Remarks on the connection between the additive separation of the Hamilton–Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions

S. Benenti,[a] C. Chanu, and G. Rastelli
Department of Mathematics, University of Turin, 10123 Torino, Italy

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The fundamental elements of the variable separation theory are revisited, including the Eisenhart and Robertson theorems, Kalnins–Miller theory, and the intrinsic characterization of the separation of the Hamilton–Jacobi equation, in a unitary and geometrical perspective. The general notion of complete integrability of first-order normal systems of PDEs leads in a natural way to completeness conditions for separated solutions of the Schrödinger equation and to the Robertson condition. Two general types of multiplicative separation for the Schrödinger equation are defined and analyzed: they are called ''free'' and ''reduced'' separation, respectively. In the free separation the coordinates are necessarily orthogonal, while the reduced separation may occur in nonorthogonal coordinates, but only in the presence of symmetries (Killing vectors). © *2002 American Institute of Physics.*
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I. INTRODUCTION

With a smooth real function V (potential energy) on a Riemannian manifold $(Q_n, \mathbf{g})$ (configuration manifold) we associate two differential equations, the time-independent Hamilton–Jacobi equation

$$\tfrac{1}{2}\nabla W \cdot \nabla W + V = E, \tag{1.1}$$

and the corresponding steady-state Schrödinger equation

$$-\frac{\hbar^2}{2}\Delta\psi + (V-E)\psi = 0. \tag{1.2}$$

In these equations, E is a constant parameter (the energy constant), ∇ is the gradient operator

$$(\nabla W)^i = g^{ij}\partial_j W,$$

and Δ is the Laplace–Beltrami operator

$$\Delta\psi = g^{ij}\nabla_i\nabla_j\psi,$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection. Besides the well-known physical connection between these two equations (we consider here the time-independent case only), there is an interesting mathematical connection due to the phenomenon of the *separation of variables*.

It is well known that, in many interesting cases, these equations admit *local separated solutions* of the form

[a]Electronic mail: benenti@dm.unito.it

$$W(\underline{q},\underline{c})=\sum_{i=1}^{n} W_i(q^i,\underline{c}) \tag{1.3}$$

for the Hamilton–Jacobi equation, and of the form

$$\psi(\underline{q},\underline{c})=\prod_{i=1}^{n} \psi_i(q^i,\underline{c}), \tag{1.4}$$

or

$$\psi(\underline{q},\underline{c})=e^{R(\underline{q})}\prod_{i=1}^{n} \psi_i(q^i,\underline{c}), \tag{1.5}$$

for the Schrödinger equation, where $\underline{q}=(q^i)$ is a suitable coordinate system on Q, and $\underline{c}$ denotes a set of constant parameters, whose number depends on an appropriate definition of *separation*. Note that in (1.5) the function $R(\underline{q})$ does not depend on the constant parameters: this kind of separation is called *R-separation*.

It happens that for solutions of this kind, Eqs. (1.1) and (1.2) become equivalent to a system of ordinary differential *separated equations*, each one involving a single coordinate. The interesting fact is that, in most cases, the separation of variables occurs simultaneously for both equations and in the same coordinate system. Although this fact is easy to illustrate for basic important examples, its general description and motivation is rather difficult and subtle to understand. Let us consider, for instance, the particular case of the orthogonal separation, where $g^{ij}=0$ for $i\neq j$. For this kind of separation we usually refer to three classical theorems of Stäckel, Robertson, and Eisenhart.[1–3]

Theorem 1.1: (Stäckel, 1893) *The Hamilton–Jacobi equation is separable in orthogonal coordinates $\underline{q}$ if and only if the diagonal components g^{ii} of the metric tensor and the potential V have the form*

$$g^{ii}=\varphi^i_{(n)}, \quad V=\phi_i g^{ii}=\phi_i\varphi^i_{(n)}, \tag{1.6}$$

where $\varphi^i_{(n)}$ is the last row of the inverse $[\varphi^i_{(j)}]$ of an $n\times n$ Stäckel matrix $[\varphi^{(j)}_i]$ and ϕ_i are functions of the coordinate corresponding to the index only.

A *Stäckel matrix* is a regular $n\times n$ matrix whose components $\varphi^{(j)}_i(q^i)$ are functions depending on the coordinate corresponding to the lower index only. A function V of the kind (1.6) is then called a *Stäckel multiplier*.

Theorem 1.2: (Robertson, 1927) *The Schrödinger equation is separable in orthogonal coordinates $\underline{q}$ if and only if in these coordinates the Hamilton–Jacobi equation is separable (i.e., the Stäckel requirements are satisfied) and moreover, the following condition is satisfied*

$$\sqrt{\Pi^n_{i=1}g^{ii}}\det(\varphi^{(j)}_i)=\Pi^n_{i=1}f_i(q^i), \tag{1.7}$$

where $f_i(q^i)$ are functions of the corresponding coordinate only.

This additional requirement is called the *Robertson condition*. It has a meaningful geometrical interpretation:

Theorem 1.3: (Eisenhart, 1934) *The Robertson condition is satisfied if and only if the Ricci tensor is diagonal:*

$$R_{ij}=0, \quad i\neq j.$$

However, while it can be easily seen that requirements (1.6) and (1.7) are sufficient for reducing the Schrödinger equation to separated equations, the proof given by Robertson (and

accepted by Eisenhart) that they are also necessary for the separation is not satisfactory. This is due to the fact that for the separability of the Schrödinger equation these authors assumed (apparently) the existence of a single solution of the kind

$$\psi(\underline{q}) = \Pi_{i=1}^{n} \psi_i(q^i),$$

without any reference to the presence and to the role of constant (real or complex) parameters [as is done in analytical mechanics for a complete solution of the Hamilton–Jacobi (HJ) equation]. Indeed, as we shall see in the following, the existence of a single separated solution of the Schrödinger equation does not imply the separation of the HJ equation for the same reason that the existence of a single solution of a first-order differential system does not imply, in general, its complete integrability. In other words, while for the HJ equation one looks for a separated solution (1.3) containing n constant parameters $\underline{c} = (c_i)$ satisfying the *completeness condition*

$$\det\left[\frac{\partial^2 W}{\partial q^i \partial c_j}\right] = \det\left[\frac{\partial p_i}{\partial c_j}\right] \neq 0, \quad p_i = W_i', \tag{1.8}$$

a similar requirement for a separated solution (1.4) of the Schrödinger equation does not appear explicitly either in the celebrated works cited previously or in many other standard reference books.[4,5] In fact, also for the Schrödinger equation we are not interested in a single separated solution, but in a parametrized family of local solutions, in order to build up a global one satisfying suitable boundary or normalization conditions. In conclusion: the statements of Robertson and Eisenhart (Theorems 1.2 and 1.3) are meaningless without a proper definition of *separation* of the Schrödinger equation.

A first ''precise'' definition of separation has been proposed by Koornwinder[6] within a rather general context, and strongly related to the basic properties of the Stäckel matrices. However, the systematic use of these matrices hides many interesting intrinsic features of the separation connected, for instance, with the existence of Killing tensors and of second-order symmetries of the Schrödinger equation. Other definitions of separation have been introduced by Olevsky[7] and more recently by Zhdanov and Zhalij[8] (both for the case $n=3$).

A crucial contribution to this matter is due to Kalnins and Miller.[9,10] Their approach is based on a definition of *regular* additive separation of a generalized Hamilton–Jacobi equation of any order, which is equivalent to the complete integrability of a suitable first-order differential system. In such a way they give an extension of the classical Levi-Civita separability conditions.[11] As a second step, they relate the definition of multiplicative separation of the Schrödinger equation to the additive separation of a suitable second-order Hamilton–Jacobi equation. A similar approach has been followed earlier by Agostinelli.[12] Kalnins and Miller begin their analysis of the separation of the Schrödinger equation by assuming that the coordinates are orthogonal, while in Agostinelli's paper (as well as in that of Koornwinder) it is shown that for the regular separation of the Schrödinger equation (in the sense of Kalnins and Miller) the coordinates are in fact necessarily orthogonal. Furthermore, Agostinelli shows that the nonorthogonal separation occurs when some of the separated factors ψ_i in (1.4) are of a special kind and the corresponding coordinates are ignorable. However, Agostinelli's approach is heavily coordinate dependent and somehow unsatisfactory, since at that time the geometrical theory of the variable separation of the Hamilton–Jacobi equation was not yet developed.

These remarks show that a revisitation and a resetting of all this matter is needed, from the very beginning, in light of the recent developments of the separability theory. The crucial question for a correct definition of separation for the Schrödinger equation is, as we said previously, how many constant parameters should enter a multiplicative separated solution and which conditions should they satisfy; in other words, as well as for the Hamilton–Jacobi equation, we need a *completeness condition* for a solution of the Schrödinger equation.

In the present paper it is shown that a completeness condition follows necessarily and in a ''natural'' way (i.e., without any consideration concerning the particular kind of equation we are

dealing with) from the general theory of the first-order normal differential systems (recalled in Sec. II, from a geometrical view point) applied to the general theory of the additive separation of Kalnins and Miller. The resulting definition of separation is quite different from those previously proposed in Refs. 6–8. First of all because it is not unique. Indeed, following our general approach we are led, again in a "natural" way, to consider at least two types of separation for the Schrödinger equation, which we have called *free* and *constrained*, respectively. These terms are motivated by the fact that one can impose "constraints" on some of the factors $\psi_i(q^i)$ of a separated solution; in other words, one can assume that some of these factors have a special form or satisfy a special kind of equations. One can, for instance, impose that $n-m$ factors have the form $\psi_\alpha = \exp(c_\alpha q^\alpha)$, for $\alpha = m+1,\ldots,n$, where c_α are arbitrary constants (see Refs. 12 and 13). We call *reduced separation* a constrained separation of this type. The remarkable fact is that, as a consequence of these assumptions, in the free separation the coordinates are necessarily orthogonal and the number of essential parameters entering the completeness condition is $2n$, while the reduced separation may occur in nonorthogonal coordinates and the number of the essential parameters reduces to $m+n<2n$, including (c_α). This is apparently in contrast to a common "ansatz" which considers only n parameters in a separated solution.[6,13,14] In fact, as we shall see, up to a transformation of the $2n$ parameters entering the completeness condition for the free separation, half of them are true "separation constants" corresponding to constants of motion (first integrals), while the remaining n are integration constants. This is quite reasonable since we are dealing with second-order equations. For a correct definition of "completeness" both groups of constant parameters are jointly involved, but in the process of integration by separation of variables all these essential constants automatically find their own place. This explains the curious fact that, in spite of the absence of the notion of "complete separable solution," the method of separation of variables in the Schrödinger equation has been applied with success, at least in the elementary cases. Thus, the content of the present paper is essentially theoretical and propedeutical to a further analysis of remaining topics on the separation: the relationship between first integrals and symmetry operators, the *R*-separation, the extensions to the case in which a vector potential is present, and to the case of a Lorentzian metric. As we shall see, a conspicuous and rich matter is hidden behind the usual approach to the separation of the Schrödinger equation.

II. FIRST-ORDER DIFFERENTIAL SYSTEMS AND THE GENERAL DEFINITION OF SEPARATION

Let us recall some classical basic facts concerning normal systems of first-order partial differential equations, from a geometrical viewpoint and in a way suitable for our purposes.

Let us consider a trivial fibration $\pi: M = Q \times Z \to Q$, where Z is an N-dimensional linear space (over $\mathbb{C}$ or $\mathbb{R}$) with coordinates $\underline{z} = (z_A)$ (capital indices $A,B,\ldots$ will run from 1 to N) and Q is an n-dimensional real differentiable manifold, with local coordinates $\underline{q} = (q^i)$ (latin indices $h,i,j,\ldots$ will run from 1 to n). A *connection* over this fibration is a regular distribution C over the tangent bundle TZ transversal and complementary to the fibers of π. This means that C is a subbundle of TM such that at each point $x \in M$ the set $C_x = C \cap T_x M$ is an n-dimensional subspace transversal to the fiber at the point z. A vector field D over M is *horizontal* if $D(M) \subset C$ i.e., if $D(x) \in C_x$ for each $x \in M$. A vector field V over M is *vertical* if it is tangent to the fibers. The only field which is simultaneously vertical and horizontal is the zero-vector field. If we consider local coordinates $(\underline{q}, \underline{z}) = (q^i, z_A)$ of M, then the distribution C is locally spanned by the following n horizontal vector fields (interpreted as derivations):

$$D_i = \frac{\partial}{\partial q^i} + C_{iA} \frac{\partial}{\partial z_A}, \tag{2.1}$$

called *generators* of C. The functions C_{iA} are the *coefficients* of the connection in these coordinates. It can be seen that the Lie brackets of two generators (as well as of any two horizontal vector fields) are vertical. Hence, by the Frobenius theorem, the distribution C is completely integrable (i.e., the connection C is *flat*) if and only if the generators commute,

$$[D_i, D_j] = 0. \tag{2.2}$$

The complete integrability of C means that there exists a foliation of integral manifolds transversal to the fibers and locally represented by equations of the kind

$$z_A = f_A(\underline{q}, \underline{c}), \tag{2.3}$$

where the N functions f_A depend on N constant parameters $\underline{c} = (c_A)$ which must be uniquely determined by assigning at any fixed point $q_0 \in Q$ (belonging to the domain of the coordinates) any arbitrary set of values of $\underline{z}$. This means that the functions (2.3) satisfy the *completeness condition*

$$\det\left[\frac{\partial \underline{z}}{\partial \underline{c}}\right] \neq 0. \tag{2.4}$$

Moreover, since the integral manifolds are tangent to the generators, equations

$$D_i(z_A - f_A(\underline{q}, \underline{c})) = 0$$

must be identically satisfied. Due to (2.1), these equations are equivalent to

$$\partial_i z_A = C_{iA}(\underline{q}, \underline{z}). \tag{2.5}$$

This proves that

Theorem 2.1: *A first-order differential system in the* normal form (2.5) *is* completely integrable, *i.e., it admits a local* complete solution (2.3) *satisfying the completeness condition* (2.4) *if and only if the generators* D_i *commute*, $[D_i, D_j] = 0$.

