On the correspondence between classical and quantum observables

(A naive approach to the exact quantization problem)

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1. Citations and comments

Van Hove (1950), p. 75:

"Dans la quantification des systèmes dynamiques de la Mécanique classique, on admet qu'aux grandeurs les plus simples de la théorie classique correspondent des opérateurs vérifiant la propriété

$$(a_1 f_1 + a_2 f_2)\widehat{\ } = a_1 \widehat{f}_1 + a_2 \widehat{f}_2, \qquad a_1, a_2 \in \mathbb{R} \tag{1.1}$$

et on utilise celle-ci pour construire ces opérateurs en fonction des opérateurs fandamentaux P_i, Q_i de la théorie quantique (voir par example Dirac, 1935, p. 88-91). Il est bien connu que par ce procédé on n'arrive pas à associer un opérateur bien déterminé à chaque grandeur classique de façon à ce que (1.1) ait lieu sans exceptions. Des incertitudes apparaissent concernant l'order des facteurs dans les produits et elles donnent lieu à des ambiguités qui semblent inévitables. Il faut signaler à ce propos un résultat de Groenewold, 1945, p. 45) qui prouve q'une correspondance univoque $f \mapsto \widehat{f}$ entre polynômes en les grandeurs classiques p_i, q_i et polynômes en les opérateurs quantiques P_i, Q_i ne peut vérifier sans exceptions la relations

$$\tfrac{h}{2\pi}\, i\, \{f_1, f_2\}\widehat{\ } = \widehat{f}_1 \widehat{f}_2 - \widehat{f}_2 \widehat{f}_1. \tag{1.2}$$

Schwinger (2000), Section 2.4:

"It is a convenient fiction to assert that every Hermitian operators symbolizes a physical quantity".

Fock (1978), p. 22:

"A certain linear operator is related to each physical quantity".

Comment 1. The quantization theory has reached a remarkable development in the case of a Euclidean configuration space. Main references are Folland, 1989; Martinez, 2001; Robert, 1987; Taylor, 1996. However, in these monographs the "quantization" is not an "exact" homomorphism of Lie algebras, but the commutator corresponds to the Poisson bracket up to higher orders of $\hbar$. The compatibility between the possible conditions to be imposed on a "quantization mapping" is discussed in the introductory chapter of Folland and in Landsman, 1998. On the contrary, for a generic Riemannian configuration manifold there are only preliminary results. The first attempt of quantizing the classical observables on a generic symplectic manifold dates back to Segal, 1963. The main reference on this topic is Woodhouse, 1991. For further references and comments on the "geometric quantization problem" see Caianiello, Marmo, Scarpetta, 1985.

Comment 2. We cannot deal with the variables q^i and p_i in the same way, unless the configuration manifold is a Euclidean space.

2. Classical and quantum observables

Let Q be the n-dimensional configuration manifold of a classical mechanical system. Local coordinates of Q will be denoted by $\underline{q} = (q^i)$. A **classical observable** (C-observable) of this system is any smooth real-valued function F on the cotangent bundle T^*Q, the "phase space" of the system. A **classical state** of the system is then a point of T^*Q. Canonical coordinates of T^*Q corresponding to configuration coordinates $\underline{q}$ will be denoted by $(\underline{q}, \underline{p}) = (q^i, p_i)$. The space of classical observables is endowed with the Lie-algebra structure given by the Poisson bracket defined by

$$\{F, G\} = \partial^i F \partial_i G - \partial_i F \partial^i G. \tag{2.1}$$

Notation:

$$\partial_i = \frac{\partial}{\partial q^i}, \qquad \partial^i = \frac{\partial}{\partial p_i}.$$

As it is well known this definition does not depend on the choice of the canonical coordinates $(\underline{q}, \underline{p})$, since it derives from the canonical symplectic structure of T^*Q. Note that this definition is equivalent to the following rules

$$\{q^i, q^j\} = 0, \qquad \{p_i, q^j\} = \delta_i^j, \qquad \{p_i, p_j\} = 0. \tag{2.2}$$

To prove the equivalence of (2.1) and (2.2) we need to recall that a Poisson bracket is, by definition, a bi-derivation hence, it is a "local" operation i.e., invariant under restrictions to open subsets.

A **quantum observable** of the system is a linear operator $\widehat{F}$ on the **quantum state space**. This is a linear space of complex-valued generalized functions (including distributions) ψ on Q. We have no need, for the present, of special assumptions about this space and the space of operators. The space of quantum observables has a natural Lie-algebra structure given by the commutator

$$[\widehat{F}, \widehat{G}] = \widehat{F}\widehat{G} - \widehat{G}\widehat{F}. \tag{2.3}$$

In dealing with quantum observables we are interested in the eigenstate equation

$$\widehat{F}\psi = \lambda\,\psi. \tag{2.4}$$

An **eigenfunction** ψ represents a **stationary quantum state** of the system. An **eigenvalue** λ represents the measured value of the quantity L in the state ψ. We are interested in real eigenvalues.

3. The exact quantization rule

An **exact quantization rule** or **quantization mapping** on a set of classical observables $\mathcal{O}$ is an injective mapping $F \mapsto \widehat{F}$ from $\mathcal{O}$ into the space of quantum observables satisfying the following two rules

$$\boxed{\begin{array}{l} (aF + bG)\widehat{\ } = a\widehat{F} + b\widehat{G}, \quad a, b \in \mathbb{C} \\ \{F, G\}\widehat{\ } = \gamma\,[\widehat{F}, \widehat{G}] \end{array}} \tag{3.1}$$

Here $\gamma \in \mathbb{C}$ is a suitable **universal constant**.

To find an exact quantization rule for a sufficiently wide class of classical observables of a given mechanical system means to solve the "quantization problem" (Folland, p. 15) for that system.

4. Basic classical observables

We consider two special classes of classical observables, denoted by $\mathcal{B}_0$ and $\mathcal{B}_1$. Both correspond to objects defined on the configuration manifold Q. $\mathcal{B}_0$ is the space of smooth functions constant on the fibers of T^*Q. These observables, which we denote by small letters $f, g, \dots$ are identified with smooth real functions on Q. $\mathcal{B}_1$ is the space of linear homogenous functions on the fibers of T^*Q in one-to-one correspondence with smooth vector fields $\mathbf{X}$ on Q,

$$P(\mathbf{X}) = P_{\mathbf{X}} = X^i\, p_i. \tag{4.1}$$

We call these observables **basic observables** for the following reason.

