooth function $F\colon \mathcal{Q} \to \mathbb{R}$ we associate a function δF on the tangent bundle $T\mathcal{Q}$ defined by

$$\delta F(v) = \langle v, dF \rangle. \tag{20}$$

The coordinate representation of this function is

$$\delta F = \frac{\partial F}{\partial q^i}\, \delta q^i. \tag{21}$$

This function is linear on each fibre of $T\mathcal{Q}$. Thus, from the expression (1) of the virtual work it follows that the equilibrium states defined by (9) are characterized by the following **variational equation**

$$\boxed{f_i\, \delta q^i = \delta(G_R + \bar{V})} \tag{22}$$

Although the two symbols d and δ have different meaning, they have the same formal properties (linearity, Leibniz rule, etc.). Thus, equation (22) is formally equivalent to equation (18). However, while (18) has a pure mathematical character, equation (22) has a physical meaning: it states that a force $f = (f_i)$ is an equilibrium force (i.e., when applied to the system it is able to maintain the system in equilibrium) if and only if the corresponding virtual work, for any virtual displacement $v = (\delta q^i)$, is given by the value of the function $\delta(G_R + \bar{V})$. Hence, equation (22) represents a generalized version of the classical **virtual work principle** of D'Alembert-Lagrange.

Remark 6. We can think of more general kinds of control relations. For another general approach to this matter see [Tulczyjew, 1989]. The definition of control relation proposed here is suitable for dealing with the applications illustrated below. For a further approach to thermostatics see [Dubois, Dufour, 1974, 1976, 1978].

Remark 7. There are special cases of control relations.

Case 1, **complete control without constraint**. In this case $\bar{\mathcal{Q}} = \Sigma = \mathcal{Q}$ and ϕ is the identity. It follows that $\mathcal{E} = \bar{\mathcal{E}}$ is the Lagrangian submanifold generated by a potential energy $V: \mathcal{Q} \to \mathbb{R}$.

Case 2, **pure constraint**: $\Sigma \subset \mathcal{Q} = \bar{\mathcal{Q}}$ and ϕ is the identity. In this case we have $R = \Delta_\Sigma$ and $\mathcal{E} = (\Sigma, V)\hat{} \subset T^*\mathcal{Q}$: $\mathcal{E}$ is generated by the potential energy V over the constraint Σ. We can interpret this case in another way: $\Sigma = \mathcal{Q} = \bar{\mathcal{Q}}$. It follows that $\mathcal{E}$ is the Lagrangian submanifold of $T^*\Sigma$ generated by the restriction $V|\Sigma$ of the potential energy to the constraint. In other words, we look at Σ as the configuration manifold of the system.

Case 3, **pure fibration**. We have no constraint, but there are internal degrees of freedom (internal or hidden variables) of $\bar{\mathcal{Q}}$ which are not controlled.

Case 4. The constraint Σ is such that the restriction of the fibration (or the surjective submersion) $\phi: \bar{\mathcal{Q}} \to \mathcal{Q}$ to Σ is a fibration (or a surjective submersion) $\phi: \Sigma \to \mathcal{Q}$. In this case we can replace $\bar{\mathcal{Q}}$ with Σ and the control relation reduces to the case 3 of pure fibration.

Let us consider some basic examples.

Example 1. Let us consider a point P free to move in the plane $\mathbb{R}^2 = (x, y) = (\mathbf{x})$ and submitted to internal forces with potential energy V. Let us act on it by imposing its position $\mathbf{x}$. In this case $\bar{\mathcal{Q}} = \mathcal{Q} = \mathbb{R}^2$ (ϕ is the identity) and we have no constraint. In this control, we first impose the position of P_1, and then we measure the force f we have to apply for maintaining the point in that position. Then, according to (22), the equilibrium states are described by equation

$$f\delta x + g\delta y = \delta V,$$

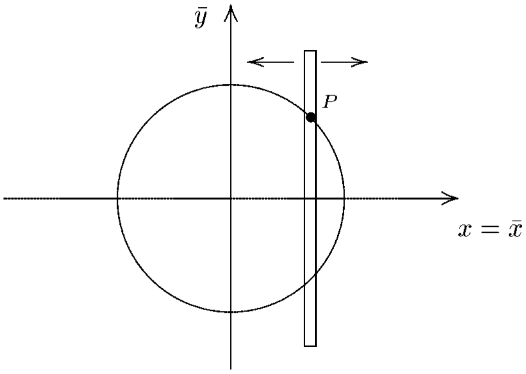
which yields equations

$$f = \frac{\partial V}{\partial x}, \qquad g = \frac{\partial V}{\partial y}.$$

These equations give the components (f, g) of the force $\mathbf{f}$ to be applied for maintaining the point at the assigned position. Thus, the set of the equilibrium states is the Lagrangian submanifold of $T^*\mathbb{R}^2$ generated by the function V. This is a case of complete control (case 1).

Example 2. A point P on the plane $(\bar{x}, \bar{y})$ is constrained to the unit circle $\mathbb{S}_1$, $\bar{x}^2 + \bar{y}^2 = 1$. We control only its coordinate $x = \bar{x}$, by moving a bar parallel to the y-axis along which P can slide freely. We can consider $\bar{\mathcal{Q}} = \mathbb{R}^2 = (\bar{x}, \bar{y})$, $\Sigma = \mathbb{S}_1$, $\mathcal{Q} = \mathbb{R} = (x)$ and the projection onto the x-axis as the fibration $\phi: \mathbb{R}^2 \to \mathbb{R}$. Since this fibration is defined by equation $x = \bar{x}$, the generating family of the canonical lift $\widehat{R}$ is, according to (13),

$$G_R(x, \bar{x}, \bar{y}; \lambda, \mu) = \lambda\,(x - \bar{x}) + \mu\,(\bar{x}^2 + \bar{y}^2 - 1). \tag{23}$$



In the case of no active force, $\bar{V} = 0$, the variational equation (22) reads

$$f\,\delta x = \delta G_R \tag{24}$$

and it is equivalent to equations

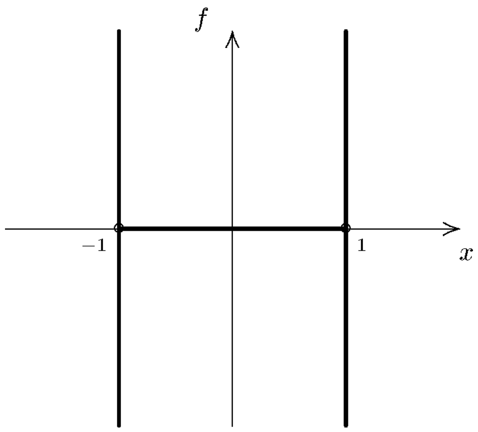
$$\begin{cases} f = \lambda \\ 0 = -\lambda + 2\,\mu\,\bar{x} \\ 0 = \mu\,\bar{y} \end{cases} \qquad \begin{cases} x - \bar{x} = 0 \\ \bar{x}^2 + \bar{y}^2 - 1 = 0. \end{cases}$$

These equations are equivalent to

$$\begin{cases} f = 2\,\mu\,x \\ \mu\,\bar{y} = 0 \\ x^2 + \bar{y}^2 - 1 = 0. \end{cases} \tag{25}$$

We observe that $\mu \neq 0 \quad \Rightarrow \quad \bar{y} = 0, \quad x = \pm 1$. So that $x \neq \pm 1 \quad \Rightarrow \quad \bar{y} \neq 0, \quad \mu = 0, \quad f = 0$. Moreover, $x = \pm 1 \quad \Rightarrow \quad \bar{y} = 0, \quad f = \pm 2\mu, \quad \mu \in \mathbb{R}$.

The equilibrium set $\mathcal{E}$ is then represented in the plane $(x, f) = \mathbb{R}^2 = T^*\mathbb{R}$ by the following picture:



The generating family (23) of $\widehat{R}$ can be reduced in this case to a generating family of $\mathcal{E}$,

$$G_{\mathcal{E}}(x; \bar{y}, \mu) = \mu\,(x^2 + \bar{y}^2 - 1) \tag{26}$$

with extra variables $(\bar{y}, \mu)$. Indeed, the variational equation

$$f\,\delta x = \delta G_{\mathcal{E}}$$

is equivalent to equations (25). It can be seen that this is a Morse family except at the two "circled" points $(\pm 1, 0)$, in accordance with the fact that without these two points $\mathcal{E}$ is a Lagrangian submanifold.

Example 3. In the preceding example assume that a gravitational constant force (parallel to the $\bar{y}$-axis) acts on P. The potential energy is $\bar{V} = g\,\bar{y}$, $g > 0$. In this case equation (24) is replaced by

$$f\,\delta x = \delta G_R + \delta \bar{V} \tag{27}$$

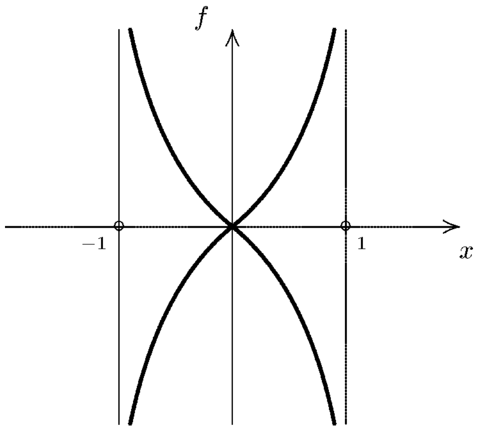
and equations (25) by

$$\begin{cases} f = 2\,\mu\,x \\ 2\,\mu\,\bar{y} + g = 0 \\ x^2 + \bar{y}^2 - 1 = 0. \end{cases} \tag{28}$$

Since the second equation implies $\mu\bar{y} \neq 0$, thus $\bar{y} \neq 0$, the last equation shows that $x \pm 1$ are incompatible values. This means that for $x = \pm 1$ the force f cannot assume a finite value. Indeed, since $\mu = -g/\bar{y}$ and $\bar{y} = \pm\sqrt{1 - x^2}$, we have

$$f = \pm\frac{gx}{\sqrt{1 - x^2}}$$

and the above graph of f is replaced by the following one



The sign of the force at a given point $x \neq \pm 1$ depends on the position on the constraint of the point P, which is not controlled.

Example 4. On the plane $\mathbb{R}^2 = (x, y) = (\mathbf{x})$ we consider a point $P_1 = (x_1)$ moving on the x-axis and tied elastically to a point $P_2 = (x_2, y_2)$ free to move on the plane. Thus, $\bar{Q} = \mathbb{R}^2 \times \mathbb{R} = (x_2, y_2, x_1)$ is the configuration manifold of the holonomic system made of these two points. Let us act simultaneously on both points by imposing their positions. We are in the case of a complete control, $\mathcal{Q} = \bar{\mathcal{Q}}$. Then the set of the equilibrium states is the regular Lagrangian submanifold $\bar{\mathcal{E}}$ generated by the potential energy

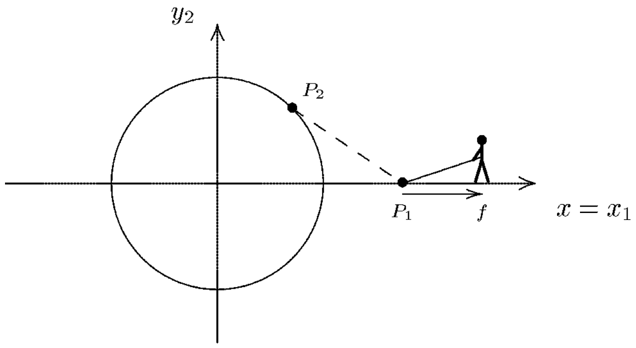
$$\bar{V} = \tfrac{k}{2} (\mathbf{x}_1 - \mathbf{x}_2)^2 = \tfrac{k}{2} \left[(x_1 - x_2)^2 + y_2^2 \right]$$

and described by equations

$$f_1 = \frac{\partial \bar{V}}{\partial x_1}, \qquad f_2 = \frac{\partial \bar{V}}{\partial x_2}, \qquad g_2 = \frac{\partial \bar{V}}{\partial y_2},$$

which provide the external forces $\mathbf{f}_1 = (f_1)$ and $\mathbf{f}_2 = (f_2, g_2)$ needed for maintaining the system in equilibrium.

Example 5. Let us act on the system of Example 4 by constraining the point P_2 to move on the circle $\mathbb{S}_1$ of radius $\mathbf{1}$ and centered at the origin and by controlling only the position of the point P_1 on the x-axis. This is a particular case of the so-called **Zeeman machine** [Poston, Stewart, 1978] where the point P_1 is free to move in the plane (see also [Dubois, Dufour, 1976].



The control configuration manifold is now $\mathcal{Q} = \mathbb{R} = (x)$, the constraint is $\Sigma = \mathbb{S}_1 \times \mathbb{R}$ and the fibration ϕ is just the Cartesian projection onto the x-axis. Thus, the control relation R is represented by equations

$$\begin{cases} x_2^2 + y_2^2 - 1 = 0 \\ x - x_1 = 0, \end{cases}$$

and its canonical lift $\widehat{R}$ is generated by the family

$$G_R(x, x_1, x_2, y_2; \lambda, \mu) = \lambda(x - x_1) + \mu(x_2^2 + y_2^2 - 1).$$

Then the set of the equilibrium states $\mathcal{E} \subset T^*\mathcal{Q} = (x, f)$ of the system under this control is described by the variational equation

$$f\,\delta x = \delta(G_R + \bar{V}) = \delta\Big(\lambda(x - x_1) + \mu(x_2^2 + y_2^2 - 1) + \tfrac{k}{2}\left[(x_1 - x_2)^2 + y_2^2\right]\Big),$$

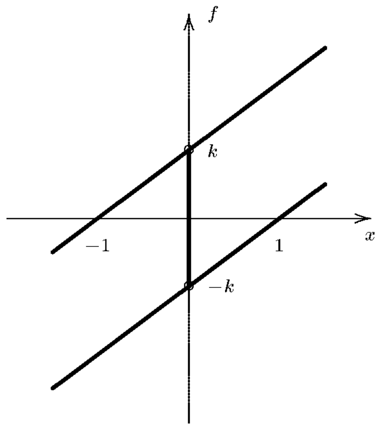
which is equivalent to equations

$$\begin{cases} f = \lambda \\ 0 = -\lambda + k(x_1 - x_2) \\ 0 = 2x_2\mu - k(x_1 - x_2) \\ 0 = 2y_2\mu + ky_2 \\ 0 = x - x_1 \\ 0 = x_2^2 + y_2^2 - 1. \end{cases}$$

These equations are reducible to

$$\begin{cases} f = k(x - x_2) \\ (2\mu + k)x_2 = kx \\ (2\mu + k)y_2 = 0 \\ x_2^2 + y_2^2 = 1. \end{cases}$$

For $y_2 = 0$ we get $x_2 = \pm 1$ and $f = k(x \pm 1)$. For $y_2 \neq 0$ we get $2\mu + k = 0$, thus $x = 0$ and $f = -kx_2$, with $|x_2| \leq 1$. The set $\mathcal{E}$ of the equilibrium states is then represented by the following picture:



Example 6. For the same system of Example 5 we can think of another control relation. We can consider the configuration manifold $\bar{\mathcal{Q}} = \mathbb{S}_1 \times \mathbb{R} = (\theta, x)$. If we control the positions of both points, then the equilibrium states are represented by the regular Lagrangian submanifold $\bar{\Lambda}$ generated by the potential energy

$$\bar{V} = \tfrac{k}{2}\,(\mathbf{x}_1 - \mathbf{x}_2)^2 = \tfrac{k}{2}\,\left[(x - \cos\theta)^2 + \sin^2\theta\right]$$

and described by equations

$$\begin{cases} f = \dfrac{\partial \bar{V}}{\partial x} = k(x - \cos\theta), \\ \tau = \dfrac{\partial \bar{V}}{\partial \theta} = k\,x\,\sin\theta, \end{cases}$$

where θ is the angle between $\mathbf{x}_2$ and the x-axis, and τ is the torque applied to the point P_2. If we control only the point P_1, leaving the point P_2 free on $\mathbb{S}_1$, then the control manifold is $\mathcal{Q} = \mathbb{R}$ (the x-axis) and the control relation is given by the trivial fibration $\phi\colon \mathbb{S}_1 \times \mathbb{R} \to \mathbb{R}$ only. The equilibrium states of the system form the set $\mathcal{E} \subset T^*\mathcal{Q} \simeq \mathbb{R}^2 = (x, f)$ represented by equations (we put $\tau = 0$ in the equations above)

$$\begin{cases} f = \dfrac{\partial \bar{V}}{\partial x_1} = k(x - \cos\theta), \\ 0 = \dfrac{\partial \bar{V}}{\partial \theta} = k\,x\,\sin\theta. \end{cases}$$

We get the same set $\mathcal{E}$ as above. Since

$$\frac{\partial^2 \bar{V}}{\partial \theta^2} = k\,x\,\cos\theta, \qquad \frac{\partial^2 \bar{V}}{\partial \theta \partial x} = k\,\sin\theta,$$

the generating family $\bar{V}(x;\theta)$ is a Morse family except for $x = 0$, $\sin\theta = 0$, that is over the points $(0, \pm k)$. In accordance with the theory, by excluding these two points the set $\mathcal{E}$ is a Lagrangian submanifold. It is made of five branches (open segments and half-lines). The "vertical" segment is the set of the singular points, in accordance with the fact that the caustic is represented by equations

$$0 = \frac{\partial^2 \bar{V}}{\partial \theta^2} = k\,x\,\cos\theta, \qquad 0 = \frac{\partial \bar{V}}{\partial \theta} = k\,x\,\sin\theta.$$

Example 7. Two points P_1 and P_2 are constrained to the x-axis and the y-axis, respectively. They are linked by a rigid rod of length a. The point P_2 is tied elastically to the origin by a spring. Let b be the length at rest of the spring. We act only on the point P_1. An interpretation of this static system is the following: the extended configuration manifold is $\bar{Q} = \mathbb{R}^2 = (x_1, y_2)$, the constraint Σ is represented by the rod i.e., by equation $x_1^2 + y_2^2 = a^2$, the control manifold is $Q = \mathbb{R} = (x)$, the fibration ϕ is represented by equation $x = x_1$; y_2 is considered as an internal variable. The internal potential energy is $\bar{V}(x_1, y_2) = \frac{k}{2}\,(b - y_2)^2$. The generating family of the control relation R is

$$G_R(x, x_1, y_2; \lambda, \mu) = \lambda(x - x_1) + \mu(x_1^2 + y_2^2 - a^2),$$

and the set $\mathcal{E} \subset T^*Q$ of the equilibrium states of the system under this control is described by equation

$$f\,\delta x = \delta(G_R + \bar{V}) = \delta\Big(\lambda(x - x_1) + \mu(x_1^2 + y_2^2 - a^2) + \tfrac{k}{2}\,(b - y_2)^2\Big)$$

which yield equations

$$\begin{cases} f = \lambda \\ 0 = x - x_1 \\ 0 = -\lambda + 2\,\mu\,x_1 \\ 0 = x_1^2 + y_2^2 - a^2 \\ 0 = 2\,\mu\,y_2 - k\,(b - y_2). \end{cases}$$

These equations reduce to

$$\begin{cases} f = 2\,\mu\,x \\ x^2 + y_2^2 = a^2 \\ 2\,\mu\,y_2 = k\,(b - y_2). \end{cases} \tag{29}$$

For $y_2 \neq 0$ we find

$$\mu = \tfrac{k}{2}\left(\frac{b}{y_2} - 1\right)$$

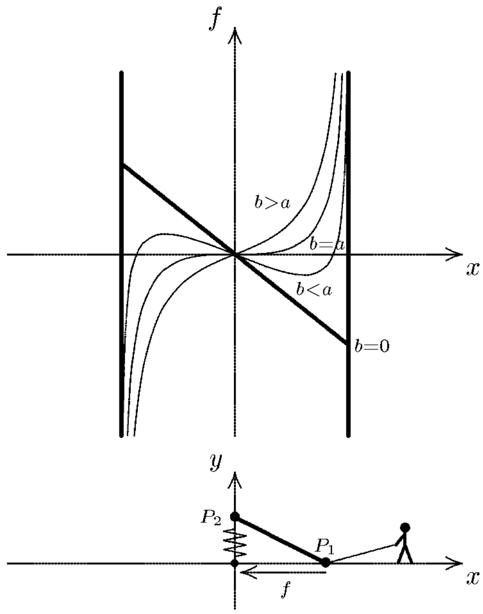
thus,

$$f = k\,x\left(\frac{b}{\sqrt{a^2 - x^2}} - 1\right).$$

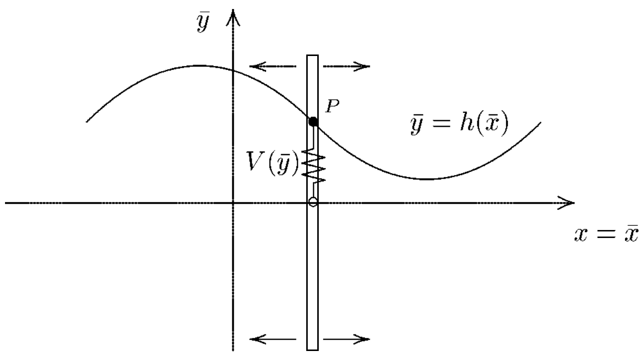
For $y_2 = 0$, the third equation has a meaning only for $b = 0$ (ideal spring). For $b = 0$ equations (29) become

$$\begin{cases} f = 2\,\mu\,x \\ x^2 + y_2^2 = a^2 \\ (2\,\mu + k)\,y_2 = 0 \end{cases}$$

so that, for $y_2 = 0$ i.e., for $x = \pm a$, the extra variable μ is not determined and we find that f may assume any arbitrary value. For $y_2 = 0$, we find $\mu = -k/2$ thus, $f = -kx$. The set $\mathcal{E}$ is then represented by the following picture, for all possible values of b. For $b = 0$ is not a submanifold.



Example 8. Let us consider Example 2 modified as follows: (i) the point P is constrained on a curve $\bar{y} = h(\bar{x})$, and (ii) it is submitted to a force parallel the $\bar{y}$-axis with potential energy $V(\bar{y})$.



The equilibrium set $\mathcal{E}$ is then described by the variational equation

$$f\,\delta x = \delta\big[V(\bar{y}) + \mu\big(\bar{y} - h(x)\big)\big],$$

which yields equations

$$\bar{y} = h(x), \quad f = -\,\mu\, h'(x), \quad V'(\bar{y}) + \mu = 0.$$

It follows that $\mathcal{E}$ is described by equation

$$f(x) = F\big(h(x)\big)\, h'(x), \quad F = V'. \tag{30}$$

If for instance $V(\bar{y}) = \frac{k}{2}\,\bar{y}^2$ (ideal spring), then

$$f(x) = k\, h(x)\, h'(x). \tag{31}$$

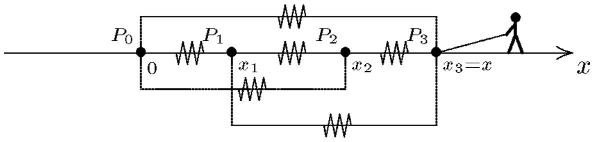
In this way, we can construct any kind of (smooth) force function $f(x)$ (at least in a neighborhood of a point x_0) by taking a curve $h(x)$ which is a solution of the differential equation (30) or (31). For instance, if we want a repulsive linear force

$$f(x) = -\,kx$$

in the neighborhood of $x = 0$, then equation (31) reads $xdx = -\,h\,dh$ and leads to solutions of the kind $h^2(x) = c^2 - x^2$. The curve which realizes such a force is then any circle centered at the origin.

Example 9. The static control of n-body systems. Let us consider a static system made of four points $(P_i) = (P_0, P_1, P_2, P_3)$ on a straight line (the x-axis), with interacting forces with potentials $V_{ij}(r_{ij})$ which are even functions of the distances $r_{ij} = x_i - x_j$. We consider for simplicity the case of four points, but the following discussion can be easily extended to the generic case of n points. Assume that the point P_0 is constrained at the origin, so that $x_0 = 0$, and that we act only on the last point P_3. The total potential energy is

$$\begin{aligned}\bar{V} = &V_{01}(x_1 - x_0) + V_{02}(x_2 - x_0) + V_{03}(x_3 - x_0)+ \\ &+ V_{12}(x_2 - x_1) + V_{13}(x_3 - x_1) + V_{23}(x_3 - x_2).\end{aligned}$$



The generating family of the canonical lift of the control relation is

$$G_R = \lambda(x - x_3) + \mu x_0.$$

Equation (27) now reads

$$\begin{aligned}f\delta x = \delta(G_R + \bar{V}) = &\delta\lambda\,(x - x_3) + \lambda(\delta x - \delta x_3) + \mu\,\delta x_0 + x_0\,\delta\mu \\ &+ f_{01}(x_1 - x_0)\,(\delta x_1 - \delta x_0) + f_{02}(x_2 - x_0)\,(\delta x_2 - \delta x_0) \\ &+ f_{03}(x_3 - x_0)\,(\delta x_3 - \delta x_0) + f_{12}(x_2 - x_1)\,(\delta x_2 - \delta x_1) \\ &+ f_{13}(x_3 - x_1)\,(\delta x_3 - \delta x_1) + f_{23}(x_3 - x_2)\,(\delta x_3 - \delta x_2),\end{aligned}$$

where $f_{ij} = V'_{ij}$ are the odd functions representing the internal interacting forces. This is equivalent to the following system,

$$\begin{cases} f = \lambda \\ x = x_3 \\ x_0 = 0 \\ 0 = \mu - f_{01}(x_1 - x_0) - f_{02}(x_2 - x_0) - f_{03}(x_3 - x_0) \\ 0 = f_{01}(x_1 - x_0) - f_{12}(x_2 - x_1) - f_{13}(x_3 - x_1) \\ 0 = f_{02}(x_2 - x_0) + f_{12}(x_2 - x_1) - f_{23}(x_3 - x_2) \\ 0 = \; - \lambda + f_{03}(x_3 - x_0) + f_{13}(x_3 - x_1) + f_{23}(x_3 - x_2). \end{cases} \tag{32}$$

Due to the first three equations, from the last one we get the expression of the controlling force,

$$f = f_{03}(x) + f_{13}(x - x_1) + f_{23}(x - x_2), \tag{33}$$

which depends only on the interacting forces between the point P_3 and the points (P_0, P_1, P_2). The remaining forces are internal forces. The fourth equation (32) gives the expression of μ as a reaction force at the fixed point P_0,

$$\mu = f_{01}(x_1) + f_{02}(x_2) + f_{03}(x_3).$$

The remaining two equations (32) read

$$\text{(34)} \qquad \begin{cases} f_{01}(x_1) = f_{12}(x_2 - x_1) + f_{13}(x - x_1) \\ f_{23}(x - x_2) = f_{02}(x_2) + f_{12}(x_2 - x_1). \end{cases}$$

For any fixed value of x, equations (34) define a subset $D_x \subseteq \mathbb{R}^2 = (x_1, x_2)$. By replacing this subset of values of (x_1, x_2) in equation (33) we get a set $F_x \subseteq \mathbb{R}$ of forces f associated with the controlled value of x. The union $\mathcal{E} = \cup_{x \in \mathbb{R}} F_x$ of all these sets gives the equilibrium states of the system. In general, it is a very complicated subset of $\mathbb{R}^2 = (x, f)$.

Remark 8. In the model of control of static systems we are considering, we have not introduced and discussed the notion of **stability** of an equilibrium state. Example 5 (the Zeeman machine) suggests the following definition: an equilibrium state of $\mathcal{E}$ is stable if it corresponds to stable states on the constraint manifold Σ.

5.2 Simple closed thermostatic systems

Let us consider a system of particles (atoms, molecules) in a closed vessel. Let us act on it by means of an external device. The energy transfered to the system in a "quasistatic process" c, made of slow transformations of equilibrium states, is defined by

$$E_c = \int_c (\delta Q - P dV)$$

where P is the pressure, V is the volume, and δQ is a 1-form representing the heat absorbed by the system. If we *postulate* that this 1-form admits an integrating factor,

$$\text{(1)} \qquad \delta Q = T dS,$$

where T is the absolute temperature and S is the entropy, then the integral E_c can be written

$$\text{(2)} \qquad E_c = \int_c \theta, \qquad \theta = T dS - P dV.$$

This suggests to consider, following [Tulczyjew, 1977a], the four-dimensional space

$$\mathcal{S} = (S, V, P, T) = \mathbb{R}^4$$

as the **space of states** (or **state manifold**). A quasi-static process c is one-dimensional path in this space. Actually, in accordance with their physical meaning, the variables (S, V, P, T) will assume only positive values, so that the physically interesting states are located in $\mathbb{R}^4_+$. We call (S, V) **extensive observables** and (T, P) **intensive observables**. A fundamental advantage of considering this four-dimensional space is that the 1-form θ induces a symplectic form

$$\omega = d\theta = dT \wedge dS + dV \wedge dP. \tag{3}$$

In this way, as a consequence of the first principle of thermodynamics and formula (1), the state manifold is endowed with a canonical symplectic structure.[1] The corresponding Poisson bracket is

$$\{F, G\} = \frac{\partial F}{\partial T}\frac{\partial G}{\partial S} - \frac{\partial F}{\partial P}\frac{\partial G}{\partial V} - \frac{\partial F}{\partial S}\frac{\partial G}{\partial T} + \frac{\partial F}{\partial V}\frac{\partial G}{\partial P}. \tag{4}$$

Definition 1. The **equilibrium states**, which are physically admissible, form a subset

$$\mathcal{E} \subseteq \mathcal{S}$$

called the **constitutive set**. We say that the system is **simple** if $\mathcal{E}$ is an **exact Lagrangian submanifold** i.e., the restriction of the 1-form θ to the vectors tangent to $\mathcal{E}$ (which is closed, since $\mathcal{E}$ is Lagrangian) is an exact form:

$$\theta|\mathcal{E} = dW, \tag{5}$$

where $W : \mathcal{E} \to \mathbb{R}$ is a smooth function which we call **intrinsic potential energy** of the system.

Remark 1. This is equivalent to assume that $\mathcal{E}$ is a smooth 2-dimensional manifold and that the integral E_c is zero for all quasistatic "cycles" over $\mathcal{E}$. It is also equivalent to assume the existence of an **internal energy** (cf. (11) below). This definition is in accordance with that of [Carathéodory, 1909].

As we shall see below, the intrinsic potential energy W will be represented by four other functions, called **thermostatic potentials** (or "thermodynamical potentials" in the classical literature) related to various **control modes** with which we act on the system.

[1] The common geometrical setting of Thermodynamics is odd-dimensional, in terms of contact manifolds [Hermann, 1973] [Mrugała, 1995]. However, the even-dimensional framework, in terms of symplectic manifolds and Lagrangian submanifolds, seems to be more "symmetric" and elegant. A remarkable example of this structural symmetry is the general setting of the Legendre transformation and the definition of thermodynamical potentials illustrated in §5.5.