On the basis of these general considerations, we can reformulate the definition of separation of Kalnins and Miller[9,10] as follows. Let us consider a partial differential equation

$$\mathcal{H}(\underline{q}, u, u_i, u_{ij}, \ldots, u_{ij\cdots h}) = E, \tag{2.6}$$

in the independent variables $\underline{q} = (q^i)$ and in the unknown function $u(\underline{q})$. Here $u_{ij\cdots h}$ denote the partial derivatives of u with respect to these variables, E is a constant parameter, and $\mathcal{H}$ is a smooth real function of the variables $\underline{q}$, u and its partial derivatives up to a degree l. A *separable solution* of this equation is a solution of the form

$$u = \sum_{i=1}^{n} S_i(q^i, E), \tag{2.7}$$

i.e., a sum of functions depending on a single variable. For separable solutions all the mixed partial derivatives (corresponding to distinct indices) vanish identically, so that Eq. (2.6) gets the simpler form

$$\mathcal{H}_s(\underline{q}, u, \underline{u}^{(1)}, \underline{u}^{(2)}, \ldots, \underline{u}^{(l)}) = E, \tag{2.8}$$

where $\underline{u}^{(1)} = (u_i^{(1)}) = (u_i)$, $\underline{u}^{(2)} = (u_i^{(2)}) = (u_{ii})$, etc., and $\mathcal{H}_s$ is the function we get by replacing $u_{i\cdots j} = 0$ in $\mathcal{H}$, for at least two distinct indices. Note that $\mathcal{H}_s$ in (2.8) is in general a function different from $\mathcal{H}$ in (2.6). However, for the sake of simplicity, in the following discussion we shall use the same symbol $\mathcal{H}$. Since for any solution of this equation the left-hand side of (2.8) reduces to a constant, the total derivatives of $\mathcal{H}$ with respect to the coordinates must vanish identically,

$$\frac{\partial \mathcal{H}}{\partial q^i} + \frac{\partial \mathcal{H}}{\partial u} u_i + \frac{\partial \mathcal{H}}{\partial u_i} u_i^{(2)} + \frac{\partial \mathcal{H}}{\partial u_i^{(2)}} u_i^{(3)} + \cdots + \frac{\partial \mathcal{H}}{\partial u_i^{(l)}} u_i^{(l+1)} = 0. \tag{2.9}$$

Hence, for

$$\frac{\partial \mathcal{H}}{\partial u_i^{(l)}} \neq 0$$

we can define the function

$$R_i(\underline{q},u,\underline{u}^{(1)},\underline{u}^{(2)},\ldots,\underline{u}^{(l)}) = -\left(\frac{\partial \mathcal{H}}{\partial u_i^{(l)}}\right)^{-1} \times\left(\frac{\partial \mathcal{H}}{\partial q^i}+\frac{\partial \mathcal{H}}{\partial u}u_i^{(1)}+\frac{\partial \mathcal{H}}{\partial u_i}u_i^{(2)}+\frac{\partial \mathcal{H}}{\partial u_i^{(2)}}u_i^{(3)}+\cdots+\frac{\partial \mathcal{H}}{\partial u_i^{(l-1)}}u_i^{(l)}\right), \tag{2.10}$$

so that Eq. (2.9) becomes

$$u_i^{(l+1)} = R_i(\underline{q},u,\underline{u}^{(1)},\underline{u}^{(2)},\ldots,\underline{u}^{(l)}).$$

Now, let us consider the following first-order differential system where $j \neq i$ and $\partial_i = \frac{\partial}{\partial q i}$:

$$\begin{aligned} &\partial_i u = u_i^{(1)}, \\ &\partial_i u_i^{(1)} = u_i^{(2)}, \quad \partial_i u_j^{(1)} = 0, \\ &\partial_i u_i^{(2)} = u_i^{(3)}, \quad \partial_i u_j^{(2)} = 0, \\ &\cdots \quad \cdots \\ &\partial_i u_i^{(l)} = R_i, \quad \partial_i u_j^{(l)} = 0. \end{aligned} \tag{2.11}$$

This is a normal system of the kind (2.5) in the unknown $N = n \cdot l + 1$ functions

$$\underline{z} = (z_A) = (u,\underline{u}^{(1)},\underline{u}^{(2)},\ldots,\underline{u}^{(l)}).$$

The comparison with (2.1) and (2.5) shows that the corresponding generators are

$$D_i = \partial_i + u_i^{(1)}\frac{\partial}{\partial u} + u_i^{(2)}\frac{\partial}{\partial u_i^{(1)}} + u_i^{(3)}\frac{\partial}{\partial u_i^{(2)}} + \cdots + R_i\frac{\partial}{\partial u_i^{(l)}}. \tag{2.12}$$

We remark that a separable solution (2.7) corresponds to a solution of this system. Due to Theorem 2.1 we can introduce the following

Definition 2.2: We say that Eq. (2.6) is *separable* in the coordinates $\underline{q}$ if it admits a *complete separable* solution, i.e., a solution of the form

$$u = \sum_{i=1}^{n} S_i(q^i,\underline{c}) \tag{2.13}$$

depending on $N = nl + 1$ constant parameters $\underline{c} = (c_A)$ and satisfying the *completeness condition*

$$\det\left[\frac{\partial \underline{z}}{\partial \underline{c}}\right] = \det\left[\frac{\partial u}{\partial c_A}\middle|\frac{\partial u_i}{\partial c_A}\middle|\frac{\partial u_i^{(2)}}{\partial c_A}\middle|\cdots\middle|\frac{\partial u_i^{(l)}}{\partial c_A}\right] \neq 0. \tag{2.14}$$

It follows that

Theorem 2.3: *Equation (2.6) is separable (i.e., it admits a complete separable solution) in the coordinates $\underline{q}$ if and only if the first-order system (2.11) is completely integrable, i.e., if and only if the separability conditions*

$$[D_i, D_j] = 0 \tag{2.15}$$

are identically satisfied for D_i defined by (2.12) and R_i defined by (2.10).

Remark 2.4: In the following applications we shall deal with equations of the kind (2.6) or (2.8) where $\mathcal{H}$ does not depend on u. In this case the dependent variables are

$$\underline{z} = (z_A) = (\underline{u}^{(1)}, \underline{u}^{(2)}, \ldots, \underline{u}^{(l)}),$$

and the first line of system (2.11) disappears, as well as the term $\partial/\partial u$ in the generators (2.12). Moreover, the number of the constants $\underline{c} = (c_A)$ entering a complete solution (2.13) is $n \cdot l$, and the completeness condition (2.14) reduces to

$$\det\left[\frac{\partial \underline{z}}{\partial \underline{c}}\right] = \det\left[\frac{\partial u_i}{\partial c_A} \middle| \frac{\partial u_i^{(2)}}{\partial c_A} \middle| \cdots \middle| \frac{\partial u_i^{(l)}}{\partial c_A}\right] \neq 0. \tag{2.16}$$

Remark 2.5: The case in which the complete integrability conditions (2.15) are identically satisfied corresponds to the case of *regular separation* of Kalnins and Miller.[9] They call *non-regular* a type of separation in which these conditions are nonidentically satisfied and, consequently, separable solutions may exist, but depending on a reduced number of constants. However, the meaning of this non-regular separation is rather obscure. As we shall see (Sec. V), in dealing with the Schrödinger equation we are led to introduce, in natural way, an alternative definition of the regular separation: the reduced separation.

Remark 2.6: Definition 2.2 and Theorem 2.3 are the basic statements for the theory of variable separation (additive, multiplicative, or any other kind) provided the kind of separation we are dealing with can be transformed, by a suitable transformation, into an additive separation. This is in fact the case of the multiplicative separation: the transformation is $u = \ln \psi$.

III. THE LEVI-CIVITA SEPARABILITY CONDITIONS AND THEIR CONSEQUENCES

Let us apply the general theory illustrated in Sec. II to the case of the Hamilton–Jacobi equation corresponding to a Hamiltonian $H: T^*Q \to \mathbb{R}$ over a cotangent bundle T^*Q. In this case we deal with the cotangent fibration $\pi: M = T^*Q \to Q$ of a configuration manifold Q. This is (in general) a nontrivial fibration, but our previous considerations can be applied, since they have a local character.

The differential equation (2.6) is now

$$H(\underline{q}, \underline{u}^{(1)}) = E, \tag{3.1}$$

with $\mathcal{H} = H$ and, according to the standard notation,

$$u = W, \quad \underline{u}^{(1)} = (u_i) = (p_i) = (\partial_i W).$$

Now $l = 1$ and $\underline{z} = (\underline{u}^{(1)}) = (p_i)$. In a complete solution we have exactly n constants $\underline{c} = (c_i)$ and the completeness condition (2.16) takes the form (1.8),

$$\det\left[\frac{\partial p_i}{\partial c_j}\right] \neq 0. \tag{3.2}$$

The differential system (2.11) reads

$$\partial_i p_i = R_i, \quad \partial_i p_j = 0 \quad (j \neq i), \tag{3.3}$$

where the functions R_i (2.10) are

$$R_i = -\frac{\partial_i H}{\partial^i H} \quad \left(\partial_i = \frac{\partial}{\partial q^i}, \partial^i = \frac{\partial}{\partial p_i}\right).$$

Since the generators D_i (2.14) are

$$D_i = \partial_i + R_i \partial^i,$$

the separability conditions $[D_i, D_j] = 0$ (2.15) are equivalent to the well-known *Levi-Civita separability conditions*

$$\partial^i \partial^j H \partial_i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H - \partial_i \partial^j H \partial^i H \partial_j H = 0, \quad i \neq j, \text{ n.s.}, \tag{3.4}$$

where "n.s." means that there is no summation over the repeated indices. As a consequence, Theorem 2.3 reduces to the celebrated *Levi-Civita theorem*,

Theorem 3.1: *The Hamilton–Jacobi equation (3.1) is separable in the coordinates* $\underline{q} = (q^i)$ *i.e., it admits a solution of the kind (2.3) satisfying the completeness condition (3.2), if and only if (3.4) are identically satisfied.*

This theorem holds for any kind of Hamiltonian. We can apply it to a *natural Hamiltonian*

$$H = G + V = \tfrac{1}{2} g^{ij}(\underline{q}) p_i p_j + V(\underline{q}),$$

where G is the geodesic Hamiltonian, $V: Q \to \mathbb{R}$ a potential energy. The corresponding Hamilton–Jacobi equation is now Eq. (1.1), which in any local canonical coordinate system becomes

$$\tfrac{1}{2} g^{ij} \partial_i W \partial_j W + V = E.$$

Moreover, we can consider two special types of coordinates: *orthogonal* and *standard* coordinates (Definition 3.3). It is straightforward to prove that

Proposition 3.2: If the coordinates are orthogonal ($g^{ij} = 0$ *for* $i \neq j$)*, then the Levi-Civita separability conditions are equivalent to*

$$\begin{array}{c} g^{ii} g^{jj} \partial_i \partial_j g^{hh} - g^{ii} \partial_i g^{jj} \partial_j g^{hh} - g^{jj} \partial_j g^{ii} \partial_i g^{hh} = 0, \\ g^{ii} g^{jj} \partial_i \partial_j V - g^{ii} \partial_i g^{jj} \partial_j V - g^{jj} \partial_j g^{ii} \partial_i V = 0, \end{array} \quad i \neq j, \text{ n.s.} \tag{3.5}$$

These equations do not involve the momenta $\underline{p}$. An equivalent form of these equations were first established by Eisenhart.[3] It can be shown that their general solutions have the Stäckel form (1.6).[15] It follows that the Hamilton–Jacobi equations split into separated equations of the form

$$\tfrac{1}{2} p_i^2 = \varphi_i^{(j)} c_j - \phi_i,$$

where $\underline{c} = (c_j)$ are arbitrary constants satisfying the completeness condition. Equation (3.5) can also be written in the form

$$\begin{array}{c} \partial_i \partial_j g^{hh} - \partial_i \ln g^{jj} \partial_j g^{hh} - \partial_j \ln g^{ii} \partial_i g^{hh} = 0, \\ \partial_i \partial_j V - \partial_i \ln g^{jj} \partial_j V - \partial_j \ln g^{ii} \partial_i V = 0, \end{array} \quad i \neq j, \text{ n.s.} \tag{3.5$'$}$$

Note that the first equations (3.5′) characterize the orthogonal separation of the pure geodesic Hamiltonian.

Definition 3.3: A *standard coordinate system* is a coordinate system $(q^i) = (q^a, q^\alpha)$ with $a = 1,...,m$ and $\alpha = m+1,...,n$, such that (i) the metric tensor assumes the semidiagonal *standard form*

$$\mathbf{G} = g^{aa} \partial_a \otimes \partial_a + g^{\alpha\beta} \partial_\alpha \otimes \partial_\beta, \tag{3.6}$$

and (ii) the coordinates (q^{α}) are *ignorable*,

$$\partial_{\alpha} g^{ij} = 0, \quad \partial_{\alpha} V = 0. \tag{3.7}$$

We call *essential* the coordinates (q^{a}).
It is straightforward to prove that

Proposition 3.4: In standard coordinates the Levi-Civita separability conditions are equivalent to the following:

$$\begin{gathered} g^{aa} g^{bb} \partial_a \partial_b g^{cc} - g^{aa} \partial_a g^{bb} \partial_b g^{cc} - g^{bb} \partial_b g^{aa} \partial_a g^{cc} = 0, \\ g^{aa} g^{bb} \partial_a \partial_b g^{\alpha\beta} - g^{aa} \partial_a g^{bb} \partial_b g^{\alpha\beta} - g^{bb} \partial_b g^{aa} \partial_a g^{\alpha\beta} = 0, \\ g^{aa} g^{bb} \partial_a \partial_b V - g^{aa} \partial_a g^{bb} \partial_b V - g^{bb} \partial_b g^{aa} \partial_a V = 0, \end{gathered} \tag{3.8}$$

with $a \neq b$ not summed.