Proposition 4.1. *The canonical PB $\{\cdot,\cdot\}$ on the phase space T^*Q is characterized by the following commutation relations*

$$\boxed{\{f, g\} = 0, \qquad \{P_{\mathbf{X}}, f\} = \langle \mathbf{X}, df\rangle, \qquad \{P_{\mathbf{X}}, P_{\mathbf{Y}}\} = P_{[\mathbf{X},\mathbf{Y}]}} \tag{4.2}$$

where $\langle \mathbf{X}, df\rangle = X^i\partial_i f$ is the derivative of f with respect to $\mathbf{X}$, and $[\mathbf{X},\mathbf{Y}]$ is the Lie bracket of vector fields,

$$[\mathbf{X},\mathbf{Y}]^k = X^i\partial_i Y^k - Y^i\partial_i X^k.$$

This means that these commutation relations define a unique PB which coincides with the canonical one.

Proof. Definition (2.1) obviously implies (4.2). Conversely, we consider local coordinates (q^i) as functions on Q and the partial derivatives ∂_i as vector fields. Then from (4.2) we derive the rules (2.2). ∎

An obvious consequence is

Proposition 4.2. *The quantization problem is solved for the space $\mathcal{B}_0$ by setting*

$$\widehat{f}\psi = f\psi. \tag{4.3}$$

A less obvious consequence is the following

Proposition 4.3. *If we assume the rule* (4.3) *then any quantization rule on a subset of observables $P_{\mathbf{X}}$ which is closed with respect to the sum and the Lie bracket of vector fields is necessarily of the form*

$$\boxed{\widehat{P}_{\mathbf{X}}(\psi) = \tfrac{1}{\gamma}\,\langle \mathbf{X}, d\psi\rangle + A_{\mathbf{X}}\,\psi} \tag{4.4}$$

where $A\colon \mathbf{X} \mapsto A_{\mathbf{X}}$ is a mapping from vector fields to functions on Q such that

$$A_{\mathbf{X}+\mathbf{Y}} = A_{\mathbf{X}} + A_{\mathbf{Y}} \tag{4.5}$$

and

$$\langle \mathbf{X}, dA_{\mathbf{Y}}\rangle - \langle \mathbf{Y}, dA_{\mathbf{X}}\rangle = A_{[\mathbf{X},\mathbf{Y}]}. \tag{4.6}$$

Proof. Definition (4.1) implies

$$P_{\mathbf{X}+\mathbf{Y}} = P_{\mathbf{X}} + P_{\mathbf{Y}}, \qquad P_{f\mathbf{X}} = f\,P_{\mathbf{X}}.$$

Because of $(3.1)_1$, $(P_{\mathbf{X}} + P_{\mathbf{Y}})\widehat{} = \widehat{P}_{\mathbf{X}} + \widehat{P}_{\mathbf{Y}}$. Then

$$\widehat{P}_{\mathbf{X}+\mathbf{Y}} = \widehat{P}_{\mathbf{X}} + \widehat{P}_{\mathbf{Y}}. \tag{4.7}$$

Because of $(3.1)_2$, $\{P_{\mathbf{X}}, f\}\widehat{} = \gamma\,(\widehat{P}_{\mathbf{X}}\,\widehat{f} - \widehat{f}\,\widehat{P}_{\mathbf{X}})$. From $(4.2)_2$ it follows that

$$\widehat{P}_{\mathbf{X}}\widehat{f} - \widehat{f}\widehat{P}_{\mathbf{X}} = \tfrac{1}{\gamma}\,\langle \mathbf{X}, df\rangle\widehat{}. \tag{4.8}$$

Due to (4.3), this is equivalent to

$$\widehat{P}_{\mathbf{X}}(f\psi) - f\,\widehat{P}_{\mathbf{X}}(\psi) = \tfrac{1}{\gamma}\,\langle \mathbf{X}, df\rangle\,\psi.$$

By interchanging f with ψ we get

$$\widehat{P}_{\mathbf{X}}(\psi f) - \psi\,\widehat{P}_{\mathbf{X}}(f) = \tfrac{1}{\gamma}\,\langle \mathbf{X}, d\psi\rangle\,f.$$

Subtracting term by term these last equations we get

$$\psi\,\widehat{P}_{\mathbf{X}}(f) - f\,\widehat{P}_{\mathbf{X}}(\psi) = \tfrac{1}{\gamma}\,\big(\langle \mathbf{X}, df\rangle\,\psi - \langle \mathbf{X}, d\psi\rangle\,f\big).$$

By choosing $f = 1$ we obtain

$$\psi\,\widehat{P}_{\mathbf{X}}(1) - \widehat{P}_{\mathbf{X}}(\psi) = -\tfrac{1}{\gamma}\,\langle \mathbf{X}, d\psi\rangle,$$

i.e.,

$$\widehat{P}_{\mathbf{X}}(\psi) = \tfrac{1}{\gamma}\,\langle \mathbf{X}, d\psi\rangle + \widehat{P}_{\mathbf{X}}(1)\,\psi. \tag{4.9}$$

Because of $(3.1)_2$, $\{P_{\mathbf{X}}, P_{\mathbf{Y}}\}\widehat{} = \gamma(\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} - \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}})$. From $(4.2)_3$ it follows that

$$\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} - \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}} = \tfrac{1}{\gamma}\,\widehat{P}_{[\mathbf{X},\mathbf{Y}]}. \tag{4.10}$$

Let us set

$$A_{\mathbf{X}} = \widehat{P}_{\mathbf{X}}(1)$$

and re-write (4.9) in the form (4.4). Because of (4.4) we get

$$\begin{aligned}\widehat{P}_{\mathbf{X}}\big(\widehat{P}_{\mathbf{Y}}(\psi)\big) &= \tfrac{1}{\gamma}\,\langle \mathbf{X}, d + \widehat{P}_{\mathbf{Y}}(\psi)\rangle + A_{\mathbf{X}}\,\widehat{P}_{\mathbf{y}}(\psi)\\ &= \tfrac{1}{\gamma}\,\langle \mathbf{X}, d\big(\tfrac{1}{\gamma}\,\langle \mathbf{y}, d\psi\rangle + A_{\mathbf{Y}}\,\psi\big)\rangle + A_{\mathbf{X}}\,\big(\tfrac{1}{\gamma}\,\langle \mathbf{Y}, d\psi\rangle + A_{\mathbf{Y}}\,\psi\big).\end{aligned}$$

and

$$\begin{aligned}(\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} - \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}})(\psi) &= \tfrac{1}{\gamma^2}\,\langle [\mathbf{X},\mathbf{Y}], d\psi\rangle + \tfrac{1}{\gamma}\,\langle \mathbf{X}, d(A_{\mathbf{Y}}\psi)\rangle - \tfrac{1}{\gamma}\,\langle \mathbf{Y}, d(A_{\mathbf{X}}\psi)\rangle\\ &\quad + \tfrac{1}{\gamma}\,A_{\mathbf{X}}\,\langle \mathbf{Y}, d\psi\rangle - \tfrac{1}{\gamma}\,A_{\mathbf{Y}}\,\langle \mathbf{X}, d\psi\rangle\\ &= \tfrac{1}{\gamma^2}\,\langle [\mathbf{X},\mathbf{Y}], d\psi\rangle + \tfrac{1}{\gamma}\,\langle \mathbf{X}, dA_{\mathbf{Y}}\rangle\psi - \tfrac{1}{\gamma}\,\langle \mathbf{Y}, dA_{\mathbf{X}}\rangle\psi\end{aligned}$$