Remark 2. Being a Lagrangian submanifold of a 4-dimensional symplectic manifold, the constitutive set $\mathcal{E}$ is represented (we assume globally) by <u>two</u> independent **equations of state** or **constitutive equations**,

$$\begin{cases} E_1(S, V, P, T) = 0 \\ E_2(S, V, P, T) = 0 \end{cases} \tag{6}$$

with functions (E_1, E_2) in involution on $\mathcal{E}$:

$$\{E_1, E_2\}|\mathcal{E} = 0. \tag{7}$$

Remark 3. In §5.4 the matter of this remark will be included in a more general framework. Let us consider the extensive variables (S, V) as global coordinates of a configuration manifold $\mathcal{Q}_1 = \mathbb{R}^2_+$. Let (S, V, p_S, p_V) be the corresponding canonical coordinates on the cotangent bundle $T^*\mathcal{Q}_1$. The Liouville form is

$$\theta_{\mathcal{Q}_1} = p_S\, dS + p_V\, dV. \tag{8}$$

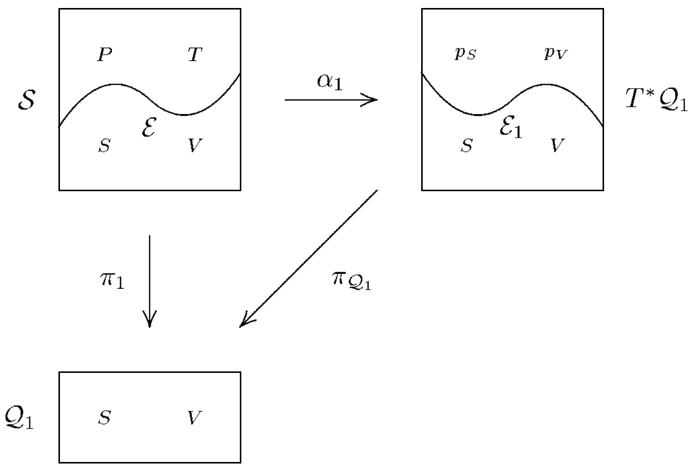
Thus, the injective mapping

$$\alpha_1 : \mathcal{S} \to T^*\mathcal{Q}_1$$

defined by

$$p_S = T, \qquad p_V = -P, \tag{9}$$

is a symplectomorphism onto the open subset of $T^*\mathcal{Q}_1$ defined by $p_S > 0$, $p_V > 0$.



It follows that the image $\mathcal{E}_1 = \alpha_1(\mathcal{E})$ of $\mathcal{E}$ by α_1 is a Lagrangian submanifold of the cotangent bundle $T^*\mathcal{Q}_1$. A generating function $U(S,V)$ of $\mathcal{E}_1$ is called **internal energy**. It may be a generating function, possibly a Morse family, $U(S,V;\underline{u})$, with supplementary variables $(\underline{u})$. This means that $\mathcal{E}_1$ is completely described by equation

$$p_S\, dS + p_V\, dV = dU(S,V) \tag{10}$$

so that, due to equations (9), $\mathcal{E}$ is described by equation

$$TdS - PdV = dU(S,V), \tag{11}$$

which is equivalent to equations

$$T = \frac{\partial U}{\partial S} = U_S, \qquad P = -\frac{\partial U}{\partial V} = -U_V. \tag{12}$$

If U is a Morse family, then to these equations we add equation

$$0 = \frac{\partial U}{\partial \underline{u}}.$$

Let us consider the fibration $\pi_1: \mathcal{S} \to \mathcal{Q}_1: (S,V,P,T) \mapsto (S,V)$. We use the following general property (for the proof, see §6.6).

Theorem 1. *Let $\alpha_1: \mathcal{S} \to T^*\mathcal{Q}_1$ be a symplectomorphism. If a Lagrangian submanifold $\mathcal{E} \subset \mathcal{S}$ is the image of a section of $\pi_1 = \pi_{\mathcal{Q}_1} \circ \alpha_1: \mathcal{S} \to \mathcal{Q}_1$, then it admits a global generating function $U: \mathcal{Q}_1 \to \mathbb{R}$ if and only if it admits a global function $W: \mathcal{E} \to \mathbb{R}$ such that $dW = \theta_1|\mathcal{E}$, where $\theta_1 = \alpha_1^*\theta_{\mathcal{Q}_1}$. The link between these two functions is $W = \pi_1^* U = U \circ \pi_1$.*

In other words, *if the constitutive set $\mathcal{E}$ is a section of π_1, then it admits an internal energy $U: \mathcal{Q}_1 \to \mathbb{R}$ if and only if it admits an intrinsic potential energy $W: \mathcal{E} \to \mathbb{R}$*. In the present case $\mathcal{S} \simeq \mathbb{R}^4_+$ and $\mathcal{Q}_1 \simeq \mathbb{R}^2_+$. Thus, both these conditions are satisfied for a section of π_1.

5.3 The ideal gas

Let us apply the general considerations of the preceding section to the special case of an ideal gas. The well known constitutive equation of an ideal gas is

$$\boxed{E_1(P,V,T) \doteq PV - nRT = 0} \tag{1}$$

where $n > 0$ is the mole number and $R > 0$ is a physical constant. This equation summarizes the Boyle, Gay-Lussac and Avogadro's laws. In accordance with the assumption that the constitutive set $\mathcal{E}$ is a 2-dimensional manifold, this equation is not sufficient to describe the behaviour of the gas: a second constitutive equation is needed. We can derive this second equation by the following "symplectic" argument. Let us assume that the second constitutive equation has the form

$$F(S, V, P, T) \doteq S - f(P, V, T) = 0 \tag{2}$$

i.e., that the entropy can be expressed as a function of the remaining observables. Then $\mathcal{E}$ is Lagrangian if and only if the function F is in involution with E: $\{F, E_1\} = 0$ (we assume that this condition is satisfied everywhere, not only on $\mathcal{E}$). Due to the definition (4), §5.2, of the Poisson bracket, this conditions is equivalent to the partial differential equation

$$nRF_S + VF_V - PF_P = 0,$$

where $F_S = \partial F/\partial S$, etc., thus, to

$$Vf_V - Pf_P = nR. \tag{3}$$

This is a Hamilton-Jacobi equation in the cotangent bundle of the configuration manifold of variables (P, V). An evident solution is the function

$$f = a + (nR + c)\,\log V + c\,\log P, \tag{4}$$

where (a, c) are constant parameters. This is a complete solution with respect to the non-additive constant c, since the matrix

$$\left[\frac{\partial^2 f}{\partial V \partial c} \quad \Big| \quad \frac{\partial^2 f}{\partial P \partial c}\right] = \left[\frac{1}{V} \quad \Big| \quad \frac{1}{P}\right]$$

has maximal rank everywhere. However, since in equation (3) the temperature T is not involved, we can consider (a, c) as functions of T. So, with the "natural" choice of (4) among the solutions of equation (3), we find the second constitutive equation of an ideal gas:

$$\boxed{F(S, V, P, T) \doteq S - a(T) - \big(R + c(T)\big)\,\log V - c(T)\,\log P = 0} \tag{5}$$

We can write this equation in the form

$$\frac{S}{c} = \frac{a}{c} + \big(\frac{nR}{c} + 1\big)\,\log V + \log P,$$

and by setting

$$\gamma = 1 + \frac{nR}{c}, \qquad k = \exp\left(-\frac{a}{c}\right), \tag{6}$$

we get

$$\boxed{E_2 \doteq PV^\gamma - k \exp\frac{S}{c} = 0} \tag{7}$$

with

$$nR = c(\gamma - 1). \tag{8}$$

This is the standard form of the second constitutive equation of an ideal gas, involving the entropy, as it appears in books of thermodynamics, with

$$c = nc_V, \qquad k = Kn^\gamma. \tag{9}$$

Remark 1. We emphasize that the second equation of state (7) has been derived from the first one (1) simply by assuming that (i) the constitutive set $\mathcal{E}$ is a Lagrangian submanifold and, (ii) that the entropy is a function of the other observables.

The matrix

$$\begin{bmatrix} \dfrac{\partial E_1}{\partial T} & \dfrac{\partial E_1}{\partial P} & \dfrac{\partial E_1}{\partial S} & \dfrac{\partial E_1}{\partial V} \\ \dfrac{\partial F}{\partial T} & \dfrac{\partial F}{\partial P} & \dfrac{\partial F}{\partial S} & \dfrac{\partial F}{\partial V} \end{bmatrix} = \begin{bmatrix} -nR & V & 0 & P \\ -a' - c'\log(VT) & -\dfrac{c}{P} & 1 & -\dfrac{nR+c}{V} \end{bmatrix}$$

has maximal rank everywhere, so that equations (1) and (5) are independent, and this confirms that they define a submanifold of codimension 2.

Let us assume that the functions a and c, thus γ and k, do not depend on T. Since

$$\det \begin{bmatrix} \dfrac{\partial E_1}{\partial T} & \dfrac{\partial E_1}{\partial P} \\ \dfrac{\partial E_2}{\partial T} & \dfrac{\partial E_2}{\partial P} \end{bmatrix} = \det \begin{bmatrix} -nR & V \\ 0 & V^\gamma \end{bmatrix} = -nRV^\gamma \neq 0,$$

the Lagrangian submanifold $\mathcal{E}$ is a section of the fibration π_1. Let us show that it admits an internal energy $U(S, V)$ thus, that the thermostatic system described by the two constitutive equations (1) and (6) is simple.

Theorem 1. *If $\mathcal{E}$ is described by the two constitutive equations* (1) *and* (6) *with* a *and* $c =$ *constant, then the internal energy is (up to an additive constant)*

$$\boxed{U(S,V) = \frac{k}{\gamma - 1} V^{1-\gamma} \exp \frac{S}{c}} \tag{10}$$

Proof. This follows from the integration of the exact form (11) of §5.2. By solving equations (1) and (6) with respect to P and T, we get

$$P = k\,V^{-\gamma} \exp \frac{S}{c}, \qquad T = \frac{PV}{nR} = \frac{kV^{1-\gamma}}{nR} \exp \frac{S}{c}.$$

We observe that $T = U_S$ and $P = -U_V$ with U given by (10). ■

Remark 2. Van der Waals gas. A similar analysis can be done for a Van der Waals gas. The first constitutive equation is assumed to be

$$E_1 \doteq P + \frac{na}{V^2} - \frac{nRT}{V - nb} = 0. \tag{11}$$

It follows that the internal energy is

$$U(S,V) = \frac{nkc_V}{R} \exp \frac{S - nR \log\left(\frac{V}{n} - b\right)}{nc_V} - \frac{na}{V}. \tag{12}$$

5.4 Control modes and the Legendre transformation

Definition 1. Let $(\mathcal{S}, \omega)$ be a symplectic manifold. A **control mode** on $\mathcal{S}$ is a symplectomorphism

$$\alpha_c \colon \mathcal{S} \to A_c \subseteq T^*\mathcal{Q}_c \tag{1}$$

into an open submanifold A_c of a cotangent bundle $T^*\mathcal{Q}_c$ such that the mapping

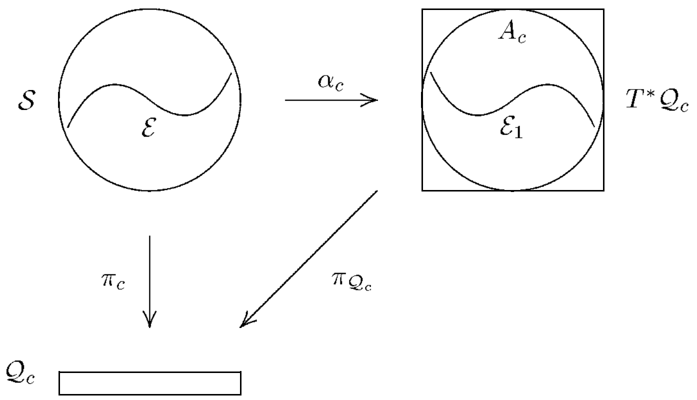
$$\pi_c = \pi_{\mathcal{Q}_c} \circ \alpha_c \colon \mathcal{S} \to \mathcal{Q}_c \tag{2}$$

is onto. We call $\mathcal{Q}_c$ the **control manifold** and the 1-form on $\mathcal{S}$

$$\theta_c = \alpha_c^* \theta_{\mathcal{Q}_c}, \tag{3}$$

the **control form**. Note that

$$d\theta_c = \omega. \tag{4}$$



Definition 2. Let $\mathcal{E} \subset \mathcal{S}$ be a Lagrangian submanifold. Since α_c is a symplectomorphism, the set

$$\mathcal{E}_c = \alpha_c(\mathcal{E}) \tag{5}$$

is a Lagrangian submanifold of T^*Q_c. A generating function G_c of $\mathcal{E}_c$ is called a **generating function of $\mathcal{E}$ with respect to the control mode** α_c.

Remark 1. In the applications to the control theory of static systems, $\mathcal{S}$ represents the space of the states and $\mathcal{E}$ the equilibrium states. The generating function G_c is the **potential energy** with respect to the control mode α_c. Functions on $\mathcal{S}$ are called **observables**.

Remark 2. The fibres of π_c are Lagrangian submanifolds, so that observables which are constant on the fibres are in involution. Conversely, if F^i are n independent global observables in involution, then they define a control mode. Indeed, equations $F^i = q^i$ (constant) define a Lagrangian foliation and the set of all the admissible constant values $(q^i) \in \mathbb{R}^n$ forms a control manifold $Q_c \subseteq \mathbb{R}^n$. It can be seen that if we choose a section of the corresponding projection π_c, then we can define a symplectomorphism into T^*Q_c.

Remark 3. From Remark 2 we derive the following "physical" rule: *we cannot "control" simultaneously and independently n observables of a static system if they are not in involution.* Here, "to control" means "to force the observables to assume arbitrary values" (at least in a suitable domain). For example, for a point P in the plane (see Example 1, §5.1) we cannot control simultaneously the position x and the force f along the x-axis (these two observables are not in involution).

Remark 4. It is interesting the case of the simultaneous existence of two control modes on $\mathcal{S}$, α_1 and α_2. Then the transition from the generating functions G_1

to G_2 of Lagrangian submanifolds of $\mathcal{S}$ is called the **Legendre transformation** [Tulczyjew, 1977]. A symplectic diffeomorphism from a symplectic manifold to a cotangent bundle has been called "special symplectic structure" [Lawruk, Sniatycki, Tulczyjew, 1975]. If X is a vector space, then the direct sum $X \oplus X^*$ is endowed with a canonical symplectic form. A symplectic isomorphism from a symplectic vector space (A, α) to a direct sum $X \oplus X^*$ has been called "frame" (see for instance [Leray, 1981]). A special important case of Legendre transformation is that connecting the Hamiltonian description and the Lagrangian description of dynamics. Other important special cases are related to the control of thermostatic systems. For another general approach to the catastrophe theory in thermodynamics see [Dubois, Dufour, 1978].

The general setting of the Legendre transformation is the following.

[1] Assume that a symplectic manifolds $\mathcal{S}$ is symplectomorphic to two distinct cotangent bundles,

$$T^*Q_2 \xleftarrow{\alpha_2} \mathcal{S} \xrightarrow{\alpha_1} T^*Q_1.$$

[2] The graph of the symplectomorphism $\alpha_2 \circ \alpha_1^{-1}: T^*Q_1 \to T^*Q_2$ is a symplectic relation

$$R_{21} \subset T^*Q_2 \times T^*Q_1$$

which may admit a global generating family L_{21} (possibly a Morse family) over the product $Q_2 \times Q_1$.

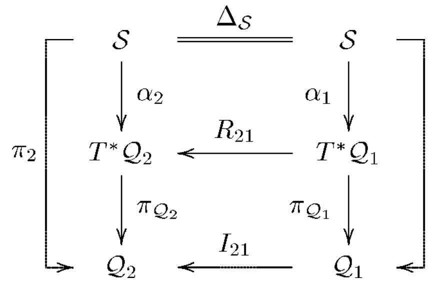
[3] Let $\mathcal{E} \subset \mathcal{S}$ be a Lagrangian submanifold and let G_1 and G_2 be the generating families of $\mathcal{E}_1 = \alpha_1(\mathcal{E})$ and $\mathcal{E}_2 = \alpha_2(\mathcal{E})$, respectively. Since

$$\mathcal{E}_2 = R_{21} \circ \mathcal{E}_1,$$

it follows that the generating families of $\mathcal{E}_1$ and $\mathcal{E}_2$ are related by the composition law

$$G_2 = L_{21} \oplus G_1. \tag{6}$$

Remark 5. The relation R_{21} is the image by $\alpha_2 \times \alpha_1: \mathcal{S} \times \mathcal{S} \to T^*Q_2 \times T^*Q_1$ of the diagonal $\Delta_{\mathcal{S}}$ i.e., of the identity mapping on $\mathcal{S}$, as illustrated in the following diagram:



In this diagram we have considered the surjective submersions

$$\pi_i = \pi_{\mathcal{Q}_i} \circ \alpha_i, \qquad i = 1, 2,$$

and the set

(7) $$I_{21} = \pi_{\mathcal{Q}_2} \times \pi_{\mathcal{Q}_1}(R_{21}) = (\pi_2 \times \pi_1)(\Delta_{\mathcal{S}}) \subseteq \mathcal{Q}_2 \times \mathcal{Q}_1$$

as a relation from $\mathcal{Q}_1$ to $\mathcal{Q}_2$. It is convenient to introduce the mappings

(8) $$\alpha_{21}: \mathcal{S} \to T^*\mathcal{Q}_2 \times T^*\mathcal{Q}_1, \quad \pi_{21}: \mathcal{S} \to \mathcal{Q}_2 \times \mathcal{Q}_1,$$

defined by

(9) $$\alpha_{21}(x) = \big(\alpha_2(x), \alpha_1(x)\big), \qquad \pi_{21}(x) = \big(\pi_2(x), \pi_1(x)\big), \qquad x \in \mathcal{S}.$$

Then we have

(10) $$R_{21} = \alpha_{21}(\mathcal{S}), \qquad I_{21} = \pi_{21}(\mathcal{S}).$$

We observe that α_{21} is a one-to-one mapping: from

$$\big(\alpha_2(x), \alpha_1(x)\big) = \big(\alpha_2(x'), \alpha_1(x')\big)$$

it follows that $\alpha_i(x) = \alpha_i(x')$, $i = 1, 2$. Thus $x = x'$, since α_i are one-to-one.
Remark 6. Since $d(\theta_2 - \theta_1) = \omega - \omega = 0$, the 1-form $\theta_2 - \theta_1$ on $\mathcal{S}$ is closed. Thus, there exist local functions $W_{21}: \mathcal{S} \to \mathbb{R}$ such that

(11) $$\theta_2 - \theta_1 = dW_{21}.$$

In the most interesting examples of Legendre transformation the set I_{21} is a submanifold of $\mathcal{Q}_2 \times \mathcal{Q}_1$ and the function W_{21} is globally defined on $\mathcal{S}$. This is the case illustrated by the following
Theorem 1. *Assume that:* (i) *$\mathcal{S}$ is connected and $I_{21} = \pi_{21}(\mathcal{S})$ is a submanifold;* (ii) *the mapping $\pi = \pi_{21}|\mathcal{S}: \mathcal{S} \to I_{21}$ is a surjective submersion;* (iii) *there exists a global function $W_{21}: \mathcal{S} \to \mathbb{R}$ satisfying* (11)*;* (iv) *there exists a function $E_{21}: I_{21} \to \mathbb{R}$ such that*

(12) $$W_{21} = E_{21} \circ \pi = \pi^* E_{21}.$$

Then,

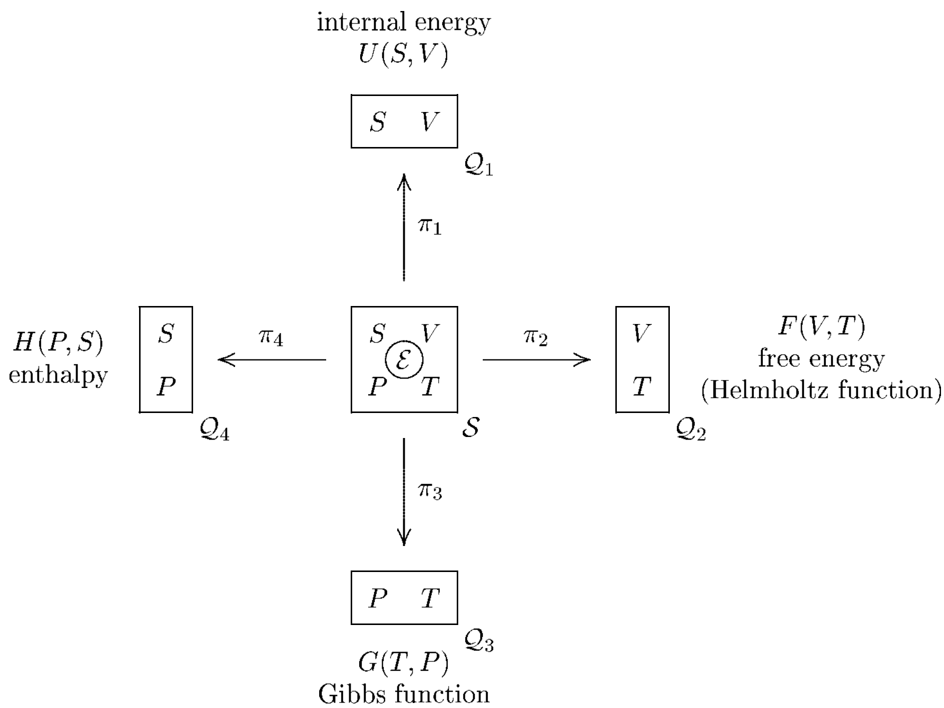
(13) $$R_{21} = (\widehat{I_{21}, E_{21}})$$

i.e., the symplectic relation R_{21} is generated by the function E_{21} on the relation I_{21}.

This theorem recalls, although in a different form, the results of [Tulczyjew, 1977] (for further comments on the Legendre transformation see [Tulczyjew, Urbanski, 1999]). It is a corollary of a general theorem, Theorem 4, §6.6, about the exact Lagrangian submanifolds over constraints. In order to apply this theorem we observe that if $\mathcal{S}$ is connected, then the Lagrangian submanifold R_{21} is connected and maximal, since it is the graph of a symplectomorphism, $\alpha_2 \circ \alpha_1^{-1}$.

5.5 Thermostatic potentials

For a simple and closed thermostatic system we have four fundamental control modes. They correspond to the four possible pairs of the fundamental observables (S, V, P, T) which are in involution i.e., $\mathcal{Q}_1 = (S, V)$, $\mathcal{Q}_2 = (V, T)$, $\mathcal{Q}_3 = (T, P)$, $\mathcal{Q}_4 = (P, S)$ [Tulczyjew, 1977]. The corresponding generating families of the Lagrangian set $\mathcal{E}$ of the equilibrium states are called **thermostatic potentials**.



The corresponding control 1-forms θ_i, for which $d\theta_i = \omega$, are

$$(1) \qquad \begin{cases} \theta_1 = T\,dS - P\,dV \\ \theta_2 = -P\,dV - S\,dT \\ \theta_3 = V\,dP - S\,dT \\ \theta_4 = T\,dS + V\,dP. \end{cases}$$

The generating families may be globally defined. They may be, in particular, Morse families or ordinary generating functions. This depends on the constitutive set $\mathcal{E}$.

Let us consider the Legendre transformation from α_1 to α_2. Since

$$\theta_2 - \theta_1 = -S\,dT - T\,dS = -d(ST),$$

we have, in accordance with the notation of §5.4,

$$(2) \qquad \begin{cases} W_{21}(S,V,P,T) = -ST \\ \mathcal{Q}_1 = (S,V_1), \quad \mathcal{Q}_2 = (V_2,T) \\ \pi_{21}(S,V,P,T) = \big((V,T),(S,V)\big) \\ I_{21} = \{\big((V_2,T),(S,V_1)\big) \mid V_2 = V_1\} \\ E_{21}\big((V_2,T),(S,V_1)\big) = -TS. \end{cases}$$

Then, all requirements of Theorem 1 of §5.4 are fulfilled. Since I_{21} is a submanifold defined by equation $V_2 - V_1 = 0$, it follows that the generating function of R_{21} is the Morse family

$$(3) \qquad L_{21}(V_2,T,S,V_1;\lambda) = -TS + \lambda(V_2 - V_1)$$

with supplementary variable $\lambda \in \mathbb{R}$. Thus, if $G_1 = U(S,V)$ is the generating function of $\mathcal{E}_1 = \alpha_1(\mathcal{E})$, then the generating function G_2 of $\mathcal{E}_2 = \alpha_2(\mathcal{E})$ is the Morse family

$$G_2(V_2,T;S,V_1) = U(S,V_1) + L_{21} = U(S,V_1) - TS + \lambda(V_2 - V_1)$$

with supplementary variables (S,V_1). However, since one of the associated equations is $V_2 - V_1 = 0$, this generating function is reducible to (we do not change the symbol for simplicity)

$$(4) \qquad G_2(V,T;S) = U(S,V) - TS,$$

with supplementary variable S. In conclusion, $\mathcal{E}_2$ is described by the variational equation

$$(5) \qquad -P\,\delta V - S\,\delta T = \delta(U(S,V) - TS)$$

equivalent to equations

$$\text{(6)} \qquad \begin{cases} P = -U_V(S,V) \\ 0 = U_S(S,V) - T. \end{cases}$$

However, if this second equation is solvable with respect to S, so that we can express S as a function $S = S(V,T)$, then we can eliminate the supplementary variable S in G_2 and get the classical Helmholtz function

$$\text{(7)} \qquad \begin{cases} F(V,T) = U\big(S(V,T), V\big) - T\,S(V,T) \\ S = S(V,T) \quad \Longleftrightarrow \quad T = U_S(S,V). \end{cases}$$

For this function we can use the notation

$$\text{(8)} \qquad \boxed{F(V,T) = \mathrm{stat}_S\big(U(S,V) - TS\big)}$$

Remark 1. It can be shown that: (i) The constitutive set $\mathcal{E}$ of an ideal gas, defined by equations (1) and (6) of §5.3, is a regular Lagrangian submanifold with respect to all control modes, so that it admits the four thermostatic potentials as ordinary generating functions. (ii) The constitutive set of a Van der Waals gas, Remark 1, §5.3, is regular with respect to the fibrations π_1 and π_2, so that it admits an internal energy $U(S,V)$ and a free energy $F(V,T)$, but it is singular with respect to the fibrations π_3 and π_4, so that the corresponding thermostatic potentials, the Gibbs function and the enthalpy, are generating families $G(T,P;V)$, $H(P,S;V)$ with supplementary variable V [Fasana, 2003].

5.6 Simple open thermostatic systems

If we act on a thermostatic system also by adding or subtracting particles, then we say that the system is **open**. In this case the energy transfered to the system by the external device in a "quasistatic process" c is given by the integral

$$E_c = \int_c \theta,$$

of the 1-form

$$\text{(1)} \qquad \theta = T\,dS - P\,dV + \mu\,dn.$$

The quantity μ represents the **chemical potential**, and the molar number n is assumed to have continuous values. If we set

$$\text{(2)} \qquad V = nv, \qquad S = ns,$$

then v and s are the **molar volume** and the **molar entropy**, respectively.

The states of the system are represented in the manifold

$$\mathcal{S} = (s, v, P, T, n, \mu) = \mathbb{R}^6,$$

endowed with the symplectic form

$$\begin{aligned} \omega = d\theta &= dT \wedge dS + dV \wedge dP + d\mu \wedge dn \\ &= n(dT \wedge ds + dv \wedge dP) + (d\mu - vdP + sdT) \wedge dn. \end{aligned} \tag{3}$$

We say that the system is **simple** if the set of the equilibrium states $\mathcal{E} \subset \mathcal{S}$ is an exact Lagrangian submanifold: there exists a function $W: \mathcal{E} \to \mathbb{R}$ such that $\theta|\mathcal{E} = dW$.

For an open system we have eight fundamental control modes, corresponding to the triples of the fundamental observables which are in involution (now we include the observables (n, μ)).

$\boxed{1}$ Let us consider the control mode associated with the control manifold

$$\mathcal{Q}_1 = (s, v, n)$$

and the control form $\theta_1 = \theta$. For a simple system we introduce the **molar internal energy** $u(s, v)$, so that the total internal energy is

$$U(S, V, n) = n\, u(s, v), \tag{4}$$

and the constitutive set $\mathcal{E}$ is described by the variational equation

$$T\,\delta S - P\,\delta V + \mu\,\delta n = \delta U, \tag{5}$$

equivalent to the system of equations

$$T = u_s(s, v), \qquad P = -\,u_v(s, v), \qquad \mu = u(s, v) + P\ - T\,s. \tag{6}$$

Note that the observables T and P do not depend on n, in accordance with their character of "intensive observables".

Example 1. For an ideal gas (cf. §5.2)

$$u(s, v) = \frac{K}{\gamma - 1}\, v^{1-\gamma} \exp \frac{s}{c}, \tag{7}$$

and for a Van der Waals gas

$$u(s, v) = \frac{K}{\gamma - 1}\, (v - b)^{1-\gamma} \exp \frac{s}{c} - \frac{a}{v}. \tag{8}$$

2 Let us consider the control mode associated with the control manifold

$$\mathcal{Q}_2 = (v, T, n)$$

and the control form

(9) $$\theta_2 = -P\,dV - S\,dT + \mu\,dn = (\mu - Pv)\,dn - nP\,dv - ns\,dT.$$

The constitutive set $\mathcal{E}$ is then described by the variational equation

(10) $$(\mu - Pv)\,\delta n - nP\,\delta v - S\,\delta T = \delta F$$

where F is the free energy,

(11) $$F(V, T, n) = n\,f(v, T)$$

and $f(v, T)$ is the **molar free energy**. Equation (10) yields equations

(12) $$P = -f_v(v, T), \qquad s = -f_T(v, T), \qquad \mu = f(v, T) + Pv.$$

The two descriptions (6) and (12) of $\mathcal{E}$ are equivalent if and only if the molar free energy is related to the molar internal energy by the Legendre transformation.

In order to perform the Legendre transformation according to Theorem 1, §5.4, we list the following ingredients:

$$\theta_2 - \theta_1 = -P\,dV - S\,dT + \mu\,dn - (T\,dS - P\,dV + \mu\,dn) = -d(ST)$$
$$W_{21}(s, v, P, T, n, \mu) = -ST = -nsT$$
$$\mathcal{Q}_1 = (s, v_1, n_1), \qquad \mathcal{Q}_2 = (v_2, T, n_2)$$
$$\pi_{21}(s, v, P, T, n, \mu) = \big((v, T, n), (s, v, n)\big)$$
$$I_{21} = \{\big((v_2, T, n_2), (s, v_1, n_1)\big) \mid v_2 = v_2,\ n_2 = n_1\}$$
$$E_{21}\big((v_2, T, n_2), (s, v_1, n_1)\big) = -n_1 sT \qquad \text{(or equivalently, } = -n_2 sT).$$

Thus,

$$L_{21}\big((v_2, T, n_2), (s, v_1, n_1)); \lambda_1, \lambda_2\big) = -n_1 sT + \lambda_1(v_2 - v_1) + \lambda_2(n_2 - n_1)$$
$$G_2(v_2, T, n_2; s, v_1, n_1, \lambda_1, \lambda_2) = n_1\,u(s, v_1) - n_1 sT + \lambda_1(v_2 - v_1) + \lambda_2(n_2 - n_1).$$

This last generating family, with supplementary variables $s, v_1, n_1, \lambda_1, \lambda_2$ is reducible to

(13) $$G_2(v, T, n; s) = n\,\big(u(s, v) - sT\big)$$

with the supplementary variable s only. This means that $\mathcal{E}_2$ is described by the variational equation

$$(\mu - Pv)\,\delta n - nP\,\delta v - ns\,\delta T = \delta G_2,$$

and the vanishing of the coefficient of δs yields equation

$$T = u_s(s, v). \tag{14}$$

If this equation is solvable with respect to s, we can remove s from G_2 defined in (13) and get an ordinary generating function of the kind (11), with

$$f(v, T) = \mathrm{stat}_s\big[u(s, v) - sT\big]. \tag{15}$$

This formula, relating the molar free energy to the molar internal energy, is in accordance with (8) of §5.5.

5.7 Composite thermostatic systems

Let us consider a general scheme where: (i) there are two symplectic manifolds, $\mathcal{S}$ and $\bar{\mathcal{S}}$, representing the states of two physical systems; (ii) there are two control modes $\alpha_c: \mathcal{S} \to T^*\mathcal{Q}_c$ and $\bar{\alpha}_c: \bar{\mathcal{S}} \to T^*\bar{\mathcal{Q}}_c$; (iii) there is a control relation $R_c \subseteq \mathcal{Q}_c \times \bar{\mathcal{Q}}_c$, where $\mathcal{Q}_c$ plays the role of control manifold; (iv) there is a Lagrangian submanifold $\bar{\mathcal{E}} \subset \bar{\mathcal{S}}$ representing the equilibrium states of the system $\bar{\mathcal{S}}$; this is assumed to be a Lagrangian submanifold generated by a function $\bar{V}: \bar{\mathcal{Q}}_c \to \mathbb{R}$ with respect to the control mode $\bar{\alpha}_c$, so that $\bar{\alpha}_c(\bar{\mathcal{E}}) = d\bar{V}(\bar{\mathcal{Q}}_c)$.