These equations can be written in the equivalent form

$$\begin{gathered} \partial_a \partial_b g^{cc} - \partial_a \ln g^{bb} \partial_b g^{cc} - \partial_b \ln g^{aa} \partial_a g^{cc} = 0, \\ \partial_a \partial_b g^{\alpha\beta} - \partial_a \ln g^{bb} \partial_b g^{\alpha\beta} - \partial_b \ln g^{aa} \partial_a g^{\alpha\beta} = 0, \\ \partial_a \partial_b V - \partial_a \ln g^{bb} \partial_b V - \partial_b \ln g^{aa} \partial_a V = 0. \end{gathered} \tag{3.8$'$}$$

Their general solutions are still of Stäckel type, but involving an $m \times m$ Stäckel matrix $[\varphi_a^{(b)}]$ of functions depending on the essential coordinates only,

$$g^{aa} = \varphi^{a}_{(m)}, \quad g^{\alpha\beta} = \phi_a^{\alpha\beta} g^{aa}, \quad V = \phi_a g^{aa}, \tag{3.9}$$

with functions $(\phi_a^{\alpha\beta}, \phi_a)$ depending on the essential coordinate corresponding to the lower index only. Then the Hamilton–Jacobi equation splits into separated equations of the kind

$$p_{\alpha} = c_{\alpha}, \quad \tfrac{1}{2} p_{\alpha}^2 = \varphi_a^{(b)} c_b - \phi_a^{\alpha\beta} c_{\alpha} c_{\beta} - \phi_a,$$

where $c = (c_i) = (c_a, c_{\alpha})$ are n arbitrary constants.

Remark 3.5: The above-given definition of standard coordinates needs some comments. (i) In a given standard coordinate system (q^{i}) the distinction between "ignorable" and "essential" coordinates is in general not univocal. Indeed, an ignorable coordinate may be orthogonal to the other ones and considered as essential; conversely, an essential coordinate may be ignorable, i.e., it may satisfy (3.7). However, in the process of integration by separation of variables of the Hamilton–Jacobi equation it is useless to consider an ignorable coordinate as "essential," since it corresponds to a linear homogeneous first integral and thus to a trivial separated equation. (ii) The distinction between ignorable and essential coordinates becomes univocal and assumes a full meaning when related to a given separable Killing web,[16,17] that is when related to the geometrical characterization of the separation (see Sec. VII).

Remark 3.6: It is known that there is no loss of generality in considering separable standard coordinates. Indeed, the analysis of the Levi-Civita separability conditions shows that any separable coordinate system admits an equivalent standard system (q^{a}, q^{α}) where the number of essential coordinates is minimalized, i.e., it coincides with the number of second-class coordinates, according to the classification of Levi-Civita. This number is invariant within an equivalence class of separable coordinates. A standard coordinate system in which the essential coordinates (q^{a}) are exactly those of second class has been called *normal*.[17] For our present purposes we do not need to take into account the classification of Levi-Civita and the subtle distinction between "normal" and "standard" coordinates. We need only to refer to the above-given definition of standard coordinates and to the corresponding separability conditions (3.8) or (3.8$'$).

Definition 3.7: We say that a symmetric two-tensor $\mathbf{K}=(K^{ij})$ has a *standard form* or that it is a *standard tensor* with respect to a standard coordinate system if it assumes the form

$$\mathbf{K}=K^{aa}\partial_a\otimes\partial_a+K^{\alpha\beta}\partial_\alpha\otimes\partial_\beta=\lambda^a g^{aa}\partial_a\otimes\partial_a+K^{\alpha\beta}\partial_\alpha\otimes\partial_\beta, \tag{3.10}$$

where (λ^a) and $K^{\alpha\beta}$ do not depend on the ignorable coordinates (q^α). Then the matrix $[K^{ij}]$ has a form similar to (3.6).

Note that a tensor may be simultaneously in standard form with respect to nonequivalent separable standard coordinate systems.

IV. THE FREE SEPARATION OF THE SCHRÖDINGER EQUATION

In this section we shall show that a convenient and precise "ansatz" for the multiplicative separation of the Schrödinger equation (1.2) is given by the following

Definition 4.1: A *complete separated solution* of the Schrödinger equation is a solution of the form $\psi(\underline{q},\underline{c})=\Pi_i^n\ \psi_i(q^i,\underline{c})$, depending on $2n$ parameters $\underline{c}=(c_I)$ satisfying the *completeness condition*

$$\det\begin{bmatrix}\dfrac{\partial u_i}{\partial c_I}\\[2ex] \dfrac{\partial v_i}{\partial c_I}\end{bmatrix}\neq 0,\quad u_i=\frac{\psi_i'}{\psi_i},\quad v_i=\frac{\psi_i''}{\psi_i}. \tag{4.1}$$

When such a solution exists we say that the Schrödinger equation is *separable* (or *freely separable*) in the coordinates $\underline{q}=(q^i)$.

The completeness condition (4.1) means that the $2n$ constants (c_I) can be uniquely determined by assigning arbitrary values to the $2n$ ratios $(\underline{u},\underline{v})$, at any fixed point. Hence, no restriction is imposed on the values that the functions $(\psi_i,\psi_i',\psi_i'')$ can assume at any given point of the domain of the coordinates, for $\psi\neq 0$. For this reason we call this kind of separation *free*. As we shall see in Sec. V, we can in fact consider another kind of separation in which such a "freedom" is lost. As we shall see in the following (Remark 4.8), the $2n$ parameters appearing in a free separated solution have a different role: in the process of integration n of them are related to constants of motion in involution or, equivalently, to second-order commuting symmetries of the Schrödinger operator, so that they can be interpreted as "separation constants," while the remaining n parameters arise as "integration constants."

By assuming Definition 4.1 we shall prove

Theorem 4.2: *The Schrödinger equation is freely separable in a coordinate system $\underline{q}$ if and only if: (i) these coordinates are orthogonal, (ii) the corresponding Hamilton–Jacobi equation is separable, (iii) the following conditions are satisfied:*

$$\partial_i\Gamma_j=0\quad (i\neq j), \tag{4.2}$$

where

$$\Gamma_i=g^{hj}\Gamma_{hj,i},\quad \Gamma_{hj,i}=\tfrac{1}{2}(\partial_h g_{ji}+\partial_j g_{ih}-\partial_i g_{hj}). \tag{4.3}$$

Since $\Gamma_{hj,i}$ are the Christoffel symbols of the Levi-Civita connection, we call the functions Γ_i the *contracted Christoffel symbols*, associated with the coordinates $\underline{q}$. From their definition (4.3) it follows that

$$\Gamma_i=\tfrac{1}{2}\partial_i\ln\det[g^{hj}]-g_{ik}\partial_h g^{hk}. \tag{4.4}$$

We call the whole set of equations (4.2) the *Robertson condition for the free separation* of the Schrödinger equation. In this form the Robertson condition means that each contracted Christoffel

symbol Γ_i is a function of the corresponding coordinate q^i only (see Remark 4.7). As we know, an equivalent form of the Robertson condition is the diagonalization of the Ricci tensor. We shall discuss this equivalence in Sec. VI.

In order to justify all the above-given statements, let us start from the local coordinate expression of the Schrödinger equation (1.2),

$$g^{ij}\partial_i\partial_j\psi-\Gamma^k\partial_k\psi+\frac{2}{\hbar^2}(E-V)\psi=0, \quad \Gamma^k=g^{ki}\Gamma_i. \tag{4.5}$$

Since the constant factor $2/\hbar^2$ is inessential for our consideration, from now on we shall replace $(2/\hbar^2)V$ with V and $(2/\hbar^2)E$ by E.

For a separated solution of the kind (1.4) we have

$$\partial_i\psi=\frac{\psi_i'}{\psi_i}\psi, \quad \partial_i\partial_j\psi=\frac{\psi_i'\psi_j'}{\psi_i\psi_j}\psi \quad (i\neq j), \quad \partial_i\partial_i\psi=\frac{\psi_i''}{\psi_i}\psi. \tag{4.6}$$

Here the prime denotes the derivative operator on a function of a single variable. We remark that in these formulas the fraction ψ/ψ_i is the product $\psi_1\cdots\hat{\psi}_i\cdots\psi_n$, without the factor ψ_i. Thus, expressions (4.6) also hold at the points where $\psi_i=0$. If we set $u=\ln\psi$, then

$$u_i=\partial_i u=\frac{\partial_i\psi}{\psi}=\frac{\psi_i'}{\psi_i} \tag{4.7}$$

are functions of the corresponding variable q^i and moreover,

$$\partial_i\psi=\psi u_i,$$

$$\partial_i\partial_j\psi=\psi u_i u_j \quad (i\neq j),$$

$$\partial_i\partial_i\psi=\psi(u_i^2+u_i^{(2)}), \quad u_i^{(2)}=u_{ii}=\partial_i\partial_i u.$$

It follows that for a separated solution Eq. (4.5) assumes the form

$$(g^{ij}u_iu_j+g^{ii}u_i^{(2)}-\Gamma^i u_i+E-V)\psi=0, \tag{4.8}$$

where the sum over the repeated indices is understood. Thus, for $\psi\neq 0$ we get a partial differential equation of the kind (2.8) with $l=2$,

$$\mathcal{S}(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)})=-E,$$

by setting (see Ref. 12)

$$\begin{gathered}\mathcal{S}(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)})=g^{ij}u_iu_j+g^{ii}u_i^{(2)}-\Gamma^i u_i-V,\\ \underline{u}^{(1)}=(u_i), \quad \underline{u}^{(2)}=(u_{ii}).\end{gathered} \tag{4.9}$$

As a consequence, by applying Definition 2.2 to the present case and Theorem 2.3 related to the general equation (2.8), taking into account Remark 2.4, the completeness condition (2.16), and expression (4.7), we are led to the following definition and theorem.

Definition 4.3: A complete separated solution of the Schrödinger equation (4.5) is a solution of the form (1.4) depending on $2n$ constant parameters $\underline{c}=(c_I)$ satisfying the completeness condition

$$\det\begin{bmatrix} \dfrac{\partial u_i}{\partial c_I} \\ \dfrac{\partial u_i^{(2)}}{\partial c_I} \end{bmatrix} \neq 0, \quad u_i = \frac{\psi_i'}{\psi_i}, \quad u_i^{(2)} = u_{ii} = \left(\frac{\psi_i'}{\psi_i}\right)'. \tag{4.10}$$

Theorem 4.4: *The Schrödinger equation (4.5) is separable in the coordinates $\underline{q}$ if and only if the following first-order differential system is completely integrable*

$$\begin{aligned} &\partial_i u_i^{(1)} = u_i^{(2)}, \quad &&\partial_i u_j^{(1)} = 0, \\ &\partial_i u_i^{(2)} = R_i, \quad &&\partial_i u_j^{(2)} = 0, \end{aligned} \quad (i \neq j), \tag{4.11}$$

where

$$R_i(\underline{q}, \underline{u}^{(1)}, \underline{u}^{(2)}) = -\left(\frac{\partial \mathcal{S}}{\partial u_i^{(2)}}\right)^{-1}\left(\frac{\partial \mathcal{S}}{\partial q^i} + \frac{\partial \mathcal{S}}{\partial u_i} u_i^{(2)}\right), \tag{4.12}$$

$\mathcal{S}$ being defined as in (4.9) i.e., if and only if the commutation relations

$$[D_i, D_j] = 0, \tag{4.13}$$

are identically satisfied for

$$D_i = \partial_i + u_i^{(2)} \frac{\partial}{\partial u_i} + R_i \frac{\partial}{\partial u_i^{(2)}}.$$

Remark 4.5: If we replace the $2n$ variables $(\underline{u}^{(1)}, \underline{u}^{(2)})$ with the variables

$$(\underline{u}, \underline{v}) = (u_i, v_i)$$

where

$$v_i = u_i^{(2)} + u_i^2 = \frac{\psi_i''}{\psi_i}, \tag{4.14}$$

then the completeness condition (4.10) becomes equivalent to (4.1), so that Definition 4.3 is equivalent to Definition 4.1.

This transformation of dependent variables turns out to be convenient for the analysis of the integrability (or separability) conditions (4.13). We note first of all that

$$\frac{\partial \mathcal{S}}{\partial u_i^{(2)}} = g^{ii},$$

so that the definition (4.12) becomes

$$R_i(\underline{q}, \underline{u}^{(1)}, \underline{u}^{(2)}) = -(g^{ii})^{-1}\left(\frac{\partial \mathcal{S}}{\partial q^i} + \frac{\partial \mathcal{S}}{\partial u_i} u_i^{(2)}\right).$$

In the new variables we have

$$\mathcal{S}(\underline{q}, \underline{u}, \underline{v}) = g^{ij} u_i u_j + g^{ii}(v_i - u_i^2) - \Gamma^i u_i - V,$$

$$R_i(\underline{q}, \underline{u}, \underline{v}) = -\frac{1}{g^{ii}}\left(\frac{\partial \mathcal{S}}{\partial q^i} + (v_i - u_i^2)\frac{\partial \mathcal{S}}{\partial u_i}\right), \tag{4.15}$$

$$D_i=\partial_i+(v_i-u_i^2)\frac{\partial}{\partial u_i}+R_i\frac{\partial}{\partial v_i},$$

and system (4.11) becomes equivalent to

$$\begin{array}{ll}\partial_i u_i=v_i-u_i^2, & \partial_i u_j=0,\\ \partial_i v_i=R_i, & \partial_i v_j=0,\end{array}\quad (i\neq j). \tag{4.16}$$

Using (4.15), a straightforward calculation shows that

$$[D_i,D_j]=\left((v_i-u_i^2)\frac{\partial R_j}{\partial u_i}+R_i\frac{\partial R_j}{\partial v_i}+\frac{\partial R_j}{\partial q^i}\right)\frac{\partial}{\partial v_j}-\left((v_j-u_j^2)\frac{\partial R_i}{\partial u_j}+R_j\frac{\partial R_i}{\partial v_j}+\frac{\partial R_i}{\partial q^j}\right)\frac{\partial}{\partial v_i}.$$

Hence, the integrability conditions (4.13) become equivalent to

$$(v_i-u_i^2)\frac{\partial R_j}{\partial u_i}+R_i\frac{\partial R_j}{\partial v_i}+\frac{\partial R_j}{\partial q^i}=0\quad (i\neq j). \tag{4.17}$$