Because of (4.10),

$$\begin{aligned}(\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} - \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}})(\psi) &= \tfrac{1}{\gamma}\,\widehat{P}_{[\mathbf{X},\mathbf{Y}]}(\psi)\\ &= \tfrac{1}{\gamma^2}\,\langle [\mathbf{X},\mathbf{Y}], d\psi\rangle + \tfrac{1}{\gamma}\,A_{[\mathbf{X},\mathbf{Y}]}\,\psi.\end{aligned}$$

Then (4.6) follows. On the other hand, from (4.4) it follows that

$$\begin{aligned}\widehat{P}_{\mathbf{X}+\mathbf{Y}}(\psi) &= A_{\mathbf{X}+\mathbf{Y}}\,\psi + \tfrac{1}{\gamma}\,\langle \mathbf{X}+\mathbf{Y}, d\psi\rangle\\ &= A_{\mathbf{X}+\mathbf{Y}}\,\psi - A_{\mathbf{X}}\,\psi - A_{\mathbf{Y}}\,\psi + \widehat{P}_{\mathbf{X}}(\psi) + \widehat{P}_{\mathbf{Y}}(\psi),\end{aligned}$$

i.e.,

$$\big(\widehat{P}_{\mathbf{X}+\mathbf{Y}} - \widehat{P}_{\mathbf{X}} - \widehat{P}_{\mathbf{X}}\big)(\psi) = \big(A_{\mathbf{X}+\mathbf{Y}} - A_{\mathbf{X}} - A_{\mathbf{Y}}\big)\,\psi.$$

Due to (4.7) the left hand side vanishes, so that we get (4.5). ∎

Now we have to put together the observables $\mathcal{B}_0$ and $\mathcal{B}_1$. We have two "natural" possible definitions of the operators $(fP_{\mathbf{X}})\widehat{}$. Since $fP_{\mathbf{X}} = P_{f\mathbf{X}}$ we can take

$$(fP_{\mathbf{X}})\widehat{} = \widehat{P}_{f\mathbf{X}} = \widehat{f}\,\widehat{P}_{\mathbf{X}}. \tag{4.11}$$

Otherwise we can consider the "symmetrization rule"

$$(4.12)\qquad (f P_{\mathbf{X}})\widehat{\ } = \tfrac{1}{2}\,(\widehat{f}\,\widehat{P}_{\mathbf{X}} + \widehat{P}_{\mathbf{X}}\,\widehat{f}).$$

Proposition 4.4. *If we assume the rule* (4.11) *then the mapping* $A\colon \mathbf{X} \mapsto A_{\mathbf{X}}$ *is a closed 2-form.*
Proof. Due to (4.11),

$$(4.13)\qquad A_{f\mathbf{X}} = \widehat{P}_{f\mathbf{X}}(1) = \widehat{f}\widehat{P}_{\mathbf{X}}(1) = f\,A_{\mathbf{X}}.$$

This formula together with (4.5) shows that we can consider $A_{\mathbf{X}}$ as the evaluation of a 1-form A over a vector field $\mathbf{X}$,

$$A_{\mathbf{X}} = \langle A, \mathbf{X}\rangle.$$

For two commuting vector fields, equation (4.6) yields

$$(4.14)\qquad \langle \mathbf{X}, dA_{\mathbf{Y}}\rangle = \langle \mathbf{Y}, dA_{\mathbf{X}}\rangle$$

since

$$A_{[\mathbf{X},\mathbf{Y}]} = \langle A, \mathbf{0}\rangle = 0.$$

This means that A is closed. Indeed, let us consider the commuting vectors $\partial_i = \partial/\partial q^i$ associated with any local coordinate system and set

$$A_{\partial_i} = A_i.$$

These are the components of the 1-form $A = A_i\,dq^i$ in these coordinates. Then (4.14) is equivalent to

$$\partial_i A_j = \partial_j A_i,$$

i.e., to $dA = 0$. ■

Proposition 4.5. *If we assume the rule* (4.12)*, then*

$$(4.15)\qquad A_{f\mathbf{X}} = f\,A_{\mathbf{X}} + \tfrac{1}{2\gamma}\,\langle \mathbf{X}, df\rangle.$$

Proof.

$$(4.16)\qquad \begin{aligned} A_{f\mathbf{X}} &= \widehat{P}_{f\mathbf{X}}(1) = (f P_{\mathbf{X}})\widehat{\ }(1) = \tfrac{1}{2}\big(\widehat{f}\,\widehat{P}_{\mathbf{X}}(1) + \widehat{P}_{\mathbf{X}}\,\widehat{f}(1)\big)\\ &= \tfrac{1}{2}\big(f\,\widehat{P}_{\mathbf{X}}(1) + \widehat{P}_{\mathbf{X}}(f)\big) = \tfrac{1}{2}\big(f\,A_{\mathbf{X}} + \tfrac{1}{\gamma}\,\langle \mathbf{X}, df\rangle + A_{\mathbf{X}}\,f\big)\\ &= f\,A_{\mathbf{X}} + \tfrac{1}{2\gamma}\,\langle \mathbf{X}, df\rangle. \qquad ■ \end{aligned}$$

Remark 4.1. In this case the mapping $\mathbf{X} \mapsto A_{\mathbf{X}}$ is not a 1-form, unless we consider subspaces of vector fields and functions such that

$$(4.17)\qquad \langle \mathbf{X}, df\rangle = \{P_{\mathbf{X}}, f\} = 0.$$

Remark 4.2. The two rules (4.11) and (4.12) are compatible (equivalent) if and only if restricted to a subspace of $\mathcal{B}_0 + \mathcal{B}_1$ for which (4.17) holds. Indeed, from (4.11) and (4.12) it follows that $\frac{1}{2}\big(f\widehat{P}_{\mathbf{X}}(\psi) + \widehat{P}_{\mathbf{X}}(f\psi)\big) = f\,\widehat{P}_{\mathbf{X}}(\psi)$, so that

$$\widehat{P}_{\mathbf{X}}(f\psi) = f\,\widehat{P}_{\mathbf{X}}(\psi).$$

Due to (4.4) this is equivalent to $\langle \mathbf{X}, df\rangle = 0$ <u>whatever</u> A. Hence, if we want these two rules to be simultaneously satisfied, then the quantization rule on the observables $f + P_{\mathbf{K}} \in \mathcal{B}_0 + \mathcal{B}_1$ can be defined only on subspaces for which (4.17) holds. Note that (4.17) means that f is a first integral of the dynamical system $\mathbf{X}$. As a consequence, if these subspaces are such that the vector fields $\mathbf{X}$ span the at each point

$q \in Q$ the tangent space T_qQ, then the functions f must be constant. So, in this case the observables are of the kind $c + P_{\mathbf{X}}$ and the space of vector fields $\mathbf{X}$ must be invariant under the mapping $\mathbf{X} \mapsto c\mathbf{X}$ with $c \in \mathbb{R}$. Vector fields of these kind are, for instance, (i) the left or right-invariant vector fields on a Lie group, and (ii) the Killing vector fields on a Riemannian manifold.