We consider the set of the equilibrium states (or the constitutive set) $\mathcal{E}_c \subset \mathcal{S}$ of the system $\bar{\mathcal{S}}$ under the control relation R_c. By considering the principle expressed by formula (8) of §5.1 and the notion of control mode described in §5.4, we assume that this set is defined by

$$\boxed{\mathcal{E}_c = \alpha_c^{-1}\big(\widehat{R}_c \circ \bar{\alpha}_c(\bar{\mathcal{E}})\big) \subset \mathcal{S}} \tag{1}$$

This scheme is illustrated by the following diagram,

$$
\begin{array}{ccccc}
\mathcal{E}_c \hookrightarrow \mathcal{S} & & & & \bar{\mathcal{S}} \hookleftarrow \bar{\mathcal{E}} \\
\Big\downarrow \alpha_c & & & & \bar{\alpha}_c \Big\downarrow \\
T^*\mathcal{Q}_c & & \xleftarrow{\ \widehat{R}_c\ } & & T^*\bar{\mathcal{Q}}_c \hookleftarrow d\bar{V}(\bar{\mathcal{Q}}_c) \\
\Big\downarrow & & & & \Big\downarrow \\
\mathcal{Q}_c & & \xleftarrow{\ R_c\ } & & \bar{\mathcal{Q}}_c
\end{array}
\tag{2}
$$

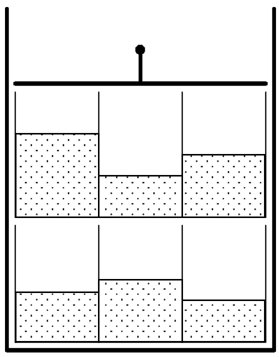
Let us apply this mathematical model to a closed thermostatic system of n moles subdivided into N open subsystems in equilibrium. The state manifold of this **composite system** is

$$\bar{\mathcal{S}} = \times_{i=1}^{N} \mathcal{S}_i = \times_{i=1}^{N} (s_i, v_i, T_i, P_i, n_i, \mu_i). \tag{3}$$

We control this system by acting only on macroscopic observables of the space

$$\mathcal{S} = (S, V, P, T). \tag{4}$$

The subsystems are assumed to be open; this means that transfer of particles between the subsystems is allowed.



$\boxed{1}$ Let us study the control of the <u>volume</u> and the <u>temperature</u>. The control manifolds are

$$\mathcal{Q}_2 = (V, T), \qquad \bar{\mathcal{Q}}_2 = \times_{i=1}^{N} (v_i, T_i, n_i). \tag{5}$$

The control relation $R_2 \subseteq Q_2 \times \bar{Q}_2$ is defined by the fibration $\phi\colon \bar{\mathcal{Q}}_2 \to \mathcal{Q}_2$ described by equations

$$V = \sum_i n_i v_i, \qquad T = T_1, \tag{6}$$

and by the constraint $\Sigma_2 \subset \bar{\mathcal{Q}}_2$ described by equations

$$\sum_i n_i = n, \qquad T_i = T_j. \tag{7}$$

This means that all the subsystems have the same temperature T (we can control the temperature by putting the system in heat bath at the temperature T) and that we do not add or subtract matter to the whole system (the pot containing the whole system is closed and has a controlled volume V).

We assume that the system is **homogeneous**: this means that the constitutive set

$$\bar{\mathcal{E}} \subset \bar{\mathcal{S}}$$

of the complete control of the N subsystems is generated by the function (free energy)

$$F = \sum_i n_i\, f(v_i, T_i), \tag{8}$$

where the function f is the same for all subsystems.

From the virtual work principle stated by formula (1) it follows that

Theorem 1. *The constitutive set $\mathcal{E}_2$ is described by equations*

$$\begin{cases} \sum_i n_i = n \\ S = -\sum_i n_i\, f_T(v_i, T) \\ V = \sum_i n_i v_i \\ -P = f_v(v_1, T) = \ldots = f_v(v_N, T) \\ P v_i + f(v_i, T) = P v_j + f(v_j, T). \end{cases} \tag{9}$$

This means that $(S, V, P, T) \in \mathcal{E}_2$ if and only if these equations are satisfied for some values of (v_i, n_i).

Proof. The generating family of the control relation is

$$G_{R_2} = \lambda_1(T - T_1) + \lambda_2(V - \textstyle\sum_i n_i v_i) + \lambda_3\,(\sum_i n_i - n) + \sum_{i \neq j} \lambda_{ij}\,(T_i - T_j). \tag{10}$$

The control form is

$$\theta_2 = -\,S\,dT - P\,dV.$$

From the principle (1), written for $c = 2$, and from the composition rule of generating families, it follows that $\mathcal{E}_2$ is described by the variational equation

$$\boxed{-S\,\delta T - P\,\delta V = \delta G_{R_2} + \delta F} \tag{11}$$

thus by equation

$$\begin{aligned} -S\,\delta T - P\,\delta V = \delta\big[&\lambda_1(T - T_1) + \lambda_2(V - \textstyle\sum_i n_i v_i) + \lambda_3\,(\sum_i n_i - n) + \\ &\textstyle\sum_{i\neq j} \lambda_{ij}(T_i - T_j)\big] + \delta\big(\sum_i n_i\, f(v_i, T_i)\big). \end{aligned} \tag{12}$$

The vanishing of the coefficients of $\delta\lambda_1$ and $\delta\lambda_{ij}$ of this last equation yields equations $T = T_1$ and $T_i = T_j$, respectively. Note that the temperatures T_i play the role of supplementary variables in the generating family $G_R \oplus F$. Hence, equation (12) is reducible to

$$-S\,\delta T - P\,\delta V = \delta\Big(\lambda_2\,(V - \textstyle\sum_i n_i v_i) + \lambda_3\,(\sum_i n_i - n)\Big) + \delta\big(\sum_i n_i\, f(v_i, T)\big). \tag{13}$$

The coefficients of $(\delta T, \delta V, \delta\lambda_2, \delta\lambda_3, \delta n_i, \delta v_i)$ yield, respectively, the following equations

$$\begin{cases} S = -\sum_i n_i\, f_T(v_i, T) \\ P = -\lambda_2 \\ 0 = V - \sum_i n_i v_i \end{cases} \qquad \begin{cases} 0 = \sum_i n_i - n \\ 0 = -\lambda_2\, v_i + \lambda_3 + f(v_i, T) \\ 0 = \lambda_2\, n_i + n_i\, f_v(v_i, T). \end{cases} \tag{14}$$

By eliminating the Lagrangian multipliers we obtain equations (9). ■

Remark 1. The control relation considered above fits with Case 4 of §5.1. Indeed, the fibration ϕ reduces to a fibration over the constraint Σ. This means that we can replace $\bar{Q}_2$ by $\Sigma_2 = (\times_i (v_i, n_i), T_\circ) \simeq \mathbb{R}_+^{2N} \times \mathbb{R}_+$, being all subsystems at the same temperature $T = T_\circ$. The control relation is now described by equations

$$R_2 \subseteq Q_2 \times \bar{Q}_2 : \quad \begin{cases} V = \sum_i n_i v_i, \quad T = T_\circ & \text{(fibration)} \\ n = \sum_i n_i & \text{(constraint)} \end{cases} \tag{15}$$

and equation (12) is replaced by equation

$$\begin{aligned} -S\,\delta T - P\,\delta V = \delta\Big(&\lambda_1(T - T_\circ) + \lambda_2(V - \textstyle\sum_i n_i v_i) + \lambda_3\,(\sum_i n_i - n)\Big) \\ &+ \delta\big(\textstyle\sum_i n_i\, f(v_i, T_\circ)\big), \end{aligned} \tag{16}$$

which reduces to

$$-S\,\delta T - P\,\delta V = \delta\Big(\lambda_2\,(V - \textstyle\sum_i n_i v_i) + \lambda_3\,(\sum_i n_i - n)\Big) + \delta\big(\sum_i n_i\, f(v_i, T)\big). \tag{17}$$

Then we find again equations (9).

Let us study equations (9) of $\mathcal{E}_2$.

Theorem 2. *Let* $(S, V, P, T) \in \mathcal{E}_2$. *For each pair* (a, b) *of values of the molar volumes* v_i *satisfying equations* (9), *we have* ([Janeczko, 1983])

$$\int_a^b \big(P + f_v(v, T)\big)\, dv = 0. \tag{18}$$

Proof. For constant values of P and T, a primitive of the function $P + f_v(v, T)$ is $Pv + f(v, T)$. Because of the last equation (9), this primitive takes equal values at the end points of the interval of integration. ∎

Theorem 3. *For each equilibrium state described by equations* (9), *the molar volumes* (v_i) *are determined by the points* $(v, y) \in \mathbb{R}^2$ *of the graph of the function* $y = f(v, T)$ *having a common tangent line.*

Proof. For $v_i \neq v_j$, from the last equation (9) it follows that

$$P(v_i - v_j) = f(v_j, T) - f(v_i, T),$$

thus,

$$P = -\frac{f(v_i, T) - f(v_j, T)}{v_i - v_j}.$$

Because of the equations $(9)_4$,

$$\frac{f(v_i, T) - f(v_j, T)}{v_i - v_j} = f_v(v_k, T),$$

for all v_k. ∎

Remark 2. The number of molar volumes (v_i) resulting from this theorem is the number of **phases** which may coexist in an equilibrium state. Note that the last equation (9) and the last equation (12) of §5.6 show that the subsystems have a common value of the chemical potential,

$$\mu_i = \mu_j. \tag{19}$$

Remark 3. The two theorems stated above give an explanation of the so called **Maxwell convention** or **Maxwell rule** of the "equal areas" (see the discussion in [Poincaré, 1892], [Fermi, 1936] and [Huang, 1987]) and of the phenomenon of the **coexistence of phases**. A first "symplectic" approach to this matter can be found in [Janeczko, 1983a,b]. In the present approach, the Maxwell rule is a theorem following from the general variational principle expressed by formula (1).

Remark 4. If for all values of T the function $f(v, T)$ is a convex function of v, then any tangent line to its graph is tangent at a single point. This means that for each equilibrium state described by equations (9) all v_i assume the same value depending on T: $v_i = v(T)$. It follows that $\mathcal{E}_2$ is a Lagrangian submanifold, generated by the function

$$F(V,T) = n\, f(\tfrac{V}{n}, T). \tag{20}$$

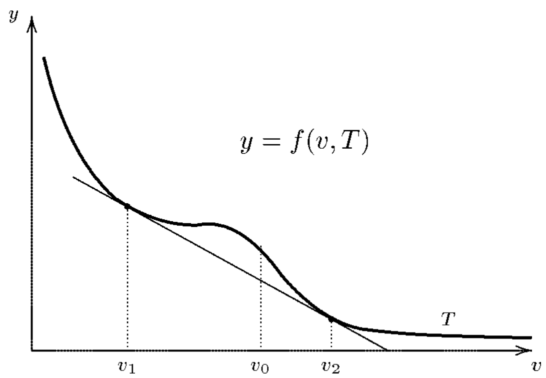
Indeed, in accordance with the expression of the control form θ_2 in (1) of §5.5, equations (9) reduce to equations

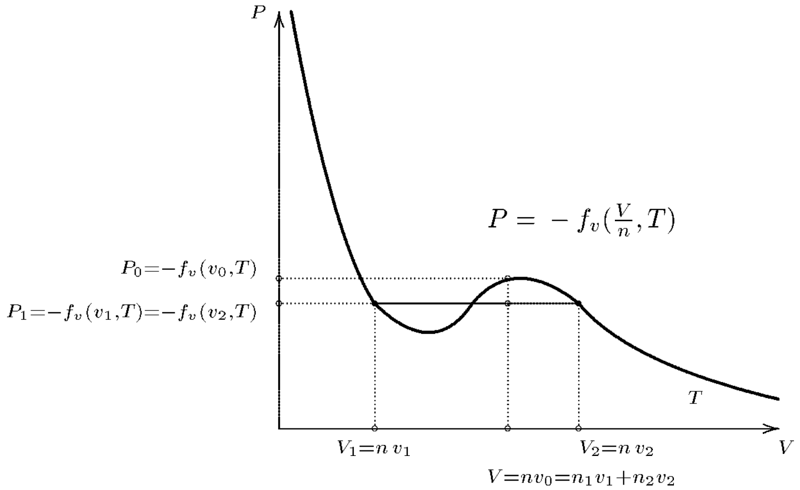
$$P = -F_V = -f_v(\tfrac{V}{n}, T), \qquad S = -F_T = -n\, f_T(\tfrac{V}{n}, T). \tag{21}$$

In this case the thermostatic system (S, V, P, T) behaves as a simple closed system. In all other cases $\mathcal{E}_2$ may not be a submanifold, and we have coexistence of phases i.e., there are states corresponding to different values of the molar volumes v_i. The pictures below illustrate the case in which for a certain value of T the graph of $y = f(v, T)$ admits a tangent to two distinct points $v_1 \neq v_2$. Then the graph of the corresponding isotherm

$$P = -f_v(\tfrac{V}{n}, T) \tag{22}$$

in the (V, P)-plane is of the kind of Van der Waals for $T < T_c$.





If $V_1 = nv_1$ and $V_2 = nv_2$, then

$$(23)\quad \int_{V_1}^{V_2} (P - P_1)\, dV = \int_{V_1}^{V_2} \left(f_v(\tfrac{V}{n}, T) - P_1\right) dV = n \int_{v_1}^{v_2} \left(f_v(v, T) - P_1\right) dv = 0$$

because of formula (18), Theorem 2. This is the Maxwell rule: all points on the horizontal segment defined by $P = P_1 = -f_v(v_1, T) = -f_v(v_2, T)$ correspond to further equilibrium states. The geometrical construction of this rule, illustrated by the above pictures is known as the **Maxwell construction** (cf. [Huang, 1987]).
Remark 5. Assume that for each value of T any tangent to the graph of $y = f(v, T)$ admits at most two tangent points. This is the case of a Van der Waals gas: for $T < T_c$, the critical temperature, we are in the situation considered in Remark 4. Then $\mathcal{E}_2$ is the union of two sets,

$$\mathcal{E}_2 = \mathcal{E}_2^{(1)} \cup \mathcal{E}_2^{(2)}.$$

The first set $\mathcal{E}_2^{(1)}$ corresponds to the case of a single phase: $v_i = v = V/n$. It is the Lagrangian submanifold described by equations (14). The second set $\mathcal{E}_2^{(2)}$ represents the equilibrium states with the coexistence of two phases $v_1 < v_2$ (as in the pictures above). For these equilibrium states the values of the volume V belong to the open interval

$$V_1 < V < V_2, \qquad V_i = nv_i.$$

Indeed, for $V \leq V_1$ or $V \geq V_2$ we have necessarily a single phase and the corresponding states belong to $\mathcal{E}_2^{(1)}$. In accordance with equations (8) the states of $\mathcal{E}_2^{(2)}$ are then described by equations

$$P = -f_v(v_1, T), \qquad S = -n_1 f_T(v_1, T) - n_2 f_T(v_2, T).$$

The first equation shows that P has a unique value determined by T, when v_1 is expressed as a function of T itself. About the second equation we observe that the value of S depends on the mole numbers (n_1, n_2) of the two phases. However, these two numbers are determined by the value of V. Indeed, by solving the linear equations

$$\begin{cases} n_1 v_1 + n_2 v_2 = V \\ n_1 + n_2 = n, \end{cases}$$

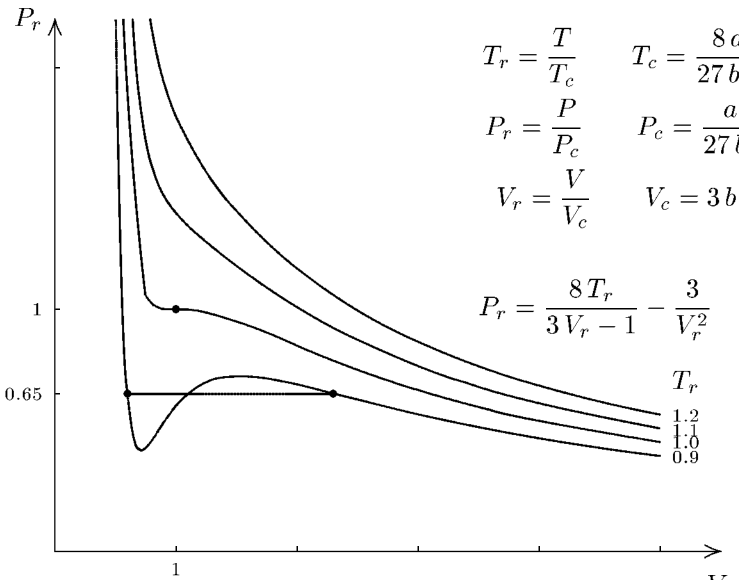
we find

$$n_1 = \frac{nv_2 - V}{v_2 - v_1} = \frac{V_2 - V}{v_2 - v_1}, \qquad n_2 = \frac{V - nv_1}{v_2 - v_1} = \frac{V - V_1}{v_2 - v_1}. \tag{24}$$

Thus,

$$S = \frac{1}{v_1 - v_2}\left[(V_2 - V)\, f_T(v_1, T) + (V - V_1)\, f_T(v_2, T)\right]. \tag{25}$$

For a complete description of the set $\mathcal{E}_2^{(2)}$ it remains to express v_1 and v_2 as functions of T.



Isotherms of Van der Waals in reduced coordinates.

Remark 6. Starting from the expression of the molar internal energy, formula (8) of §5.6, and by performing the suitable Legendre transformation, it can be shown that for a Van der Waals gas the Helmholtz molar function is

$$f(v,T) = c_V\,T \log \frac{eK}{RT} - RT\,\log(v-b) - \frac{a}{v}. \tag{26}$$

By studying the graph of $y = f(v,T)$, it can be shown that all v_i assume the same values for $T \geq T_c$ or two distinct values for $T < T_c$, where T_c is the critical temperature.

$\boxed{2}$ Let us consider the control of the <u>entropy</u> and the <u>volume</u>. The control manifolds are

$$Q_1 = (S,V), \qquad \bar{Q}_1 = \times_{i=1}^{N}(s_i, v_i, n_i). \tag{27}$$

The control relation is defined by equations

$$R_1 \subseteq Q_1 \times \bar{Q}_1 \qquad \begin{cases} S = \sum_i n_i s_i, \;\; V = \sum_i n_i v_i & \text{(fibration)} \\ n = \sum_i n_i & \text{(constraint)}. \end{cases} \tag{28}$$

The control form is

$$\theta_1 = TdS - PdV.$$

The generating function of $\bar{\mathcal{E}}$ is the total internal energy

$$U = \sum_i n_i\, u(s_i, v_i). \tag{29}$$

Theorem 4. *The constitutive set*

$$\mathcal{E}_1 = \alpha_1^{-1}\big(\widehat{R}_1 \circ \bar{\alpha}_1(\bar{\mathcal{E}})\big) \subset \mathcal{S} \tag{30}$$

is described by equations

$$\begin{cases} \sum_i n_i = n \\ S = \sum_i n_i\, s_i \\ V = \sum_i n_i\, v_i \\ -P = u_v(s_1, v_1) = \ldots = u_v(s_N, v_N) \\ T = u_s(s_1, v_1) = \ldots = u_s(s_N, v_N) \\ u(s_i, v_i) + Pv_i - Ts_i = u(s_j, v_j) + Pv_j - Ts_j. \end{cases} \tag{31}$$

This means that $(S, V, P, T) \in \mathcal{E}_1$ if and only if these equations are satisfied for some values of (s_i, v_i, n_i). The proof of this theorem follows the same pattern of that of Theorem 1. Equation (11) is replaced by

$$T\,\delta S - P\,\delta V = \delta(G_{R_2} + U). \tag{32}$$

We can state a theorem similar to Theorem 3.

Theorem 5. *For each equilibrium state described by equations* (31)*, the molar volumes* (v_i) *and the molar entropy are determined by the points* $(s, v, y) \in \mathbb{R}^3$ *of the graph of the function* $y = u(s, v)$ *(called the* **Gibbs surface***) having a common tangent plane.*

Proof. Let us consider two pairs (s_1, v_1) and (s_2, v_2) satisfying equations (31). Due to the last equation,

$$u(s_1, v_1) - u(s_2, v_2) = T\,(s_1 - s_2) - P\,(v_1 - v_2),$$

and by applying equations $(31)_{4,5}$ we get

$$\begin{aligned} u(s_1, v_1) - u(s_2, v_2) &= (s_1 - s_2)\, u_s(s_1, v_1) + (v_1 - v_2)\, u_v(s_1, v_1) \\ &= (s_1 - s_2)\, u_s(s_2, v_2) + (v_1 - v_2)\, u_v(s_2, v_2) \end{aligned}$$

On the other hand, the equation of the tangent plane to the Gibbs surface at a point $P_1 = \big(s_1, v_1, u(s_1, v_1)\big)$ is

$$(s - s_1)\, u_s(s_1, v_1) + (v - v_1)\, u_v(s_1, v_1) = y - u(s_1, v_1).$$

By setting $(s, v) = (s_2, v_2)$ we get equation above. ∎

Remark 6. The comparison of equations (9) and (31) shows that $\mathcal{E}_1 = \mathcal{E}_2$ i.e., that the equilibrium states under the two control relations coincide. Indeed, $(9)_2$ and $(31)_2$ are equivalent because of $(12)_3$ of §5.6,

$$s_i = -\, f_T(v_i, T).$$

The remaining equations are, respectively,

$$\text{(33)} \qquad \begin{cases} -P = f_v(v_i, T) = f_v(v_j, T) \\ Pv_i + f(v_i, T) = Pv_j + f(v_j, T) \end{cases}$$

$$\text{(34)} \qquad \begin{cases} -P = u_v(s_i, v_i) = u_v(u_j, v_j) \\ T = u_s(s_i, v_i) = u_s(s_j, v_j) \\ u(s_i, v_i) + Pv_i - Ts_i = u(s_j, v_j) + Pv_j - Ts_j. \end{cases}$$

The Legendre transformation (15) of §5.6 can be written

$$\text{(35)} \qquad f(v, T) = \big(u(s, v) - sT\big)\Big|_{s=s(v,T)}$$

where the function $s = s(v, T)$ is the inverse of $T = u_s(s, v)$. It follows that for any fixed value of T,

$$f_v(v, T) = \big(u_v(s, v) + u_s(s, v)\, \tfrac{\partial s}{\partial v} - T\, \tfrac{\partial s}{\partial v}\big)\big|_{s=s(v,T)} \big(u_s(s, v)\big)\big|_{s=s(v,T)}.$$

This shows that $(33)_1$ is equivalent to $(34)_1$ and $(34)_2$. Finally, equations $(33)_2$ and $(34)_3$ are equivalent because of (35).

Chapter 6

Supplementary topics

6.1 Symplectic relations generated by a submanifold

A submanifold $\Sigma \subseteq Q$ generates three smooth relations: (i) the **zero-relation**

$$\Sigma \times \{0\} \subset Q \times \{0\}, \qquad (q, 0) \in \Sigma \times \{0\} \quad \Leftrightarrow \quad q \in \Sigma,$$

(ii) the **injection relation**

$$R_\Sigma \subset \sigma \times Q, \qquad (q, q') \in R_\Sigma \quad \Leftrightarrow \quad q = q' \in \Sigma,$$

and (iii) the **diagonal relation**

$$\Delta_\Sigma \subset Q \times Q, \qquad (q, q') \in \Delta_\Sigma \quad \Leftrightarrow \quad q = q' \in \Sigma.$$

Their canonical prolongations define three symplectic relations between cotangent bundles,

$$(1) \qquad \begin{cases} \widehat{\Sigma} \subset T^*Q \times \{0\}, \\ \widehat{R}_\Sigma \subset T^*\Sigma \times T^*Q, \\ \widehat{\Delta}_\Sigma \subset T^*Q \times T^*Q. \end{cases}$$

The first one is the canonical lift of Σ interpreted as a zero-relation. The third one has been examined in §3.5. It is a remarkable fact that [Benenti, 1988]

Theorem 1. *The symplectic relations $\widehat{R}_\Sigma$ and $\widehat{\Delta}_\Sigma$ are, respectively, the reduction relation and the characteristic relation of the coisotropic submanifold*

$$C = T^*_\Sigma Q = \{p \in T^*Q \mid \pi_Q(p) \in \Sigma\}$$

made of the covectors based on points of Σ,

$$R_C = \widehat{R}_\Sigma, \qquad D_C = \widehat{\Delta}_\Sigma,$$

Indeed, the characteristics of C are the equivalence classes of the following equivalence relation:

$$(2) \qquad p \sim p' \iff \pi_Q(p) = \pi_Q(p') = q \in \Sigma, \;\; \langle v, p - p' \rangle = 0, \;\; \forall v \in T_q\Sigma.$$

Remark 1. The canonical lift $\widehat{\Sigma}$ is invariant under the characteristic relation i.e.,

$$(3) \qquad D_C \circ \widehat{\Sigma} = \widehat{\Sigma}.$$

Indeed,

$$\begin{aligned} D_C \circ \widehat{\Sigma} &= \{p \in T^*Q \mid \exists\, p', \; (p,p') \in D_C, \; p' \in T^\circ\Sigma\} \\ &= \{p \in T^*Q \mid \exists\, p', \; p - p' \in T^\circ\Sigma, \; p' \in T^\circ\Sigma\} = T^\circ\Sigma = \widehat{\Sigma}\,. \end{aligned}$$

The same holds for the canonical lift with a function:

$$(4) \qquad D_C \circ \widehat{(\Sigma, F)} = \widehat{(\Sigma, F)}.$$

This fact can be interpreted in the following way: the Lagrangian submanifold $\widehat{(\Sigma, F)}$ is the geometrical solution of the Hamilton-Jacobi $C = T^*_\Sigma Q$ determined by the initial data (Σ, F).

Remark 2. We can consider Σ as the zero-section of $T^*\Sigma$. Then, Σ is a Lagrangian submanifold of $T^*\Sigma$. Hence, its inverse image $R_C^\top \circ \Sigma$ is a Lagrangian submanifold of T^*Q. This Lagrangian submanifold coincides with the canonical lift of Σ,

$$(5) \qquad R_C^\top \circ \Sigma = \widehat{\Sigma}.$$

Indeed, since $R_C = \widehat{R}_\Sigma$ and a covector $p' \in \Sigma \subset T^*\Sigma$ is a zero-covector, we have

$$\begin{aligned} &R_C^\top \circ \Sigma = \\ &= \{p \in T^*Q \mid \exists p' \in \Sigma \text{ s.t. } (p', p) \in \widehat{R}_\Sigma\} \\ &= \{p \in T^*Q \mid \exists p' \in \Sigma \text{ s.t. } \pi_Q(p') = \pi_Q(p) = q, \; \langle v, p - p' \rangle = 0, \; \forall v \in T_q\Sigma\} \\ &= \{p \in T^*Q \mid \langle v, p \rangle = 0 \;\; \forall v \in T_q\Sigma\} = \widehat{\Sigma}. \end{aligned}$$

In a similar way it can be proved that if $F\colon \Sigma \to \mathbb{R}$ and $dF(\Sigma) \subset T^*\Sigma$ is the Lagrangian submanifold generated by F, then

$$(6) \qquad R_C^\top \circ dF(\Sigma) = \widehat{(\Sigma, F)}.$$

Remark 3. The characteristics of $C = T^*_\Sigma \mathcal{Q}$ are vertical submanifolds (i.e., their tangent vectors are vertical) and the rays are the points of Σ.

6.2 The canonical lift of reductions and diffeomorphisms

The definition of canonical lift of a relation, §2.9, can be applied to (the graph of) a mapping, in particular to a diffeomorphism or to a surjective submersion. These two last extensions are special cases of the canonical lift of a reduction. Let $R \subset \mathcal{Q}_2 \times \mathcal{Q}_1$ be a differentiable reduction i.e., the graph of a surjective submersion $\rho: A \to \mathcal{Q}_2$ from a submanifold $A \subseteq \mathcal{Q}_1$ onto $\mathcal{Q}_2$. Let $V(\rho) \subset T\mathcal{Q}_1$ be the subbundle of the vertical vectors i.e., of the vectors tangent to the fibres of ρ,

(1) $$V(\rho) = \{v \in T\mathcal{Q}_1 \mid T\rho(v) = 0$$

and $V^\circ(\rho) \subset T^*\mathcal{Q}_1$ the subbundle of the covectors annihilating the vertical vectors,

(2) $$V^\circ(\rho) = \{p \in T^*\mathcal{Q}_1 \mid \langle v, p \rangle = 0,\ \forall v \in V(\rho) \cap T_q\mathcal{Q}_1,\ q = \pi_{\mathcal{Q}_1}(p)\}.$$

It can be proved that [Benenti, 1983b]:
Theorem 1. *The canonical lift of a reduction* $R \subset \mathcal{Q}_2 \times \mathcal{Q}_1$, $R = \mathrm{graph}(\rho: A \to \mathcal{Q}_2)$ *is a symplectic reduction* $\widehat{R} \subset T^*\mathcal{Q}_2 \times T^*\mathcal{Q}_1$, *whose inverse image* $C = \widehat{R}^\top \circ T^*_A\mathcal{Q}_2 \subset T^*\mathcal{Q}_1$ *is the coisotropic submanifold* $C = V^\circ(\rho)$ *made of the covectors annihilating the vectors tangent to the fibres of* ρ*. The underlying surjective submersion* $\widehat{\rho}: C \to T^*\mathcal{Q}_2$ *is defined by equation*

(3) $$\langle T\rho(v), \widehat{\rho}(p) \rangle = \langle v, p \rangle,$$

where $v \in T_q\mathcal{Q}_1$ *and* $q = \pi_{\mathcal{Q}_1}(p)$, *and*

(4) $$\widehat{\rho}^*\theta_{\mathcal{Q}_2} = \theta_{\mathcal{Q}_1} | T^*_A\mathcal{Q}_1.$$

Theorem 2. *The composition of two reductions* $S \circ R$ *is a reduction and* $\widehat{S \circ R} = \widehat{S} \circ \widehat{R}$.
Remark 1. A diffeomorphism $\rho: \mathcal{Q}_1 \to \mathcal{Q}_2$ is a special case of reduction. By equation (3) we can see that its canonical lift $\widehat{\rho}: T^*\mathcal{Q}_1 \to T^*\mathcal{Q}_2$ is the symplectomorphism defined by

(5) $$\langle v, \widehat{\rho}(p) \rangle = \langle T\rho^{-1}(v), p \rangle.$$

It preserves the Liouville forms,

(6) $$\widehat{\rho}^*\theta_{\mathcal{Q}_2} = \theta_{\mathcal{Q}_1}.$$

Note that the pair $(\rho, \widehat{\rho})$ is a fibre-bundle isomorphism,

$$\pi_{\mathcal{Q}_2} \circ \widehat{\rho} = \rho \circ \pi_{\mathcal{Q}_1}. \tag{7}$$

6.3 Basic observables

There is a one-to-one correspondence between the vector fields X on a manifold $\mathcal{Q}$ and the first-degree homogeneous functions P_X on $T^*\mathcal{Q}$ defined by

$$P_X : T^*\mathcal{Q} \to \mathbb{R} : p \mapsto \langle X, p \rangle = X^i(\underline{q})\, p_i.$$

Moreover, any function $f: \mathcal{Q} \to \mathbb{R}$ can be interpreted as a function $f: T^*\mathcal{Q} \to \mathbb{R}$ constant on the fibres (we use the same symbol for simplicity). We call these functions the **basic observables**. On the basic observables we define an internal operation $\{\cdot, \cdot\}$ by setting

$$\begin{cases} \{f, g\} = 0 \\ \{P_X, f\} = Xf \\ \{P_X, P_Y\} = P_{[X,Y]}. \end{cases} \tag{1}$$

We observe that these rules are fulfilled by the PB associated with the canonical symplectic form on $T^*\mathcal{Q}$ (§2.2). Conversely, assuming the rules (1) as fundamental, we can extend the operation $\{\cdot, \cdot\}$ in a unique way to a PB on functions over $T^*\mathcal{Q}$. The resulting PB coincides with the canonical one. Hence, equations (1) characterize the canonical Poisson bracket on a cotangent bundle, and provide a direct definition which avoids the use of the canonical symplectic form.