Because of (4.15),

$$\frac{\partial R_i}{\partial q^i}=-\frac{1}{g^{jj}}\partial_i g^{jj}R_j-\frac{1}{g^{jj}}\left[\partial_i\partial_j\mathcal{S}+\frac{\partial^2\mathcal{S}}{\partial q^i\partial u_j}(v_j-u_j^2)\right]$$

and, for $i\neq j$,

$$\frac{\partial R_j}{\partial v_i}=-\frac{1}{g^{jj}}\left[\frac{\partial^2\mathcal{S}}{\partial q^j\partial v_i}+\frac{\partial^2\mathcal{S}}{\partial v_i\partial u_j}(v_j-u_j^2)\right]=-\frac{1}{g^{jj}}\partial_j g^{ii}\quad (i\neq j).$$

Moreover,

$$\frac{\partial\mathcal{S}}{\partial u_i}=2g^{ik}u_k-2g^{ii}u_i-\Gamma^i.$$

Thus,

$$\frac{\partial^2\mathcal{S}}{\partial u_i\partial u_j}=2g^{ij}\quad (i\neq j).$$

It follows that

$$\frac{\partial R_j}{\partial u_i}=-\frac{1}{g^{jj}}\left[\frac{\partial^2\mathcal{S}}{\partial u_i\partial q^j}+2g^{ij}(v_j-u_j^2)\right]\quad (i\neq j).$$

Due to these last equations, the integrability conditions (4.17) become equivalent to

$$2g^{ij}(v_i-u_i^2)(v_j-u_j^2)+\frac{\partial^2\mathcal{S}}{\partial q^i\partial u_j}(v_j-u_j^2)+\frac{\partial^2\mathcal{S}}{\partial q^j\partial u_i}(v_i-u_i^2)+\partial_i\partial_j\mathcal{S}+R_i\partial_j g^{ii}+R_j\partial_i g^{jj}=0$$

$$(i\neq j). \tag{4.18}$$

Note that in this form they are symmetric in the distinct indices (i,j). We remark that these are algebraic equations in the variables $(\underline{u},\underline{v})$ and that they must be identically satisfied for all values of these variables, due to the completeness condition. A closer analysis of expressions (4.15) shows that (4.18) are of second degree in $\underline{v}$, and that the corresponding second-degree homogeneous polynomial is given by the first term $g^{ij}v_iv_j$, with $i\neq j$ not summed. This implies

$$g^{ij}=0 \quad (i\neq j) \tag{4.19}$$

and shows that

Proposition 4.6: In the free separation of the Schrödinger equation the coordinates are necessarily orthogonal.

In orthogonal coordinates, Eqs. (4.18) and (4.15) assume the simpler form

$$\frac{\partial^2\mathcal{S}}{\partial q^i\partial u_j}(v_j-u_j^2)+\frac{\partial^2\mathcal{S}}{\partial q^j\partial u_i}(v_i-u_i^2)+\partial_i\partial_j\mathcal{S}+R_i\partial_j g^{ii}+R_j\partial_i g^{jj}=0 \quad (i\neq j) \tag{4.20}$$

and

$$\mathcal{S}=g^{ii}v_i-\Gamma^i u_i-V, \quad R_i=\frac{1}{g^{ii}}[\Gamma^i(v_i-u_i^2)-\partial_i\mathcal{S}]. \tag{4.21}$$

It follows that

$$R_i=\frac{1}{2g^{ii}}[2\Gamma^i(v_i-u_i^2)-\partial_i g^{kk}v_k+\partial_i\Gamma^k u_k+\partial_i V].$$

By inserting this last expression of R_i into (4.20) we conclude that the integrability conditions of system (4.16) are equivalent to the orthogonality conditions (4.19) and to the following:

$$\begin{aligned}&2(u_j^2-v_j)(\partial_i\Gamma^j-\Gamma^j\partial_i\ln g^{jj})+2(u_i^2-v_i)(\partial_j\Gamma^i-\Gamma^i\partial_j\ln g^{ii})\\&\quad+v_k(\partial_i\partial_j g^{kk}-\partial_i\ln g^{jj}\partial_j g^{kk}-\partial_j\ln g^{ii}\partial_i g^{kk})-u_k(\partial_i\partial_j\Gamma^k-\partial_i\ln g^{jj}\partial_j\Gamma^k-\partial_j\ln g^{ii}\partial_i\Gamma^k)\\&\quad-(\partial_i\partial_j V-\partial_i\ln g^{jj}\partial_j V-\partial_j\ln g^{ii}\partial_i V)=0 \quad (i\neq j).\end{aligned} \tag{4.22}$$

Since these last equations are polynomial in the variables $(\underline{u},\underline{v})$, they are identically satisfied if and only if all the coefficients vanish, namely:

$$\begin{gathered}\partial_i\partial_j g^{kk}-\partial_i\ln g^{jj}\partial_j g^{kk}-\partial_j\ln g^{ii}\partial_i g^{kk}=0,\\ \partial_i\partial_j V-\partial_i\ln g^{jj}\partial_j V-\partial_j\ln g^{ii}\partial_i V=0,\\ \partial_j\Gamma^i-\Gamma^i\partial_j\ln g^{ii}=0,\\ \partial_i\partial_j\Gamma^k-\partial_i\ln g^{jj}\partial_j\Gamma^k-\partial_j\ln g^{ii}\partial_i\Gamma^k=0.\end{gathered} \quad (i\neq j) \tag{4.23}$$

These equations are in fact redundant. Indeed, it can be seen that due to the first and third equations, the last equation is identically satisfied. Moreover, the third equation is equivalent to

$$\partial_j\Gamma_i=0 \quad (i\neq j),$$

where, in orthogonal coordinates,

$$\Gamma_i=g_{ii}\Gamma^i=\tfrac{1}{2}\Sigma_{k\neq i}\partial_i\ln g^{kk}-\tfrac{1}{2}\partial_i\ln g^{ii}=\partial_i\ln(g_{ii}\sqrt{\Pi_k g^{kk}}). \tag{4.24}$$

Finally, we recognize in the first two equations of (4.23) the necessary and sufficient conditions (3.5') for the orthogonal separation of the Hamilton–Jacobi equation. Thus, due to Theorem 4.4, Theorem 4.2 is proved.

Remark 4.7: After Theorem 4.2, the integration of the Schrödinger equation by separation of variables is accomplished as follows. Due to the Stäckel form (1.6) of the metric tensor components and of the potential, the Schrödinger equation (4.8) takes the form

$$\varphi^i_{(n)}(u_i'+u_i^2-\Gamma_i u_i-\phi_i)=-E, \quad u_i=\frac{\psi_i'}{\psi_i}.$$

This equation is interpreted as the last one of a system of n equations involving all the remaining rows of the matrix $[\varphi^i_{(j)}]$ and n constants $\underline{a}=(a_i)$,

$$\varphi^i_{(j)}(u_i'+u_i^2-\Gamma_i u_i-\phi_i)=-a_j, \quad a_n=E, \tag{4.25}$$

which is equivalent to the following system of separated equations:

$$u_i'+u_i^2-\Gamma_i u_i-\phi_i+\varphi_i^{(j)}a_j=0. \tag{4.26}$$

These are first-order Riccati equations in the unknown functions $u_i(q^i)$, depending on the n constant parameters $\underline{a}=(a_j)$. Their integration yields functions

$$u_i=u_i(q^i,\underline{a},b_i), \tag{4.27}$$

each one depending on a further constant b_i. The complete separated solution is then given, up to an inessential constant factor, by

$$\psi=\exp\int u_i\,\mathrm{d}q^i. \tag{4.28}$$

We note that (as happens for any Riccati equation) (4.26) are equivalent to the linear second-order equations

$$\psi_i''-\Gamma_i\psi_i'-(\phi_i-\varphi_i^{(j)}a_j)\psi_i=0 \tag{4.29}$$

in the original unknown functions ψ_i.

Remark 4.8: The $2n$ constants $\underline{c}=(c_I)=(\underline{a},\underline{b})=(a_i,b_i)$, appearing in a separated solution (4.28) as a result of the process of integration by separation of variables illustrated previously, play a different role. While $\underline{b}$ arise as integration constants from the integration of the first-order Riccati equations (4.26), or of the second-order linear equations (4.29), the constants $\underline{a}$ have two interpretations. (i) They are the constant values taken by the quadratic first integrals in involution related to the separation of the Hamilton–Jacobi equation (see Theorem 7.14), whose expressions in orthogonal separable coordinates are

$$H_j=\tfrac{1}{2}\varphi^i_{(j)}(p_i^2+2\phi_i). \tag{4.30}$$

(ii) They are the eigenvalues of the second-order symmetry operators of the Schrödinger equation corresponding to H_j,

$$\hat{H}_j\psi=\psi^i_{(j)}(-\partial_i^2\psi+\Gamma_i\partial_i\psi+\phi_i\psi). \tag{4.31}$$

The link between H_j and $\hat{H}_j$ will be discussed in a following paper.[18] If we apply these operators to a separated solution (1.4) then, due to (4.6), (4.7) and (4.14), we get

$$\hat{H}_j\psi=\varphi^i_{(j)}(\Gamma_i u_i-v_i+\phi_i)\psi, \tag{4.32}$$

which shows that the eigenvalues are

$$a_j=\varphi^i_{(j)}(\Gamma_i u_i-v_i+\phi_i). \tag{4.33}$$

Remark 4.9: The values of the constants $\underline{a}$ are determined through (4.25), which are equivalent to (4.33), by the initial values at a point q_0 of (u_i,u_i') i.e., by the initial values of $(\psi_i,\psi_i',\psi_i'')$, for $\psi_i(q_0)\neq 0$. The values of $\underline{b}$ are then determined by reversing (4.27) at the initial point q_0: $u_i(q_0)=u_i(q_0^i,\underline{a},b_i)$. If instead of the first-order equations (4.26) we consider the equivalent second-order equations (4.29), the resulting integration constants are $2n$, but only half

of them are essential, since the functions ψ_i are determined up to a constant factor. Thus, also in this case we reduce to n essential constants (b_i). See Sec. IX for an example.

V. THE REDUCED SEPARATION OF THE SCHRÖDINGER EQUATION

In Sec. IV we have considered a kind of multiplicative separation of the Schrödinger equation involving $2n$ constant parameters $\underline{c}$ which can be uniquely determined by assigning the values, at any chosen point, of the functions $u_i=\psi_i'/\psi_i$ and $v_i=\psi_i''/\psi_i$. This means that we have no restriction on the values that the functions $(\psi_i,\psi_i',\psi_i'')$ can assume at any given point of the domain of the coordinates, for $\psi\neq 0$. As we have seen, this freedom implies the orthogonality of the separable coordinates.

However, we can think of a kind of separation in which *constraints* are "*a priori*" imposed on some of the factors ψ_i.

Although it is not interpreted in this sense, a usual constraint appearing in the literature is represented by the following supplementary conditions (see for instance Refs. 9, 10, and 12):

$$\psi_\alpha'=\kappa_\alpha\psi_\alpha, \quad \alpha=m+1,...,n, \tag{5.1}$$

where (κ_α) are arbitrary (real or complex) constants. This means that, up to an inessential multiplicative constant,

$$\psi_\alpha(q^\alpha)=\exp(\kappa_\alpha q^\alpha). \tag{5.2}$$

In this way we define a kind of *constrained separation*, which we call *reduced separation*. As we shall see, for this separation the coordinates are not necessarily orthogonal and the number of essential constants appearing in a separated solution is $n+m<2n$ (the case $m=n$ corresponds to the free separation). We base our approach on a definition similar to Definition 4.1.

Definition 5.1: A *reduced separated solution* of the Schrödinger equation is a solution (1.4) where the factors ψ_α are of the type (5.2) for $\alpha=m+1,...,n$ and where all factors ψ_α depend on further $2m$ parameters $\underline{c}=(c_A)$ $(A=1,...,2m)$ satisfying the *completeness condition*

$$\det\begin{bmatrix}\dfrac{\partial u_a}{\partial c_A}\\[2ex] \dfrac{\partial v_a}{\partial c_A}\end{bmatrix}\neq 0, \quad u_a=\frac{\psi_a'}{\psi_a}, \quad v_a=\frac{\psi_a''}{\psi_a} \quad (a=1,...,m). \tag{5.3}$$

When such a solution exists we say that the Schrödinger equation is *reductively separable* in the coordinates (q^a,q^α). The coordinates (q^α) and (q^a) are called *constrained* and *free* coordinates, respectively.

The completeness condition (5.3) means that the $2m$ constant parameters (c_A) can be uniquely determined by assigning arbitrary values of the $2m$ ratios (u_a,v_a) at any given point. Hence, we have no restriction on the values that the functions $(\psi_a,\psi_a',\psi_a'')$ can assume at any given point of the domain of the coordinates, for $\psi\neq 0$. Moreover, we remark that the total number of constant parameters in a reduced separated solution is $m+n$. Indeed, besides the $2m$ constants (c_A), also the $n-m$ constants (κ_α) are present, although they are not involved in the completeness condition.

In this section we shall prove the following theorem (analogous to Theorem 4.2).

Theorem 5.2: *The Schrödinger equation is reductively separable in a coordinate system* $\underline{q}=(q^a,q^\alpha)$ *if and only if: (i) the constrained coordinates* (q^α) *are ignorable*,

$$\partial_\alpha g^{ij}=0, \quad \partial_\alpha V=0,$$

(ii) the free coordinates (q^a) *are orthogonal*,

$$g^{ab}=0, \quad a\neq b,$$

(iii) there exists a coordinate transformation leaving the coordinates (q^a) *invariant, preserving the constraints and the separation, in which the metric tensor assumes the standard form (3.6) and such that (iv) the Hamilton–Jacobi equation is separable and (v) the following conditions are satisfied:*

$$\partial_a\Gamma_b=0 \quad (a\neq b), \quad \Gamma_a=g^{ij}\Gamma_{ij,a}. \tag{5.4}$$

We call Eq. (5.4) the *Robertson condition for the reduced separation* of the Schrödinger equation. As will be discussed in Sec. VI, this is in fact equivalent to $R_{ab}=0$ for $a\neq b$.