Remark 4.3. The rule (4.4) for $A = 0$ implies the rule

$$X^i p_i \mapsto \tfrac{1}{\gamma}\, X^i \partial_i$$

and in particular

$$p_i \mapsto \tfrac{1}{\gamma}\, \partial_i.$$

5. Quadratic observables

Let us denote by $\mathcal{B}_2$ the space quadratic observables associated with symmetric two tensors on Q:

$$P_{\mathbf{K}} = P(\mathbf{K}) = K^{ij}\, p_i\, p_j.$$

Let us use the notation

$$\langle \mathbf{K}, A\rangle = (K^{ij} A_j), \qquad \mathbf{K}(A,B) = K^{ij} A_i B_j$$

for two 1-forms A and B.

Proposition 5.1. *If we assume the rule* (4.4) *where A is a 1-form, then a quantization rule on the observables of the kind $f + P_{\mathbf{K}}$ is necessarily of the form*

$$\widehat{P}_{\mathbf{K}}(\psi) = \tfrac{2}{\gamma}\, \mathbf{K}(A, d\psi) + \widehat{P}_{\mathbf{K}}(1)\, \psi. \tag{5.1}$$

Proof. From $\{P_{\mathbf{K}}, f\} = 2K^{ij} p_j \partial_i f = 2P_{\mathbf{K}\nabla f}$ and $\widehat{\{P_{\mathbf{K}}, f\}} = \gamma(\widehat{P}_{\mathbf{K}} \widehat{f} - \widehat{f}\widehat{P}_{\mathbf{K}})$ it follows that

$$\gamma(\widehat{P}_{\mathbf{K}} \widehat{f} - \widehat{f}\widehat{P}_{\mathbf{K}}) = 2\, \widehat{P}_{\langle \mathbf{K}, df\rangle}$$

and, because of (4.4),

$$\gamma\big(\widehat{P}_{\mathbf{K}}(f\psi) - f\, \widehat{P}_{\mathbf{K}}(\psi)\big) = \tfrac{2}{\gamma}\, \mathbf{K}(df, d\psi) + 2\, \mathbf{K}(df, A)\, \psi. \tag{5.2}$$

By interchanging f with ψ we get

$$\gamma\big(\widehat{P}_{\mathbf{K}}(f\psi) - \psi\, \widehat{P}_{\mathbf{K}}(f)\big) = \tfrac{2}{\gamma}\, \mathbf{K}(d\psi, df) + 2\, \mathbf{K}(d\psi, A)\, f, \tag{5.3}$$

and subtracting term by term,

$$\psi\, \widehat{P}_{\mathbf{K}}(f) - f\, \widehat{P}_{\mathbf{K}}(\psi) = \tfrac{2}{\gamma}\, K^{ij} A_j (\psi \partial_i f - f \partial_i \psi).$$

For $f = 1$ we get formula (5.1). ∎

Remark 5.1. Formula (5.1) shows that $\widehat{P}_{\mathbf{K}}$ is a first-order operator on ψ for $A \neq 0$ or a zero-order operator for $A = 0$. This is not acceptable, if we require the operator $\widehat{P}_{\mathbf{G}}$ associated with a metric tensor to be of second order (more precisely, to be equal to the Laplace operator, up to a constant factor). However, if in the proof above we consider $f = c$ (constant), then all the equalities involved are identically satisfied ($0 = 0$) and we cannot derive formula (5.1). A second possibility is to consider only tensors and functions such that

$$\{P_{\mathbf{K}}, f\} = 0, \qquad \text{i.e.} \quad \mathbf{K}\nabla f = 0. \tag{5.4}$$

Then (5.2) implies

$$\widehat{P}_{\mathbf{K}}(f\psi) = f\,\widehat{P}_{\mathbf{K}}(\psi) = \psi\,\widehat{P}_{\mathbf{K}}(f) = \widehat{P}_{\mathbf{K}}(\psi f). \tag{5.5}$$

From (5.3) it follows that

$$\mathbf{K}(d\psi, df) = \gamma K^{ij}\partial_i\psi A_j\, f,$$

thus, for $f \neq 0$,

$$K^{ij}\partial_i\psi A_j = 0.$$

This formula is not acceptable except for $A = 0$. But for $A = 0$ we cannot derive formula (5.1).
Thus,

Proposition 5.2. *A quantization rule can be defined only on special subsets of the observables $f + P_{\mathbf{K}}$. If we choose $A = 0$, then we are forced to consider only tensors $\mathbf{K}$ and functions f such that $\mathbf{K}df = 0$. If we choose $A \neq 0$, then we are forced to consider only $f = c$.*

Let us consider on Q a symmetric connection with covariant derivative ∇_i. Then with any symmetric contravariant 2-tensor $\mathbf{K}$ we associate the **pseudo-Laplacian** operator

$$\Delta_{\mathbf{K}}\psi = \nabla_i(K^{ij}\partial_i\psi).$$

We denote by

$$\delta\mathbf{X} = \nabla_i X^i$$

the divergence of any vector field. We shall use the same symbol δ for the co-differential operator on contravariant skew-symmetric tensors.

Let us consider the homogeneous observables $P_{\mathbf{K}}$ in the special case where $\mathbf{K}$ is the symmetric product of two vector fields,

$$\mathbf{K} = \mathbf{X}\odot\mathbf{Y} = \tfrac{1}{2}\,(\mathbf{X}\otimes\mathbf{Y} + \mathbf{Y}\otimes\mathbf{X}), \qquad K^{ij} = X^{(i}Y^{j)} = \tfrac{1}{2}\,(X^iY^j + Y^iX^j).$$

Proposition 5.3. *If for the observables $P_{\mathbf{X}}$ and $P_{\mathbf{Y}}$ we assume the quantization rule* (4.4) *with $A = 0$ i.e.,*

$$\widehat{P}_{\mathbf{X}}\psi = \tfrac{1}{\gamma}\,\langle\mathbf{X}, d\psi\rangle, \tag{5.6}$$

and the symmetrization rule

$$(P_{\mathbf{X}}P_{\mathbf{Y}})\widehat{\ } = \tfrac{1}{2}\,(\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} + \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}}), \tag{5.7}$$

then

$$\widehat{P}_{\mathbf{K}} = \tfrac{1}{\gamma^2}\,\Delta_{\mathbf{K}} - \tfrac{1}{2\gamma}\,(\delta\mathbf{X}\,\widehat{P}_{\mathbf{Y}} + \delta\mathbf{Y}\,\widehat{P}_{\mathbf{X}}), \qquad \mathbf{K} = \mathbf{X}\odot\mathbf{Y} \tag{5.8}$$

where $\Delta_{\mathbf{K}}$ and δ are associated with any symmetric connection.