6.4 Canonical lift of vector fields

Let X be a vector field on $\mathcal{Q}$. We denote by $\widehat{X}$ the Hamiltonian vector field on $T^*\mathcal{Q}$ generated by the function P_X,

$$i_{\widehat{X}} d\theta_{\mathcal{Q}} = -\, dP_X. \tag{1}$$

We call $\widehat{X}$ the **canonical lift** of X. If X^i are the components of X in a coordinate system (q^i) then the components $\widehat{X}^i = \langle \widehat{X}, dq^i \rangle$ and $\widehat{X}_i = \langle \widehat{X}, dp_i \rangle$ in the canonical coordinates (q^i, p_i) are

$$\widehat{X}^i = X^i, \qquad \widehat{X}_i = -\, \frac{\partial X^j}{\partial q^i}\, p_j. \tag{2}$$

The canonical lift of vector fields has the following properties:
(i) The vector field $\widehat{X}$ is projectable onto X i.e.,

$$T\pi_Q \circ \widehat{X} = X \circ \pi_Q, \tag{3}$$

(ii) The restriction of $\widehat{X}$ to the zero-section of T^*Q coincides with X.
(iii) The following equations hold

$$i_{\widehat{X}}\theta_Q = P_X, \qquad d_{\widehat{X}}\theta_Q = 0. \tag{4}$$

(iv) The mapping $X \mapsto \widehat{X}$ is a Lie-algebra homomorphism,

$$(aX + bY)\widehat{\ } = a\widehat{X} + b\widehat{Y} \quad (a, b \in \mathbb{R}), \qquad [\widehat{X}, \widehat{Y}] = [X, Y]\widehat{\ }. \tag{5}$$

(v) For each smooth function F on T^*Q,

$$d_{\widehat{X}}F = \{P_X, F\}. \tag{6}$$

The following theorem shows that the above definition of canonical lift of vector fields is strictly related to the basic definition of the canonical lift of relations.
Theorem 1. *If X is a complete vector field with one-parameter group $\varphi_t^X : Q \to Q$, $t \in \mathbb{R}$, then its canonical lift $\widehat{X}$ is complete and its one-parameter group*

$$\varphi_t^{\widehat{X}} : T^*Q \to T^*Q$$

is the canonical lift of φ_t^X,

$$\varphi_t^{\widehat{X}} = (\widehat{\varphi_t^X}). \tag{7}$$

Proof. Let us put $\varphi_t^X = \varphi_t$ for simplicity. Due to the definition of canonical lift of a diffeomorphism, formula (5) of §6.2, we can write

$$\langle v, \widehat{\varphi}_t(p_0)\rangle = \langle T\varphi_t^{-1}(v), p_0\rangle \tag{8}$$

for all $p_0 \in T^*_{q_0}Q$ and $v \in T_qQ$, with $q = \varphi_t(q_0)$. Due to the functorial properties of T, $\widehat{\varphi}_t$ is a one-parameter group of transformations on T^*Q. Let $\widehat{X}$ be the corresponding (complete) vector field, and let (X^i, X_i) be its components in standard canonical coordinates $(\underline{q}, \underline{p}) = (q^i, p_i)$. Let us consider a local coordinate representation of $\widehat{\varphi}_t$,

$$\begin{cases} q^i = \varphi^i(t, \underline{q}_0), \\ p_i = \varphi_i(t, \underline{q}_0, \underline{p}_0). \end{cases}$$

Then (see appendix A.6),

$$\text{(9)} \qquad \begin{cases} X^i(\underline{q}_0, \underline{p}_0) = \dot{\varphi}^i(0, \underline{q}_0), \\ X_i(\underline{q}_0, \underline{p}_0) = \dot{\varphi}_i(0, \underline{q}_0, \underline{p}_0). \end{cases}$$

On the other hand, definition (8) is equivalent to equation

$$v^i \varphi_i(t, \underline{q}_0, \underline{p}_0) = p_{0i} \, \frac{\partial \varphi^i(-t, \underline{q}_0)}{\partial q_0^j} \, v^j,$$

for all (v^i), thus to equation

$$\text{(10)} \qquad \varphi_i(t, \underline{q}_0, \underline{p}_0) = p_{0j} \, \frac{\partial \varphi^j(-t, \underline{q}_0)}{\partial q_0^i}.$$

From (9) and (10) it follows that

$$X_i(\underline{q}_0, \underline{p}_0) = p_{0j} \left. \frac{\partial \dot{\varphi}^j(-t, \underline{q}_0)}{\partial q_0^i} \right|_{t=0} = -\, p_{0j} \, \frac{\partial X^j}{\partial q_0^i} = -\, \frac{\partial P_X}{\partial q_0^i}.$$

Due to equations (2), this is sufficient to prove that the vector field (X^i, X_i) is the canonical lift of $X = (X^i)$. ■

6.5 Regular distributions and Frobenius theorem

A **regular distribution** on a manifold Q is a subbundle Δ of the tangent bundle TQ i.e, a submanifold of TQ such that for each point $q \in Q$,

$$\Delta_q = \Delta \cap T_q Q$$

is a subspace of constant dimension r, called the **rank** of the distribution. Hence, a distribution is a mapping which assigns at each point $q \in Q$ a subspace $\Delta_q \subset T_q Q$ of constant dimension r, in such a way that the union Δ of all Δ_q is a submanifold.

A vector field X on Q is said to be **compatible** with the distribution Δ if its image is contained in Δ i.e.,

$$\text{(1)} \qquad X(q) \in \Delta_q, \qquad \forall q \in Q.$$

A one-form θ on Q is a **characteristic form** of Δ if it annihilates all vectors of Δ,

$$\text{(2)} \qquad \langle \theta, v \rangle = 0, \qquad \forall v \in \Delta.$$

A regular distribution Δ of rank r can be locally described in three equivalent ways:

(i) By **equations** i.e., by $m = n - r$ independent homogenous linear equations

$$\theta_i^a(\underline{q})\,\dot{q}^i = 0, \qquad a = r+1, \dots, n. \tag{3}$$

(ii) By a **basis of characteristic forms** i.e., by $n - r$ pointwise independent one-forms θ^a annihilating the vectors of Δ $(a = r+1, \dots, n)$. If equations (3) hold, then such a basis is given by the one-forms

$$\theta^a = \theta_i^a\, dq^i. \tag{4}$$

(iii) By a **basis of generators** i.e., by r pointwise independent vector fields X_α $(\alpha = 1, \dots, r)$ spanning at each point where they are defined the subspaces Δ_q. We have

$$\langle \theta^a, X_\alpha \rangle = 0. \tag{5}$$

The set of all vector fields compatible with Δ form a subspace $\mathcal{X}_\Delta$ of the space $\mathcal{X}(Q)$. We say that the distribution is **involutive** if $\mathcal{X}_\Delta$ is a Lie subalgebra i.e., if it is closed in the Lie bracket,

$$X, Y \in \mathcal{X}_\Delta \quad \Longrightarrow \quad [X, Y] \in \mathcal{X}_\Delta.$$

It follows that if (X_α) is a basis of generators, then the distribution Δ is involutive if and only if

$$[X_\alpha, X_\beta] = F_{\alpha\beta}^\gamma\, X_\gamma, \tag{6}$$

where $F_{\alpha\beta}^\gamma$ are functions on the domain of definition of the local generators.

If (θ^a) is a basis of local characteristic forms, then the distribution Δ is involutive if and only if

$$d\theta^a \wedge \theta^{r+1} \wedge \ldots \wedge \theta^n = 0. \tag{7}$$

Let us prove the equivalence of conditions (6) and (7) for a distribution of rank $r = n - 1$ (the proof for the general case is similar). In this case we have a single characteristic form θ and condition (7) becomes

$$d\theta \wedge \theta = 0. \tag{8}$$

If (X, Y, Z) are three vector fields, then the following identity can proved, by using the fundamental properties of the derivations i_X and d_X (§A.15):

$$i_X i_Y i_Z (d\theta \wedge \theta) = i_X\theta(d_Y i_Z\theta - d_Z i_Y\theta - i_{[Y,Z]}\theta + \text{c.p.}$$

(c.p. = cyclic permutations of the vector fields). For each $X, Y, \in \mathcal{X}_\Delta$ and $Z \notin \mathcal{X}_\Delta$ we get

$$i_X i_Y i_Z (d\theta \wedge \theta) = -\, i_Z \theta i_{[X,Y]} \theta.$$

This shows that $d\theta \wedge \theta = 0$ if and only if $i_{[X,Y]}\theta = 0$ i.e., $[X, Y] \in \mathcal{X}_\Delta$.

An **integral manifold** of a regular distribution Δ is a submanifold $S \subseteq \mathcal{Q}$ such that $T_q S = \Delta_q$. A **maximal integral manifold** is a connected integral manifold which is not properly contained in another connected integral manifold. A maximal integral manifold may be an immersed submanifold. A distribution is said to be **completely integrable** if for all $q \in \mathcal{Q}$ there exists an integral manifold containing q.

Theorem 1. (i) *A regular distribution is completely integrable if and only if it is involutive.* (ii) *If a regular distribution is completely integrable, then for each point $q \in \mathcal{Q}$ there is a unique maximal integral manifold containing q.*

This is known as **Frobenius' theorem**.

An **integral function** of a distribution Δ is a smooth function $F\colon \mathcal{Q} \to \mathbb{R}$ such that

$$\langle v, dF \rangle = 0, \quad \forall\, v \in \Delta. \tag{9}$$

Locally, the integral functions are the solutions of the linear homogeneous partial differential equations

$$\langle X_\alpha, dF \rangle = X^i_\alpha\, \partial_i F = 0. \tag{10}$$

Theorem 2. *A regular distribution is completely integrable if and only if in a neighborhood of any point $q \in \mathcal{Q}$ there exists a* **basis of $n - r$ integral functions** *(u^a).*

This means that the differentials (du^a) are pointwise independent and any other integral function is functionally dependent on (u^a). Note that a completely integrable distribution may have no global integral function.

Proof. If Δ is completely integrable, then the foliation of its integral manifolds can be locally parametrized by coordinates $(u^i) = (u^\alpha, u^a)$ such that the differentials du^a form a basis of local characteristic forms or, in other words, such that the integral manifolds are locally described by equations $u^a =$ constant. As a consequence, the derivations $\partial/\partial u^\alpha$ form a basis of local generators. Coordinates of this kind are said to be **adapted** to the distribution. In adapted coordinates equations (10) read

$$\frac{\partial F}{\partial u^\alpha} = 0. \tag{11}$$

The most general solution of these equation is a function depending only on the coordinates (u^a). Conversely, if (u^a) is a basis of integral functions, then locally we can find other functions (u^α) in such a way that (u^a, u^α) is a coordinate system. Since $\partial u^a/\partial u^\alpha = 0$, the r derivations $(X_\alpha = \partial/\partial u^\alpha)$ form a basis of generators which are tangent to the r-dimensional submanifolds $u^a = \text{const}$. ∎

There is a remarkable symplectic interpretation of all these concepts, which leads to a simple proof of Frobenius' theorem.

Let $\Delta^\circ \subset T^*\mathcal{Q}$ be the subbundle of the covectors annihilating the vectors of Δ,

$$\Delta^\circ = \{p \in T^*\mathcal{Q} \mid \langle v, p\rangle = 0, \ \ \forall\, v \in \Delta_q, \ q = \pi_{\mathcal{Q}}(p)\}. \tag{12}$$

Lemma 1. *The distribution Δ is involutive if and only if Δ° is a coisotropic submanifold.*

Proof. It follows from the definition (12) that a vector field X is compatible with Δ if and only if

$$P_X | \Delta^\circ = 0. \tag{13}$$

Hence, Δ° is described by equations $P_X = 0$ for X variable in the space $\mathcal{X}_\Delta$. It follows that Δ° is coisotropic if and only if

$$\{P_X, P_Y\} | \Delta^\circ = 0 \tag{14}$$

for all $X, Y \in \mathcal{X}_\Delta$. Due to (14), this equation is equivalent to

$$P_{[X,Y]} | \Delta^\circ = 0.$$

This shows that (14) holds if and only if $[X, Y] \in \mathcal{X}_\Delta$. ∎

Lemma 2. *If Δ° is coisotropic then the canonical lift $\widehat{X}$ of a vector field $X \in \mathcal{X}_\Delta$ is a characteristic vector field of Δ°.*

Proof. The Hamiltonian of $\widehat{X}$ is P_X. If $X \in \mathcal{X}_\Delta$ then (13) holds. This shows that the Hamiltonian of $\widehat{X}$ is constant on Δ°. Hence, $\widehat{X}$ is characteristic (Theorem 1, §1.4). ∎

Lemma 3. *If Δ° is coisotropic then* (i) *the corresponding rays are r-dimensional submanifolds of $\mathcal{Q}$ tangent to Δ i.e., integral manifolds of Δ and* (ii) *they coincide with the characteristics lying on the zero-section of $T^*\mathcal{Q}$.*

Proof. Let (X_α) be a local basis of generators of Δ. Due to Lemma 2, the canonical lifts $\widehat{X}_\alpha$ are pointwise independent and span the characteristic distribution of Δ°. Since they project onto the r independent vectors X_α, the characteristics projects onto r-dimensional submanifolds tangent to these vectors. Hence, the rays are the integral manifolds of Δ. This proves item (i). Item (ii) follows from the fact that on the zero-section, identified with $\mathcal{Q}$, we have $\widehat{X}_\alpha | \mathcal{Q} = X_\alpha$. ∎

Proof of Frobenius' theorem. If we assume that Δ is involutive, then Δ° is coisotropic (Lemma 1) and the corresponding rays are r-dimensional integral manifolds of Δ (Lemma 3). Thus, Δ is completely integrable. Conversely, if Δ is completely integrable, then any basis of generators is tangent to the integral manifolds, so that also their Lie brackets are tangent, and (6) holds. This proves item (i) of Theorem 1. Item (ii) follows from item (ii) of Lemma 3, since two distinct maximal characteristics of a coisotropic submanifold have empty intersection. ∎

Remark 1. An involutive (completely integrable) distribution provides an example of Hamilton-Jacobi equation i.e., of a coisotropic submanifold $C = \Delta^\circ$ of codimension $r \geq 1$. The complete solution $W(\underline{q}, \underline{u})$ is a solution of a system of r independent linear homogeneous equations,

$$X^i_\alpha \partial_i W = 0, \tag{15}$$

depending on $n - r$ parameters $\underline{u} = (u^a)$, which is in fact a parametrized family of integral functions of the distribution. Assume that the set U of the maximal integral manifolds of Δ has a differentiable structure such that the canonical projection $\pi: Q \to U$ is a submersion. Then it can be proved that *the reduced symplectic manifold $\mathcal{S} = T^*Q/\Delta^\circ$ is symplectomorphic to the cotangent bundle T^*U and the symplectic reduction R_{Δ° is isomorphic to the canonical lift of the graph of π.*

6.6 Exact Lagrangian submanifolds

Let $\Lambda \subset T^*Q$ be a Lagrangian submanifold. Since Λ is isotropic, the pull-back of the canonical symplectic form $d\theta_Q$ to Λ is the zero two-form: $(d\theta_Q)|\Lambda = 0$. Since the differential operator commutes with the pull-back, this means that the pull-back of the Liouville form θ_Q to Λ is a closed one-form,

$$d(\theta_Q|\Lambda) = 0.$$

Hence, for each $p \in \Lambda$ there is an open neighborhood $U_p \subseteq \Lambda$ and a function $W_p: U_p \to \mathbb{R}$ such that $\theta_Q|U_p = dW_p$. We call these functions the **local potentials** of Λ. We say that a Lagrangian submanifold $\Lambda \subset T^*Q$ is **exact** if it admits a **global potential** i.e., if there exists a function $W: \Lambda \to \mathbb{R}$ such that

$$\theta_Q|\Lambda = dW.$$

If $\iota: \Lambda \to T^*Q$ is the immersion of Λ, then this equation can be written

$$\iota^*\theta_Q = dW.$$

Let $\pi: \Lambda \to Q$ be the restriction of the cotangent fibration $\pi_Q: T^*Q \to Q$ to Λ. Then,

$$\pi = \pi_Q \circ \iota.$$

We observe that π is a differentiable function, since it is the composition of two differentiable functions, and that $T\pi_Q(v) = T\pi(v)$ for all $v \in T\Lambda$.

Theorem 1. *If Λ is generated by a function $G: Q \to \mathbb{R}$, then it is exact and $W = G \circ \pi = \pi^* G$ is a global potential.*

Proof. If $v \in T_p\Lambda$, then $\langle v, \theta_Q \rangle = \langle T\pi_Q(v), p \rangle = \langle T\pi(v), dG \rangle = \langle v, \pi^* dG \rangle = \langle v, dW \rangle$. ∎

Note that this theorem follows directly from formula (7) of §2.8, with $\Sigma = Q$: $\theta_Q|\Lambda = d\pi^* G$. Conversely,

Theorem 2. *Let $\Lambda \subset T^*Q$ be an exact Lagrangian submanifold, with global potential W, such that:* (i) *$\pi: \Lambda \to Q$ is a surjective submersion;* (ii) *there exists a function $G: Q \to \mathbb{R}$ such that $W = \pi^* G = G \circ \pi$. Then Λ is the Lagrangian submanifold generated by the function G, $\Lambda = dG(Q)$.*

Proof. Let $p \in \Lambda$, $q = \pi(p)$, $v \in T_p\Lambda$ and $u = T\pi(v)$. Then,

$$\langle v, \theta_Q \rangle = \langle T\pi(v), p \rangle = \langle u, p \rangle$$

and

$$\langle v, dW \rangle = \langle v, d\pi^* G \rangle = \langle v, \pi^* dG \rangle = \langle T\pi(v), dG \rangle = \langle u, dG \rangle.$$

Since $\langle v, \theta_Q \rangle = \langle v, dW \rangle$ for all $v \in T\Lambda$, it follows that

$$\langle u, p \rangle = \langle u, dG \rangle \tag{1}$$

for all $u \in T\pi(T_p\Lambda)$. Since π is a submersion, $T\pi(T_p\Lambda) = T_qQ$. Thus, $p = d_qG$. ∎

Remark 1. For the sake of simplicity we consider only C^∞ Lagrangian submanifold, so that a global potential is a C^∞ function. However, there are cases in which this theorem holds with a generating function G which is not C^∞. An example is the Lagrangian submanifold $q = p^3$ of $T^*\mathbb{R}$ (Example 1, §2.4). Its parametric equations are $p = \lambda$, $q = \lambda^3$. The global potential is $W(\lambda) = \frac{3}{4}\lambda^4$, and the projection π is represented by equation $q = \lambda^3$. It is a one-to-one mapping, but it is not a diffeomorphism. The generating function is $G(q) = \frac{3}{4}q^{\frac{4}{3}}$, and this function does not admit the second derivative for $q = 0$.

Remark 2. Assumption (i) in Theorem 2 does not imply (ii). An example is the curve $\Lambda \subset T^*\mathbb{S}_1 \sim \mathbb{S}_1 \times \mathbb{R}$ defined by parametric equations $\mathbf{u} = (\cos\lambda, \sin\lambda) \in \mathbb{S}_1$ and $\lambda \in \mathbb{R}$.

Remark 3. Let us replace assumptions (i) and (ii) by: *π is a diffeomorphism.* Then the function $G = (\pi^{-1})^* W = W \circ \pi^{-1}$ is a C^∞ generating function of Λ. Indeed, $T_p\pi: T_p\Lambda \to T_qQ$ is an isomorphism for each $p \in \lambda$ and moreover,

$$\langle u, dG \rangle = \langle u, (\pi^{-1})^* dW \rangle = \langle T\pi^{-1}(u), dW \rangle = \langle v, dW \rangle = \langle v, \theta_Q \rangle = \langle u, p \rangle.$$

Remark 4. We can replace assumption (i) by: *π is surjective and Λ is connected.* In this case condition (1) still holds for all $u \in T\pi(T_p\Lambda)$. This shows that if p is a regular point i.e., $T\pi(T_p\Lambda) = T_q\mathcal{Q}$, then $p = d_qG$. This means, in particular, that Λ cannot have two distinct regular points p over a same point $q \in \mathcal{Q}$ (i.e., on a same fibre of $T^*\mathcal{Q}$). Let us consider the caustic $\Gamma \subset \mathcal{Q}$ of Λ and a point $q \in \Gamma$. (i) Assume that q is an isolated point of Γ: there exists an open and connected neighborhood N_q of q not containing caustic points except q. Then Λ is generated by G over $N_q - \{q\}$. Since Λ is connected, the exceptional point q is included by continuity. (ii) Assume that in any open neighborhood N of q there are points which do not belong to the caustic. Then, in these points we have that Λ is the image of dG and, by an argument similar to that of (i), we conclude that it is the image of dG in a neighborhood of q. (iii) The case in which there exists an open neighborhood N of q all contained in Γ is not possible. Indeed, a caustic cannot contain open subsets.[1] To see this, let us consider a Morse family $F(q^i, u^\alpha)$, at least of class C^2, generating Λ in the neighborhood of a singular point. The caustic is the projection into $\mathcal{Q}$ of the intersection of the singular set Ξ, described by equations $\partial_\alpha F = 0$, with the set described by equation $\det\left[\partial_\alpha\partial_\beta F\right] = 0$. Hence, it is contained in the projection of Ξ. Since F is a Morse family, Ξ is (locally) a submanifold of dimension equal to the dimension of $\mathcal{Q}$. It projects locally onto open subsets of $\mathcal{Q}$ if and only if it is locally a section of the trivial fibration $\mathcal{Q} \times U \to \mathcal{Q}$. But this is the case in which it is completely reducible to an ordinary generating function of class C^2, and this is against our assumption that it generates a neighborhood of Λ containing a singular point. Note that in this last part of this proof we need the existence of a Morse family of class C^2. This is certainly satisfied if Λ is of class C^2.

Let us consider the case of a Lagrangian submanifold over a submanifold $\Sigma \subset \mathcal{Q}$.

Theorem 3. *The Lagrangian submanifold* $\Lambda = (\widehat{\Sigma, G})$ *generated by a function* $G: \Sigma \to \mathbb{R}$ *on a submanifold* $\Sigma \subset \mathcal{Q}$ *is exact with global potential* $W = \pi^*G$*, where* $\pi: \Lambda \to \Sigma$ *is the restriction of* $\pi_\mathcal{Q}$ *to* Λ.

Proof. As we have seen in §2.8, $\theta_\mathcal{Q}|\Lambda = d\pi^*G$. ∎

Conversely,

Theorem 4. *Let* $\Lambda \subset T^*\mathcal{Q}$ *be an exact Lagrangian submanifold, with global potential* W*, which projects onto a submanifold* $\Sigma = \pi_\mathcal{Q}(\Lambda) \subseteq \mathcal{Q}$*. Assume that:* (i) *the restriction* $\pi: \Lambda \to \Sigma$ *of* $\pi_\mathcal{Q}$ *to* Λ *is a submersion,* (ii) *there exists a function* $G: \Sigma \to \mathbb{R}$ *such that* $\pi^*G = W$*,* (iii) Λ *is connected and maximal i.e., it is not properly contained in a larger Lagrangian submanifold satisfying properties* (i) *and* (ii)*. Then* Λ *is generated by* G *on the constraint* Σ*:* $\Lambda = (\widehat{\Sigma, G})$.

Proof. Since Λ projects onto Σ, it is made of covectors based on points of the submanifold Σ. Hence, it is contained in the coisotropic submanifold $C = T^*_\Sigma\mathcal{Q}$. By the absorption principle it follows that it is made of characteristics of C. Because

[1] A caustic is a closed subset [Abraham, Robbins, 1967] (for further comments and references on the Lagrangian singularities see e.g. [Marmo, Morandi, Mukunda, 1990]).

of (i) it is the union of maximal characteristics. Then its image by the reduction R_C, $\Lambda_0 = R_C \circ \Lambda \subset T^*\Sigma$, is a Lagrangian submanifold. Let $\rho: C \to T^*\Sigma$ be the surjective submersion underlying R_C. Since R_C is a canonical lift, then $\rho^*\theta_\Sigma = \theta_Q|C$ and for each $v \in T\Lambda$ we have

$$\langle T\rho(v), \theta_\Sigma \rangle = \langle v, \rho^*\theta_\Sigma \rangle = \langle v, \theta_Q|C \rangle = \langle v, dW \rangle. \tag{2}$$

If v is tangent to a characteristic, then $T\rho(v) = 0$ and $\langle v, dW \rangle = 0$. This shows that the function W is constant on the characteristics (contained in Λ) so that it reduces to a function W_0 on Λ_0. Let us consider the restriction of ρ to Λ, $\rho|\Lambda: \Lambda \to \Lambda_0$. It is a surjective submersion such that

$$W = (\rho|\Lambda)^* W_0. \tag{3}$$

It follows that

$$\langle v, dW \rangle = \langle v, (\rho|\Lambda)^* dW_0 \rangle = \langle T(\rho|\Lambda)(v), dW_0 \rangle.$$

Thus, because of (2), for each vector $u \in T\Lambda_0$ we have $\langle u, \theta_\Sigma \rangle = \langle u, dW_0 \rangle$. This shows that $dW_0 = \theta_\Sigma|\Lambda_0$ i.e., that Λ_0 is exact with potential function W_0. The projection $\pi = \pi_Q|\Lambda: \Lambda \to \Sigma$ is the composition of $\rho|\Lambda$ with $\pi_\Sigma|\Lambda_0: \Lambda_0 \to \Sigma$,

$$\pi = \pi_\Sigma|\Lambda_0 \circ \rho|\Lambda. \tag{4}$$

Since π is a surjective submersion (by assumption) as well $\rho|\Lambda$, $\pi_\Sigma|\lambda_0$ is also a surjective submersion. From $W = \pi^* G$ and (3), (4) it follows that

$$(\rho|\Lambda)^* W_0 = W = \pi^* G = (\rho|\Lambda)^* (\pi_\Sigma|\Lambda_0)^* G.$$

This shows that $W_0 = (\pi_\Sigma|\Lambda_0)^* G$. Hence, to the Lagrangian submanifold $\Lambda_0 \subset \Sigma$ we can apply Theorem 2, so that $\Lambda_0 = dG(\Sigma)$. Due to (6) of §6.1, $\Lambda = R_C^\top \circ \Lambda_0 = R_C^\top \circ dG(\Sigma) = \widehat{(\Sigma, G)}$. ∎

Remark 5. If in Theorem 4 the last assumption (iii) is not fulfilled, then we can conclude only that Λ is an open subset of $\widehat{(\Sigma, G)}$.

6.7 Dual pairings

Let A and B be (real, finite-dimensional) vector spaces. A **dual pairing** between A and B is a bi-linear mapping

$$\langle \, | \, \rangle: A \times B \to \mathbb{R}: (a, b) \mapsto \langle a | b \rangle,$$

satisfying the following regularity conditions

$$\begin{cases} \langle a|b\rangle = 0,\ \forall\, a \in A & \Rightarrow \quad b = 0, \\ \langle a|b\rangle = 0,\ \forall\, b \in B & \Rightarrow \quad a = 0. \end{cases}$$

With each subspace (or subset) $K \subseteq A$ we associate a subspace $K^\P \subseteq B$, which we call the **polar** of K in the dual pairing $\langle\,|\,\rangle$, defined by

$$K^\P = \{b \in B \,|\, \langle a|b\rangle = 0,\ \forall\, a \in K\}. \tag{1}$$

By the same symbol $H^\P$ we denote the polar of a subspace $H \subseteq B$.

A first example of dual pairing is the evaluation $\langle\,,\,\rangle$ between vectors of a space A and the covectors of the dual space $B = A^*$. We denote by $K^\circ \subset A^*$ the polar of $K \subset A$ in this canonical dual pairing (§1.2),

$$K^\circ = \{b \in A^* \,|\, \langle a, b\rangle = 0,\ \forall\, a \in K\}. \tag{2}$$

All dual pairings are *isomorphic* to this one, as shown by the following

Theorem 1. *Let $\langle\,|\,\rangle: A\times B \to \mathbb{R}$ be a dual pairing. The linear mapping $\psi: B \to A^*$ defined by*

$$\langle a, \psi(b)\rangle = \langle a, b\rangle \tag{3}$$

is an isomorphism, and for each subspace $K \subseteq A$,

$$\psi(K^\P) = K^\circ. \tag{4}$$

Proof. Assume that $\psi(b) = 0$. From (3) it follows that $\langle a|b\rangle = 0$, $\forall a \in A$, and this implies $b = 0$, because of the regularity condition. Hence, the kernel of ψ is the zero vector only, and the mapping is injective. It follows in particular that $\dim B \leq \dim A^* = \dim A$. We can define in a similar way a linear mapping $\psi': A \to B^*$, and by the regularity condition (which operates on both sides of the dual pairing) we conclude that it is injective, thus $\dim A \leq \dim B^* = \dim B$. It follows that $\dim A = \dim B$, and ψ is an isomorphism. Formula (4) is a direct consequence of (2) and (3). ∎

From this theorem and its proof it follows that,

Theorem 2. *In a dual pairing $\langle\,|\,\rangle: A \times B \to \mathbb{R}$ the spaces A and B have the same dimension.*

Due to Theorem 1 and formula (4), the polar operator ¶ has formal properties similar to those of $^\circ$ (here 0_A and 0_B denotes the zero vectors of A and B, respectively),

$$
\begin{aligned}
A^{\P} = 0_B, \quad B^{\P} = 0_A, \quad 0_B^{\P} = A, \quad 0_A^{\P} = B,\\
\dim K + \dim K^{\P} = \dim A = \dim B,\\
K^{\P} \subseteq L^{\P} \quad \Leftrightarrow \quad L \subseteq K,\\
(K + L)^{\P} = K^{\P} \cap L^{\P},\\
K^{\P} + L^{\P} = (K \cap L)^{\P},\\
K^{\P\P} = K.
\end{aligned} \tag{5}
$$

A second remarkable example of dual pairing is

$$\langle \, | \, \rangle \colon A \times A \to \mathbb{R} \colon (a, a') \mapsto \alpha(a', a), \tag{6}$$

where (A, α) is a symplectic vector space. We have denoted by $K^{\S}$ the polar of $K \subseteq A$ in this dual pairing (§1.2). The isomorphism $\flat \colon A \to A^*$ is the isomorphism ψ of Theorem 1.

The notion of dual pairing turns out to be useful in various applications. For instance, in the proof of the basic functorial rule (3) of §1.7. We use three lemmas.

Lemma 1. *Let $R \subseteq B \oplus A$ be a linear relation. A linear relation $R^{\bullet} \subseteq B^* \oplus A^*$ is defined by*

$$R^{\bullet} = \{(g, f) \in B^* \oplus A^* \mid \langle a, f\rangle = \langle b, g\rangle, \ \forall\, (b, a) \in R\}. \tag{7}$$

The subspace $R^{\bullet}$ is the polar of R in the dual pairing

$$\langle \, | \, \rangle \colon (B \oplus A) \times (B^* \oplus A^*) \to \mathbb{R} \colon \big((b, a), (g, f)\big) \mapsto \langle b, g\rangle - \langle a, f\rangle. \tag{8}$$

The proof is straightforward.