In order to justify Definition 5.1 and prove Theorem 5.2 we begin by observing that the constraints (5.1) imply that the functions u_i, defined as in (4.7) and labeled with Greek indices (running from $m+1$ to n) are constant

$$u_\alpha=\frac{\psi'_\alpha}{\psi_\alpha}=\kappa_\alpha, \quad u^{(2)}_\alpha=0,$$

so that, for such a separated solution, Eq. (4.8) becomes equivalent (for $\psi\neq 0$) to

$$g^{ab}u_au_b+g^{aa}u^{(2)}_a+2g^{a\alpha}u_a\kappa_\alpha+g^{\alpha\beta}\kappa_\alpha\kappa_\beta-\Gamma^au_a-\Gamma^\alpha\kappa_\alpha-V+E=0,$$

with summation over the repeated indices a, $b=1,\ldots,m$ and α, $\beta=m+1,...,n$. This equation can be written in the form

$$\mathcal{S}(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)},\kappa_\alpha)=-E, \tag{5.5}$$

by setting

$$\begin{gathered}\mathcal{S}(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)},\kappa_\alpha)=g^{ab}u_au_b+g^{aa}u^{(2)}_a+2g^{a\alpha}u_a\kappa_\alpha+g^{\alpha\beta}\kappa_\alpha\kappa_\beta-\Gamma^au_a-\Gamma^\alpha\kappa_\alpha-V,\\ \underline{u}^{(1)}=(u_a), \quad \underline{u}^{(2)}=(u_{aa}).\end{gathered} \tag{5.6}$$

Equation (5.5) is of the type (2.8). The constants (κ_α) play the role of independent constant parameters and the relevant dependent variables are $\underline{z}=(\underline{u}^{(1)},\underline{u}^{(2)})=(u_a,u_{aa})$. Their number is $2m$. By applying the method of Sec. II, we have for a solution of this equation

$$\sum_{a=1}^{m}\left(\frac{\partial\mathcal{S}}{\partial q^i}+\frac{\partial\mathcal{S}}{\partial u^{(1)}_a}\frac{\partial u^{(1)}_a}{\partial q^i}+\frac{\partial\mathcal{S}}{\partial u^{(2)}_a}\frac{\partial u^{(2)}_a}{\partial q^i}\right)=0,$$

thus (no sum over the index a)

$$\frac{\partial\mathcal{S}}{\partial q^a}+\frac{\partial\mathcal{S}}{\partial u^{(1)}_a}\frac{\partial u^{(2)}_a}{\partial q^a}+\frac{\partial\mathcal{S}}{\partial u^{(2)}_a}\frac{\partial u^{(2)}_a}{\partial q^a}=0, \quad \frac{\partial\mathcal{S}}{\partial q^\alpha}=0.$$

As a consequence, by adapting Definition 2.2 to the present case, we are led to the following

Definition 5.3: A *reduced separable solution* of the Schrödinger equation is a solution of the form (1.4)–(5.2), depending on $2m$ parameters $\underline{c}=(c_A)$ $(A=1,...,2m)$ satisfying the completeness condition

$$\det\begin{bmatrix} \dfrac{\partial u_a}{\partial c_A} \\ \dfrac{\partial u_a^{(2)}}{\partial c_A} \end{bmatrix} \neq 0, \quad u_a = \frac{\psi_a'}{\psi_a}, \quad u_a^{(2)} = u_{aa} = \left(\frac{\psi_a'}{\psi_a}\right)'. \tag{5.7}$$

Hence, by applying Theorem 2.3 and recalling (2.5) and (2.2), we get

Proposition 5.4: The Schrödinger equation is reductively separable in the coordinates $\underline{q} = (q^a, q^\alpha)$ *if and only if*

$$\partial_\alpha \mathcal{S} = 0, \tag{5.8}$$

$\mathcal{S}$ *being defined as in (5.6), and the first-order differential system*

$$\begin{aligned} &\partial_a u_a^{(i)} = u_a^{(2)}, \quad \partial_i u_a^{(1)} = 0, \\ &\partial_a u_a^{(2)} = R_a, \quad \partial_i u_a^{(2)} = 0, \end{aligned} \quad (i \neq a) \tag{5.9}$$

is completely integrable for

$$R_a(\underline{q}, \underline{u}^{(1)}, \underline{u}^{(2)}) = -\left(\frac{\partial \mathcal{S}}{\partial u_a^{(2)}}\right)^{-1}\left(\frac{\partial \mathcal{S}}{\partial q^a} + \frac{\partial \mathcal{S}}{\partial u_a} u_a^{(2)}\right).$$

The complete integrability conditions for this system are

$$[D_i, D_j] = 0, \quad \begin{cases} D_a = \partial_a + u_a^{(2)} \dfrac{\partial}{\partial u_a} + R_a \dfrac{\partial}{\partial u_a^{(2)}}, \\ D_\alpha = \partial_\alpha. \end{cases} \tag{5.10}$$

From (5.8) and (5.6) it follows that

$$\partial_\alpha g^{ij} = 0, \quad \partial_\alpha \Gamma^i = 0, \quad \partial_\alpha V = 0,$$

since $\mathcal{S}$ is a polynomial function in the variables $(\underline{u}, \underline{u}^{(2)}, \kappa_\alpha)$ assuming arbitrary values. This proves that

Proposition 5.5: In the reduced separation of the Schrödinger equation the constrained coordinates (q^α) *are ignorable.*

It follows that the significant part of the integrability conditions (5.10) is that related to the free (and nonignorable) coordinates, $[D_a, D_b] = 0$ (for $a \neq b$). For examining these conditions, as in the case of the free separation, it is convenient to deal with the new $2m$ variables

$$(\underline{u}, \underline{v}) = (u_a, v_a), \quad v_a = u_{aa} + u_a^2 = \frac{\psi_\alpha''}{\psi_\alpha}.$$

Remark 5.6: Under such a transformation the completeness condition (5.7) is equivalent to (5.3), so that Definition 5.3 is equivalent to Definition 5.1.

Furthermore, the differential system (5.9) becomes equivalent to

$$\begin{aligned} &\partial_i u_a = 0, \\ &\partial_i v_a = 0, \\ &\partial_a u_a = v_a - u_a^2, \\ &\partial_a v_a = R_a, \end{aligned} \quad (i \neq a = 1, \ldots, m), \tag{5.11}$$

where

$$R_a = -\frac{1}{g^{aa}}\left(\frac{\partial \mathcal{S}}{\partial q^a} + \frac{\partial \mathcal{S}}{\partial u_a}(v_a - u_a^2)\right),$$

$$\mathcal{S}(\underline{q},\underline{u},\underline{v}) = g^{aa}(v_a - u_a^2) + g^{ab}u_a u_b + 2g^{a\alpha}u_a\kappa_\alpha + g^{\alpha\beta}\kappa_\alpha\kappa_\beta - \Gamma^a u_a - \Gamma^\alpha\kappa_\alpha - V,$$

$$D_a = \partial_a + (v_a - u_a^2)\frac{\partial}{\partial u_a} + R_a\frac{\partial}{\partial v_a},$$

$$D_\alpha = \partial_\alpha .$$

The complete integrability conditions $[D_a, D_b] = 0$ of system (5.11) are then equivalent to equations similar to (4.17),

$$(v_a - u_a^2)\frac{\partial R_b}{\partial u_a} + R_a\frac{\partial R_b}{\partial v_a} + \frac{\partial R_b}{\partial q^a} = 0 \quad (a \neq b).$$

A calculation similar to that of Sec. IV shows that

Proposition 5.7: In a reduced separation the free coordinates are orthogonal,

$$g^{ab} = 0, \quad a \neq b,$$

and the integrability conditions of system (5.11) are equivalent to

$$\frac{\partial^2 \mathcal{S}}{\partial q^a \partial u_b}(v_b - u_b^2) + \frac{\partial^2 \mathcal{S}}{\partial q^b \partial u_a}(v_a - u_a^2) + \partial_a\partial_b\mathcal{S} + R_a\partial_b g^{aa} + R_b\partial_a g^{bb} = 0 \quad (a \neq b), \tag{5.12}$$

where

$$\begin{aligned} \mathcal{S} &= g^{aa}v_a + 2g^{a\alpha}u_a\kappa_\alpha + g^{\alpha\beta}\kappa_\alpha\kappa_\beta - \Gamma^a u_a - \Gamma^\alpha\kappa_\alpha - V, \\ R_a &= \frac{1}{g^{aa}}((\Gamma^a - 2g^{a\alpha}\kappa_\alpha)(v_a - u_a^2) - \partial_a S). \end{aligned} \tag{5.13}$$

From these last equations we can derive the following

Proposition 5.8: The reduced separation of the Schrödinger equation always occurs in a standard coordinate system $(q^i) = (q^a, q^\alpha)$, *for which (3.6) and (3.7) hold, and such that the following equations are satisfied for* $a \neq b$:

$$\begin{aligned} &\partial_a\partial_b g^{cc} - \partial_b \ln g^{aa}\partial_a g^{cc} - \partial_a \ln g^{bb}\partial_b g^{cc} = 0, \\ &\partial_a\partial_b g^{\alpha\beta} - \partial_b \ln g^{aa}\partial_a g^{\alpha\beta} - \partial_a \ln g^{bb}\partial_b g^{\alpha\beta} = 0, \\ &\partial_a\partial_b V - \partial_b \ln g^{aa}\partial_a V - \partial_a \ln g^{bb}\partial_b V = 0, \\ &\partial_b \Gamma_a = 0, \end{aligned} \tag{5.14}$$

where

$$\Gamma_a = g_{aa}\Gamma^a = \frac{1}{2}\sum_{c \neq a}\partial_c \ln g^{cc} - \frac{1}{2}\partial_a \ln g^{aa} + \frac{1}{2}\partial_a \ln \det[g^{\alpha\beta}]. \tag{5.15}$$

Proof: By setting

$$\hat{\Gamma}^a = \Gamma^a - 2g^{a\alpha}\kappa_\alpha, \quad \hat{V} = V + \Gamma^\alpha\kappa_\alpha - g^{\alpha\beta}\kappa_\alpha\kappa_\beta,$$

(5.13) assume the form

$$\mathcal{S}=g^{aa}v_a-\hat{\Gamma}^a u_a-\hat{V},\quad R_a=\frac{1}{g^{aa}}((\hat{\Gamma}^a(v_a-u_a^2)-\partial_a\mathcal{S}),\tag{5.16}$$

so that the integrability conditions (5.12) become

$$\begin{aligned}&2(u_b^2-v_b)(\partial_a\hat{\Gamma}^b-\hat{\Gamma}^b\partial_a\ln g^{bb})+2(u_a^2-v_a)(\partial_b\hat{\Gamma}^a-\hat{\Gamma}^a\partial_b\ln g^{aa})+v_c(\partial_a\partial_b g^{cc}-\partial_a\ln g^{bb}\partial_b g^{cc}\\&\quad-\partial_b\ln g^{aa}\partial_a g^{cc})-u_c(\partial_a\partial_b\hat{\Gamma}^c-\partial_a\ln g^{bb}\partial_b\hat{\Gamma}^c-\partial_b\ln g^{aa}\partial_a\hat{\Gamma}^c)\\&\quad-(\partial_a\partial_b\hat{V}-\partial_a\ln g^{bb}\partial_b\hat{V}-\partial_b\ln g^{aa}\partial_a\hat{V})=0.\end{aligned}\tag{5.17}$$

We remark that (5.12) and (5.16) are similar to (4.20) and (4.21), so that (5.17) are similar to (4.22) and, since (u_a,v_a) can assume arbitrary values, we get equations similar to (4.23),

$$\begin{gathered}\partial_a\partial_b g^{cc}-\partial_a\ln g^{bb}\partial_b g^{cc}-\partial_b\ln g^{aa}\partial_a g^{cc}=0,\\\partial_a\partial_b\hat{V}-\partial_a\ln g^{bb}\partial_b\hat{V}-\partial_b\ln g^{aa}\partial_a\hat{V}=0,\\\partial_a\hat{\Gamma}^b-\hat{\Gamma}^b\partial_a\ln g^{bb}=0,\\\partial_a\partial_b\hat{\Gamma}^c-\partial_a\ln g^{bb}\partial_b\hat{\Gamma}^c-\partial_b\ln g^{aa}\partial_a\hat{\Gamma}^c=0.\end{gathered}\tag{5.18}$$

The first equations are just the first equations in (5.14). Since also the constant parameters (κ_α) assume arbitrary values, the fourth equations (5.18) are equivalent to

$$\begin{gathered}\partial_a\partial_b\Gamma^c-\partial_a\ln g^{bb}\partial_b\Gamma^c-\partial_b\ln g^{aa}\partial_a\Gamma^c=0,\\\partial_a\partial_b g^{c\alpha}-\partial_a\ln g^{bb}\partial_b g^{c\alpha}-\partial_b\ln g^{aa}\partial_a g^{c\alpha}=0,\end{gathered}\tag{5.19}$$

while the second equations (5.18) are equivalent to the second and third equations of (5.14) and

$$\partial_a\partial_b\Gamma^\alpha-\partial_a\ln g^{bb}\partial_b\Gamma^\alpha-\partial_b\ln g^{aa}\partial_a\Gamma^\alpha=0.\tag{5.20}$$

Finally, the third equations (5.18) are equivalent to

$$\partial_a\Gamma^b-\Gamma^b\partial_a\ln g^{bb}=0,\quad \partial_a g^{b\alpha}-g^{b\alpha}\partial_a\ln g^{bb}=0,$$

which can be written in the form

$$\partial_a\left(\frac{\Gamma^b}{g^{bb}}\right)=0,\quad \partial_a\left(\frac{g^{b\alpha}}{g^{bb}}\right)=0,\quad a\neq b.\tag{5.21}$$

These last equations show that

$$\Gamma^a=g^{aa}f_a,\quad g^{a\alpha}=g^{aa}f_a^\alpha,\tag{5.22}$$

where (f_a,f_a^α) are functions of the coordinate corresponding to the lower index only. As a consequence, due to the first equations (5.18), (5.19) are identically satisfied and do not add further information. But (5.22) have another important consequence which allows a remarkable simplification of our analysis (a similar argument has been used in Refs. 15, 17 in the discussion of the separation of the HJ equations). Indeed, let us consider a coordinate transformation of the kind

$$\mathrm{d}x^a=\mathrm{d}q^a,\quad \mathrm{d}x^a=\mathrm{d}q^\alpha-f_a^\alpha\,\mathrm{d}q^a.\tag{5.23}$$