Proof.

$$\begin{aligned}(P_{\mathbf{X}}P_{\mathbf{Y}})\widehat{\ }\psi &= \tfrac{1}{2}\,(\widehat{P}_{\mathbf{X}}\widehat{P}_{\mathbf{Y}} + \widehat{P}_{\mathbf{Y}}\widehat{P}_{\mathbf{X}})\psi \\ &= \tfrac{1}{2\gamma^2}\Big(X^k\partial_k(Y^i\partial_i\psi) + Y^k\partial_k(X^i\partial_i\psi)\Big) = \dots\end{aligned}$$

If we introduce any symmetric connection, then

$$\begin{aligned}\dots &= \tfrac{1}{2\gamma^2}\Big(X^k\nabla_k(Y^i\nabla_i\psi) + Y^k\nabla_k(X^i\nabla_i\psi)\Big) \\ &= \tfrac{1}{2\gamma^2}\Big(\nabla_k(X^kY^i\nabla_i\psi + Y^kX^i\nabla_i\psi) - \nabla_kX^k\,Y^i\nabla_i\psi - \nabla_kY^k\,X^i\nabla_i\psi\Big) \\ &= \tfrac{1}{\gamma^2}\,\Big(\nabla_k(X^{(k}Y^{i)}\nabla_i\psi - \tfrac{1}{2}\big(\delta\mathbf{X}\,\langle\mathbf{Y}, d\psi\rangle + \delta\mathbf{Y}\,\langle\mathbf{X}, d\psi\rangle\big)\Big) \\ &= \tfrac{1}{\gamma^2}\,\Big(\nabla_k(X^{(k}Y^{i)}\nabla_i\psi - \tfrac{1}{2}\big(\delta\mathbf{X}\,\langle\mathbf{Y}, d\psi\rangle + \delta\mathbf{Y}\,\langle\mathbf{X}, d\psi\rangle\big)\Big) \\ &= \tfrac{1}{\gamma^2}\,\Big(\Delta_{\mathbf{K}}\psi - \tfrac{\gamma}{2}\big(\delta\mathbf{X}\,\widehat{P}_{\mathbf{Y}}\psi + \delta\mathbf{Y}\,\widehat{P}_{\mathbf{X}}\psi\big)\Big) \qquad \blacksquare\end{aligned}$$

Remark 5.2. The quantization rule (5.8) is not invariant under the transformation

$$\mathbf{X} \mapsto f\mathbf{X}, \qquad \mathbf{Y} \mapsto \frac{1}{f}\mathbf{Y},$$

which leaves $\mathbf{K}$ invariant, unless f = constant.

Remark 5.3. If $\delta\mathbf{X} = \delta\mathbf{Y} = 0$, then the quantization rule reduces to

$$\widehat{P}_{\mathbf{K}} = \tfrac{1}{\gamma^2}\,\Delta_{\mathbf{K}}, \qquad \mathbf{K} = \mathbf{X} \odot \mathbf{Y}. \tag{5.9}$$

This formula can be extended to any linear combination with constant coefficients of symmetric products of "solenoidal" vector fields. We recall that all these conclusions come from the assumptions (5.6) and (5.7).

Remark 5.4. The very restrictive conditions imposed by of the above remarks are fulfilled by a Killing tensor $\mathbf{K}$ on a space of constant curvature and of the Levi-Civita connection. Indeed, in these spaces any Killing tensor is reducible to a linear combination of symmetric products of Killing vectors, and any Killing tensor satisfies $\delta\mathbf{X} = 0$.

6. Separable systems

Let us consider a natural Hamiltonian system

$$H = G + V = \tfrac{1}{2}\, g^{ij}(\underline{q})\, p_i p_j + V(\underline{q}),$$

where g^{ij} are the contravariant components of a metric tensor $\mathbf{G}$. It is known that such a system is **orthogonally separable** i.e., the corresponding Hamilton-Jacobi equations admits a complete additive separated complete solution is some orthogonal coordinate system $\underline{q}$ (see below) if and only if it admits an involutive n-dimensional space $\mathcal{H}$ of quadratic first integrals in involution of the kind

$$H_{\mathbf{K}} = \tfrac{1}{2}\, P_{\mathbf{K}} + V_{\mathbf{K}},$$

where $\mathbf{K}$ are Killing 2-tensors forming a **Killing-Stäckel algebra** and $V_{\mathbf{K}}$ are functions on the n-dimensional configuration manifold Q (canonically extended to the cotangent bundle T^*Q. This space include the Hamiltonian itself, $H_{\mathbf{G}} = H$. A Killing-Stäckel algebra $\mathcal{K}$ is a n-dimensional linear space of Killing 2-tensors in involution with n eigenvectors in common (thus, commuting as linear operators). Then, it can be proved that (see Benenti, Chanu, Rastelli for details)

Proposition 6.1. *An exact quantization mapping is defined on the space $\mathcal{H}$ by setting*

$$\widehat{P}_{\mathbf{K}}\psi = -\tfrac{1}{\gamma^2}\,\Delta_{\mathbf{K}}\psi = -\tfrac{1}{\gamma^2}\,\delta(\mathbf{K}\nabla\psi) = -\tfrac{1}{\gamma^2}\,\nabla_i(K^{ij}\partial_j\psi), \tag{6.1}$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection, provided that

$$\delta(\mathbf{KR} - \mathbf{RK}) = 0, \qquad \forall \mathbf{K} \in \mathcal{K}, \tag{6.2}$$

where $\mathbf{R}$ is the Ricci tensor.