Lemma 2. *Let (A, α) and (B, β) be symplectic vector spaces and let $R \subseteq B \oplus A$ be a linear relation. If $\flat_A \colon A \to A^*$ and $\flat_B \colon B \to B^*$ are the natural isomorphisms defined by the symplectic forms α and β, respectively* (cf. (2), §1.2), *then*

$$R^{\bullet} = (\flat_B \times \flat_A)(R^{\S}), \tag{9}$$

where $R^{\bullet} \subseteq B^* \oplus A^*$ is defined by (7) *and $R^{\S} \subseteq B \oplus A$ is defined in* (2) *of* §1.7.

Proof. Due to the definition of $\flat$, (2) of §1.2, we can re-write definition (7) as follows (here, we denote by the same symbol $\sharp$ the inverse mappings of $\flat_A$ and $\flat_B$),

$$R^{\bullet} = \{(g, f) \in B^* \oplus A^* \mid \alpha(f^{\sharp}, a') - \beta(g^{\sharp}, b') = 0, \ \forall\, (b', a') \in R\}.$$

This is equivalent to

$$R^\bullet = \{(b^\flat, a^\flat) \in B^* \oplus A^* \mid \alpha(a, a') - \beta(b, b') = 0,\ \forall\, (b', a') \in R\},$$

and, because of definition of $R^\S$, (2) of §1.7, this equation is equivalent to (9). ■

The reason why we consider $R^\bullet$ is explained by the following

Lemma 3. *If $R \subseteq B \oplus A$ and $S \subseteq C \oplus B$ are linear relations, then*

$$(S \circ R)^\bullet = S^\bullet \circ R^\bullet. \tag{10}$$

Proof. Because of the definition (7) we have

$$\begin{cases} R^\bullet = \{(g, f) \in B^* \oplus A^* \mid \langle a, f\rangle = \langle b, g\rangle,\ \forall\, (b, a) \in R\}, \\ S^\bullet = \{(h, g) \in C^* \oplus B^* \mid \langle b, g\rangle = \langle c, h\rangle,\ \forall\, (c, b) \in S\}, \end{cases} \tag{7'}$$

$$(S \circ R)^\bullet = \{(h, f) \in C^* \oplus A^* \mid \langle c, h\rangle = \langle a, f\rangle,\ \forall\, (c, a) \in S \circ R\}, \tag{11}$$

and

$$\begin{aligned} S^\bullet \circ R^\bullet &= \{(h, f) \in C^* \oplus A^* \mid \exists\, g \in B^*,\ (h, g) \in S^\bullet,\ (g, f) \in R^\bullet\} \\ &= \{(h, f) \in C^* \oplus A^* \mid \exists\, g \in B^*,\ \langle c, h\rangle = \langle b, g\rangle,\ \langle b', g\rangle = \langle a, f\rangle, \\ &\quad \forall\, (c, b) \in S,\ \forall\, (b', a) \in R\}. \end{aligned} \tag{12}$$

(i) Let $(h, f) \in S^\bullet \circ R^\bullet$. For any arbitrary element $(c, a) \in S \circ R$ there exists $b \in B$ such that $(c, b) \in S$ and $(b, a) \in R$. It follows from (12), with $b = b'$, that $\langle c, h\rangle = \langle a, f\rangle$. Because of (11), $(h, f) \in (S \circ R)^\bullet$. This proves the inclusion $S^\bullet \circ R^\bullet \subseteq (S \circ R)^\bullet$. (ii) To prove the inverse inclusion we consider the following dual pairing

$$\begin{aligned} &(C \oplus B \oplus B \oplus A) \times (C^* \oplus B^* \oplus B^* \oplus A^*) \to \mathbb{R}: \\ &\big((c, b, b', a), (h, g, g', f)\big) \mapsto \langle c, h\rangle - \langle b, g\rangle + \langle b', g'\rangle - \langle a, f\rangle. \end{aligned} \tag{13}$$

We denote by ¶ the corresponding dual operator. For this dual pairing we have

$$(S \oplus R)^\P = S^\bullet \oplus R^\bullet. \tag{14}$$

Indeed, due to (13),

$$\begin{aligned} (S \oplus R)^\P = \{(h, g, g', f) \mid \langle c, h\rangle - \langle b, g\rangle + \langle b', g'\rangle - \langle a, f\rangle = 0, \\ \forall\, (c, b) \in S,\ \forall\, (b', a) \in R\} \end{aligned}$$

and, due to (7′),

$$S^\bullet \oplus R^\bullet = \{(h, g, g', f) \mid \langle c, h\rangle = \langle b, g\rangle,\ \langle b', g'\rangle = \langle a, f\rangle,\ \forall\,(c, b) \in S,\ \forall\,(b', a) \in R\}.$$

This second expression shows that $S^\bullet \oplus R^\bullet \subseteq (S \oplus R)^\P$. On the other hand, by the dimensional property of the dual polar operators, we have

$$\begin{aligned}\dim(S^\bullet \oplus R^\bullet) &= \dim S^\bullet + \dim R^\bullet = \mathrm{codim} S + \mathrm{codim} R\\ &= \dim C + 2\,\dim B + \dim A - \dim S - \dim R.\end{aligned}$$

and

$$\dim(S \oplus R)^\P = \mathrm{codim}(S \oplus R)^\P = \dim C + 2\,\dim B + \dim A - \dim S - \dim R.$$

Thus, $\dim(S^\bullet \oplus R^\bullet) = \dim(S \oplus R)^\P$ and (14) is proved. Let us consider the following two subspaces of $C \oplus B \oplus B \oplus A$,

$$L = \{(c, b, b, a)\}, \qquad K = (S \oplus R) \cap L. \tag{15}$$

We remark that

$$K = \{(c, b, b, a) \mid (c, b) \in S,\ (b, a) \in R\} \tag{16}$$

so that

$$(c, b, b, a) \in K \quad \Rightarrow \quad (c, a) \in S \circ R. \tag{17}$$

The polar $L^\P$ is made of elements of the kind $(0, g, g, 0)$ with $g \in B^*$. Indeed,

$$\begin{aligned}L^\P &= \{(h, g, g', f) \mid \langle c, h\rangle - \langle b, g\rangle + \langle b', g'\rangle - \langle a, f\rangle = 0,\ \forall\,(c, b, a) \in C \times B \times A\}\\ &= \{(h, g, g', f) \mid h = 0,\ f = 0,\ g = g'\}.\end{aligned}$$

Furthermore, from one of the rules (5) and from (14) we derive

$$K^\P = S^\bullet \oplus R^\bullet + L^\P. \tag{18}$$

If $(h, f) \in (S \circ R)^\bullet$ and $g \in B^*$, then $(h, g, g, f) \in K^\P$. Indeed, $(h, f) \in (S \circ R)^\bullet$ means

$$\langle c, h\rangle = \langle a, f\rangle,\ \forall\,(c, a) \in S \circ R,$$

and because of (16) and (17), for any $(c, b, b, a) \in K$ we have, in the dual pairing (13),

$$\langle (c, b, b, a) \,|\, (h, g, g, f)\rangle = \langle c, h\rangle - \langle b, g\rangle + \langle b, g\rangle - \langle a, f\rangle = 0.$$

It follows from (18) that there exists elements $\bar{g} \in B^*$ and $(h', g', g'', f') \in S^\bullet \oplus R^\bullet$ such that

$$(h, g, g, f) = (h', g', g'', f') + (0, \bar{g}, \bar{g}, 0).$$

From this equality we see that $g' = g''$, $h' = h$ and $f' = f$. Since $(h', g') \in S^\bullet$ and $(g'', f') \in R^\bullet$, we conclude that if $(h, f) \in (S \circ R)^\bullet$ then there exists a $g' \in B^*$ such that $(h, g') \in S^\bullet$ and $(g', f) \in R^\bullet$, that is $(h, f) \in S^\bullet \circ R^\bullet$. This proves $(S \circ R)^\bullet \subseteq S^\bullet \circ R^\bullet$. ■

Now we can prove Theorem **1** of §1.7.

Proof. Let (A, α), (B, β) and (C, γ) be symplectic vector spaces and let $R \subseteq B \oplus A$ and $S \subseteq C \oplus B$ be linear relations. Then we have

$$\begin{aligned}
&(\flat_C \times \flat_B)(S) \circ (\flat_B \times \flat_A)(R) \\
&= \{(f, g) \in C^* \oplus A^* \mid \exists\, h \in B^*,\ (f, h) \in (\flat_C \times \flat_B)(S), (h, g) \in (\flat_B \times \flat_A)(R)\} \\
&= \{(f, g) \in C^* \oplus A^* \mid \exists\, b \in B,\ (f^\sharp, b) \in S, (b, g^\sharp) \in R\} \\
&= \{(f, g) \in C^* \oplus A^* \mid (f^\sharp, g^\sharp) \in S \circ R\} \\
&= (\flat_C \times \flat_A)(S \circ R).
\end{aligned}$$

This proves the identity

$$(\flat_C \times \flat_B)(S) \circ (\flat_B \times \flat_A)(R) = (\flat_C \times \flat_A)(S \circ R),$$

which holds for any two relations R and S. We can write it for $R^\S$ and $S^\S$,

$$(\flat_C \times \flat_B)(S^\S) \circ (\flat_B \times \flat_A)(R^\S) = (\flat_C \times \flat_A)(S^\S \circ R^\S).$$

Because of (9) and (10), it follows that

$$(\flat_C \times \flat_A)(S^\S \circ R^\S) = S^\bullet \circ R^\bullet = (S \circ R)^\bullet = (\flat_C \times \flat_A)(S \circ R)^\S.$$

Since $\flat_C \times \flat_A$ is an isomorphism, this proves the functorial rule (3) of §1.7. ■

6.8 Lagrangian splittings and canonical basis

A **Lagrangian splitting** of a symplectic vector space (A, α) is an ordered pair (L, M) of Lagrangian subspaces such that

(1) $$L \cap M = 0.$$

This condition is equivalent to

(2) $$L + M = A.$$

Indeed, from $A^{\S} = 0$, $L^{\S} = L$ and $M^{\S} = M$ it follows that

$$(L \cap M)^{\S} = L^{\S} + M^{\S} = L + M.$$

Hence, a Lagrangian splitting is a decomposition of A as a direct sum of two Lagrangian subspaces,

$$A = L \oplus M.$$

Proposition 1. *Let (L, M) be a Lagrangian splitting of (A, α). The mapping*

$$\langle\ |\ \rangle : L \times M \to \mathbb{R} : (l, m) \mapsto \alpha(m, l) \tag{3}$$

is a dual pairing.

Proof. Equation $\langle l|m\rangle = 0$ i.e., $\alpha(l, m) = 0$, for each $l \in L$, means that $m \in L^{\S}$, that is $m \in L$. Since $L \cap M = 0$, it follows that $m = 0$. ∎

Let $\psi : M \to L^*$ be the isomorphism associated with this dual pairing. It is defined by

$$\langle l, \psi(m)\rangle = \alpha(m, l). \tag{4}$$

Let (e_i) be an ordered basis of the subspace L and let (ε^i) be its dual basis in the dual space L^*: $\langle e_i, \varepsilon^j\rangle = \delta_i^j$. It follows from (4) that the vectors $f^j = \psi^{-1}(\epsilon^j)$ form a basis of M such that (e_i, f^j) is a **canonical basis** of (A, α) i.e.,

$$\alpha(e_i, e_j) = 0, \qquad \alpha(f^i, f^j) = 0, \qquad \alpha(e_i, f^j) = \delta_i^j. \tag{5}$$

Conversely, let (e_i, f^j) be a canonical basis and let I be a subset of the set $I_n = \{1, 2, \ldots, n\}$, $n = \frac{1}{2}\dim(A)$. Let us denote by L_I the subspace of L spanned by the vectors $(e_i, f^{\bar{i}})$ with $i \in I$ and $\bar{i} \in \bar{I}$, the complementary set of I in I_n. Then,

Proposition 2. *For each Lagrangian subspace L there exists a subset $I \subset I_n$ such that (L, L_I) is a Lagrangian splitting.*

For the proof we use the following

Lemma 1. *If E is a n-dimensional vector space with a basis (e_i) and $S \subset E$ is a subspace, then there exists a subset $I \subset I_n$ such that*

$$E = S \oplus S_I \quad \begin{cases} S_I \cap S = 0 \\ S + S_I = E. \end{cases}$$

where $S_I = \operatorname{span}(e_i,\ i \in I)$.

Proof. Assume that S is defined by the $m = n - r$ independent linear equations

$$S_{\alpha i} x^i = 0.$$

Up to an inessential re-ordering of the basis we assume that the submatrix $[S_{\alpha\beta}]$ $(\alpha, \beta = 1, \ldots, m)$ is regular with inverse matrix $[S^{\alpha\beta}]$, $S^{\alpha\beta} S_{\beta\gamma} = \delta^\alpha_\gamma$. From

$$S_{\alpha\beta} x^\beta + S_{\alpha\bar\beta} x^{\bar\beta} = 0 \qquad (\bar\alpha, \bar\beta = m+1, \ldots, n)$$

it follows that the vectors $x \in S$ are characterized by equations

$$x^\beta = -\, S^{\beta\alpha} S_{\alpha\bar\beta} x^{\bar\beta}.$$

Let us consider the subspace $S' = \mathrm{span}(e_\alpha)$ made of vectors $y = y^\alpha e_\alpha$. Let $v = v^\alpha e_\alpha + v^{\bar\alpha} e_{\bar\alpha}$ be any vector of E. Let us set

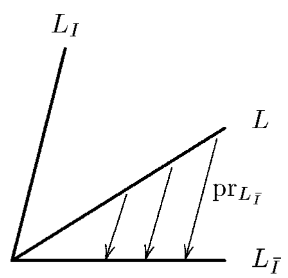
$$\begin{cases} x^\beta = -\, S^{\beta\alpha} S_{\alpha\bar\beta}\, v^{\bar\beta} \\ x^{\bar\beta} = v^{\bar\beta} \\ y^\beta = v^\beta + S^{\beta\alpha} S_{\alpha\bar\beta}\, v^{\bar\beta}. \end{cases}$$

Then, $x = x^\beta e_\beta + x^{\bar\beta} e_{\bar\beta} \in S$ and $y = y^\beta e_\beta \in S'$ and moreover, $x + y = v$. This shows that $S + S' = E$. Let $y = y^\alpha e_\alpha = x \in S$. Then from $y^\alpha e_\alpha = x^\beta e_\beta + x^{\bar\beta} e_{\bar\beta}$ it follows that $x^{\bar\beta} = 0$ hence, $x^\beta = 0$. This shows that $S \cap S' = 0$. Note that $S' = S_I$ with $I = \{1, \ldots, m\}$. ∎

Proof. The subspace $E = L_{I_n} = \mathrm{span}(e_i)$ is Lagrangian (it is isotropic due to $(5)_1$ and of dimension n). The subspace $S = L \cap E$ is isotropic (it is the intersection of two isotropic subspaces). There exists at least a subset $I \subset I_n$ such that $S_I \cap S = 0$ and $S_I + S = E$, where $S_I = \mathrm{span}(e_i;\ i \in I)$ (Lemma 1). Note that $S_I \subseteq L_I$ is isotropic. As a consequence,

$$\begin{aligned} S \subseteq L \ \wedge\ S_I \subseteq L_I \quad &\Rightarrow \\ &\Rightarrow \quad L \subset S^\S \ \wedge\ L_I \subseteq S_I^\S \\ &\Rightarrow \quad L \cap L_I \subseteq S^\S \cap S_I^\S = (S + S_I)^\S = E^\S = E \\ &\Rightarrow \quad L \cap L_I = E \cap L \cap L_I = (E \cap L) \cap (E \cap L_I) = S \cap S_I = 0. \quad ∎ \end{aligned}$$

Now we remark that also $(L_{\bar I}, L_I)$ is a Lagrangian splitting, having a common element with (L, L_I). As a consequence, if we consider the two projections with respect to the complementary subspaces $(L_I, L_{\bar I})$, then L projects isomorphically onto $L_{\bar I}$.



Hence, we can prove (cf. [Arnold, 1967] and [Mishchenko, Shatalov, Sternin, 1978, §2.1)

Proposition 3. *Let*

$$\mathbf{L} = \begin{bmatrix} \mathbf{Q} \\ \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} Q^i_k \\ \\ P_{jk} \end{bmatrix} \tag{6}$$

be a $2n \times n$ *matrix with maximal rank* n *such that*

$$Q^i_k\, P_{ih} - P_{ik}\, Q^i_h = 0. \tag{7}$$

Then there exists a subset $I \subseteq I_n$ *such that the* $n \times n$ *submatrix*

$$\mathbf{S}_I = \begin{bmatrix} \mathbf{Q}^{\bar{I}} \\ \\ \mathbf{P}_I \end{bmatrix} = \begin{bmatrix} Q^{\bar{I}}_k \\ \\ P_{Ik} \end{bmatrix}, \quad I \subseteq \{1, 2, \ldots, n\}. \tag{8}$$

is regular.

Proof. Let (A, α) be a $2n$-dimensional symplectic vector space and let (e_i, f^j) be a canonical basis. For each $v \in A$ we have the representation $v = v^i e_i + v_j f^j$. Let $L \subset A$ be the subspace described by parametric equations

$$v^i = Q^i_k\, \lambda^k, \qquad v_j = P_{jk}\, \lambda^k, \qquad (\lambda^k) \in \mathbb{R}^n, \tag{9}$$

with the matrix (6) of maximal rank. It follows that L is a subspace of dimension n. Condition (7) is equivalent to the isotropy of this subspace:

$$\alpha(u, v) = \alpha(u^i e_i + u_j f^j, v^i e_i + v_j f^j) = u^i v_i - u_i v^i = (Q^i_k P_{jk} - P_{ik} Q^i_h)\, \lambda^k_u\, \lambda^h_v.$$

Hence, L is a Lagrangian subspace and equations (9) describe an isomorphism $\mathbb{R}^n \to L$. On the other hand, due to Proposition 2, there is an isomorphism $\mathbb{R}^n \to L_{\bar{I}}$. Since $v \in L_{\bar{I}}$ if and only if $v = \sum_{a \in \bar{I}} v^a e_a + \sum_{\alpha \in I} v_\alpha f^\alpha$, this last isomorphism is described by equations

$$v^a = Q^a_k\, \lambda^k, \qquad v_\alpha = P_{\alpha k}\, \lambda^k,$$

it follows that

$$\det \begin{bmatrix} Q^a_k \\ \\ P_{\alpha k} \end{bmatrix} \neq 0. \qquad \blacksquare$$

Appendix A

Notation and basic notions of calculus on manifolds

A.1 Tangent vectors and tangent bundles

Let $\mathcal{Q}$ be a differentiable (real) manifold of dimension n. We denote by:

– $\underline{q} = (q^i)$, $(i = 1, \dots, n)$, any coordinate system on an open domain of $\mathcal{Q}$.

– $T_q\mathcal{Q}$ the **tangent space** of $\mathcal{Q}$ at a point $q \in \mathcal{Q}$, the linear n-dimensional space of the tangent vectors based (or applied, or attached) at q.

– $T\mathcal{Q}$ the **tangent bundle** of Q, the set of all **tangent vectors** of $\mathcal{Q}$; it is a differentiable manifold of dimension $2n$.

– $\tau_\mathcal{Q}: T\mathcal{Q} \to \mathcal{Q}$ the **tangent fibration** of $\mathcal{Q}$, which maps a tangent vector $v \in T\mathcal{Q}$ to the point $q \in \mathcal{Q}$ such that $v \in T_q\mathcal{Q}$. A **fibre** is a tangent space: $\tau_\mathcal{Q}^{-1}(q) = T_q\mathcal{Q}$.

– $\mathcal{F}(\mathcal{Q}) = C^\infty(\mathcal{Q}, \mathbb{R})$ the ring of all smooth real functions on $\mathcal{Q}$. A (real) function $f: \mathcal{Q} \to \mathbb{R}$ is **smooth** if any local representative $y = f(\underline{q})$ in coordinates $\underline{q}$ is of class C^∞.

A **curve** on $\mathcal{Q}$ is a smooth mapping $\gamma: I \to \mathcal{Q}$, where I is an open interval of real numbers containing $0 \in \mathbb{R}$. We say that the curve is **based** at the point $q = \gamma(0)$.

We consider two definitions of tangent vector:

(i) A tangent vector v at a point $q \in \mathcal{Q}$ is a **derivation** on $\mathcal{F}$, that is a mapping $v: \mathcal{F}(\mathcal{Q}) \to \mathbb{R}$ such that

$$\begin{cases} v(aF + bG) = a\,v(F) + b\,v(G) \quad a, b \in \mathbb{R}, \quad \text{(linearity)} \\ v(FG) = v(F)\,G(q) + F(q)\,v(G) \qquad \text{(Leibniz rule).} \end{cases}$$

We use the notation

$$v(F) = \langle v, dF \rangle.$$

(ii) A tangent vector is an **equivalence class** $[\gamma]$ of curves. Two curves γ and γ' are **equivalent** if

$$\begin{cases} \gamma(0) = \gamma'(0) \\ D(F \circ \gamma)(0) = D(F \circ \gamma')(0), \quad \forall\, F \in \mathcal{F}(Q). \end{cases}$$

D is the symbol of derivative of a real-valued function on $\mathbb{R}$.

A link between these two definitions is given by

$$v = [\gamma] \quad \Longleftrightarrow \quad v(F) = D(F \circ \gamma)(0).$$

This correspondence does not depend on the choice of the representative curve γ.

If $q \in Q$ and $\underline{q}$ is a local coordinate system on a domain containing the point q, then the **components** of v with respect to these coordinates are the numbers defined by

$$v^i = \langle v, dq^i \rangle$$

or

$$v^i = D\gamma^i(0).$$

In the first definition, a coordinate q^i is interpreted as a function. In the second definition,

$$q^i = \gamma^i(t), \qquad t \in I,$$

are the **parametric equations** of the curve γ. It follows that

$$v(F) = \langle v, dF \rangle = v^i\, \partial_i F(q), \qquad \partial_i = \frac{\partial}{\partial q^i}.$$

We denote by

$$(q^i, \delta q^i) \qquad \text{or} \qquad (q^i, \dot{q}^i)$$

the coordinates on TQ corresponding to coordinates $\underline{q} = (q^i)$ on Q. They are defined as follows: if $v \in T_q Q$ with q in the domain of the coordinates, then $q^i(v)$ are the coordinates of q and $\delta q^i(v) = v^i$ are the components of the vector in these coordinates.

There is a mapping $\delta : \mathcal{F}(Q) \to \mathcal{F}(TQ)$, from functions on Q to functions on TQ, defined by

$$\delta F = \frac{\partial F}{\partial q^i}\, \delta q^i.$$

The function δF is linear on the fibres of TQ.

With each curve $\gamma : I \to Q$ we associate a curve $\dot{\gamma} : I \to TQ$, called the **tangent lift**, or **tangent prolongation** of γ, defined by

$$\langle \dot{\gamma}, F \rangle = D(\gamma \circ F).$$

Its local parametric equations are

$$\begin{cases} q^i = \gamma^i(t) \\ \dot{q}^i = D\gamma^i(t). \end{cases}$$

Mechanical interpretation. If $\mathcal{Q}$ represents the n-dimensional **configuration manifold** of a holonomic mechanical system with n degrees of freedom, then a curve $\gamma: I \to \mathcal{Q}$ represents a **motion** of the system ($t \in I$ is the time). Coordinates (q^i) on $\mathcal{Q}$ are called **Lagrangian coordinates**. A tangent vector $v \in T_q\mathcal{Q}$ represents a **virtual displacement** or a **virtual velocity** of the system at the configuration q. The curve $\dot{\gamma}(t)$ represents the velocity of the system at each instant t.

A.2 The tangent functor

A mapping $\varphi: \mathcal{Q}_1 \to \mathcal{Q}_2$ between two manifolds is smooth if any local representative $\underline{q}_2 = \varphi(\underline{q}_1)$ is of class C^∞. A smooth mapping $\varphi: \mathcal{Q}_1 \to \mathcal{Q}_2$ generates a smooth mapping

$$T\varphi: T\mathcal{Q}_1 \to T\mathcal{Q}_2$$

between the corresponding tangent bundles, called the **tangent prolongation** or **tangent mapping** of φ. It is defined by

$$\langle T\varphi(v), dF\rangle = \langle v, d(F \circ \varphi)\rangle, \quad F \in \mathcal{F}(\mathcal{Q}_2),$$

or by

$$T\varphi([\gamma]) = [\varphi \circ \gamma].$$

If $v \in T_q\mathcal{Q}_1$, then $T\varphi(v) \in T_{\varphi(q)}\mathcal{Q}_2$. Thus,

$$\tau_{\mathcal{Q}_2} \circ T\varphi = \varphi \circ \tau_{\mathcal{Q}_1}.$$

This means that the following diagram is commutative,

$$\begin{array}{ccc} T\mathcal{Q}_1 & \xrightarrow{T\varphi} & T\mathcal{Q}_2 \\ {\scriptstyle \tau_{\mathcal{Q}_1}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathcal{Q}_2}} \\ \mathcal{Q}_1 & \xrightarrow{\varphi} & \mathcal{Q}_2 \end{array}$$

The tangent mapping $T\varphi$ is locally represented by equations

$$\begin{cases} q_2^\alpha = \varphi^\alpha(\underline{q}_1) \\ \dot{q}_2^\alpha = \dfrac{\partial \varphi^\alpha}{\partial q_1^i}\, \dot{q}_1^i, \end{cases}$$

where the first set of equations is the local representative of φ in coordinates $\underline{q}_1 = (q_1^i)$ and $\underline{q}_2 = (q_2^\alpha)$ of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, respectively.

The functorial rules

$$T\mathrm{id}_{\mathcal{Q}} = \mathrm{id}_{T\mathcal{Q}}, \qquad T(\varphi \circ \psi) = T\varphi \circ T\psi$$

hold. The operator T which associates with each manifold $\mathcal{Q}$ its tangent bundle $T\mathcal{Q}$ and with each mapping φ between manifolds its tangent prolongation $T\varphi$ is a covariant functor, called the **tangent functor**.

We denote by $T_q\varphi\colon T_q\mathcal{Q}_1 \to T_{\varphi(q)}\mathcal{Q}_2$ the restriction of $T\varphi$ to a tangent space $T_q\mathcal{Q}_1$. It is a linear mapping.

A.3 Special mappings

A smooth mapping $\varphi\colon \mathcal{Q}_1 \to \mathcal{Q}_2$ is:

(i) A **diffeomorphism** if it is one-to-one and also its inverse φ^{-1} is smooth.

(ii) A **transformation** if it is a diffeomorphism and $\mathcal{Q}_1 = \mathcal{Q}_2$.

(iii) An **immersion** if $T_q\varphi$ is injective for all $q \in \mathcal{Q}_1$; a **submersion** if $T_q\varphi$ is surjective; a **subimmersion** if $T_q\varphi$ has a constant rank. In these three cases we have respectively,

$$\begin{cases} \mathrm{rank}\,[\partial_i\varphi^\alpha] = n_1 \leq n_2 \\ \mathrm{rank}\,[\partial_i\varphi^\alpha] = n_2 \leq n_1 \\ \mathrm{rank}\,[\partial_i\varphi^\alpha] = \mathrm{const.} \end{cases}$$

for any local representation of φ, where $n_1 = \dim(\mathcal{Q}_1)$, $n_2 = \dim(\mathcal{Q}_2)$.

(iv) An **embedding** if it is an immersion and if, in addition, it is a homeomorphism onto its image $\varphi(\mathcal{Q}_1) \subseteq \mathcal{Q}_2$ equipped with the topology induced by the topology of $\mathcal{Q}_2$.

(v) A **fibration** if for each $q \in \mathcal{Q}_2$ there exists a neighborhood $U \subset \mathcal{Q}_2$ of q and a manifold F such that the set $\varphi^{-1}(U)$ is an open subset of $\mathcal{Q}_1$ diffeomorphic to the product $U \times F$ in such a way that the restriction of φ to $\varphi^{-1}(U)$ coincides with the canonical projection of $U \times F$ over U. This is illustrated by the commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_1 \supset \varphi^{-1}(U) & \longrightarrow & U \times F \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{pr}_U} \\ \mathcal{Q}_2 \supset U & \xrightarrow{\mathrm{id}_U} & U \end{array}$$

It can be proved that a fibration is a surjective submersion. The manifold $\mathcal{Q}_1$ is called **fibre bundle** on the **base manifold** $\mathcal{Q}_2$. If for all $q \in \mathcal{Q}_2$ the corresponding manifolds F are diffeomorphic (this happens for instance when the manifold $\mathcal{Q}_2$

is connected), then F is called the **fibre** of the fibration. A fibration $\varphi: \mathcal{Q}_1 \to \mathcal{Q}_2$ is **trivial** if the commutative diagram above holds for $U = \mathcal{Q}_2$. This means that, up to a diffeomorphism, $\mathcal{Q}_1 = \mathcal{Q}_2 \times F$.

A mapping $\sigma: \mathcal{Q}_2 \to \mathcal{Q}_1$ such that $\varphi \circ \sigma = \mathrm{id}_{\mathcal{Q}_2}$ (i.e., $\sigma(p) \in \varphi^{-1}(p)$) is a **section** of the fibration.

The tangent fibration $\tau_{\mathcal{Q}}: T\mathcal{Q} \to \mathcal{Q}$ is an example of fibration. When the tangent fibration is trivial, then the manifold $\mathcal{Q}$ is said to be **parallelizable** and $T\mathcal{Q} = \mathcal{Q} \times \mathbb{R}^n$.

A.4 Submanifolds

A **submanifold** S of a manifold $\mathcal{Q}$ is a subset of $\mathcal{Q}$ which has a structure of differentiable manifold such that the canonical injection $S \to \mathcal{Q}$ is an embedding. An equivalent definition is the following: $S \subseteq \mathcal{Q}$ is a submanifold if for every point for all $q \in S$ there exists an **adapted chart** of domain U and coordinates $(q^i) = (q^\alpha, q^a)$, such that the points $U \cap S$ are described by equations $q^a = 0$. If $\alpha = 1, \dots, m$, $m \leq n = \dim(\mathcal{Q})$ then S is a manifold of dimension m and of **codimension** $n - m$.

An **immersed submanifold** is the image of an injective immersion $\varphi: S \to \mathcal{Q}$ (which is not necessarily an embedding).

The distinction between immersed and embedded submanifolds is needed, for instance, in the discussion of foliations or (in particular) of the orbits of vector fields.

Let $(S^a) = (S^1, \dots, S^k)$ be k (real, smooth) functions on an open domain U of $\mathcal{Q}$. They are said to be **independent** at a point $q \in U$ if their differentials dS^a are linearly independent at q. This means that the $n \times k$ matrix $[\partial_i S^a]$ has maximal rank ($= k$) at q. Locally, a submanifold $S_m \subset \mathcal{Q}_n$ can be represented by **independent equations**

$$S^a(\underline{q}) = 0, \quad a = 1, \dots, k = n - m.$$

The equations are independent if the functions S^a are independent at each point of S.

A vector v is **tangent to a submanifold** S if

$$v(F) = 0$$

for all functions F constant on S, or equivalently, if it is represented by a curve γ on S (its orbit $\gamma(I)$ lies in S). With each submanifold S we associate its **tangent prolongation** $TS \subset T\mathcal{Q}$, made of all vectors tangent to S. This is a submanifold of dimension $2(n - k)$ locally described by equations

$$\begin{cases} S^a(\underline{q}) = 0 \\ \partial_i S^a(\underline{q})\, \dot{q}^i = 0. \end{cases}$$

The **rank** of a mapping $F: \mathcal{Q} \to \mathbb{R}^N$ at a point $q \in \mathcal{Q}$ is the rank of the $N \times n$ matrix

$$\left[\frac{\partial F^I}{\partial q^i}\right], \qquad I = 1, \ldots, N,$$

at that point. It can be proved that (for further details and references see e.g. [Libermann, Marle, 1987], Appendix 1),

Theorem 1. *If $F: \mathcal{Q} \to \mathbb{R}^N$ has constant rank r in a neighborhood of $S = F^{-1}(0)$, then* (i) *S is a submanifold of codimension r and* (ii) $T_q S = \mathrm{Ker}(T_q F)$.