For the components $g_x^{ij}=\mathrm{d}x^i\cdot\mathrm{d}x^j$ of the metric tensor in the new coordinates $(x^i)=(x^a,x^\alpha)$ we have $g_x^{ab}=g^{ab}$ and

$$g_x^{a\alpha}=\mathrm{d}x^a\cdot\mathrm{d}x^\alpha=\mathrm{d}q^a\cdot(\mathrm{d}q^\alpha-f_b^\alpha\mathrm{d}q^b)=g^{a\alpha}-f_b^\alpha g^{ab}=g^{a\alpha}-f_a^\alpha g^{aa}=0.$$

This coordinate transformation is compatible with the separation, in the sense that in the new coordinates the solution of the Schrödinger equation is still separable and the constraint equations (5.1) hold (with the same constants κ_α). Indeed, the essential coordinates remain unchanged ($x^a=q^a$ up to inessential additive constants) and moreover,

$$\frac{\mathrm{d}\psi_\alpha}{\mathrm{d}x^\alpha}=\frac{\partial\psi_\alpha\partial q^\beta}{\partial q^\beta\partial x^\alpha}=\frac{\mathrm{d}\psi_\alpha\mathrm{d}q^\alpha}{\mathrm{d}q^\alpha\mathrm{d}x^\alpha}=\frac{\mathrm{d}\psi_\alpha}{\mathrm{d}q^\alpha}.$$

Hence, *without loss of generality* we can assume

$$g^{a\alpha}=0, \tag{5.24}$$

so that the metric tensor takes the standard form (3.6), with ignorable coordinates (q^α). For a metric of this kind,

$$\Gamma_a=g_{ai}\Gamma^i=g_{aa}\Gamma^a=(g^{aa})^{-1}\Gamma^a.$$

Thus, the first equations (5.21) are equivalent to the fourth equations (5.14). Moreover, $\Gamma^\alpha=0$ (see Sec. VI) so that (5.20) are identically satisfied. We conclude that the integrability conditions (5.12) of system (5.11), up to a coordinate transformation preserving the separation and reducing the metric tensor in the standard form, are equivalent to (5.14). ■

We recognize in the first three lines of system (5.14) the necessary and sufficient conditions (3.8′) for the separation of the Hamilton–Jacobi equation in standard coordinates. Hence, we have proved the following

Proposition 5.9: Up to a coordinate transformation of the kind (5.23), the reduced separation of the Schrödinger equation always occurs in standard separable coordinates $\underline{q}=(q^a,q^\alpha)$ ($a=1,...,m,\alpha=m+1,...,n$) *for which* $\partial_b\Gamma_a=0$ *for* $a\neq b$.

As a conclusion, from Propositions 5.5, 5.7, 5.8, and 5.9, we derive Theorem 5.2.

Remark 5.10: Let us see how the integration by separation of variables is performed when the items in Theorem 5.2 are satisfied (this will give a further proof of the sufficiency of these items for the separation) and how the $m+n$ constants arise in a reduced separated solution. Due to the factorization

$$\psi=\Pi_{a=1}^m\psi_a\cdot\Pi_{\alpha=m+1}^n\psi_\alpha,$$

and to the constraints (5.1), since $\Gamma^\alpha=0$, the Schrödinger equation (4.5) becomes equivalent to the following *reduced Schrödinger equation:*

$$g^{aa}\partial_a^2\tilde{\psi}+\Gamma^a\partial_a\tilde{\psi}+(E-V+g^{\alpha\beta}\kappa_a\kappa_\beta)\tilde{\psi}=0,\quad \tilde{\psi}=\Pi_{a=1}^m\psi_a. \tag{5.25}$$

The additional condition $\partial_b\Gamma_a=0$ for $a\neq b$ means that the contracted Christoffel symbols Γ_a (with indices $a=1,...,m$) are functions of the corresponding coordinate q^a only. By a method similar to that illustrated in Remark 4.7, due to expressions (3.9), the integration of the Schrödinger equation is reduced to the integration of m separated Riccati equations,

$$u_a'+u_a^2-\Gamma_a u_a-\phi_a+\phi_a^{\alpha\beta}\kappa_\alpha\kappa_\beta+\varphi_a^{(b)}\tilde{a}_b=0 \tag{5.26}$$

parametrized by n constants $(\tilde{a}_b,\kappa_\alpha)$, with $\tilde{a}_m=E$. Its solutions $u_a=u_a(q^a;\tilde{a}_b,\kappa_\alpha,\tilde{b}_a)$ give rise, separately, to other m constants $(\tilde{b}_a)$ and generate, by a further integration, the reduced separated solution (summation over the indices)

$$\psi=\exp\left(\kappa_\alpha q^\alpha+\int u_a\,\mathrm{d}q^a\right)=\Pi_{a=1}^m\psi_a\cdot\Pi_{\alpha=m+1}^n\psi_\alpha.$$

Equations (5.26) are equivalent to the m linear second-order equations

$$\psi_a''-\Gamma_a\psi_a'+(\phi_a^{\alpha\beta}\kappa_\alpha\kappa_\beta+\varphi_a^{(b)}\tilde{a}_b-\phi_a)\psi_a=0 \tag{5.27}$$

in the functions ψ_a. In the reduced separation the number of the constants appearing in a complete solution (1.4), as a result of this process of integration, is $m+n<2n$, but only the $2m$ constants $(\tilde{a}_a,\tilde{b}_a)$ are involved in the completeness condition.

Remark 5.11: A coordinate transformation of the kind

$$q^a=\bar{q}^a,\quad q^\alpha=\sum_i F_i^\alpha(\bar{q}^i)=\sum_a F_a^\alpha(q^a)+\sum_\beta F_\beta^\alpha(\bar{q}^\beta),$$

with

$$\det[f_\beta^\alpha]\neq 0,\quad f_\beta^\alpha=(F_\beta^\alpha)',$$

composed by any transformation over single coordinates, is the most general transformation leading to nonstandard separable coordinates $(\bar{q}^a,\bar{q}^\alpha)$ for the Hamilton–Jacobi equation.[15,17] The same transformation applied to standard separable coordinates, in general, does not preserve (5.24), $g^{a\alpha}\neq 0$, and the coordinates $(\bar{q}^\alpha)$ are no longer ignorable. However, this transformation preserves the multiplicative separation of the Schrödinger equation. Indeed, the factors $\psi_a(q^a)$ remain unchanged, while the factors $\psi_\alpha(q^\alpha)$ are transformed as follows:

$$\psi_\alpha(q^\alpha)=\exp(\kappa_\alpha q^\alpha)=\exp(\kappa_\alpha F_\beta^\alpha(\bar{q}^\beta))\cdot\exp(\kappa_\alpha F_a^\alpha(q^a))=\prod_\beta\tilde{\psi}_\beta(\bar{q}^\beta)\cdot\prod_a\tilde{\psi}_a(q^a),$$

where

$$\tilde{\psi}_\beta(\bar{q}^\beta)=\exp(\kappa_\alpha F_\beta^\alpha(\bar{q}^\beta)),\quad \tilde{\psi}_a(q^a)=\exp(\kappa_\alpha F_a^\alpha(q^a)),$$

and we finally get a solution of the kind

$$\psi=\prod_\beta\tilde{\psi}_\beta(\bar{q}^\beta)\prod_a\psi_a(q^a)\tilde{\psi}_a(q^a).$$

We observe that

$$\tilde{\psi}_\beta'=\kappa_\alpha f_\beta^\alpha(\bar{q}^\beta)\tilde{\psi}_\beta,$$

where $f=F'$. This shows that a constraint of the kind (5.1) is still satisfied if and only if $f_\beta^\alpha=\delta_\beta^\alpha c'_\beta$ with arbitrary constants (c'_β). It shows also that a constraint of the kind

$$\psi'_\beta=\kappa_\alpha f_\beta^\alpha(q^\beta)\psi_\beta,\quad \det[f_\beta^\alpha]\neq 0$$

could be considered, but that it is equivalent to (5.1).

Remark 5.12: A remark analogous to Remark 4.8 is in order. The constants $\tilde{a}_b$ have two interpretations: (i) They are the constant values taken by the quadratic first integrals in involution related to the nonorthogonal separation of the Hamilton–Jacobi equation in standard coordinates (cf. Theorem 7.25) whose expressions are

$$H_a=\tfrac{1}{2}\varphi_{(a)}^b(p_b^2+\phi_a^{\alpha\beta}p_\alpha p_\beta+\phi_a). \tag{5.28}$$

(ii) They are the eigenvalues of the second-order symmetry operators corresponding to H_a of the reduced Schrödinger equation (5.25):

$$\hat{H}_b\tilde{\psi}=\varphi^a_{(b)}(-\partial_a^2\tilde{\psi}+\Gamma_a\partial_a\tilde{\psi}+(\phi_a-\phi_a^{\alpha\beta}\kappa_\alpha\kappa_\beta)\psi). \tag{5.29}$$

About the constants κ_α introduced in the constraint equations (5.1), we observe that they correspond to the eigenvalues of the first-order symmetry operators of the Schrödinger equation

$$\hat{H}_\alpha\psi=-i\hbar\partial_\alpha\psi. \tag{5.30}$$

Indeed, from (5.1) it follows that

$$\partial_\alpha\psi=\kappa_\alpha\psi.$$

Thus,

$$\hat{H}_\alpha\psi=-i\hbar\kappa_\alpha\psi. \tag{5.31}$$

As will be shown in Ref. 18, these operators are related to the linear first integrals

$$H_\alpha=p_\alpha \tag{5.32}$$

corresponding to the ignorable coordinates and thus to the Killing vectors characterizing the separation in standard coordinates (see Sec. VII). From (5.31) we observe that κ_α must be pure imaginary, since the operators $\hat{H}_\alpha$ are self-adjoint. This is in agreement with the fact that the choice of the constraint equations (5.2) does not change the state represented by ψ.

VI. THE ROBERTSON CONDITION

In this section we analyze the Robertson condition in a standard separable coordinate system. We shall extend to the reduced separation the analysis on the Robertson condition done by Eisenhart[3] in the case of the orthogonal separation. This extension has already been discussed in Ref. 19; here we give an improved and simplified version.

For a metric tensor in the standard form (3.6), where (q^α) are ignorable, the nonvanishing Christoffel symbols are

$$\begin{gathered}\Gamma_{\alpha\beta a}=-\tfrac{1}{2}\partial_a g_{\alpha\beta},\quad \Gamma^a_{\alpha\beta}=-\tfrac{1}{2}g^{aa}\partial_a\partial_{\alpha\beta},\\ \Gamma_{\alpha a\beta}=\tfrac{1}{2}\partial_a g_{\alpha\beta},\quad \Gamma^\beta_{\alpha a}=\tfrac{1}{2}\sum_\gamma g^{\beta\gamma}\partial_a g_{\alpha\gamma},\\ \Gamma_{aba}=\tfrac{1}{2}\partial_b g_{aa},\quad \Gamma^a_{ac}=\tfrac{1}{2}g^{aa}\partial_c g_{aa},\\ \Gamma_{aab}=-\tfrac{1}{2}\partial_b g_{aa}\quad (a\neq b),\quad \Gamma^b_{aa}=-\tfrac{1}{2}g^{bb}\partial_b g_{aa}\quad (a\neq b).\end{gathered} \tag{6.1}$$

It follows that

$$\Gamma^\alpha=g^{ij}\Gamma^\alpha_{ij}=0,\quad \Gamma^i_{i\alpha}=0, \tag{6.2}$$

and from (4.3),

$$\Gamma_a=g_{aa}\Gamma^a=-\partial_a\ln g^{aa}+\tfrac{1}{2}\partial_a\ln\det[g^{ij}]=-\partial_a\ln g^{aa}+\tfrac{1}{2}\partial_a\ln\det[g^{\alpha\beta}]+\tfrac{1}{2}\sum_{c=1}^{m}\partial_a\ln g^{cc}. \tag{6.3}$$

Moreover,

$$\Gamma^i_{ia}=\Gamma^b_{ba}+\Gamma^\alpha_{\alpha a}=\tfrac{1}{2}g^{bb}\partial_a g_{bb}+\tfrac{1}{2}g^{\alpha\beta}\partial_a g_{\alpha\beta},$$

so that

$$\Gamma^i_{ia}=-\tfrac{1}{2}\partial_a \ln\det[g^{ij}]=-\Gamma_a-\partial_a \ln g^{aa}. \tag{6.4}$$

Let us consider the Riemann tensor and the Ricci tensor defined as follows:

$$R^l_{ijk}=\partial_i\Gamma^l_{jk}-\partial_j\Gamma^l_{ik}+\Gamma^r_{jk}\Gamma^l_{ir}-\Gamma^r_{ik}\Gamma^l_{jr},\quad R_{jk}=R^l_{ljk}.$$

By a straightforward calculation it can be seen that the Riemann tensor components which are needed for the computation of the nondiagonal Ricci tensor components, R_{ab}, $a\neq b$, have the following expressions:

$$R^c_{cab}=\tfrac{3}{4}\partial_a\partial_b \ln g^{cc}-\tfrac{1}{4}g_{cc}(\partial_a\partial_b g^{cc}-\partial_b \ln g^{aa}\partial_a g^{cc}-\partial_a \ln g^{bb}\partial_b g^{cc})\quad (c\neq a,b;c \text{ n.s.}),$$

$$R^a_{aab}=0, \tag{6.5}$$

$$R^\alpha_{\alpha ab}=\tfrac{3}{4}\partial_a(g_{\alpha\beta}\partial_b g^{\alpha\beta})-\tfrac{1}{4}g_{\alpha\beta}(\partial_a\partial_b g^{\alpha\beta}-\partial_b \ln g^{aa}\partial_a g^{\alpha\beta}-\partial_a \ln g^{bb}\partial_b g^{\alpha\beta})\quad (\alpha \text{ n.s.})$$

and that

$$R_{a\alpha}=0. \tag{6.6}$$

A remarkable fact is that, due to the separability conditions (3.8′), expressions (6.5) reduce to

$$R^c_{cab}=\tfrac{3}{4}\partial_a\partial_b \ln g^{cc},\quad R^\alpha_{\alpha ab}=\tfrac{3}{4}\partial_a(g_{\alpha\beta}\partial_b g^{\alpha\beta}),$$

so that

$$R_{ab}=\frac{3}{4}\partial_a\partial_b\left(\sum_{c\neq a,b}\ln g^{cc}+\ln\det[g^{\alpha\beta}]\right)\quad (a\neq b). \tag{6.7}$$

For the case $m=n$ the second term on the right-hand side of (6.7) disappears and we find a formula first stated by Eisenhart[20] and related to the orthogonal separation.