In (6.2) $\mathbf{KR} - \mathbf{RK}$ is the commutator of $\mathbf{K}$ and $\mathbf{R}$ interpreted as linear operators on vector fields or 1-forms. The obstruction (6.2) has been called the **pre-Robertson condition**. In any separable orthogonal coordinate system this condition is equivalent to

$$\partial_i R_{ij} - \Gamma_i\, R_{ij} = 0, \qquad i \neq j \;\; \text{(no sum over the indices)} \tag{6.3}$$

where Γ_i are the contracted Christoffel symbols,

$$\Gamma_i = g_{ij} g^{hk} \Gamma^j_{hk}.$$

The pre-Robertson condition is obviously satisfied if

$$R_{ij} = 0, \qquad i \neq j. \tag{6.4}$$

This is the well known **Robertson condition**, found by EISENHART. Indeed, following a precise definition of multiplicative separation of the Schrödinger equation,

$$-\tfrac{1}{\gamma^2}\,\Delta\psi + (V - E)\psi = 0, \tag{6.5}$$

corresponding to a suitable **completeness condition** it can be shown that

Proposition 6.2. *The Schödinger equation admits a solution of the form*

$$\psi = \prod_{i=1}^{n} \psi_i(q^i, c_A),$$

where (c_A) *are* $2n$ *constant parameters such that*

$$\det \begin{bmatrix} \psi_i\, \dfrac{\partial \psi_i'}{\partial c_A} - \psi_i'\, \dfrac{\partial \psi_i}{\partial c_A} \\ \psi_i\, \dfrac{\partial \psi_i''}{\partial c_A} - \psi_i''\, \dfrac{\partial \psi_i}{\partial c_A} \end{bmatrix} \neq 0 \tag{6.6}$$

if and only if the corresponding Hamilton-Jacobi equation is (additively) separable i.e., it admits a solution of the kind

$$W = \sum_{i=1}^{n} W_i(q^i; a_j) \tag{6.7}$$

where (a_j) *are* n *constant parameters such that*

$$\det \left[\frac{\partial W_i'}{\partial a_j} \right] \neq 0. \tag{6.8}$$

and moreover $R_{ij} = 0$ *for* $i \neq j$.

It follows that a separable natural Hamiltonian system whose Schrödinger equation is separable is "exactly quantizable" in the sense that there exists a quantization rule on the whole space $\mathcal{H}$ of the first integrals in involution $P_{\mathbf{K}}$. In this case the corresponding operators commute,

$$[\widehat{P}_{\mathbf{K}_1}, \widehat{P}_{\mathbf{K}_2}] = 0.$$

When $\mathbf{R} = \kappa\,\mathbf{G}$ (Einstein spaces) both Robertson and pre-Robertson conditions are satisfied. Then we can assert that

Proposition 6.2. *On Einstein spaces any orthogonally separable Hamiltonian system is exactly quantizable.*

It can be shown that the same property holds for non-orthogonal separable systems (BENENTI, CHANU, RASTELLI, 2002), where also linear first integrals (i.e., Killing vectors) are involved. Open questions are: (i) Assume that a Hamiltonian system is integrable with quadratic and linear first integrals in involution; under which conditions it is exactly quantizable i.e., the linear operators corresponding to these first integrals commute? (ii) Conversely, if it is exactly quantizable, when it is separable?

7. The normalization conditions

Since an eigenfunction is determined up to a constant factor, we impose a **normalization condition**. This condition depends on the kind of the **spectrum** of the linear operator and it requires the presence on Q of a volume n-form η, so that Q is required to be orientable.

Let us write the eigenstate equation of a linear operator L in the form

$$L\psi(q,\lambda) = \lambda\,\psi(q,\lambda), \tag{7.1}$$

pointing out the fact that ψ depends not only on the point $q \in Q$, but also on the eigenvalue λ. Then we can consider two cases: (i) If the eigenvalue λ belongs to a **discrete spectrum** (i.e., to a denumerable set of eigenvalues), the normalization condition is

$$\int_Q |\psi|^2 \eta = 1, \qquad |\psi|^2 = \psi^* \psi, \tag{7.2}$$

(ii) If the eigenvalue λ belongs to a **continuous spectrum** i.e., to an interval $I \subset \mathbb{R}$ of eigenvalues, the normalization condition is

$$\lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda} \int_Q |\Delta\Psi|^2\, \eta = 1, \tag{7.3}$$

where, by definition,

$$\Delta\Psi(q,\lambda,\Delta\lambda) = \int_\lambda^{\lambda+\Delta\lambda} \psi(q,x)dx. \tag{7.4}$$

Such a function is called **proper differential** (cf. Fock, p. 36). These normalization conditions can be extended to the case of simultaneous eigenfunctions of commuting operators and to the case where ψ is defined through a distribution (namely, a Dirac delta). Thus, the only assumption we need is that the eigenfunctions and the operators belong to suitable linear spaces, to be specified later on, for which the above equations and integrals exist.

From these normalization conditions it follows that an eigenfunction ψ is determined up to a unitary constant factor $e^{i\theta}$. Thus a stationary state is represented by a class of equivalence of normalized eigenfunctions, being equivalent two functions ψ and ψ' such that $\psi' = e^{i\theta}\,\psi$.

8. The universal constant γ

Proposition 8.1. *Assume that* (i) *on the space of linear operators a linear endomorphism* $\dagger\colon L \mapsto L^\dagger$ *is defined such that*

$$(LM)^\dagger = M^\dagger L^\dagger, \qquad z^\dagger = z^*, \tag{8.1}$$

and that (ii) *we give a physical meaning only to those operators such that*

$$L^\dagger = L, \tag{8.2}$$

then γ *is pure imaginary* (cf. Fock, p. 44).

Proof. If we assume that $[L,M]^\dagger = [L,M]$ for two operators such that $L^\dagger = L$ and $M^\dagger = M$, then (3.1) and (8.1) imply $\gamma^* = -\gamma$. ∎

This proposition is rather vague, since we have not defined such an operator † (for this, we need a definition of a "Hermitian bilinear form" on the space of linear operators; an operator satisfying (8.2) is "self-adjoint" or "Hermitian"), but it gives a first formal reason for the universal constant γ to be an imaginary number. However, it does not show that it must be imaginary-positive (or negative). To the same conclusion we are led by the following two remarks.

Remark 8.1. If A is exact, $A = dS$, then

$$\widehat{P}_{\mathbf{X}}(\psi) = \tfrac{1}{\gamma}\,\langle \mathbf{X}, d\psi\rangle + \langle \mathbf{X}, dS\rangle\,\psi \tag{8.3}$$

and the eigenstate equation

$$\widehat{P}_{\mathbf{X}}(\psi) = \lambda\psi$$

reads

$$\langle \mathbf{X}, d\psi\rangle = \gamma\,(\lambda - \langle \mathbf{X}, dS\rangle)\,\psi. \tag{8.4}$$

This shows that

$$\psi' = e^{\gamma S}\,\psi \tag{8.5}$$

is an eigenfunction corresponding to the operator $\widehat{P}_{\mathbf{X}}$ for $A = 0$,

$$\langle \mathbf{X}, d\psi'\rangle = \gamma\,\lambda\,\psi'. \tag{8.6}$$

We conclude that *if the universal constant γ is pure imaginary, then ψ and ψ' are equivalent and possibly simultaneously normalizable.*