By using this last theorem one can prove:

Theorem 2. *Let N and C be submanifolds of $\mathcal{S}$ such that $N = f^{-1}(0)$ and $C = g^{-1}(0)$, where $f: \mathcal{S} \to \mathbb{R}^k$ and $g: \mathcal{S} \to \mathbb{R}^c$ are subimmersions (i.e., with constant rank). If the function $(f, g): \mathcal{S} \to \mathbb{R}^{k+c}: x \mapsto \big(f(x), g(x)\big)$ is a subimmersion in a neighborhood of the set $N \cap C$, then N and C have clean intersection.*

Proof. We have $T_x N = \mathrm{Ker}(T_x f)$ and $T_x C = \mathrm{Ker}(T_x g)$. Moreover, due to a general property of the linear mappings, $\mathrm{Ker}(T_x f) \cap \mathrm{Ker}(T_x g) = \mathrm{Ker}\big(T_x(f, g)\big)$. Since $N \cap C = (f, g)^{-1}(0)$ and the mapping (f, g) is a subimmersion, it follows that $N \cap C$ is a submanifold and $T_x(N \cap C) = \mathrm{Ker}\big(T_x(f, g)\big)$. Furthermore, $T_x(N \cap C) = \mathrm{Ker}(T_x f) \cap \mathrm{Ker}(T_x g) = T_x N \cap T_x C$. ∎

Remark 4. Theorem 2 can be interpreted as follows. Assume that N and C are described by equations

$$f^\alpha(\underline{x}) = 0, \qquad g^a(\underline{x}) = 0,$$

where $\underline{x} = (x^A)$ are coordinates on $\mathcal{S}$. If the matrices

$$\left[\frac{\partial f^\alpha}{\partial x^A}\right], \qquad \left[\frac{\partial g^a}{\partial x^A}\right], \qquad \left[\frac{\partial f^\alpha}{\partial x^A} \,\middle|\, \frac{\partial g^a}{\partial x^A}\right]$$

have constant rank in neighborhoods of N, C and $N \cap C$, respectively, then N and C have clean intersection.

A.5 Vector fields

A **vector field** on a manifold $\mathcal{Q}$ is a section of the tangent bundle $T\mathcal{Q}$, that is a smooth mapping $X: \mathcal{Q} \to T\mathcal{Q}$ which assigns to each point $q \in \mathcal{Q}$ a vector $X(q)$ at that point. Such a section is locally described by equations

$$\dot{q}^i = X^i(\underline{q}).$$

The functions X^i are the **components** of the vector field X in the coordinates (q^i).

There is an equivalent definition: a vector field is a **derivation** on $\mathcal{F}(\mathcal{Q})$ i.e., a mapping $X: \mathcal{F}(\mathcal{Q}) \to \mathcal{F}(\mathcal{Q})$ such that

$$\begin{cases} X(aF + bG) = a\,X(F) + b\,X(G) \quad a, b \in \mathbb{R}, \quad \text{(linearity)} \\ X(FG) = X(F)\,G + F\,X(G) \qquad \text{(Leibniz rule)}. \end{cases}$$

We use the notation

$$X(F) = \langle X, dF \rangle.$$

This function is called the **derivative** of F with respect to X. The link between these two definitions is given by equation

$$\langle X, dF \rangle(q) = \langle X(q), dF \rangle.$$

The components of a vector field X are the derivatives of the coordinates,

$$X^i = \langle X, dq^i \rangle,$$

so that

$$\langle X, dF \rangle = X^i\, \partial_i F.$$

We denote by $\mathcal{X}(\mathcal{Q})$ the set of the smooth vector fields on $\mathcal{Q}$. It is a module over the ring $\mathcal{F}(\mathcal{Q})$ and an infinite dimensional vector space over $\mathbb{R}$, the sum and the product by a function being defined by

$$(X + Y)(q) = X(q) + Y(q), \qquad (fX)(q) = f(q)X(q).$$

A.6 Integral curves and flows

Let X be a vector field on a manifold $\mathcal{Q}$. An **integral curve** of X is a curve on $\mathcal{Q}$, $\gamma: I \to \mathcal{Q}$, such that $\dot{\gamma}(t) = X(\gamma(t))$ for all $t \in I$ i.e., $\dot{\gamma} = X \circ \gamma$. The integral curves of X are locally represented by the solutions of the first-order differential system in normal form,

$$\frac{dq^i}{dt} = X^i(\underline{q}).$$

Hence, a vector field can be interpreted as a **dynamical system**. We say that an integral curve is **based** at a point q if $\gamma(0) = q$. For smooth vector fields the Cauchy theorem asserts that for each point q there exists a unique **maximal integral curve** $\gamma_q: I_q \to \mathcal{Q}$ based on q, such that any other integral curve based at q is defined on an interval $I \subseteq I_q$. When $I_q = \mathbb{R}$ for all q, then the field is said to be **complete**.

A **flow** on a manifold $\mathcal{Q}$ is a smooth mapping

$$\varphi: \mathbb{R} \times \mathcal{Q} \to \mathcal{Q}: (t, q) \mapsto \varphi(t, q)$$

such that for all $t \in \mathbb{R}$ the mapping

$$\varphi_t: \mathcal{Q} \to \mathcal{Q}: q \mapsto \varphi(t, q)$$

is a transformation of $\mathcal{Q}$ and for all $t, s \in \mathbb{R}$,

(1) $$\varphi_t \circ \varphi_s = \varphi_{t+s}.$$

It follows that

(2) $$\begin{cases} \varphi_0 = \mathrm{id}_{\mathcal{Q}} \\ \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t \\ \varphi_{-t} = (\varphi)^{-1}. \end{cases}$$

The set of all φ_t, $t \in \mathbb{R}$, is said to be a **one-parameter group of transformations**.

A complete vector field X generates a flow φ^X defined by

(3) $$\varphi^X(t, q) = \gamma_q(t).$$

Conversely, a flow φ generates a complete vector field X by setting

(4) $$X(q) = \dot{\gamma}_q(0),$$

where $\gamma_q: \mathbb{R} \to \mathcal{Q}$ are the curves defined by

(5) $$\gamma_q(t) = \varphi(t, q).$$

These curves are the maximal integral curves of X. A non-complete vector field generates **local flows**, defined on open subsets of $\mathbb{R} \times \mathcal{Q}$.

If

(6) $$q^i = \varphi^i(t, q_0^h)$$

is a local representation of a flow φ in local coordinates (q^i) then, according to (4), the components of the associated vector field at the point q_0 are given by

$$X^i(q_0) = \dot{\varphi}^i(0, q_0),$$

where the dot represents the derivative with respect to the variable t.

If φ_t^X is the one-parameter group of transformations generated by a complete vector field X, then

$$T\varphi_t^X : TQ \to TQ$$

is a one-parameter group of transformations on TQ, generating a vector field on TQ which we denote by $\dot{X}$. The vector field $\dot{X}$ is projectable onto X. This means that the following diagram is commutative,

$$\begin{array}{ccc} TTQ & \xrightarrow{T\tau_Q} & TQ \\ {\scriptstyle \dot{X}}\uparrow & & \uparrow{\scriptstyle X} \\ TQ & \xrightarrow{\tau_Q} & Q \end{array}$$

i.e.,

$$T\tau_Q \circ \dot{X} = X \circ \tau_Q .$$

The components of $\dot{X}$ in coordinates $(q^i, \dot{q}^i)$ of TQ are $(X^i, \dot{X}^i)$ where X^i are the components of X and

$$\dot{X}^i(q_0^h, \dot{q}_0^j) = \dot{\varphi}_j^i(0, q_0^h)\, \dot{q}_0^j,$$

where

$$\varphi_j^i(t, q_0^h) = \frac{\partial \varphi^i}{\partial q_0^j},$$

being $\varphi^i(t, q_0^h)$ the local representative of φ_t^X (cf. (6)).

A.7 First integrals

A **first integral** or **integral function** of a vector field is a function F such that

$$\langle X, dF \rangle = 0.$$

The first integrals can be locally determined by integrating the first-order linear partial differential equation

$$X^i \partial_i F = 0.$$

There is an equivalent definition: a first integral is a function F which takes a constant value along any integral curve:

$$D(F \circ \gamma_q) = 0.$$

Indeed, the local expression of this condition is

$$\frac{d}{dt} F(\gamma^i(t)) = \partial_i F\, \dot{\gamma}^i(t) = \partial_i F\, X^i(t) = 0.$$

A vector field may not have global first integrals. However,
Theorem 1. *In a neighborhood of a non-singular point* $q \in \mathcal{Q}$ $(X(q) \neq 0)$ *there exist* $n-1$ *independent first integrals.*

This follows from
Theorem 2. *In a neighborhood of a non-singular point* $q \in \mathcal{Q}$ $(X(q) \neq 0)$ *there exists a coordinate system* (q^i) *such that* $X = \partial/\partial q^1$.

These coordinates are said to be **adapted** to X.

A.8 Lie bracket

The **Lie-bracket** $[X, Y]$ of two vector fields is the vector field defined by

$$[X, Y]F = X(YF) - Y(XF).$$

In local coordinates,

$$[X, Y]^i = X^k\,\partial_k Y^i - Y^k\,\partial X^i.$$

This operation satisfies the following rules,

$$\begin{cases} [X, Y] = \, -[Y, X] \quad \text{(anticommutativity)} \\ [aX + bY, Z] = a[X, Z] + b[Y, Z], \quad a, b \in \mathbb{R}, \quad (\mathbb{R}\text{-linearity}) \\ \big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0 \quad \text{(cyclic or Jacobi identity).} \end{cases}$$

Thus, the space $\mathcal{X}(\mathcal{Q})$ endowed with the Lie bracket is a Lie-algebra.

We say that two vector fields **commute** if $[X, Y] = 0$. Indeed, it can be proved that
Theorem 1. *The flows of two (complete) vector fields commute i.e.,*

$$\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X,$$

for all $t, s \in \mathbb{R}$, *if and only if* $[X, Y] = 0$.

A vector field is **tangent to a submanifold** S when all its values $X(q)$ are vectors tangent to S. This holds if and only if any integral curve intersecting S lies on S. It can be proved that:
Theorem 2. *If two vector fields* X *and* Y *are tangent to a submanifold* S, *then also* $[X, Y]$ *is tangent to* S.

A.9 One-forms

A **one-form** on a manifold $\mathcal{Q}$ is a mapping $\theta: T\mathcal{Q} \to \mathbb{R}$ linear on each tangent space $T_q\mathcal{Q}$. We use the notation

$$\theta(v) = \langle v, \theta \rangle.$$

An equivalent definition is the following: a one-form is a linear mapping from vector fields to functions, $\theta\colon \mathcal{X}(\mathcal{Q}) \to \mathcal{F}(\mathcal{Q})$. We shall use the notation

$$\theta(X) = \langle X, \theta \rangle.$$

The link between these two definition is

$$\langle X, \theta \rangle(q) = \langle X(q), \theta \rangle.$$

The linearity implies that

$$\langle X, \theta \rangle = X^i \, \theta_i,$$

where θ_i are functions called the **components** of θ (w.r. to the coordinates (q^i)). It follows that

$$\theta_i = \langle \partial_i, \theta \rangle.$$

We can define the sum of two one-forms and the product of a one-form with a function (or a number) in an obvious way.

A special case of one-form is the **differential of a function** dF. It is defined by

$$\langle X, dF \rangle = XF$$

and its components are

$$(dF)_i = \partial_i F.$$

It follows that in a coordinate system any one-form can be represented by a linear combination of the differentials dq^i,

$$\theta = \theta_i \, dq^i.$$

Thus, a one-form is also called a **linear differential form**. We call **elementary one-form** a one-form of the kind $F\,dG$, where F and G are smooth functions on $\mathcal{Q}$.

A.10 Exterior forms

Let $\times^p_{\mathcal{Q}} T\mathcal{Q}$ be the subset of the Cartesian power $(T\mathcal{Q})^p$ made of ordered sets of p tangent vectors applied to a same point. It is a manifold of dimension $(p+1)n$, if $n = \dim(\mathcal{Q})$. An **exterior form of order** p, briefly a p**-form**, on a manifold $\mathcal{Q}$ is a multilinear skew-symmetric smooth mapping from this space to $\mathbb{R}$,

$$\omega\colon \times^p_{\mathcal{Q}} T\mathcal{Q} \to \mathbb{R}\colon (v_1, \ldots, v_p) \mapsto \omega(v_1, \ldots, v_p).$$

The value $\omega(v_1, \dots, v_p)$ changes in sign by interchanging any two arguments. It follows that for linearly dependent vectors $\omega(v_1, \dots, v_p) = 0$. Thus, any p-form for $p > n$ vanishes identically.

A **zero-form** ($p = 0$) is a function $F: \mathcal{Q} \to \mathbb{R}$. For $p = 1$ we get the definition of one-form.

An equivalent definition is the following: a p-form is a multilinear and skew-symmetric smooth mapping from the Cartesian power $(\mathcal{X}(\mathcal{Q}))^p$ of the space of vector fields to $\mathcal{F}(\mathcal{Q})$,

$$\omega: (\mathcal{X}(\mathcal{Q}))^p \to \mathcal{F}(\mathcal{Q}): (X_1, \dots, X_p) \mapsto \omega(X_1, \dots, X_p).$$

The sum of two p-forms and the multiplication of a p-form with a function or a real number are defined in an obvious way. We denote by $\Phi^p(\mathcal{Q})$ the linear space of all p-forms. It is a module on the ring $\mathcal{F}(\mathcal{Q})$. In particular, $\Phi^0(\mathcal{Q}) = \mathcal{F}(\mathcal{Q})$ and $\Phi^p(\mathcal{Q}) = 0$ for $p > n$. We set $\Phi^p(\mathcal{Q}) = 0$ for $p < 0$ and denote by $\Phi(\mathcal{Q})$ the direct sum of all these spaces,

$$\Phi(\mathcal{Q}) = \oplus_{p=-\infty}^{+\infty} \Phi^p(\mathcal{Q}).$$

An **exterior** or **differential form** is an element of this space.

A.11 Exterior algebra

The **exterior product** $\varphi \wedge \psi$ of a p-form φ times a q-form ψ is the $p + q$-form defined by

$$\varphi \wedge \psi = \frac{(p+q)!}{p!\, q!} \mathsf{A}(\varphi \otimes \psi)$$

being A the **antisymmetrization operator**. On any p-linear form $\eta: \times^p_{\mathcal{Q}} T\mathcal{Q} \to \mathbb{R}$ it is defined by

$$\mathsf{A}\eta = \frac{1}{p!} \sum_{\sigma \in G_p} \varepsilon_\sigma \eta \circ \sigma,$$

where G_p is the permutation group of order p and $\varepsilon_\sigma = \pm 1$ is the signature of the permutation σ. For a 0-form (function), $\mathsf{A}f = f$. For $p < 0$ or $q < 0$, $\varphi \wedge \psi = 0$. If one of the two forms is a function, then

$$\varphi \wedge \psi = \varphi\psi.$$

By a linear extension of the exterior product to the direct sum $\Phi(\mathcal{Q})$ we get the **exterior algebra**. It is a commutative and associative graded algebra,

$$\begin{cases} \Phi^p(\mathcal{Q}) \wedge \Phi^q(\mathcal{Q}) \subset \Phi^{p+q}(\mathcal{Q}), \\ \varphi \wedge \psi = (-1)^{pq} \psi \wedge \varphi, \\ (\varphi \wedge \psi) \wedge \varphi = \varphi \wedge (\psi \wedge \varphi). \end{cases}$$

For two one-forms the exterior product is anticommutative, $\varphi \wedge \psi = -\psi \wedge \varphi$.

An **elementary p-form** is a p-form of the kind

$$\omega = F\, dG_1 \wedge \ldots \wedge dG_p,$$

where $F, G_1, \ldots, G_p$ are functions. Then the exterior product of two elementary exterior forms is obtained by applying the associative rule and the commutation rules $F \wedge dG = dG \wedge F$ and $dF \wedge dG = -dG \wedge dF$. Any p-form can be locally expressed as a sum of elementary p-forms. Indeed, in any coordinate system we have the representation

$$\omega = \frac{1}{p!}\, \omega_{i_1 \ldots i_p}\, dq^{i_1} \wedge \ldots \wedge dq^{i_p},$$

where

$$\omega_{i_1 \ldots i_p} = \omega(\partial_{i_1}, \ldots, \partial_{i_p})$$

are the components of ω.

A.12 Pull-back

Let $\alpha\colon \mathcal{Q}_1 \to \mathcal{Q}_2$ be a smooth mapping. For each $p \in \mathbb{Z}$ we define a linear mapping

$$\alpha^*\colon \Phi^p(\mathcal{Q}_2) \to \Phi^p(\mathcal{Q}_1)$$

by setting

$$\begin{cases} \alpha^*\omega(v_1, \ldots, v_p) = \omega(T\alpha(v_1), \ldots, T\alpha(v_p)), & p > 0, \\ \alpha^*\omega = \omega \circ \alpha, & p = 0, \\ \alpha^*\omega = 0, & p < 0. \end{cases}$$

By a linear extension we get a linear mapping

$$\alpha^*\colon \Phi(\mathcal{Q}_2) \to \Phi(\mathcal{Q}_1),$$

called **pull-back**, with the following properties:

$$\begin{cases} \alpha^*(\omega \wedge \psi) = \alpha^*\omega \wedge \alpha^*\psi, \\ \mathrm{id}^*_{\mathcal{Q}} = \mathrm{id}_{\Phi(\mathcal{Q})}, \\ (\beta \circ \alpha)^* = \alpha^* \circ \beta^*. \end{cases}$$

The last two properties show that the operator

$$^*\colon \quad \begin{cases} \mathcal{Q} \mapsto \Phi(\mathcal{Q}) \\ \alpha \mapsto \alpha^* \end{cases}$$

is a covariant functor from the category of the differentiable manifolds into the category of the graded algebras, the **exterior functor**.

If $\iota: S \to \mathcal{Q}$ is the canonical injection of a submanifold $S \subset \mathcal{Q}$, then the pull-back $\iota^*\omega$ of a form on $\mathcal{Q}$ is the **restriction** of ω to S and it is also denoted by $\omega|S$. In fact, it is the restriction of $\omega: \times^p_{\mathcal{Q}} T\mathcal{Q} \to \mathbb{R}$ to the submanifold $\times^p_S TS$.

If in local coordinates the mapping α is represented by equations

$$q_2^a = \alpha^a(\underline{q}_1),$$

then the pull-back of a form is obtained by replacing these functions into its local coordinate representation. It follows that the pull-back is locally represented by equations

$$(\alpha^*\omega)_{i_1\ldots i_p} = \omega_{a_1\ldots a_p} \frac{\partial \alpha^{a_1}}{\partial q^{i_1}} \cdots \frac{\partial \alpha^{a_p}}{\partial q^{i_p}}.$$

A.13 Derivations

A **derivation of degree** $r \in \mathbb{Z}$ on $\Phi(\mathcal{Q})$ is a mapping $D: \Phi(\mathcal{Q}) \to \Phi(\mathcal{Q})$ satisfying the following rules,

$$(1) \qquad \begin{cases} D\Phi^p(\mathcal{Q}) \subset \Phi^{p+r}(\mathcal{Q}) & (p \in \mathbb{Z}), \\ D(a\varphi + b\psi) = aD\varphi + bD\psi & (a, b \in \mathbb{R}), \\ D(\varphi \wedge \psi) = D\varphi \wedge \psi + (-1)^{pr}\, \varphi \wedge D\psi & \big(\varphi \in \Phi^p(\mathcal{Q})\big). \end{cases}$$

Hence, D maps a p-form to a $(p+r)$-form, it is $\mathbb{R}$-linear and satisfies a **graded Leibniz rule**. From the linearity and the Leibniz rule it follows that $Da = 0$ for any number $a \in \mathbb{R}$ interpreted as a constant 0-form.

The general theory of derivations is due to [Frölicher, Nijenhuis, 1956] and it is based on the following theorems.

Theorem 1. *Let D be a derivation. If $\varphi, \psi \in \Phi(\mathcal{Q})$ are two exterior forms such that $\varphi|U = \psi|U$ in an open subset $U \subset \mathcal{Q}$, then $D\varphi|U = D\psi|U$* (**locality of a derivation**).

Theorem 2. *Any derivation is uniquely determined by its action on $\Phi^0(\mathcal{Q})$ and $\Phi^1(\mathcal{Q})$* (i.e., on functions and one-forms).

In other words: any mapping $D: \Phi^0(\mathcal{Q}) \oplus \Phi^1(\mathcal{Q}) \to \Phi(\mathcal{Q})$ satisfying the rules (1) is extended in a unique way to a derivation of degree r on $\Phi(\mathcal{Q})$. Note that if $r = -2$, then D has image in $\Phi^{-2} \oplus \Phi^{-1} = 0 \oplus 0$, so that its extension is necessarily the zero-mapping. Thus,

Theorem 3. *Any derivation of degree $r < -1$ is trivial: $D = 0$.*

The **commutator** of two derivations D_1 and D_2, of degree r_1 and r_2 respectively, is the derivation of degree $r_1 + r_2$ defined by

$$\boxed{[D_1, D_2] = D_1 D_2 - (-1)^{r_1 r_2} D_2 D_1} \tag{2}$$

Indeed, the composition $D_1 D_2 = D_1 \circ D_2$ is linear but it does not satisfy the graded Leibniz rule, which is instead satisfied by the operator defined in (2).

There are three special important derivations: the differential, the interior product and the Lie derivative.

A.14 The differential

The **differential** is the derivation d of degree **1** whose action on functions and one-forms is defined by

$$\boxed{\langle X, df \rangle = Xf}$$

$$\boxed{d\theta(X, Y) = \langle X, d\langle Y, \theta \rangle \rangle - \langle Y, d\langle X, \theta \rangle \rangle - \langle [X, Y], \theta \rangle}$$

As a consequence, it can be proved that

$$\boxed{d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^p \varphi \wedge d\psi, \qquad \varphi \in \Phi^p(Q)}$$

and

$$\boxed{d^2 = 0}$$

For an elementary p-form $\omega = F\, dG_1 \wedge \ldots \wedge dG_p$,

$$d\omega = dF \wedge dG_1 \wedge \ldots \wedge dG_p.$$

The pull-back α^* associated with a mapping α commutes with the differential,

$$\boxed{d\alpha^* \omega = \alpha^* d\omega}$$

In particular, the differential commutes with the restriction of forms to submanifolds,

$$\boxed{d(\omega|S) = (d\omega)|S}$$

A p-form ω is said to be **closed** if $d\omega = 0$, **exact** if there exists a $(p-1)$-form ϕ, called **potential form**, such that $\omega = d\phi$. *An exact form is closed*, since $d^2 = 0$. Conversely, it can be proved that *a closed form is locally exact* (**Poincaré-Volterra lemma**).

A derivation D is called of type i_* if it is trivial on functions: $Df = 0$. It is called of type d_* if it commutes with the differential:

$$Dd = (-1)^r dD.$$

Theorem 4. (i) *Any derivation can be decomposed in a unique way as a sum of a derivation of type i_* and a derivation of type d_*.* (ii) *Any derivation of type d_* is uniquely determined by its action on functions.*

We have two fundamental derivations of type i_* and d_* associated with a vector field: the interior product and the Lie derivative.

A.15 Interior product

The **interior product** (or the **Cartan product**) w.r. to a vector field X is the derivation i_X of degree -1 and type i_* defined by the following action on functions and one-forms,

$$\boxed{\begin{array}{l} i_X f = 0 \\ i_X \theta = \langle X, \theta \rangle \end{array}}$$

It has the following properties,

$$\boxed{i_X(\varphi \wedge \psi) = i_X\varphi \wedge \psi + (-1)^p \, \varphi \wedge i_X\psi}$$

where p is the degree of φ;

$$\boxed{i_X f\varphi = f\, i_X\varphi} \qquad \boxed{i_Y i_X \omega = \omega(X, Y)}$$

for a two-form. A similar formula holds for any p-form.

$$\boxed{i_X i_Y = -i_Y i_X} \qquad \boxed{i_X^2 = 0}$$

In local coordinates,

$$\boxed{i_X dq^i = X^i} \qquad \boxed{(i_X\omega)_{i_2 \ldots i_p} = X^{i_1} \omega_{i_1 i_2 \ldots i_p}}$$

A.16 Lie derivative

The **Lie derivative** w.r. to a vector field X is the derivation of type d_* and degree 0 defined by

$$\boxed{\begin{aligned} dd_X &= d_X d \\ d_X f &= i_X df \end{aligned}}$$

For the Lie derivative there are other two (equivalent) definitions,

$$\boxed{d_X = [i_X, d] = i_X d + d i_X}$$

$$\boxed{d_X \omega = \lim_{t \to 0} \frac{1}{t} (\varphi_t^* \omega - \omega)}$$

The first one is known as **Cartan formula** (the Lie derivative is the commutator of the Cartan product and the differential). The Lie derivative has the following properties,

$$\boxed{d_X(\varphi \wedge \psi) = d_X \varphi \wedge \psi + \varphi \wedge d_X \psi}$$

$$\boxed{\begin{aligned} [d_X, d_Y] &= d_X d_Y - d_Y d_X = d_{[X,Y]} \\ [d_X, i_Y] &= d_X i_Y - i_Y d_X = i_{[X,Y]} \end{aligned}}$$

$$\boxed{d\theta(X, Y) = d_X i_Y \theta - d_Y i_X \theta - i_{[X,Y]} \theta}$$

For a two-form ω,

$$\boxed{d\omega(X, Y, Z) = i_{[X,Y]} i_Z \omega - d_X i_Y i_Z \omega + p.c.}$$

where *p.c.* means the sum of the similar terms obtained by all cyclic permutations of the vector fields.

A form $\omega \in \Phi(\mathcal{Q})$ is said to be **invariant** w.r. to a transformation $\varphi\colon \mathcal{Q} \to \mathcal{Q}$ if $\varphi^* \omega = \omega$. It can be proved that

Theorem 1. *A form ω is invariant w.r. to the group φ_t generated by a (complete) vector field X if and only if $d_X \omega = 0$.*

Appendix B

Global Hamilton principal functions of the eikonal equations on $\mathbb{S}_2$ and $\mathbb{H}_2$

S. Benenti, F. Cardin

B.1 Introduction

In $\mathbb{R}^3 = (x, y, z)$ endowed with the natural Euclidean structure we consider the unit sphere $\mathbb{S}_2$, $x^2+y^2+z^2-1=0$. In $\mathbb{R}^3$ endowed with a Minkowski metric, with z time-like coordinate, we consider the hyperboloid $\mathbb{H}_2$ of equation $z = \sqrt{1+x^2+y^2}$ made of all unit time-like vectors oriented to the future.

Both are two-dimensional Riemannian manifolds with constant curvature (positive and negative, respectively). We will show that their eikonal equations admit global Hamilton principal functions, which are not Morse families. To this end, we need to recall some basic definitions and formulae of vector calculus in $\mathbb{R}^3$.

B.2 Vector calculus in the real three-space

B.2.1 The metric tensor and the scalar product. In $\mathbb{R}^3$ we consider the ordered canonical basis $\mathbf{c}_i$,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3 = \mathbf{t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the metric tensors $\mathbf{g}_\varepsilon$, with $\varepsilon = \pm 1$, such that

$$\begin{cases} \mathbf{g}_\varepsilon(\mathbf{c}_i, \mathbf{c}_j) = 0, \quad i \neq j \\ \mathbf{g}_\varepsilon(\mathbf{c}_1, \mathbf{c}_1) = \mathbf{g}_\varepsilon(\mathbf{c}_2, \mathbf{c}_2) = 1 \\ \mathbf{g}_\varepsilon(\mathbf{c}_3, \mathbf{c}_3) = \varepsilon = \pm 1. \end{cases}$$

We denote by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{g}_\varepsilon(\mathbf{u}, \mathbf{v})$$

the scalar product of two vectors, and use the notation

$$\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}, \qquad |\mathbf{u}| = \sqrt{|\mathbf{u}^2|}.$$

Two vectors are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. In this case we use the notation $\mathbf{u} \perp \mathbf{v}$. If

$$g_{ij} = \mathbf{c}_i \cdot \mathbf{c}_j,$$

then

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

For $\varepsilon = \mathbf{1}$ the metric is positive-definite (Euclidean). For $\varepsilon = -\mathbf{1}$ the metric is hyperbolic (Minkowskian) and the vector $\mathbf{c}_3 = \mathbf{t}$ is time-like.