Another remarkable fact is that for $c=a$ the first equation (3.8′) is equivalent to

$$\partial_a\partial_b \ln g^{aa}=\partial_a \ln g^{bb}\partial_b \ln g^{aa}\quad (a\neq b),$$

so that, due to the symmetry in the indices of the right-hand side of this equation, we can write

$$\partial_a\partial_b \ln g^{aa}=\tfrac{1}{2}(\partial_a\partial_b \ln g^{aa}+\partial_a\partial_b \ln g^{bb}),$$

and from (6.3) we get

$$\partial_b\Gamma_a=\frac{1}{2}\partial_a\partial_b\left(\sum_{c\neq a,b}\ln g^{cc}+\ln\det[g^{\alpha\beta}]\right). \tag{6.8}$$

Formula (6.6) and the comparison between (6.8) and (6.7) show that

Theorem 6.1: *(i) In standard separable coordinates* (q^a,q^α),

$$R_{ab}=\tfrac{3}{2}\partial_b\Gamma_a\quad (a\neq b),\quad R_{a\alpha}=0. \tag{6.9}$$

(ii) In orthogonal separable coordinates,

$$R_{ij}=\tfrac{3}{2}\partial_j\Gamma_i, \quad i\neq j. \tag{6.10}$$

This last equation is simply the reduction of (6.9) to the case $m=n$ (no greek indices). Thus,

Theorem 6.2: *The Robertson condition in orthogonal separable coordinates* $\partial_i\Gamma_j=0$ $(i\neq j)$ *is equivalent to*

$$R_{ij}=0 \quad (i\neq j).$$

The Robertson condition in standard separable coordinates $\partial_a\Gamma_b=0$ $(a\neq b)$ *is equivalent to*

$$R_{ab}=0 \quad (a\neq b).$$

Remark 6.3: Theorem 6.2 gives a geometrical meaning of the Robertson condition for the free and the reduced separation. For the reduced separation we take the standard coordinates as required by Proposition 5.8.

Since $R_{a\alpha}=0$, it is proved that

Theorem 6.4: *The Robertson condition in standard coordinates is fulfilled if and only if the Ricci tensor assumes the standard form*.

Remark 6.5: The Robertson condition is identically satisfied, so that there is a simultaneous separation of the Schrödinger and Hamilton–Jacobi equations, for Einstein manifolds, where the Ricci tensor is proportional to the metric tensor, $\mathbf{R}=a\mathbf{G}$ (thus, in particular, for constant curvature manifolds and Ricci-flat manifolds).

VII. THE EISENHART–KILLING EQUATIONS AND THE INTRINSIC CHARACTERIZATION OF THE SEPARATION

As has been illustrated in the preceding sections, the separation of variables is apparently a strictly ''coordinate dependent'' matter. This is perhaps the reason why for long it has not been recognized as a ''modern'' theory. However, as we know today, the existence of separable coordinates for the Hamilton–Jacobi and Schrödinger equations requires the presence on the underlying Riemannian manifold of a rich intrinsic (coordinate independent) structure, described by algebraic or geometrical objects: Killing vectors, Killing tensors, and webs (sets of foliations).

The first fundamental contibution to the intrinsic theory of the variable separation dates back to Levi-Civita.[11] He pointed out first, that the separation of the geodesic Hamiltonian is a necessary condition for the separation of all the associated natural Hamiltonians with scalar and vector potentials.[21] This gives a prominent role to the separation of the geodesic Hamilton–Jacobi equation. Second, he proposed a classification of the separable coordinates into two classes and, by using his methods of ''calcolo differenziale assoluto,'' developed together with Ricci[22] a few years earlier, he proved that when all the coordinates are of first class then the manifold is necessarily locally flat. This result was later extended to the general case by Agostinelli,[23] who proved that the separation associated with r first-class coordinates corresponds to the existence of a foliation of r-dimensional locally flat submanifolds. These pioneering results have recently been incorporated within a general geometrical framework of the geodesic separation, based on the notion of *separable Killing web*.[16] However, a milestone in this way is represented by the contribution of Eisenhart,[3] which we shall revisit in the present section, with suitable modifications and extensions.

We recall that a contravariant symmetric tensor $\mathbf{K}=(K^{i\cdots j})$ is said to be a *Killing tensor* if its components satisfy the *Killing equation*

$$\nabla^{(h}K^{i\cdots j)}=0,$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection and the parentheses (...) denote the symmetrization of the indices. There is however an alternative equivalent definition of Killing tensor, which is not related to the covariant derivative but to the canonical

symplectic structure of a cotangent bundle. This definition is based on the one-to-one correspondence between the contravariant symmetric tensors and the homogeneous polynomial functions on T^*Q,

$$P(\mathbf{K}) = P_{\mathbf{K}} = K^{i\cdots j} p_i \cdots p_j .$$

We say that two tensors $\mathbf{K}$ and $\mathbf{K}'$ are *in involution* if the corresponding polynomial functions are in involution, i.e., if their Poisson bracket vanishes identically:

$$\{P(\mathbf{K}), P(\mathbf{K}')\} = 0.$$

Then a tensor $\mathbf{K}$ is a Killing tensor if and only if $P(\mathbf{K})$ is a first integral of the geodesic flow, i.e., it is in involution with the geodesic Hamiltonian $G = \frac{1}{2} P_{\mathbf{G}}$:

$$\{P_{\mathbf{K}}, G\} = 0. \tag{7.1}$$

Also this definition does not depend on the choice of the coordinates. Let us call (7.1) the *Poisson–Killing equation*.

In the theory of the separation of variables a fundamental role is played by Killing vectors, corresponding to linear first integrals and to one-parameter groups of isometries, and by Killing two-tensors, interpreted as symmetric linear operators on one-forms or vector fields. Thus, in the following discussion by "Killing tensor" we mean "Killing two-tensor."

Remark 7.1: We shall use the following notation. If $\mathbf{K} = (K^{ij})$ is a contravariant two-tensor and $\mathbf{X} = (X^i)$ a vector field, then by $\mathbf{KX}$ we denote the vector field image of $\mathbf{X}$ by $\mathbf{K}$ (as linear operator) whose components are

$$(\mathbf{KX})^i = K^{ij} g_{jh} X^h = K^{i}_{\cdot j} X^j,$$

and by $\mathbf{K}\varphi$ the one-form image by $\mathbf{K}$ of the one-form $\varphi = \varphi_i \mathrm{d}q^i$ whose components are

$$(\mathbf{K}\varphi)_i = g_{ih} K^{hj} \varphi_j = K_i^{\cdot j} \varphi_j .$$

If $\mathbf{K}$ and $\mathbf{L}$ are two such tensors then by $\mathbf{LK}$ we mean their composition as linear operators. It is a contravariant two-tensor whose components are

$$(\mathbf{LK})^{ij} = L^{ih} g_{hk} K^{kj} = L^{i}_{\cdot k} K^{kj} = L^{ih} K_h^{\cdot j} .$$

We begin by considering the orthogonal separation. The link between the orthogonal separation and the existence of Killing tensors is based on the following statement due to Eisenhart:[20]

Proposition 7.2: Let $\mathbf{K}$ *be a symmetric tensor on a Riemannian manifold* $(Q, \mathbf{g})$ *which is diagonalized in an orthogonal coordinate system* $\underline{q} = (q^i)$. *Then* $\mathbf{K}$ *is a Killing tensor if and only if its eigenvalues* (λ^i) *satisfy*

$$\partial_i \lambda^j = (\lambda^i - \lambda^j) \partial_i \ln g^{jj}. \tag{7.2}$$

Proof: In the coordinates $\underline{q}$ the components of $\mathbf{K}$ are

$$K^{ii} = \lambda^i g^{ii}, \quad K^{ij} = 0 \quad (i \neq j),$$

so that $P(\mathbf{K}) = \lambda^i g^{ii} p_i^2$. A straightforward calculation shows that (7.1) are equivalent to (7.2). ■

We call (7.2) the *Eisenhart–Killing equations*. These equations can be interpreted as a linear normal first-order differential system in the unknown functions (λ^i), of the kind considered in Sec. II. It is a crucial fact that its complete integrability conditions (as shown by a straightforward calculation) are

$$(\lambda^i - \lambda^j) S_{ij}^k = 0 \quad (i \neq j, \ \text{n.s.}), \tag{7.3}$$

where

$$S_{ij}^{k}=\partial_i\partial_j g^{kk}-\partial_i \ln g^{jj}\partial_j g^{kk}-\partial_j \ln g^{ii}\partial_i g^{kk}$$

are just the left-hand sides of the first separability conditions (3.5).

Remark 7.3: The orthogonal separability theory lies in the following rather surprising circumstance: there are three "different" first-order differential systems with the "same" complete integrability conditions, $S_{ij}^{k}=0$ $(i\neq j)$. They are: (i) system (3.3), with

$$R_i(\underline{q},\underline{p})=-\frac{\partial_i G}{\partial^i G},\quad G=\frac{1}{2}g^{ii}p_i^2,$$

related to the orthogonal separation of the geodesic Hamilton–Jacobi equation; (ii) system (4.11) with

$$R_i(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)})=-\left(\frac{\partial \mathcal{S}}{\partial u_i^{(2)}}\right)^{-1}\left(\frac{\partial \mathcal{S}}{\partial q^i}+\frac{\partial \mathcal{S}}{\partial u_i}u_i^{(2)}\right),\quad \mathcal{S}(\underline{q},\underline{u}^{(1)},\underline{u}^{(2)})=g^{ij}u_iu_j+g^{ii}u_i^{(2)}-\Gamma^i u_i,$$

related to the free (thus, orthogonal) separation of the geodesic Schrödinger equation $(V=0)$; (iii) system (7.2) related to the existence of Killing tensors diagonalized in orthogonal coordinates. Actually, as we have seen, the complete integrability of the second system requires an additional condition: the Robertson condition $R_{ij}=0$ for $i\neq j$. The same remarkable property holds for a nonconstant potential V.

According to this remark, it is convenient to introduce the following

Definition 7.4: We call *Killing–Stäckel algebra* an n-dimensional linear space $\mathcal{K}$ of Killing tensors which are (i) simultaneously diagonalized in orthogonal coordinates, or equivalently, (ii) with n common normal (i.e., orthogonally integrable or surface forming) eigenvectors.

Item (ii) in this definition is a coordinate-free translation of item (i). Thus, we can affirm that

Theorem 7.5: *The orthogonal separation of the geodesic Hamilton–Jacobi equation is equivalent to the complete integrability of the Eisenhart–Killing equations i.e., to the existence of a Killing–Stäckel algebra.*

This is a synthetic and simplified version of the classical theorem of Eisenhart on "separable systems of Stäckel."[3]

Remark 7.6: The submanifolds of codimension 1 orthogonal to the normal eigenvectors of a Killing–Stäckel algebra form an *orthogonal separable web*. A coordinate system $\underline{q}$ is called *adapted* to this web if the corresponding coordinate hypersurfaces belong to the web. Any coordinate system adapted to a separable web is orthogonal and separable. It follows that: *an orthogonal web is separable if and only if its leaves are orthogonal to the eigenvectors of a Killing–Stäckel algebra.*

Remark 7.7: A Killing–Stäckel algebra, as well as a separable orthogonal web, may not be defined on the whole configuration manifold Q. The points of Q where the requirements of Definition 7.4 are not fulfilled (or where the manifolds orthogonal to the normal eigenvectors are not defined) form the *singular set* of the algebra (or of the web).

Due to Theorem 7.5, the analysis of the orthogonal separation is now related to the analysis of the Killing–Stäckel algebras. Going back to Definition 7.4, we observe that: (i) when the Eisenhart–Killing equations are completely integrable, we can always find, locally, a solution such that $\lambda^i\neq\lambda^j$ for $i\neq j$; (ii) as shown by a straightforward calculation, two functions $P(\mathbf{K})$ and $P(\mathbf{K}')$ corresponding to two solutions of system (7.2) are in involution; (iii) $\lambda^i=1$ is a trivial solution. Thus,

Theorem 7.8: *(i) A Killing–Stäckel algebra admits locally a Killing tensor with pointwise distinct eigenvalues. (ii) All Killing tensors in a Killing–Stäckel algebra are in involution. (iii) The metric tensor* $\mathbf{G}$ *belongs to any Killing–Stäckel algebra.*

Furthermore, the presence of the term $\lambda^i-\lambda^j$ in (7.3) implies that if there exists a Killing tensor which is diagonalized in orthogonal coordinates and with pointwise simple eigenvalues,

$\lambda^i \neq \lambda^j$, then all S^k_{ij} $(i \neq j)$ vanish identically (these are functions of the coordinates only), so that the Eisenhart–Killing equations are completely integrable.[24] This proves that

Theorem 7.9: *A Killing–Stäckel algebra is uniquely determined by a Killing tensor with normal eigenvectors and pointwise simple eigenvalues*.

Remark 7.10: The "uniqueness" in this last statement has a local meaning. It means that if $\mathcal{K}$ and $\mathcal{K}'$ are two Killing–Stäckel algebras both containing a Killing tensor $\mathbf{K}$ with simple eigenvalues, then $\mathcal{K}=\mathcal{K}'$ at least in the domain of definition of $\mathbf{K}$. Then we say that $\mathbf{K}$ is a *characteristic tensor* of the Killing–Stäckel algebra $\mathcal{K}$. Note that it is not uniquely determined in $\mathcal{K}$. We also remark that there are cases in which a Killing–Stäckel algebra does not admit a "global" characteristic Killing tensor. An example is the parabolic translational web in the Euclidean three-space (see Ref. 25 for a description of this web).