Remark 8.2. A linear operator L is selfadjoint when (Born, 1960, p. 497)

$$\int_Q \phi^*\,(L\psi)\,\eta = \int_Q (L\phi)^*\,\psi\,\eta. \tag{8.7}$$

Let us consider the operator

$$L\psi = c\,\langle \mathbf{X}, d\psi\rangle = c\,X^i\partial_i\psi, \quad c \in \mathbb{C},$$

defined by a real vector field $\mathbf{X}$. Then from equation (8.7) we get

$$c\int_Q \phi^*\,(X^i\partial_i\psi)\,\eta = \int_Q (c\,X^i\partial_i\phi)^*\,\psi\,\eta,$$

$$c\int_Q X^i\partial_i(\phi^*\psi)\,\eta - c\int_Q X^i\partial_i\phi^*\,\psi\,\eta = c^*\int_Q X^i\partial_i\phi^*\,\psi\,\eta.$$

$$c\int_Q X^i\partial_i(\phi^*\psi)\,\eta = (c + c^*)\int_Q X^i\partial_i\phi^*\,\psi\,\eta. \tag{8.8}$$

Let us consider a Riemannian structure on Q and the corresponding volume element η. Then,

$$L\psi = cX^i\partial_i\psi = cX^i\nabla_i\psi = c\nabla_i(\psi X^i) - c\psi\,\nabla_i X^i,$$

and from (8.8) it follows that

$$c\int_Q \nabla_i(X^i\phi^*\psi)\,\eta - c\int_Q \psi\phi^*\,\nabla_i X^i\,\eta = (c + c^*)\left(\int_Q \nabla_i(\phi^* X^i)\psi\,\eta - \int_Q \psi\phi^*\,\nabla_i X^i\,\eta.\right.$$

Thus, (8.8) becomes equivalent to

$$c\int_Q \nabla_i(\phi^*\psi X^i)\eta = (c + c^*)\int_Q \nabla_i(\phi^* X^i)\,\psi\eta. \tag{8.9}$$

Let us consider a function ϕ^* with compact support $D \subset Q$ and vanishing on the boundary ∂D. Define $\bar{\mathbf{X}} = \phi^* \mathbf{X}$. By Gauss' theorem,

$$\int_D \delta \bar{\mathbf{X}} = \int_{\partial D} \text{flow of } \bar{\mathbf{X}},$$

we get

$$(c + c^*) \int_D \nabla_i (\phi^* X^i)\, \psi\, \eta + c^* \int_D \psi \phi^*\, \nabla_i X^i\, \eta = 0.$$

Let us consider the case $\delta \mathbf{X} = 0$, i.e.,

$$\nabla_i X^i = 0.$$

Then we get

$$(c + c^*) \int_D \nabla_i (\phi^* X^i)\, \psi\, \eta = 0.$$

This proves

Proposition 8.2. *The linear operator $L = c\, X^i \nabla_i$, for a vector field $\mathbf{X} = (X^i)$ such that $\delta \mathbf{X} = 0$, is selfadjoint if and only if $c + c^* = 0$.*

In other words, the operator $\widehat{P}_{\mathbf{X}}$, at least for $A = 0$ and for a solenoidal vector field, is selfadjoint only if the constant $\gamma = c^{-1}$ is pure imaginary.

Let us call **quantum-observable** (Q-observable) a classical observable F which admits a well defined linear operator $\widehat{F}$ with normalized eigenfunctions and real eigenvalues.

From the following elementary examples we conclude that

Proposition 8.3. *If we want that the basic observables $P_{\mathbf{X}}$ on the real line $Q = \mathbb{R}$ be Q-observable, then the universal constant must be positive-imaginary, that is of the form*

$$\gamma = \tfrac{i}{\hbar}, \qquad \hbar > 0. \tag{8.7}$$

Example 8.1. For $Q = \mathbb{R} = (x)$, and $\mathbf{X} = \partial_x$, the eigenstate equation becomes (we consider $A = 0$)

$$\frac{d\psi}{dx} = \gamma\, \lambda\, \psi.$$

Thus, the eigenfunctions are

$$\psi(x, \lambda) = c\, e^{\gamma \lambda x}, \qquad c \in \mathbb{C}.$$

We are in the case of a continuous spectrum and we apply the normalization condition (7.3), in order to find the constant c. The proper differential is given by:

$$\Delta \Psi = c \int_{\lambda}^{\lambda + \Delta\lambda} e^{\gamma t x}\, dt = c\, \tfrac{1}{\gamma x}\, \big[e^{\gamma t x}\big]_{\lambda}^{\lambda + \Delta\lambda} = \tfrac{c}{\gamma x}\, e^{\gamma \lambda x}\, \big(e^{\gamma \Delta\lambda x} - 1\big).$$

If we set

$$\gamma = a + ib,$$

then

$$\Delta\psi^* \Delta\psi = |\Delta\psi|^2 = \frac{|c|^2}{|\gamma|^2\, x^2}\, e^{2a\lambda x}\, \big[e^{2ax\Delta\lambda} + 1 - 2\, e^{ax\Delta\lambda}\, \cos(bx\Delta\lambda)\big].$$

The last term is of the kind

$$f(x) = \frac{1}{x^2}\, e^{\alpha x}\, \cos \beta x,$$

with $\alpha, \beta \in \mathbb{R}$. Its integral for $x \to +\infty$ does not have a limit, thus if we want a normalization the integral must converge. If we set $\beta = 0$ ($b = 0$) the situation is even worst, so we must have $\alpha = 0$ i.e., $a = 0$: *the universal constant γ must be pure imaginary.* Hence (in this case $\eta = dx$),

$$|\Delta\Psi|^2\, dx = \frac{2\, |c|^2}{b^2 x^2} \big(1 - \cos(bx\Delta\lambda)\big)\, dx = 4 \frac{|c|^2}{b^2 x^2}\, \sin^2 \frac{bx\Delta\lambda}{2}\, dx,$$

and

$$\int_{\mathbb{R}} |\Delta\Psi|^2\, dx = 2\, \frac{|c|^2}{b}\, \Delta\lambda \int_{-\infty}^{+\infty} \frac{\sin^2\xi}{\xi^2}\, d\xi, \qquad \xi = \tfrac{1}{2}\, b\Delta\lambda\, x.$$