Let $(\mathbf{e}_a)$ be any basis. Its dual basis $(\mathbf{e}^a)$ is defined by

$$\mathbf{e}_a \cdot \mathbf{e}^b = \delta_a^b.$$

If

$$g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b, \qquad g^{ab} = \mathbf{e}^a \cdot \mathbf{e}^b,$$

then the two symmetric matrices $[g_{ab}]$ and $[g^{ab}]$ are each one inverse, $g^{ab} g_{bc} = \delta_c^a$, and we get the well known rules of raising and lowering of indices: if $\mathbf{v} = v^a \mathbf{e}_a = v_a \mathbf{e}^a$, then

$$v_a = g_{ab} v^b, \qquad v^a = g^{ab} v_b,$$

and

$$v_a = \mathbf{v} \cdot \mathbf{e}_a, \qquad v^a = \mathbf{v} \cdot \mathbf{e}^a.$$

For the canonical basis,

$$\mathbf{c}^1 = \mathbf{c}_1, \quad \mathbf{c}^2 = \mathbf{c}_2, \quad \mathbf{c}^3 = \varepsilon\, \mathbf{c}_3,$$

$$v^1 = v_1, \quad v^2 = v_2, \quad v^3 = \varepsilon\, v_3.$$

B.2.2 The volume form. We define a volume three-form $V(\mathbf{u}, \mathbf{v}, \mathbf{w})$ by setting

$$V(\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3) = \varepsilon.$$

As a consequence,

$$V(\mathbf{c}_h, \mathbf{c}_i, \mathbf{c}_j) = \varepsilon\, \varepsilon_{hij}$$

where ε_{hij} is the Levi-Civita symbol, and

$$V(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \varepsilon\, \varepsilon_{hij}\, u^h v^i w^j = \varepsilon \begin{vmatrix} u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \\ w^1 & w^2 & w^3 \end{vmatrix},$$

For any arbitrary basis $(\mathbf{e}_a)$ we have

$$V(\mathbf{u}, \mathbf{v}, \mathbf{w}) = V_{abc}\, u^a v^b w^c = V^{abc}\, u_a v_b w_c$$

where

$$V_{abc} = V(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c), \qquad V^{abc} = V(\mathbf{e}^a, \mathbf{e}^b, \mathbf{e}^c) = g^{ad} g^{be} g^{cf}\, V_{def}.$$

If the basis $(\mathbf{e}_a)$ is oriented as $(\mathbf{c}_i)$ i.e., if

$$\mathbf{e}_a = A_a^i\, \mathbf{c}_i, \qquad \det \mathbf{A} > 0, \qquad \mathbf{A} = [A_a^i],$$

then

$$(1) \qquad V_{abc} = \varepsilon \sqrt{|g|}\, \varepsilon_{abc}, \qquad V^{abc} = \frac{1}{\sqrt{|g|}}\, \varepsilon^{abc}, \qquad g = \det[g_{ab}],$$

where ε_{abc} and ε^{abc} are Levi-Civita symbols. It follows that

$$V^{abc}\, V_{abc} = 3!\, \varepsilon, \qquad V^{abc}\, V_{dbc} = 2\, \varepsilon\, \delta_d^a, \qquad V^{abc}\, V_{dec} = \varepsilon\, \delta_{de}^{ab} = \varepsilon\, (\delta_d^a \delta_e^b - \delta_e^a \delta_d^b).$$

To prove (1) we observe that from $g_{ab} = A_a^i A_b^j\, g_{ij}$ it follows that

$$g = \det[g_{ab}] = (\det \mathbf{A})^2\, \det[g_{ij}] = (\det \mathbf{A})^2\, \varepsilon.$$

Thus, g has the same sign of ε and we can write $g = \varepsilon\, |g|$. Moreover, since $\mathbf{A}$ has positive determinant,

$$\det \mathbf{A} = \sqrt{|g|}.$$

Hence,

$$V_{abc} = A_a^i A_b^j A_c^k\, V_{ijk} = \varepsilon\, \varepsilon_{ijk}\, A_a^i A_b^j A_c^k = \varepsilon\, \varepsilon_{abc}\, \det \mathbf{A} = \varepsilon \sqrt{|g|}\, \varepsilon_{abc},$$

$$\begin{aligned} V^{abc} &= V_{def}\, g^{ad} g^{be} g^{cf} = \varepsilon\,\sqrt{|g|}\,\varepsilon_{def}\, g^{ad} g^{be} g^{cf} \\ &= \varepsilon\,\sqrt{|g|}\,\varepsilon^{abc}\;\det[g^{ab}] = \varepsilon\,\sqrt{|g|}\,\varepsilon^{abc}\,\frac{1}{g} = \frac{1}{\sqrt{|g|}}\,\varepsilon^{abc}. \end{aligned}$$

B.2.3 The cross product. By means of the volume form we define the cross product $\mathbf{u}\times\mathbf{v}$ of two vectors by setting

$$\mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = V(\mathbf{u},\mathbf{v},\mathbf{w}).$$

With respect to any basis $(\mathbf{e}_a)$ we have

$$\mathbf{e}_a\times\mathbf{e}_b\cdot\mathbf{e}_c = V_{abc}, \qquad (\mathbf{u}\times\mathbf{v})_c = \mathbf{u}\times\mathbf{v}\cdot\mathbf{e}_c = V_{abc}\,u^a\,v^b,$$

and

$$\mathbf{u}\times\mathbf{v} = V_{abc}\,u^a\,v^b\,\mathbf{e}^c = V^{abc}\,u_a\,v_b\,\mathbf{e}_c.$$

For the canonical basis, $\mathbf{c}_i\times\mathbf{c}_j = \varepsilon\,\varepsilon_{ijk}\,\mathbf{c}^k$; thus,

$$\begin{cases} \mathbf{c}_1\times\mathbf{c}_2 = \mathbf{c}_3 = \mathbf{t} \\ \mathbf{c}_2\times\mathbf{c}_3 = \varepsilon\mathbf{c}_1 \\ \mathbf{c}_3\times\mathbf{c}_1 = \varepsilon\mathbf{c}_2. \end{cases}$$

The cross product satisfies the following rules,

$$\begin{cases} \mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u} \\ \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\times\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\times\mathbf{w}\cdot\mathbf{u} \\ \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{v}\times\mathbf{w}, \end{cases}$$

whatever ε. For the double cross product we have

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \varepsilon\,(\mathbf{u}\cdot\mathbf{w}\,\mathbf{v} - \mathbf{v}\cdot\mathbf{w}\,\mathbf{u}).$$

Indeed,

$$\begin{aligned} (\mathbf{u}\times\mathbf{v})\times\mathbf{w} &= V_{abc}\,u^a\,v^b\,\mathbf{e}^c\times\mathbf{w} = V_{abc}\,u^a\,v^b\,V^{cde}\,w_d\,\mathbf{e}_c \\ &= V_{abc}\,V^{dec}\,u^a\,v^b\,w_d\,\mathbf{e}_e = \varepsilon\,\delta^{de}_{ab}\,u^a\,v^b\,w_d\,\mathbf{e}_e \\ &= \varepsilon\,(u^a\,w_a\,v^b\,\mathbf{e}_b - v^b\,w_b\,u^a\,\mathbf{e}_a. \end{aligned}$$

As a consequence,

$$(\mathbf{n}\times\mathbf{a})\cdot(\mathbf{n}\times\mathbf{b}) = (\mathbf{n}\times\mathbf{a})\times\mathbf{n}\cdot\mathbf{b} = \varepsilon\left(\mathbf{n}^2\,\mathbf{a}\cdot\mathbf{b} - \mathbf{n}\cdot\mathbf{a}\,\mathbf{n}\cdot\mathbf{b}\right)$$

and

$$(\mathbf{n}\times\mathbf{a})^2 = \varepsilon\left(\mathbf{n}^2\,\mathbf{a}^2 - (\mathbf{n}\cdot\mathbf{a})^2\right).$$

B.2.4 Rotations. With a vector $\mathbf{n}$ such that

$$\gamma = \mathbf{n}^2 = \pm 1$$

we associate the linear operators $\mathcal{R}\colon \mathbb{R}^3 \to \mathbb{R}^3$ of the kind

$$\mathcal{R}(\mathbf{v}) = \mathbf{v} + \alpha\, \mathbf{n} \times \mathbf{v} + \beta(\mathbf{n} \cdot \mathbf{v}\, \mathbf{n} - \gamma \mathbf{v}), \qquad \alpha, \beta \in \mathbb{R}.$$

It follows that

(1) $$\mathcal{R}(\mathbf{n}) = \mathbf{n},$$

(2) $$\mathbf{v} \cdot \mathcal{R}(\mathbf{v}) = (1 - \beta\gamma)\, \mathbf{v}^2 + \beta(\mathbf{n} \cdot \mathbf{v})^2,$$

(3) $$\begin{aligned}(\mathcal{R}(\mathbf{v}))^2 &= \mathbf{v}^2 + \alpha^2\,(\mathbf{n} \times \mathbf{v})^2 + \beta^2\left(\gamma(\mathbf{n} \cdot \mathbf{v})^2 + \gamma^2\,(\mathbf{v})^2 - 2\gamma\,(\mathbf{n} \cdot \mathbf{v})^2\right) \\ &\quad + 2\beta\left((\mathbf{n} \cdot \mathbf{v})^2 - \gamma\,\mathbf{v}^2\right) \\ &= \mathbf{v}^2 + \alpha^2\,\varepsilon\left(\gamma\,\mathbf{v}^2 - (\mathbf{n} \cdot \mathbf{v})^2\right) + \beta^2\left(\gamma^2\,\mathbf{v}^2 - \gamma(\mathbf{n} \cdot \mathbf{v})^2\right) \\ &\quad + 2\beta\left((\mathbf{n} \cdot \mathbf{v})^2 - \gamma\,\mathbf{v}^2\right) \\ &= \left(1 + \gamma(\varepsilon\alpha^2 + \gamma\beta^2 - 2\beta)\right)\mathbf{v}^2 + \left(2\beta - \varepsilon\alpha^2 - \gamma\beta^2\right)(\mathbf{n} \cdot \mathbf{v})^2,\end{aligned}$$

(4) $$(\mathcal{R}(\mathbf{v}))^2 = \mathbf{v}^2 \quad \Leftrightarrow \quad \left(\gamma(\varepsilon\alpha^2 + \gamma\beta^2 - 2\beta)\right)\mathbf{v}^2 + \left(2\beta - \varepsilon\alpha^2 - \gamma\beta^2\right)(\mathbf{n} \cdot \mathbf{v})^2 = 0.$$

Let us consider the case $\gamma = \mathbf{n}^2 = 1$ and $\left(\mathcal{R}_{\mathbf{n}}(\mathbf{v})\right)^2 = \mathbf{v}^2$. Then, for each vector $\mathbf{v} \perp \mathbf{n}$, $\mathbf{v}^2 \neq 0$, from (2) and (4) we obtain

(5) $$\frac{\mathbf{v} \cdot \mathcal{R}(\mathbf{v})}{\mathbf{v}^2} = 1 - \beta$$

and

(6) $$\varepsilon\alpha^2 + \beta^2 - 2\beta = 0.$$

In the Euclidean metric i.e., for $\varepsilon = 1$, equation (6) implies $\alpha^2 \leq 1$, and $|1 - \beta| \leq 1$. Let us set $1 - \beta = \cos\theta$, $\beta = 1 - \cos\theta$. Then (6) implies $\alpha^2 = \sin^2\theta$. If we choose $\alpha = \sin\theta$, then we obtain the **Rodrigues formula** for the rotations in the Euclidean three-space,

(7) $$\boxed{\mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{v}) = \mathbf{v} + \sin\theta\, \mathbf{n} \times \mathbf{v} + (1 - \cos\theta)(\mathbf{n} \cdot \mathbf{v}\, \mathbf{n} - \mathbf{v})}$$

The choice $\alpha = \sin\theta$ (instead of $\alpha = -\sin\theta$) is in accordance with the conditions

$$\mathcal{R}_{(\mathbf{n},\frac{\pi}{2})}(\mathbf{v}) = \mathbf{n}\times\mathbf{v}, \qquad \mathbf{v}\perp\mathbf{n}.$$

It follows that for all $\mathbf{v}\neq 0$ orthogonal to $\mathbf{n}$,

$$\boxed{\sin\theta = \frac{\mathbf{v}\times\mathcal{R}(\mathbf{v})}{\mathbf{v}^2}\cdot\mathbf{n}, \qquad \cos\theta = \frac{\mathbf{v}\cdot\mathcal{R}(\mathbf{v})}{\mathbf{v}^2}} \tag{8}$$

Note that θ is the **angle of rotation** i.e., the angle between $\mathbf{v}$ and $\mathcal{R}(\mathbf{v})$, for all $\mathbf{v}\perp\mathbf{n}$. The unit vector $\mathbf{n}$ is the **axis of rotation**.

In the Minkowski metric i.e., for $\varepsilon = -1$, equation (6) shows that $|1-\beta|\geq 1$ and, because of equation (5), we put $1-\beta = \cosh\chi$ i.e., $\beta = 1-\cosh\chi$. This choice corresponds to the assumption that any time-like vector $\mathbf{v}\perp\mathbf{n}$ is time-equioriented with its image $\mathcal{R}(\mathbf{v})$ i.e., $\mathbf{v}\cdot\mathcal{R}(\mathbf{v}) < 0$. In particular, $\chi = 0$ corresponds to $\mathcal{R}(\mathbf{v}) = \mathbf{v}$. Equation (6) implies $\alpha^2 = \sinh^2\chi$. If we choose $\alpha = \sinh\chi$ then we obtain the Rodrigues formula for rotations in the Minkowski three-space with a space-like axis $\mathbf{n}$,

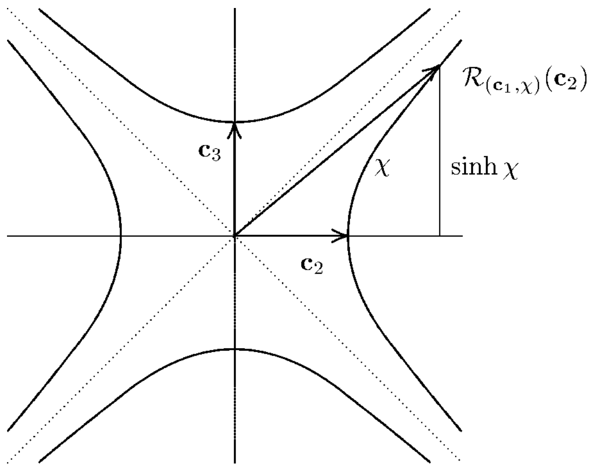
$$\boxed{\mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{v}) = \mathbf{v} + \sinh\chi\,\mathbf{n}\times\mathbf{v} + (1-\cosh\chi)(\mathbf{n}\cdot\mathbf{v}\,\mathbf{n} - \mathbf{v})} \tag{9}$$

It follows that for all non-light-like $\mathbf{v}$ orthogonal to $\mathbf{n}$,

$$\boxed{\sinh\chi = -\frac{\mathbf{v}\times\mathcal{R}(\mathbf{v})}{\mathbf{v}^2}\cdot\mathbf{n}, \qquad \cosh\chi = \frac{\mathbf{v}\cdot\mathcal{R}(\mathbf{v})}{\mathbf{v}^2}} \tag{10}$$

The choice $\alpha = \sinh\chi$ (instead of $\alpha = -\sinh\chi$) is in accordance with the condition

$$\mathcal{R}_{(\mathbf{c}_1,\chi)}(\mathbf{c}_2) = \cosh\chi\,\mathbf{c}_2 + \sinh\chi\,\mathbf{c}_3.$$



B.2.5 Standard symplectic structure of an orientable surface. Let us consider a surface $\mathcal{S} \subset \mathbb{R}^3$ described by a parametric equation

$$\mathbf{x} = \mathbf{x}(u^1, u^2) = \mathbf{x}(u^\alpha).$$

The tangent vectors

$$\mathbf{e}_\alpha = \partial_\alpha \mathbf{x}$$

are assumed to be pointwise independent, so that they form a **tangent frame**. With this frame we associate the coefficients of the **first fundamental form**

$$A_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta.$$

The **dual frame** is defined by

$$\mathbf{e}^\alpha = A^{\alpha\beta}\, \mathbf{e}_\beta, \qquad \mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta^\alpha_\beta.$$

The covariant components of a tangent vector $\mathbf{p}$ are

$$p_\alpha = \mathbf{p} \cdot \mathbf{e}_\alpha.$$

The **Christoffel symbols** and the coefficient of the **second fundamental vector valued form** are defined by:

$$\partial_\alpha \mathbf{e}_\beta = \Gamma^\gamma_{\alpha\beta}\, \mathbf{e}_\gamma + \mathbf{B}_{\alpha\beta}, \qquad \mathbf{B}_{\alpha\beta} \cdot \mathbf{e}_\gamma = 0.$$

A regular surface $\mathcal{S} \subset \mathbb{R}^3$ is orientable if it admits a global orthogonal vector field $\mathbf{n} \neq 0$. We assume that $\mathbf{n}$ is a unit vector, $\mathbf{n}^2 = \pm 1$.

Let us consider the two-form σ on tangent vectors defined by

$$\sigma(\mathbf{u}, \mathbf{v}) = \mathbf{n} \cdot \mathbf{u} \times \mathbf{v}, \qquad \mathbf{u}, \mathbf{v} \in T\mathcal{S}.$$

This is the **area two-form**. Its integral over a compact subset $U \subseteq \mathcal{S}$ gives, by definition, the area of U. By setting

$$\sigma_{\alpha\beta} = \sigma(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \mathbf{n} \cdot \mathbf{e}_\alpha \times \mathbf{e}_\beta,$$

we get

$$\sigma(\mathbf{u}, \mathbf{v}) = \mathbf{n} \cdot \mathbf{e}_\alpha \times \mathbf{e}_\beta\, u^\alpha\, v^\beta = \sigma_{\alpha\beta}\, u^\alpha v^\beta = \sigma_{12}(u^1 v^2 - u^2 v^1).$$

Since

$$v^\alpha = \langle \mathbf{v}, du^\alpha \rangle = \mathbf{v} \cdot \mathbf{e}^\alpha,$$

it follows that

$$\sigma = \tfrac{1}{2}\,\sigma_{\alpha\beta}\, du^\alpha \wedge du^\beta = \sigma_{12}\, du^1 \wedge du^2.$$

The area two-form is non degenerate thus, it is a symplectic form (a two-form on a two-dimensional surface is obviously closed).

The Hamiltonian vector field $\mathbf{X}_f$ associated with a function $f(\mathbf{r}) = f(u^\alpha)$ is defined by equations

$$\begin{aligned}
X_f^\alpha\, \sigma_{\alpha\beta} = -\,\partial_\beta f \quad &\Leftrightarrow \quad X_f^\alpha\, \mathbf{n}\cdot\mathbf{e}_\alpha \times \mathbf{e}_\beta = -\,\partial_\beta f\\
&\Leftrightarrow \quad \mathbf{n}\cdot\mathbf{X}_f \times \mathbf{e}_\beta = -\,\partial_\beta f\\
&\Leftrightarrow \quad \mathbf{X}_f\cdot\mathbf{e}_\beta \times \mathbf{n} = -\,\partial_\beta f\\
&\Leftrightarrow \quad \mathbf{X}_f\cdot\mathbf{n} \times \mathbf{e}_\beta = \partial_\beta f\\
&\Leftrightarrow \quad \mathbf{X}_f\cdot\mathbf{n} \times \mathbf{e}_\beta\, A^{\beta\alpha} = A^{\beta\alpha}\,\partial_\beta f\\
&\Leftrightarrow \quad \mathbf{X}_f\cdot\mathbf{n} \times \mathbf{e}^\alpha = (\nabla f)^\alpha\\
&\Leftrightarrow \quad \mathbf{X}_f \times \mathbf{n}\cdot\mathbf{e}^\alpha = (\nabla f)^\alpha\\
&\Leftrightarrow \quad \mathbf{X}_f \times \mathbf{n} = \nabla f,
\end{aligned}$$

where ∇ is the gradient operator on the surface. This shows that the Hamiltonian vector field $\mathbf{X}_f$ is defined by the implicit equation

$$\mathbf{X}_f \times \mathbf{n} = \nabla f. \tag{1}$$

We have

$$\nabla f \times \mathbf{n} = (\mathbf{X}_f \times \mathbf{n}) \times \mathbf{n} = \varepsilon\,(\mathbf{X}_f\cdot\mathbf{n}\,\mathbf{n} - \mathbf{n}\cdot\mathbf{n}\,\mathbf{X}_f) = -\,\varepsilon\,\mathbf{n}^2\,\mathbf{X}_f,$$

since $\mathbf{X}_f\cdot\mathbf{n} = 0$, being $\mathbf{X}_f$ tangent to the sphere. It follows that

$$\mathbf{X}_f = \varepsilon\,\mathbf{n}^2\,\mathbf{n}\times\nabla f. \tag{2}$$

This gives the explicit definition of $\mathbf{X}_f$. The PB is defined by

$$\{f,g\}_\sigma = \sigma(\mathbf{X}_f,\mathbf{X}_g) = \mathbf{n}\cdot\mathbf{X}_f\times\mathbf{X}_g = \mathbf{n}\cdot(\mathbf{n}\times\nabla f)\times(\mathbf{n}\times\nabla g).$$

Thus,

$$\boxed{\{f,g\}_\sigma = \varepsilon\,\mathbf{n}^2\,\mathbf{n}\cdot(\nabla f\times\nabla g)} \tag{3}$$

Assume that $f(\mathbf{x})$ is the restriction to $\mathbf{x}\in\mathcal{S}$ of a function $\mathcal{F}(\mathbf{x})$ on $\mathbb{R}^3$. Since the gradient ∇f of a function f on a submanifold $\mathcal{S}$ of a Riemannian manifold $\mathbb{R}^3$

is simply the orthogonal projection to the tangent space of $\mathcal{S}$ of the gradient $\nabla\mathcal{F}$ of any (local) extension $\mathcal{F}$ of f, we have

$$\nabla\mathcal{F}(\mathbf{x}) = \nabla f + h(\mathbf{x})\,\mathbf{n}.$$

It follows that on the surface

$$\mathbf{n}\cdot(\nabla f\times\nabla g) = \mathbf{n}\cdot(\nabla\mathcal{F}\times\nabla\mathcal{G})$$

and

$$\boxed{\{f,g\}_\sigma = \varepsilon\,\mathbf{n}^2\,\mathbf{n}\cdot(\nabla\mathcal{F}\times\nabla\mathcal{G})} \tag{4}$$

This formula gives the PB of functions $f(\mathbf{x})$ on the surface in terms of local extensions $\mathcal{F}(\mathbf{x})$.

B.2.6 The PB of functions of a special kind. With any (smooth) function $\mathcal{F}(\mathbf{x})$ on $\mathbb{R}^3$ we associate a function $F(\mathbf{q},\mathbf{p})$ on $T^*\mathbb{R}^3$ defined by

$$F(\mathbf{q},\mathbf{p}) = \mathcal{F}(\mathbf{x}), \qquad \mathbf{x} = \mathbf{q}\times\mathbf{p}. \tag{1}$$

Let us compute the PB of functions of this kind. In the canonical basis, we have $\mathbf{q} = q^i\mathbf{c}_i$, $\mathbf{p} = p^i\mathbf{c}_i$, and

$$x^i = V^{ijk}\,q_j\,p_k.$$

Hence,

$$\frac{\partial x^i}{\partial q^l} = V^{ijk}\,g_{jl}\,p_k, \qquad \frac{\partial x^i}{\partial p_l} = V^{ijl}\,q_j,$$

$$\frac{\partial G}{\partial q^l} = \frac{\partial\mathcal{G}}{\partial x^i}\frac{\partial x^i}{\partial q^l} = \frac{\partial\mathcal{G}}{\partial x^i}V^{ijk}\,g_{jl}\,p_k, \qquad \frac{\partial F}{\partial p_l} = \frac{\partial\mathcal{F}}{\partial x^i}\frac{\partial x^i}{\partial p_l} = \frac{\partial\mathcal{F}}{\partial x^i}V^{ijl}\,q_j.$$

$$\begin{aligned}
\frac{\partial F}{\partial p_l}\frac{\partial G}{\partial q^l} &= \frac{\partial\mathcal{F}}{\partial x^i}V^{ijl}\,q_j\,\frac{\partial\mathcal{G}}{\partial x^r}V^{rsk}\,g_{sl}\,p_k \\
&= \frac{\partial\mathcal{F}}{\partial x^i}\,\frac{\partial\mathcal{G}}{\partial x^r}\,g^{ih}\,V_{hjs}\,V^{krs}\,q^j\,p_k \\
&= \varepsilon\,\frac{\partial\mathcal{F}}{\partial x^i}\,\frac{\partial\mathcal{G}}{\partial x^r}\,g^{ih}\,(\delta^k_h\delta^r_j - \delta^k_j\delta^r_h)\,q^j\,p_k \\
&= \varepsilon\,\big(\mathbf{p}\cdot\nabla\mathcal{F}\;\mathbf{q}\cdot\nabla\mathcal{G} - \mathbf{q}\cdot\mathbf{p}\;\nabla\mathcal{F}\cdot\nabla\mathcal{G}\big).
\end{aligned}$$

$$\{F,G\} = \frac{\partial F}{\partial p_l}\frac{\partial G}{\partial q^l} - \frac{\partial G}{\partial p_l}\frac{\partial F}{\partial q^l} = \varepsilon\big(\mathbf{p}\cdot\nabla\mathcal{F}\;\mathbf{q}\cdot\nabla\mathcal{G} - \mathbf{q}\cdot\nabla\mathcal{F}\;\mathbf{p}\cdot\nabla\mathcal{G}\big).$$

Since

$$\begin{aligned}(\mathbf{q}\times\mathbf{p})\cdot(\nabla\mathcal{F}\times\nabla\mathcal{G}) &= \mathbf{q}\cdot\mathbf{p}\times(\nabla\mathcal{F}\times\nabla\mathcal{G}) = \varepsilon\,\mathbf{q}\cdot(\nabla\mathcal{G}\cdot\mathbf{p}\,\nabla\mathcal{F}-\nabla\mathcal{F}\cdot\mathbf{p}\,\nabla\mathcal{G})\\ &= \varepsilon\,(\nabla\mathcal{G}\cdot\mathbf{p}\,\nabla\mathcal{F}\cdot\mathbf{q}-\nabla\mathcal{F}\cdot\mathbf{p}\,\nabla\mathcal{G}\cdot\mathbf{q}),\end{aligned}$$

we have proved that

$$\boxed{\{F,G\} = -(\mathbf{q}\times\mathbf{p})\cdot(\nabla\mathcal{F}\times\nabla\mathcal{G})} \tag{2}$$

This formula gives the PB of functions $F(\mathbf{q},\mathbf{p})$ on $T^*\mathbb{R}^3$ of the type (1).

B.3 The Hamilton principal function of $\mathbb{S}_2$.

The basic objects are: (i) the space $\mathbb{R}^3$ endowed with the Euclidean metric; (ii) the configuration manifold $\mathcal{Q}=\mathbb{S}_2=(\mathbf{q})\subset\mathbb{R}^3$, defined by $\mathbf{q}^2=\mathbf{1}$; (iii) the cotangent bundle $T^*Q=T^*\mathbb{S}_2$: it consists of pairs $(\mathbf{q},\mathbf{p})$, where $\mathbf{p}$ is interpreted as a vector tangent to $\mathbb{S}_2$ at the point $\mathbf{q}$, by setting $\langle\mathbf{v},\mathbf{p}\rangle=\mathbf{v}\cdot\mathbf{p}$ for each vector $\mathbf{v}$ tangent to $\mathbb{S}_2$ at $\mathbf{q}$; (iv) the eikonal equation (coisotropic submanifold) $C\subset T^*Q$ defined by equation $\mathbf{p}^2=\mathbf{1}$.

The oriented geodesics on $\mathbb{S}_2$ are in one-to-one correspondence with the unit vectors $\mathbf{n}$. The geodesic corresponding to $\mathbf{n}$ is the intersection of $Q=\mathbb{S}_2$ with the plane $\Pi_{\mathbf{n}}$ orthogonal to $\mathbf{n}$ and passing through the origin. The orientation of this maximal circle is determined by the formula $\mathbf{p}=\mathbf{n}\times\mathbf{q}$, equivalent to $\mathbf{n}=\mathbf{q}\times\mathbf{p}$, where $\mathbf{q}\in Q$ and $\mathbf{p}$ is the unit vector tangent to the oriented circle. Since the oriented geodesics are in one-to-one correspondence with the characteristics of C, we have

Theorem 1. *The set $\mathcal{S}$ of the characteristics of C is a differentiable manifold diffeomorphic to the unit sphere $\mathbb{S}_2$. The one-to-one correspondence between characteristics $\gamma(\mathbf{n})$ of C and the unit vectors $\mathbf{n}\in\mathbb{S}_2$ is given by*

$$(\mathbf{q},\mathbf{p})\in\gamma(\mathbf{n}) \iff \begin{cases}\mathbf{q}^2=1 & (\mathbf{q}\in Q=\mathbb{S}_2)\\ \mathbf{p}^2=1 & (\mathbf{p}\in C)\\ \mathbf{q}\cdot\mathbf{p}=0 & (\mathbf{p}\in T^*Q)\\ \mathbf{n}=\mathbf{q}\times\mathbf{p}. & \end{cases} \tag{1}$$

It follows that two pairs $(\mathbf{q}_0,\mathbf{p}_0)$ and $(\mathbf{q}_1,\mathbf{p}_1)$ of $T^*\mathcal{Q}$ belong to a same characteristics $\gamma(\mathbf{n})$ if and only if the above equations are satisfied with $\mathbf{n}=\mathbf{q}_0\times\mathbf{p}_0=\mathbf{q}_1\times\mathbf{p}_1$ or equivalently, if and only if $\mathbf{p}_0=\mathbf{n}\times\mathbf{q}_0$ and $\mathbf{p}_1=\mathbf{n}\times\mathbf{q}_1$. Since $\mathbf{q}_0$ and $\mathbf{q}_1$ are both orthogonal to $\mathbf{n}$, we can consider the rotation with axis $\mathbf{n}$ which maps $\mathbf{q}_0$ to $\mathbf{q}_1$. The axis $\mathbf{n}$ is determined (even in the case $\mathbf{q}_0=\mathbf{q}_1$) by setting $\mathbf{n}=\mathbf{q}_0\times\mathbf{p}_0$ (or $\mathbf{n}=\mathbf{q}_1\times\mathbf{p}_1$). This proves

Theorem 2. *A pair* $\big((\mathbf{q}_0,\mathbf{p}_0),(\mathbf{q}_1,\mathbf{p}_1)\big)$ *belongs to the characteristic relation* D_C *if and only if* $\mathbf{q}_0 \in \mathbb{S}_2$ *and there exists a pair* $(\mathbf{n},\theta) \in \mathbb{S}_2 \times \mathbb{R}$ *such that*

$$\begin{cases} \mathbf{q}_1 = \mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{q}_0) \\ \mathbf{p}_1 = \mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{p}_0) \\ \mathbf{q}_0 \cdot \mathbf{n} = \mathbf{q}_1 \cdot \mathbf{n} = 0 \\ \mathbf{p}_1 = \mathbf{n} \times \mathbf{q}_1, \end{cases} \tag{2}$$

where $\mathcal{R}_{(\mathbf{n},\theta)}$ *is the rotation of axis* $\mathbf{n}$ *and angle* θ*,*

$$\mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{v}) = \mathbf{v} + \sin\theta\, \mathbf{n} \times \mathbf{v} + (1-\cos\theta)(\mathbf{n}\cdot\mathbf{v}\,\mathbf{n} - \mathbf{v}). \tag{3}$$

Then we can prove

Theorem 3. *The characteristic relation* D_C *is generated by the family*

$$S\colon (\mathbb{S}_2 \times \mathbb{S}_2) \times (\mathbb{R} \times \mathbb{R} \times \mathbb{S}_2 \times \mathbb{R}^3) \longrightarrow \mathbb{R},$$

defined by

$$\boxed{\begin{aligned} S(\mathbf{q}_1,\mathbf{q}_0;\lambda,\theta,\mathbf{n},\mathbf{v}) = \theta + \lambda\,\big((\mathbf{q}_0\cdot\mathbf{n})^2 + (\mathbf{q}_1\cdot\mathbf{n})^2 + (\mathbf{v} - \mathbf{n}\times\mathbf{q}_1)^2\big) \\ + \mathbf{v}\cdot\big(\mathbf{q}_1 - \mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{q}_0)\big) \end{aligned}} \tag{4}$$

The critical set Ξ *of* S *is described by equations*

$$\mathbf{q}_0\cdot\mathbf{n} = \mathbf{q}_1\cdot\mathbf{n} = 0, \qquad \mathbf{v} = \mathbf{n}\times\mathbf{q}_1, \qquad \mathbf{q}_1 = \mathcal{R}(\mathbf{q}_0). \tag{5}$$

This generating family is not a Morse family.

Proof. The equations generated by S are

$$\begin{cases} 0 = \dfrac{\partial S}{\partial \lambda} \\ 0 = \dfrac{\partial S}{\partial \mathbf{v}} \\ 0 = \dfrac{\partial S}{\partial \theta} \\ 0 = \dfrac{\partial S}{\partial \mathbf{n}}, \end{cases} \qquad \begin{cases} \mathbf{p}_0 = -\dfrac{\partial S}{\partial \mathbf{q}_0} \\ \mathbf{p}_1 = \dfrac{\partial S}{\partial \mathbf{q}_1}, \end{cases} \tag{6}$$

The first four equations describe the critical set Ξ. For all $\mathbf{n} \in \mathbb{S}_2$ we have

$$\frac{\partial S}{\partial \mathbf{n}} = \nabla_{\mathbf{x}} S\big|_{\mathbf{x}=n}(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}) = P_{\mathbf{n}}(\nabla_{\mathbf{x}} S\big|_{\mathbf{x}=n}), \tag{7}$$

where

$$P_{\mathbf{n}} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}.$$

Similar equations hold for $\mathbf{q}_0$ and $\mathbf{q}_1$. The first two equations (6) of the critical set read

$$0 = \frac{\partial S}{\partial \lambda} = (\mathbf{q}_0 \cdot \mathbf{n})^2 + (\mathbf{q}_1 \cdot \mathbf{n})^2 + (\mathbf{v} - \mathbf{n} \times \mathbf{q}_1)^2$$

and

$$0 = \frac{\partial S}{\partial \mathbf{v}} = 2\lambda(\mathbf{v} - \mathbf{n} \times \mathbf{q}_1) + \mathbf{q}_1 - \mathcal{R}(\mathbf{q}_0).$$

These are equivalent to equations (5). We shall see below that the remaining equations (6) of Ξ are identically satisfied. Due to equations (5), on the critical set we have

$$\begin{aligned}
\mathbf{p}_0 &= -\frac{\partial S}{\partial \mathbf{q}_0} = \frac{\partial}{\partial \mathbf{q}_0}\big(\mathbf{v} \cdot \mathcal{R}(\mathbf{q}_0)\big) = \frac{\partial}{\partial \mathbf{q}_0}\big(\mathbf{q}_0 \cdot \mathcal{R}^\top(\mathbf{v})\big) \\
&= \mathcal{R}^\top(\mathbf{v})(\mathbf{I} - \mathbf{q}_0 \otimes \mathbf{q}_0) = \mathcal{R}^\top(\mathbf{v}) - \mathcal{R}^\top(\mathbf{v}) \cdot \mathbf{q}_0\, \mathbf{q}_0 = \mathcal{R}^\top(\mathbf{v}) - \mathbf{v} \cdot \mathcal{R}(\mathbf{q}_0) \\
&= \mathcal{R}^\top(\mathbf{v}) - \mathbf{v} \cdot \mathbf{q}_1 = \mathcal{R}^\top(\mathbf{v}) - \mathbf{n} \times \mathbf{q}_1 \cdot \mathbf{q}_1 = \mathcal{R}^\top(\mathbf{v})
\end{aligned}$$

and

$$\mathbf{p}_1 = -\frac{\partial S}{\partial \mathbf{q}_1} = \mathbf{v}\,(\mathbf{I} - \mathbf{q}_1 \otimes \mathbf{q}_1) = \mathbf{v}.$$

Thus, being $\mathcal{R}^\top = \mathcal{R}^{-1}$,

$$\mathbf{p}_0 = \mathcal{R}^{-1}(\mathbf{p}_1).$$

All equations (2) have been found. We show that the last two equations (6) of the critical set are identically satisfied. On the critical set the vector $\mathbf{q}_0 \times \mathbf{v}$ is parallel to $\mathbf{n}$,

$$\mathbf{q}_0 \times \mathbf{v} = \mathbf{q}_0 \times (\mathbf{n} \times \mathbf{q}_1) = \mathbf{q}_0 \cdot \mathbf{q}_1\, \mathbf{n} - \mathbf{q}_0 \cdot \mathbf{n}\, \mathbf{q}_1 = \mathbf{q}_0 \cdot \mathbf{q}_1\, \mathbf{n}. \tag{8}$$

so that

$$\begin{aligned}
\frac{\partial S}{\partial \mathbf{n}} &= \frac{\partial}{\partial \mathbf{n}}\big(\mathbf{v} \cdot \mathcal{R}(\mathbf{q}_0)\big) \\
&= \sin\,\theta \frac{\partial}{\partial \mathbf{n}}(\mathbf{n} \cdot \mathbf{q}_0 \times \mathbf{v}) + (1 - \cos\theta)\frac{\partial}{\partial \mathbf{n}}(\mathbf{n} \cdot \mathbf{q}_0\, \mathbf{n} \cdot \mathbf{v}) \\
&= \sin\theta P_{\mathbf{n}}(\mathbf{q}_0 \times \mathbf{v}) = 0.
\end{aligned}$$

Finally, since on the critical set $\mathbf{n} \cdot \mathbf{v} = 0$,

$$\begin{aligned}
\frac{\partial S}{\partial \theta} &= 1 - \mathbf{v} \cdot \Big(\cos\theta \mathbf{n} \times \mathbf{q}_0 + \sin\theta\,(\mathbf{n} \cdot \mathbf{q}_0\, \mathbf{n} - \mathbf{q}_0)\Big) \\
&= 1 - \cos\theta\, \mathbf{v} \cdot \mathbf{n} \times \mathbf{q}_0 + \sin\theta\, \mathbf{v} \cdot \mathbf{q}_0.
\end{aligned} \tag{9}$$

Because of $(8)_2$ of § 2.4,

$$\mathbf{v}\cdot\mathbf{n}\times\mathbf{q}_0=(\mathbf{n}\times\mathcal{R}(\mathbf{q}_0))\cdot(\mathbf{n}\times\mathbf{q}_0)=\mathcal{R}(\mathbf{q}_0)\cdot\mathbf{q}_0=\mathbf{q}_0^2\,\cos\theta=\cos\theta, \tag{10}$$

being $\mathbf{q}_0$ orthogonal to $\mathbf{n}$. Moreover, because of $(8)_1$ of §B.2.4 for $\mathbf{v}^2=\mathbf{1}$,

$$\mathbf{v}\cdot\mathbf{q}_0=\mathbf{n}\times\mathcal{R}(\mathbf{q}_0)\cdot\mathbf{q}_0=\mathbf{n}\cdot\mathcal{R}(\mathbf{q}_0)\times\mathbf{q}_0=\,-\,\mathbf{q}_0^2\,\sin\theta=\,-\sin\theta. \tag{11}$$

Thus,

$$\frac{\partial S}{\partial\theta}=\mathbf{1}-\cos^2\theta-\sin^2\theta=0.$$

Since

$$S_\lambda=\frac{\partial S}{\partial\lambda}=(\mathbf{q}_0\cdot\mathbf{n})^2+(\mathbf{q}_1\cdot\mathbf{n})^2+(\mathbf{v}-\mathbf{n}\times\mathbf{q}_1)^2,$$

on the critical set we have $dS_\lambda=0$. This shows that S is not a Morse family. ∎

Theorem 4. *An equivalent reduced generating family of D_C is*

$$S'\colon(\mathbb{S}_2\times\mathbb{S}_2)\times(\mathbb{R}\times\mathbb{R}\times\mathbb{S}_2)\longrightarrow\mathbb{R}$$

$$\boxed{S'(\mathbf{q}_1,\mathbf{q}_0;\lambda,\theta,\mathbf{n})=\theta+\lambda\left((\mathbf{q}_0\cdot\mathbf{n})^2+(\mathbf{q}_1\cdot\mathbf{n})^2\right)-\mathbf{n}\times\mathbf{q}_1\cdot\mathcal{R}_{(\mathbf{n},\theta)}(\mathbf{q}_0)} \tag{12}$$

Proof. By means of equations $\mathbf{v}=\mathbf{n}\times\mathbf{q}_1$ of the critical set we can remove the supplementary variable $\mathbf{v}$ of S. Thus, we get the reduced generating family S'. ∎

Remark 1. On the critical set the generating family reduces to $S=\theta$. This function is obviously symmetric in $(\mathbf{q}_0,\mathbf{q}_1)$, in accordance with the symmetry of the characteristic relation. Also the reduced generating family S' is not a Morse family.

Remark 2. We consider the inclusion relation

$$R\subset\mathbb{S}_2\times\mathbb{R}^3=\{(\mathbf{q},\mathbf{x})\mid\mathbf{q}=\mathbf{x}\}.$$

The canonical lift $\widehat{R}\subset T^*\mathbb{S}_2\times T^*\mathbb{R}^3$ is a symplectic reduction whose inverse image $\widehat{R}^\top\circ(T^*\mathbb{S}_2)$ is the coisotropic submanifold $T^*_{\mathbb{S}_2}\mathbb{R}^3$ of the covectors $\mathbf{p}\in\mathbb{R}^3$ based at points of $\mathbb{S}_2$. The fibres of this reduction are the equivalence classes of the equivalence relation

$$\mathbf{p}\sim\mathbf{p}'\quad\Longleftrightarrow\quad\mathbf{p},\mathbf{p}'\text{ based at the same point }\mathbf{q}\in\mathbb{S}_2,\;\;\mathbf{p}-\mathbf{p}'\perp\mathbb{S}_2,$$

i.e.,

$$\mathbf{p}\sim\mathbf{p}'\quad\Longleftrightarrow\quad\mathbf{p},\mathbf{p}'\text{ based at the same point }\mathbf{q}\in\mathbb{S}_2,\;\;(\mathbf{p}-\mathbf{p}')\times\mathbf{q}=0. \tag{13}$$

A second symplectic reduction we have to consider is the characteristic reduction associated with the coisotropic submanifold C,

$$R_C \subset \mathcal{S} \times T^*Q.$$

This is the graph of the surjective submersion which maps a point of C to the characteristic which contains this point. This reduction defines a reduced symplectic form ω on the space $\mathcal{S}$ of the characteristics. We prove that

Theorem 5. *The reduced symplectic form ω is the opposite of the standard symplectic form σ on $\mathbb{S}_2$ defined by*

$$\sigma(\mathbf{u}, \mathbf{v}) = \mathbf{n} \cdot \mathbf{u} \times \mathbf{v}, \tag{14}$$

where $\mathbf{n} \in \mathbb{S}_2$ and $(\mathbf{u}, \mathbf{v})$ are vectors tangent to $\mathbb{S}_2$ at the point $\mathbf{n}$.

Proof. (I) With any arbitrary (smooth) function $\mathcal{F}(\mathbf{x})$ on $\mathbb{R}^3$ we associate a function $F(\mathbf{q}, \mathbf{p})$ on $T^*\mathbb{R}^3$,

$$F(\mathbf{q}, \mathbf{p}) = \mathcal{F}(\mathbf{q} \times \mathbf{p}), \tag{15}$$

and a function $f(\mathbf{n})$ on $\mathcal{S} = \mathbb{S}_2$,

$$f(\mathbf{n}) = \mathcal{F}(\mathbf{n}) \tag{16}$$

(this is simply the restriction of $\mathcal{F}$ to the sphere).

(II) The functions F on $T^*\mathbb{R}^3$ are constant on the fibres of the first reduction $\widehat{R}$, since on a fibre we have $\mathbf{q} \times \mathbf{p} = \mathbf{q} \times \mathbf{p}'$ and $\mathbf{q} \in \mathbb{S}_2$, cf. (13). Hence, these functions reduces to functions on $T^*\mathbb{S}_2$ by taking $\mathbf{q}^2 = 1$ and $\mathbf{p} \cdot \mathbf{q} = 0$.

(III) When restricted to the submanifold C, by taking $\mathbf{p}^2 = 1$, a function $F(\mathbf{q}, \mathbf{p})$ of the kind (15) is constant on each characteristic $\gamma(\mathbf{n})$ because of (1), so that it reduces to a function $f(\mathbf{n}) = \mathcal{F}(\mathbf{n})$, with $\mathbf{n} = \mathbf{q} \times \mathbf{p}$ (note that $\mathbf{n}^2 = 1$, since $\mathbf{q}$ and $\mathbf{p}$ are orthogonal unit vectors).

(IV) Let us consider on the sphere $\mathcal{S} = \mathbb{S}_2$ the standard symplectic form (14). By formula (3) of §B.2.5 we get the PB

$$\{f, g\}_\sigma = \mathbf{n} \cdot (\nabla f \times \nabla g). \tag{17}$$

(V) We recall that, according to the general theory of the symplectic reductions, the PB of two functions $f(\mathbf{n})$ on the reduced symplectic manifold $(\mathcal{S}, \omega)$ is defined by

$$\{f, g\}_\omega(\mathbf{n}) = \{F, G\}(\mathbf{q}, \mathbf{p}), \tag{18}$$

where $(\mathbf{q}, \mathbf{p}) \in \gamma(\mathbf{n}) \subset C \subset T^*\mathbb{S}_2$ and where F and G are any two functions on $T^*\mathbb{R}^3$ constant on the characteristics γ of C. Because of (III) the functions

F and f defined by (15) and (16) by means of any function $\mathcal{F}$, fit with this scheme. Hence, $(\mathbf{q}, \mathbf{p}) \in \gamma(\mathbf{n})$ means in particular that $\mathbf{n} = \mathbf{q} \times \mathbf{p}$, see (1). Then, by applying formulae (18), (2) of § 2.6 and (4) of § 2.5 for $\varepsilon = 1$, which reads $\{f, g\}_\sigma = \mathbf{n} \cdot \nabla\mathcal{F} \times \nabla\mathcal{G}$, we get

$$\text{(19)} \qquad \begin{aligned} \{f, g\}_\omega(\mathbf{n}) = \{F, G\}(\mathbf{q}, \mathbf{p}) &= -\,\mathbf{q} \times \mathbf{p} \cdot (\nabla\mathcal{F} \times \nabla\mathcal{G}) \\ &= -\,\mathbf{n} \cdot \nabla\mathcal{F} \times \nabla\mathcal{G} = -\,\{f, g\}_\sigma(\mathbf{n}). \end{aligned}$$

This holds for all functions $\mathcal{F}(\mathbf{x})$. We remark that any function $f(\mathbf{n})$ on the sphere $\mathcal{S}$ admits an extension $\mathcal{F}(\mathbf{x})$ to $\mathbb{R}^3$, such that $\mathcal{F}(\mathbf{n}) = f(\mathbf{n})$. We can for instance extend the function f by constant values along the half lines issued from the origin of $\mathbb{R}^3$ (the origin must be excluded, but this exclusion is irrelevant). Thus, the equality

$$\{f, g\}_\sigma(\mathbf{n}) = -\,\{f, g\}_\omega(\mathbf{n})$$

holds for all f and g on $\mathcal{S}$. This shows that $\omega = -\sigma$. ∎

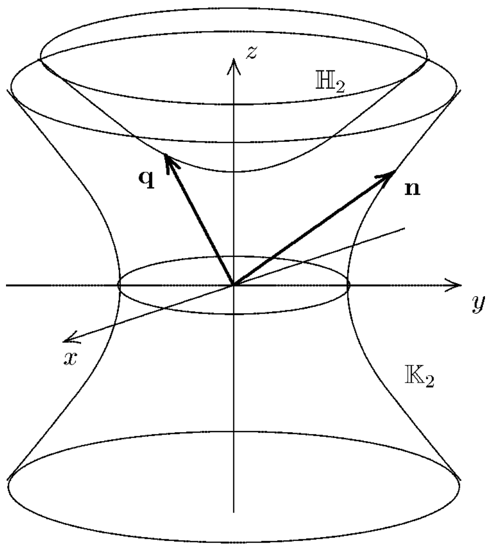
Note that in definition (14) of σ the normal vector $\mathbf{n}$ is oriented outside the sphere. If we choose it pointing to the center, then we get $\sigma = \omega$.

B.4 The Hamilton principal function of $\mathbb{H}_2$

The basic objects are: (i) The space $\mathbb{R}^3$ endowed with the Minkowskian metric, with z time-like. (ii) The configuration manifold $\mathcal{Q} = \mathbb{H}_2 = (\mathbf{q}) \subset \mathbb{R}^3$, defined by

$$\begin{cases} \mathbf{q}^2 = -1 \\ \mathbf{q} \cdot \mathbf{c}_3 < 0 \end{cases} \iff \begin{cases} x^2 + y^2 - z^2 + 1 = 0 \\ z > 0. \end{cases}$$

The Minkowskian metric induces on $\mathbb{H}_2$ a positive-definite metric and $\mathbf{q} \in \mathbb{H}_2$ implies $\mathbf{q} \perp \mathbb{H}_2$. Indeed, for any curve $\mathbf{q}(t) \in \mathbb{H}_2$ we have $\dot{\mathbf{q}} \cdot \mathbf{q} = 0$ thus, $\dot{\mathbf{q}}$ (tangent to $\mathbb{H}_2$) is space-like (every non-zero vector orthogonal to a time-like vector is space-like). (iii) The cotangent bundle $T^*Q = T^*\mathbb{H}_2 = (\mathbf{q}, \mathbf{p})$, where $\mathbf{p}$ is a vector tangent to $\mathbb{H}_2$ at $\mathbf{q}$. (iv) The eikonal equation (coisotropic submanifold) $C \subset T^*Q$ defined by equation $\mathbf{p}^2 = 1$. Note that the covectors $(\mathbf{q}, \mathbf{p}) \in T^*\mathbb{H}_2$ can be interpreted as vectors $\mathbf{p}$ tangent to $\mathbb{H}_2$ by setting $\langle \mathbf{v}, \mathbf{p} \rangle = \mathbf{v} \cdot \mathbf{p}$ for each vector $\mathbf{v}$ tangent to $\mathbb{H}_2$ at the point $\mathbf{q}$. (v) The hyperboloid $\mathbb{K}_2$ of the unit space-like vectors $\mathbf{n}$, $\mathbf{n}^2 = 1$. The metric induced on $\mathbb{K}_2$ is Lorentzian.



Theorem 1. *The set $\mathcal{S}$ of the characteristics of C is a differentiable manifold diffeomorphic to $\mathbb{K}_2$. The one-to-one correspondence between characteristics $\gamma(\mathbf{n})$ of C and the unit vectors $\mathbf{n} \in \mathcal{S}$ is given by*

$$(1) \qquad (\mathbf{q}, \mathbf{p}) \in \gamma(\mathbf{n}) \quad \Leftrightarrow \quad \begin{cases} \mathbf{q}^2 = 1 & (\mathbf{q} \in Q = \mathbb{H}_2) \\ \mathbf{p}^2 = 1 & (\mathbf{p} \in C) \\ \mathbf{q} \cdot \mathbf{p} = 0 & (\mathbf{p} \in T^*Q) \\ \mathbf{n} = \mathbf{q} \times \mathbf{p}. \end{cases}$$

Proof. The geodesics of Q are the orbits of spontaneous motions (no active force). These motions admit the first integral

$$(2) \qquad \mathbf{n} = \mathbf{q} \times \dot{\mathbf{q}}.$$

Indeed,

$$\dot{\mathbf{n}} = \mathbf{q} \times \ddot{\mathbf{q}} = \mathbf{q} \times \mathbf{R},$$

where $\mathbf{R}$ is the reaction force orthogonal to $\mathbb{H}_2$. Since also $\mathbf{q}(t) \perp \mathbb{H}_2$, it follows that $\dot{\mathbf{n}} = 0$. From (2) it follows that: (i) $\dot{\mathbf{q}} \perp \mathbf{n}$, so that, for any fixed $\mathbf{n}$, the corresponding geodesic has velocity $\dot{\mathbf{q}}$ orthogonal to $\mathbf{n}$; (ii) $\mathbf{n}$ is space-like since it is orthogonal to the time-like vector $\mathbf{q}$ ($\mathbf{n} \neq 0$, since $\dot{\mathbf{q}}$ cannot be parallel to $\mathbf{q}$ unless $\dot{\mathbf{q}} = 0$. We can consider only geodesic motions with unit velocity, $\dot{\mathbf{q}}^2 = \mathbf{1}$. Due to (2), this is equivalent to assume $\mathbf{n}^2 = 1$, that is $\mathbf{n} \in \mathbb{K}_2$. Since the characteristics are in one-to-one correspondence with the oriented geodesics, the set $\mathcal{S}$ is identified with $\mathbb{K}_2$ and equations (1) follow by replacing $\dot{\mathbf{q}}$ with $\mathbf{p}$. ■

As a consequence of Theorem **1** we have

Theorem 2. *A pair* $\big((\mathbf{q}_0, \mathbf{p}_0), (\mathbf{q}_1, \mathbf{p}_1)\big)$ *belongs to the characteristic relation* D_C *if and only if* $\mathbf{q}_0 \in \mathbb{H}_2$ *and there exists a pair* $(\mathbf{n}, \chi) \in \mathbb{K}_2 \times \mathbb{R}$ *such that*

$$(3) \qquad \begin{cases} \mathbf{q}_1 = \mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{q}_0) \\ \mathbf{p}_1 = \mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{p}_0) \\ \mathbf{q}_0 \cdot \mathbf{n} = \mathbf{q}_1 \cdot \mathbf{n} = 0 \\ \mathbf{p}_1 = \mathbf{n} \times \mathbf{q}_1, \end{cases}$$

where $\mathcal{R}_{(\mathbf{n},\chi)}$ *is the rotation of axis* $\mathbf{n}$ *and pseudo-angle* χ,

$$(4) \qquad \mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{v}) = \mathbf{v} + \sinh\chi\, \mathbf{n} \times \mathbf{v} + (1 - \cosh\chi)(\mathbf{n} \cdot \mathbf{v}\, \mathbf{n} - \mathbf{v}).$$

Then we can prove

Theorem 3. *The characteristic relation* D_C *is generated by the family*

$$S\colon (\mathbb{H}_2 \times \mathbb{H}_2) \times (\mathbb{R} \times \mathbb{R} \times \mathbb{K}_2 \times \mathbb{R}^3) \longrightarrow \mathbb{R},$$

defined by

$$(5) \qquad \boxed{\begin{aligned} S(\mathbf{q}_0, \mathbf{q}_1; \lambda, \chi, \mathbf{n}, \mathbf{v}) = \chi + \lambda \Big((\mathbf{q}_0 \cdot \mathbf{n})^2 + (\mathbf{q}_1 \cdot \mathbf{n})^2 + (\mathbf{v} - \mathbf{n} \times \mathbf{q}_1)^2_+\Big) \\ + \mathbf{v} \cdot \Big(\mathbf{q}_1 - \mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{q}_0)\Big) \end{aligned}}$$

The critical set Ξ *of* $\mathcal{S}$ *is described by equations*

$$(6) \qquad \mathbf{q}_0 \cdot \mathbf{n} = \mathbf{q}_1 \cdot \mathbf{n} = 0, \qquad \mathbf{v} = \mathbf{n} \times \mathbf{q}_1, \qquad \mathbf{q}_1 = \mathcal{R}(\mathbf{q}_0).$$

This generating family is not a Morse family.

Note that in the definition (5) by the symbol $(\mathbf{u})^2_+$ we mean the scalar product $\mathbf{u} \cdot \mathbf{u}$ in the Euclidean metric.

Proof. The proof follows the same pattern of that concerning $\mathbb{S}_2$, with the following variants. (i) In (8) of §B.3 we used the double cross product formula, thus the second and the last terms should be multiplied by ε; in fact, this has no consequence and we get again $\partial S/\partial \mathbf{n} = 0$; (ii) formula (9) of § 3 is replaced by the analogous formula with $\cosh\chi$ and $-\sinh\chi$ instead of $\cos\theta$ and $\sin\theta$,

$$\frac{\partial S}{\partial \chi} = 1 - \cosh\chi\, \mathbf{v}\cdot\mathbf{n}\times\mathbf{q}_0 - \sinh\chi\, \mathbf{v}\cdot\mathbf{q}_0;$$

(iii) formula (10) involves the scalar product of two cross products, so that the second equality is multiplied by $\varepsilon = -1$; we get

$$\mathbf{v}\cdot\mathbf{n}\times\mathbf{q}_0 = -\mathcal{R}(\mathbf{q}_0)\cdot\mathbf{q}_0 = -\mathbf{q}_0^2\cosh\chi = \cosh\chi,$$

being $\mathbf{v} = \mathbf{n}\times\mathbf{q}_1 = \mathbf{n}\times\mathcal{R}(\mathbf{q}_0)$. In the present case

$$\mathbf{v}\cdot\mathbf{q}_0 = \mathbf{n}\cdot\mathcal{R}(\mathbf{q}_0)\times\mathbf{q}_0 = \mathbf{q}_0^2\sinh\chi = -\sinh\chi.$$

As a consequence,

$$\frac{\partial S}{\partial \chi} = 1 - \cosh^2\chi + \sinh^2\chi = 0. \qquad \blacksquare$$

We have theorems similar to Theorems 4 and 5 for the sphere:
Theorem 4. *An equivalent reduced generating family is*

$$S' \colon (\mathbb{H}_2\times\mathbb{H}_2)\times(\mathbb{R}\times\mathbb{R}\times\mathbb{K}_2) \longrightarrow \mathbb{R},$$

defined by

$$\boxed{S'(\mathbf{q}_0,\mathbf{q}_1;\lambda,\chi,\mathbf{n}) = \chi + \lambda\left((\mathbf{q}_0\cdot\mathbf{n})^2 + (\mathbf{q}_1\cdot\mathbf{n})^2\right) - \mathbf{n}\times\mathbf{q}_1\cdot\mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{q}_0)} \tag{7}$$

Proof. The proof is similar to that of $\mathbb{S}_2$. ■
Note that, in (7),

$$\mathbf{n}\times\mathbf{q}_1\cdot\mathcal{R}_{(\mathbf{n},\chi)}(\mathbf{q}_0) = \cosh\chi\,\mathbf{n}\cdot\mathbf{q}_1\times\mathbf{q}_0 + \sinh\chi\,(\mathbf{n}\cdot\mathbf{q}_0\,\mathbf{n}\cdot\mathbf{q}_1 - \mathbf{q}_0\cdot\mathbf{q}_1). \tag{7'}$$

Theorem 5. *The reduced symplectic manifold is $(\mathbb{K}_2,\omega)$, where the reduced symplectic form ω coincides with the standard symplectic form σ of $\mathbb{K}_2$ defined by*

$$\sigma(\mathbf{u},\mathbf{v}) = \mathbf{n}\cdot\mathbf{u}\times\mathbf{v}. \tag{8}$$

Proof. The proof is similar, *mutatis mutandis*, to that of $\mathbb{S}_2$ till equations (19), §B.3, which now gives

$$\begin{aligned}\{f,g\}_\omega(\mathbf{n}) = \{F,G\}(\mathbf{q},\mathbf{p}) &= -\mathbf{q}\times\mathbf{p}\cdot(\nabla\mathcal{F}\times\nabla\mathcal{G})\\ &= -\mathbf{n}\cdot\nabla\mathcal{F}\times\nabla\mathcal{G}\{f,g\}_\sigma(\mathbf{n}),\end{aligned}$$

being, for $\varepsilon = -1$, $\{f, g\}_\sigma = -\mathbf{n} \cdot \nabla\mathcal{F} \times \nabla\mathcal{G}$. ∎

Note that in the case of the eikonal equation of $\mathbb{H}_2$ the reduced symplectic form coincides with the standard area two-form on $\mathbb{K}_2$ associated with the normal vector $\mathbf{n}$. Moreover, the symplectic manifold $(\mathbb{K}_2, \sigma)$ is now symplectomorphic to a cotangent bundle, as shown by the following

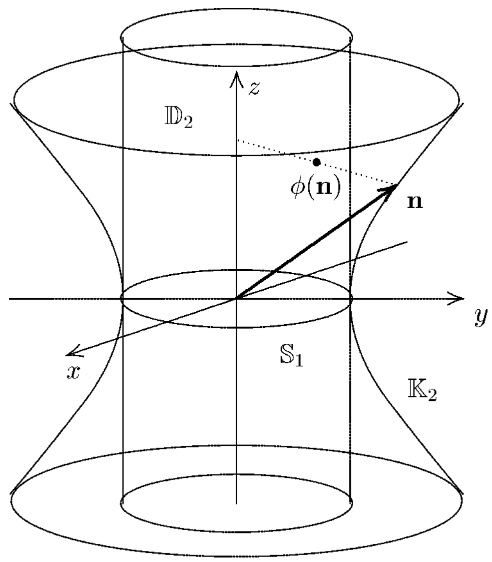
Theorem 6. *Let $\mathbb{D}_2$ be the cylinder in the Minkowski space $\mathbb{R}^3$, with axis z and intersecting the (x, y)-plane in the unit circle $\mathbb{S}_1$, $x^2 + y^2 = 1$. Then the mapping $\phi\colon \mathbb{K}_2 \to \mathbb{D}_2$ defined by*

$$\phi(\mathbf{n}) = \frac{\mathbf{n} + \mathbf{n} \cdot \mathbf{c}_3\, \mathbf{c}_3}{\sqrt{1 + (\mathbf{n} \cdot \mathbf{c}_3)^2}}$$

is a symplectomorphism from $(\mathbb{K}_2, \sigma)$ to the cotangent bundle $T^*\mathbb{S}_1 \sim \mathbb{S}_1 \times \mathbb{R}$.

Remark 1. The mapping ϕ is the radial orthogonal projection, with respect to the z-axis, from $\mathbb{K}_2$ to $\mathbb{D}_2$. Note that the vector $\mathbf{n} + \mathbf{n} \cdot \mathbf{c}_3\, \mathbf{c}_3$ is the (x, y)-component of $\mathbf{n}$, being orthogonal to $\mathbf{c}_3$, and that its square is

$$(\mathbf{n} + \mathbf{n} \cdot \mathbf{c}_3\, \mathbf{c}_3)^2 = \mathbf{n}^2 - (\mathbf{n} \cdot \mathbf{c}_3)^2 + 2\, (\mathbf{n} \cdot \mathbf{c}_3)^2 = 1 + (\mathbf{n} \cdot \mathbf{c}_3)^2.$$



Proof. Let $\mathbb{D}_2 \sim \mathbb{S}_1 \times \mathbb{R}$ be described by the parametric equation

$$\mathbf{r} = \mathbf{u}(\theta) + z\mathbf{t} = \cos\theta\, \mathbf{c}_1 + \sin\theta\, \mathbf{c}_2 + z\, \mathbf{c}_3, \qquad (u^1, u^2) = (z, \theta).$$

The associated tangent frame is

$$\mathbf{e}_1 = \mathbf{t}, \qquad \mathbf{e}_2 = \partial_\theta \mathbf{u} = \mathbf{s}.$$

We choose the orthogonal unit vector $\mathbf{n} = \mathbf{u}$. Then, in accordance with the definition of §B.2.5,

$$\sigma_{12} = \mathbf{n} \cdot \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{u} \cdot \mathbf{t} \times \mathbf{s}$$

In the Minkowskian metric i.e., for $\varepsilon = -1$,

$$\mathbf{u} \cdot \mathbf{t} \times \mathbf{s} = \mathbf{c}_1 \cdot \mathbf{c}_3 \times \mathbf{c}_2 = 1.$$

Thus, $\sigma_{12} = 1$ and the standard symplectic form on $\mathbb{D}_2$ is

$$\sigma_{\mathbb{D}_2} = \sigma_{12}\, du^1 \wedge du^2 = dz \wedge d\theta.$$

If we consider $T^*\mathbb{S}_1 \sim \mathbb{S}_1 \times \mathbb{R}$ with canonical coordinates (θ, z), then the Liouville form is $\theta_{\mathbb{S}_1} = z\, d\theta$ and the symplectic form $d\theta_{\mathbb{S}_1} = dz \wedge d\theta$ coincides with the area 2-form. Let $\mathbb{K}_2 \sim \mathbb{S}_1 \times \mathbb{R}$ be described by the parametric equation

$$\mathbf{r} = \cosh\xi\, \mathbf{u}(\theta) + \sinh\xi\, \mathbf{t}, \qquad (u^1, u^2) = (\xi, \theta).$$

The associated frame is

$$\mathbf{e}_1 = \sinh\xi\, \mathbf{u} + \cosh\xi\, \mathbf{t}, \qquad \mathbf{e}_2 = \cosh\xi \partial_\theta\, \mathbf{u} = \cosh\xi\, \mathbf{s}.$$

We choose the orthogonal unit vector $\mathbf{n} = \mathbf{r}$. It follows that

$$\begin{aligned}\sigma_{12} &= \mathbf{n} \cdot \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{r} \cdot (\sinh\xi\, \mathbf{u} + \cosh\xi\, \mathbf{t}) \times (\cosh\xi\, \mathbf{s}) \\ &= (\cosh\xi\, \mathbf{u} + \sinh\xi\, \mathbf{t}) \cdot (\sinh\xi\, \mathbf{u} + \cosh\xi\, \mathbf{t}) \times (\cosh\xi\, \mathbf{s}) \\ &= \cosh\xi\, (\cosh^2\xi\, \mathbf{u} \cdot \mathbf{t} \times \mathbf{s} + \sinh^2\xi\, \mathbf{t} \cdot \mathbf{u} \times \mathbf{s}) \\ &= \cosh\xi\, \mathbf{u} \cdot \mathbf{t} \times \mathbf{s} = \cosh\xi.\end{aligned}$$

Then, according with the above choices, the standard symplectic form is

$$\sigma_{\mathbb{K}_2} = \sigma_{12}\, du^1 \wedge du^2 = \cosh\xi\, d\xi \wedge d\theta = d\sinh\chi \wedge d\theta.$$

We look for a diffeomorphism $\phi\colon \mathbb{K}_2 \to \mathbb{D}_2$ described by equation

$$z = z(\xi) \qquad (\theta = \theta)$$

such that

$$\phi^* \sigma_{\mathbb{K}_2} = \sigma_{\mathbb{D}_2}.$$

This last condition is equivalent to equation

$$dz(\xi) \wedge d\theta = d\sinh\xi \wedge d\theta.$$

Then we can choose

$$z(\xi) = \sinh\xi.$$

The diffeomorphism ϕ so defined is the radial orthogonal projection w.r.to the z-axis restricted to $\mathbb{K}_2$. ∎

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