Since

$$\int_{-\infty}^{+\infty} \frac{\sin^2\xi}{\xi^2}\, d\xi = \pi,$$

we conclude that (cf. FOCK, p. 50),

$$\frac{1}{\Delta\lambda} \int_{\mathbb{R}} |\Delta\Psi|^2\, dx = 2\frac{|c|^2}{b}\, \pi,$$

and the normalization condition gives

$$|c|^2 = \frac{b}{2\pi},$$

thus,

$$c = \sqrt{\frac{b}{2\pi}}\, e^{i\theta}, \qquad \theta \in \mathbb{R}.$$

The factor $e^{i\theta}$ is inessential (since $e^{i\theta}\psi$ is equivalent to ψ). We conclude first, that b (the imaginary components of γ) is positive, so that the universal constant has the form (1); second, that the normalized eigenfunction is (cf. FOCK, p. 51)

$$\psi(x,\lambda) = (2\pi\hbar)^{-\frac{1}{2}}\, e^{ix\lambda/\hbar}. \quad \bullet$$

From Example 8.1 we derive the following

Proposition 8.3. *A necessary and sufficient condition for the basic C-observable p_x on the real line $\mathbb{R} = (x)$ be Q-observable is that the universal constant be of the kind* (8.7)*.*

Example 8.2. For $Q = \mathbb{R} = (x)$, and $f(x) = x$, the eigenstate equation $\widehat{f}\psi = \lambda\psi$ becomes

$$x\,\psi(x,\lambda) = \lambda\,\psi(x,\lambda). \tag{8.8}$$

This equation cannot be solved by an "ordinary" function, but by a Dirac delta:

$$\psi(x,\lambda) = \delta(x-\lambda). \tag{8.9}$$

Let us apply the normalization procedure (for a continuum spectrum) to this case. The proper differential is

$$\Delta\Psi = \int_{\lambda}^{\lambda+\Delta\lambda} \delta(x-s)\, ds = \begin{cases} 1 & \text{if} \quad x \in [\lambda, \lambda+\Delta\lambda], \\ 0 & \text{if} \quad x \notin [\lambda, \lambda+\Delta\lambda]. \end{cases}$$

Then,

$$\int_{\mathbb{R}} |\Delta\Psi|^2\, dx = \Delta\lambda,$$

and the normalization condition (7.3) is trivially satisfied. Thus, the generalized function (8.9) is a normalized eigenfunction of the operator $\widehat{x}$ over the real line. We also conclude that the basic C-observable $f(x) = x$ on the real line is Q-observable.

Now we extend the analysis of Example 8.1 to a general basic observable $P_{\mathbf{X}} = X^i p_i$.

Proposition 8.4. *If F and τ are two real functions on the manifold Q such that*

$$\langle \mathbf{X}, dF\rangle = 0, \qquad \langle \mathbf{X}, d\tau\rangle = 1, \tag{8.10}$$

then

$$\psi = F\, e^{\gamma\lambda\tau} \tag{8.11}$$

is a solution of the eigenstate equation

$$\widehat{P}_{\mathbf{X}}(\psi) = \lambda\psi \tag{8.12}$$

with $A = 0$.
Proof. The eigenstate equation (34) reads

$$\langle \mathbf{X}, d\psi \rangle = \gamma\lambda\, \psi.$$

If ψ has the form (8.11), then

$$\langle \mathbf{X}, dF \rangle\, e^{\gamma\lambda\tau} + F\,\gamma\,\lambda\,\langle \mathbf{X}, d\tau \rangle\, e^{\gamma\lambda\tau} = F\,\gamma\,\lambda\, e^{\gamma\lambda\tau}. \qquad \blacksquare$$

Remark 8.3. A function F such that $\langle \mathbf{X}, dF \rangle = 0$ is a first integral of the dynamical system $\mathbf{X}$. A function τ such that $\langle \mathbf{X}, d\tau \rangle = 1$ when restricted to an orbit of $\mathbf{X}$ coincides, up to an additive constant, with the affine parameter over the integral curves of $\mathbf{X}$ (the affine parameter is the "proper time" measured by a point running along that orbit, making its velocity equal to $\mathbf{X}$). Note that this result does not depend on the presence of any metric on Q. However, if Q is Riemannian and $\mathbf{X}$ is a Killing vector, then the scalar $I = X^i X_i$ is a first integral (it is constant along the integral curves) and the affine parameter τ is the length of the integral curves (starting from a fixed hypersurface, transversal to $\mathbf{X}$) divided by $\sqrt{I}$.

Remark 8.4. Assume that all orbits of $\mathbf{X}$ are closed. Let us denote by $T(q)$ the **period** of the orbit passing through $q \in Q$. This means that if $c_q\colon \mathbb{R} \to Q$ is the integral curve of $\mathbf{X}$ based on q, then $c_q(T) = c_q(0)$, and T is the minimal positive number for which this equation is satisfied. From (8.11) we see that, in order to have a "single valued" function ψ, the exponent $\gamma\lambda T$ must be an imaginary number and a multiple of $i\pi$. Since λ and T are real, once more we conclude that the universal constant must be an imaginary number. It follows that

$$\tfrac{\lambda}{\hbar}\, T = 2\, m\, \pi, \qquad m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

This shows that T must be a constant independent on the orbit, so that the eigenvalues are

$$\lambda = m\,\hbar \frac{2\pi}{T} = m\, h\, \nu \qquad h = 2\pi\hbar, \quad \nu = \tfrac{1}{T}, \tag{8.13}$$

where $m \in \mathbb{Z}$ (integer number). It remains to look at the normalization of the eigenfunctions. From (8.11) and $\gamma = i/\hbar$ it follows that $|\psi|^2 = F^2$. Hence, the normalization condition (7.3) is equivalent to

$$\int_Q F^2\, \eta = 1. \tag{8.14}$$

This leads to

Proposition 8.5. *A basic classical observable $P_{\mathbf{X}}$ corresponding to a vector field $\mathbf{X}$ with all closed orbits is a Q-observable if all orbits have the same period and there exists a first integral F such that* (8.14) *holds. Then, the spectrum is discrete and given by* (8.13).

Remark 8.5. The Killing vectors generating rotations in Euclidean spaces fulfill the above conditions. Are there other special vector fields satisfying these conditions (for instance, invariant vector fields on Lie groups) ?

Remark 8.6. Let $\mathbf{X}$ be a complete vector field whose flow $\varphi\colon Q \times \mathbb{R} \to Q$ defines a diffeomorphism $\Phi\colon U \times \mathbb{R} \to Q$, where U is a $n-1$-dimensional submanifold of Q transversal to the orbits of $\mathbf{X}$. We have by definition,

$$\Phi(u, \tau) = \varphi(u, \tau).$$

No orbit of $\mathbf{X}$ is closed. Then an eigenfunction of $\widehat{P}_{\mathbf{X}}$ is, cf. (8.11),

$$\psi = F\, e^{i\lambda\tau/\hbar}$$

where τ is interpreted as a coordinate on Q, assuming values on all $\mathbb{R}$ with $\tau = 0$ on U. The first integral F can be defined by

$$F(q) = f(u),$$

where f is any function on U and $q = \varphi(u, \tau)$. This is the case, for instance, of the Killing vectors generating translations in Euclidean spaces.

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