Remark 7.11: As a consequence of the preceding remarks and statements, we have at least two ways for characterizing intrinsically the orthogonal separation of a geodesic Hamilton–Jacobi equation: (A) by means of a Killing–Stäckel algebra $\mathcal{K}$, or (B) by means of a characteristic Killing tensor $\mathbf{K}$, i.e., a Killing tensor with simple eigenvalues and normal eigenvectors. This second characterization seems to be more effective that the first one, since it involves only one Killing tensor $\mathbf{K}$.[24] But difficulties could arise in checking if first, this tensor has simple eigenvalues (via the analysis of the discriminant of the corresponding characteristic equation) and second, if its eigenvectors are normal. Of course, analogous problems arise in dealing with a whole Killing–Stäckel algebra (i.e., with n independent Killing tensors) at least as it is defined in Definition 7.4, item (ii): we still have the algebraic problem of checking if n independent tensors have common eigenvectors, and the differential problem of checking if these eigenvectors are normal. However, these problems can be simultaneously solved by using the following characterization of a Killing–Stäckel algebra:

Theorem 7.12: *An n-dimensional space $\mathcal{K}$ of Killing tensors is a Killing–Stäckel algebra if and only if its elements (i) commute as linear operators*,

$$\mathbf{K}_1\mathbf{K}_2-\mathbf{K}_2\mathbf{K}_1=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2\in\mathcal{K}, \tag{7.4}$$

and (ii) are in involution,

$$\{P(\mathbf{K}_1),P(\mathbf{K}_2)\}=0, \quad \forall \mathbf{K}_1,\mathbf{K}_2\in\mathcal{K}. \tag{7.5}$$

We postpone the proof of this theorem to Sec. VIII. What is important in this characterization is that, (i) as first pointed out by Kalnins and Miller,[26] if n independent Killing tensors have n common eigenvectors and are in involution, then the eigenvectors are necessarily normal; (ii) if they commute as linear operators, then they have necessarily the same eigenvectors (this holds, however, only in a positive-definite metric); (iii) both conditions (7.4) and (7.5) require simple calculations, of algebraic and differential character, respectively.

Going back to Remark 7.11, we emphasize the advantage of dealing with a single characteristic tensor in the nongeodesic case, i.e., when a potential V is present. Indeed, it can be proved[24] that

Theorem 7.13: *The Hamilton–Jacobi equation associated with a natural Hamiltonian $H=G+V$ is orthogonally separable if and only if (i) there exists a first integral*

$$H_{\mathbf{K}}=\tfrac{1}{2}P_{\mathbf{K}}+U=\tfrac{1}{2}K^{ij}p_ip_j+U,$$

where $\mathbf{K}$ *is a tensor with pointwise simple eigenvalues and normal eigenvectors, or equivalently, (ii) if and only if there exists a Killing tensor* $\mathbf{K}$ *with pointwise simple eigenvalues and normal eigenvectors such that the one-form image of* $\mathrm{d}V$ *by* $\mathbf{K}$ *is closed:*

$$\mathrm{d}(\mathbf{K}\mathrm{d}V)=0. \tag{7.6}$$

In fact, the equivalence of items (i) and (ii) is only "local." Indeed, a function $H_{\mathbf{K}}$ is in involution with $H=G+V$ if and only if

$$\{G, P_{\mathbf{K}}\} = 0, \quad \mathrm{d}U = \mathbf{K}\,\mathrm{d}V.$$

The first equation is the Poisson–Killing equation, while the second one implies (7.6) and is implied by (7.6) only locally. However, in most of the applications this equivalence turns out to be global (that is, the closed one-form $\mathbf{K}\,\mathrm{d}V$ is exact) at least on the manifold Q deprived of the singular set of the Killing tensor $\mathbf{K}$ (where the eigenvalues are not simple).

Equation (7.6) has been called the *characteristic equation of a separable potential*. Indeed, for checking if a potential V is separable in an orthogonal separable web (thus, in any orthogonal coordinate system adapted to this web) it is sufficient to apply this equation to a single characteristic Killing tensor, and not to all elements of a basis of the corresponding Killing–Stäckel algebra. In fact, it can be proved that[24]

Theorem 7.14: *If the characteristic equation is satisfied by a characteristic Killing tensor* $\mathbf{K}$, *then it is satisfied by all elements of the Killing–Stäckel algebra* $\mathcal{K}$ *generated by* $\mathbf{K}$, *and the functions on* T^*Q,

$$H_{\mathbf{K}} = \tfrac{1}{2}P_{\mathbf{K}} + V_{\mathbf{K}}, \quad \mathrm{d}V_{\mathbf{K}} = \mathbf{K}\,\mathrm{d}V, \quad \forall \mathbf{K} \in \mathcal{K}, \tag{7.7}$$

form an n-dimensional space of quadratic first integrals in involution. A basis (H_j) *of this space is expressed in terms of Stäckel matrices by formula (4.30).*

By combining Theorem 7.12 and Theorems 7.13, 7.14, we get a further characterization of the separation, which uses n Killing tensors (the metric tensor $\mathbf{G}$ may be included) but which avoids the use of eigenvectors and their normality conditions:

Theorem 7.15: *The Hamilton–Jacobi equation associated with a natural Hamiltonian* $H = G + V$ *is orthogonally separable if and only if there exist n pointwise independent Killing tensors* $(\mathbf{K}_j)$ *one other (i) commuting as linear operators, (ii) in involution and such that (iii)*

$$\mathrm{d}(\mathbf{K}_j \mathrm{d}V) = 0$$

or equivalently, such that (iii′)

$$\mathrm{d}(\mathbf{K}\mathrm{d}V) = 0$$

for a suitable linear combination $\mathbf{K} = c^j\mathbf{K}_j$, $c^j \in \mathbb{R}$, *with simple eigenvalues.*

After Theorem 7.13, we can give an intrinsic translation of Theorem 4.2 by applying Theorem 6.2:

Theorem 7.16: *The Schrödinger equation is freely separable if and only if there exists a Killing tensor* $\mathbf{K}$ *with simple eigenvalues and normal eigenvectors, satisfying the characteristic equation (7.6) and commuting with the Ricci tensor*

$$\mathbf{KR} - \mathbf{RK} = 0. \tag{7.8}$$

Indeed, since $\mathbf{K}$ has simple eigenvalues, the commutation relation (7.8) means that $\mathbf{R}$ has the same eigenvectors of $\mathbf{K}$.

For a general treatment of the separation (orthogonal or nonorthogonal) we follow a similar way, starting from a suitable extension of the Eisenhart equations, that is from the following extension of Proposition 7.1, which can be easily proved by using the Poisson–Killing equations (7.1):

Theorem 7.17: *Let* $\mathbf{K}$ *be a symmetric tensor on a Riemannian manifold. Assume that in a coordinate system* (q^a, q^α) *both the contravariant metric tensor* $\mathbf{G}$ *and* $\mathbf{K}$ *assume the standard form (3.6) and (3.10), with* (q^α) *ignorable for both tensors, so that*

$$\mathbf{G} = g^{aa}\partial_a \otimes \partial_a + g^{\alpha\beta}\partial_\alpha \otimes \partial_\beta, \quad \mathbf{K} = \lambda^a g^{aa}\partial_a \otimes \partial_a + K^{\alpha\beta}\partial_\alpha \otimes \partial_\beta,$$

where (λ^a) *are the eigenvalues of* $\mathbf{K}$ *corresponding to the eigenforms* $(\mathrm{d}q^a)$. *Then* $\mathbf{K}$ *is a Killing tensor if and only if*

$$\partial_a \lambda^b = (\lambda^a - \lambda^b)\partial_a \ln g^{bb}, \quad \partial_a K^{\alpha\beta} = \lambda^a \partial_a g^{\alpha\beta}. \tag{7.9}$$

These equations can be interpreted as a linear normal first-order differential system in the unknown functions $(\lambda^a, K^{\alpha\beta})$. The complete integrability conditions assume the form

$$(\lambda^a - \lambda^b) S^{cc}_{ab} = 0, \quad (\lambda^a - \lambda^b) S^{\alpha\beta}_{ab} = 0 \quad (a \neq b, \text{ n.s}),$$

where

$$S^{hk}_{ab} = \partial_a \partial_b g^{hk} - \partial_a \ln g^{bb} \partial_b g^{hk} - \partial_b \ln g^{aa} \partial_a g^{hk} \quad (a \neq b, \text{ n.s.})$$

are just the left-hand sides of the first two separability conditions (3.8′).

Thus, remarks and theorems similar to those illustrated for the orthogonal separation are in order. The only difference is that now Eqs. (7.9), when completely integrable, generates a space $\mathcal{K}$ of Killing tensors of dimension $m < n$. The lost dimensions are replaced by a linear space D of Killing vectors. The resulting structure is a pair $(D, \mathcal{K})$ having the properties listed in the following

Definition 7.18: We call *separable Killing algebra* a pair $(D, \mathcal{K})$ where (I) D is an r-dimensional linear space of commuting Killing vectors, (II) $\mathcal{K}$ is a D-invariant $n-r$-dimensional linear space of Killing two-tensors with $m = n - r$ normal eigenvectors in common and orthogonal to D. We call these eigenvectors *essential*.

As has been proved in Ref. 16,

Theorem 7.19: *The separation of the geodesic Hamilton–Jacobi equation is equivalent to the existence of a separable Killing algebra.*

Remark 7.20: The m orthogonal foliations $\mathcal{S}^a$ of the one-codimensional submanifolds orthogonal to the essential eigenvectors of $\mathcal{K}$ (thus tangent to D and containing the orbits of D) form a geometrical structure called *separable Killing web*. A standard coordinate system (q^a, q^α) is related to such a structure in the following way: (i) $(\mathrm{d}q^a)$ are common (local) eigenforms of all elements of $\mathcal{K}$ corresponding to the common essential eigenvectors, or equivalently, their coordinate hypersurfaces belong to the web $\mathcal{S}^a$, (ii) the partial derivatives (∂_α), interpreted as vector fields, are commuting Killing vectors and form a local basis of D.

Furthermore, by a straightforward analysis of system (7.9) we can prove the following two statements:[16]

Theorem 7.21: *In a separable Killing algebra: (i) there are Killing tensors with distinct eigenvalues corresponding to the essential eigenvectors, (ii) all Killing tensors are in involution, (iii) the metric tensor is included, (iv) D is normal (i.e., orthogonally integrable).*

Theorem 7.22: *A separable Killing algebra is uniquely determined by a characteristic Killing pair* $(D, \mathbf{K})$ *where D is an r-dimensional space of Killing vectors in involution and* $\mathbf{K}$ *is a D-invariant Killing tensor with* $m = n - r$ *normal eigenvectors orthogonal to D and corresponding to pointwise distinct eigenvalues.*

Remark 7.23: Also for the general separation we have two equivalent intrinsic characterizations: (A) by means of a separable Killing algebra $(D, \mathcal{K})$ or, (B) by a characteristic Killing pair $(D, \mathbf{K})$, and what has been said in Remark 7.11 can be adapted to this case.

The use of a separable Killing algebra is made more effective by the following theorem analogous to Theorem 7.12 (see Sec. VIII for the proof).

Theorem 7.24: *A pair* $(D, \mathcal{K})$, *where D is an r-dimensional space of Killing vectors in involution and* $\mathcal{K}$ *is an m-dimensional space of D-invariant Killing tensors, is a separable Killing algebra if and only if: (i) the distribution* $\Delta^\perp$ *orthogonal to D is invariant under the elements of* $\mathcal{K}$ *interpreted as linear operators, (ii) the restrictions to* $\Delta^\perp$ *of the elements of* $\mathcal{K}$ *form at each point an m-dimensional space and commute; (iii) the elements of* $\mathcal{K}$ *are in involution.*

Finally, it can be proved[16] that

Theorem 7.25: *The Hamilton–Jacobi equation associated with a natural Hamiltonian* $H = G + V$ *is separable if and only if there exists a characteristic Killing pair* $(D, \mathbf{K})$ *such that V is D-invariant and the characteristic equation (7.6) is satisfied. In this case the functions*

$$H_{\mathbf{K}}=\tfrac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}}, \quad \mathrm{d}V_{\mathbf{K}}=\mathbf{K}\,\mathrm{d}V, \quad \mathbf{K}\in\mathcal{K},$$

$$\mathcal{H}_{\mathbf{X}}=P_{\mathbf{X}}, \quad \mathbf{X}\in D,$$

form an n-dimensional space of first integrals in involution. A basis (H_a,H_α) *of this space is expressed in terms of Stäckel matrices by formulas (5.28) and (5.32).*

Hence, the corresponding Hamiltonian system is integrable in the Arnold–Liouville sense. This theorem shows that the existence of m quadratic and $n-m$ linear first integrals in involution is a necessary condition for the separation.

Remark 7.26: It is important to remark that the use of a characteristic Killing pair $(D,\mathbf{K})$ provides a finer classification of the orthogonal separation. Indeed, an orthogonal web could admit two (or more) different characteristic Killing pairs, thus two (or more) different classes of separable potentials. This is the case, for instance, of the translational or rotational webs in the Euclidean three-space, as described in Ref. 25, where an orthogonal coordinate system may be interpreted in more than one way as a standard separable coordinate system (see the examples in Ref. 21).

As a consequence of Theorem 7.25, we get the following intrinsic translation of Theorems 5.2 and 6.4 related to the reduced separation of the Schrödinger equation. We observe that, **K** and **R** being in standard form, the tangent subspaces orthogonal to D are invariant subspaces for both.

Theorem 7.27: *The Schrödinger equation is reductively separable if and only if there exists a characteristic Killing pair* $(D,\mathbf{K})$ *such that: (i) the potential V is D-invariant; (ii) the characteristic equation (7.6) is satisfied; (iii) the spaces orthogonal to D are invariant under the Ricci tensor* **R**, *interpreted as a linear operator, and the restrictions to these spaces of* **R** *and* **K** *commute or equivalently, (iii′) the essential eigenvectors are eigenvectors of the Ricci tensor* **R** *(i.e., "Ricci principal directions").*

Remark 7.28: Let $\widetilde{Q}$ be the quotient of the manifold Q by the orbits of D (that is the set of the orbits of D). At least locally, it is an m-dimensional *reduced Riemannian manifold*, whose *reduced metric tensor* $\widetilde{\mathbf{G}}=(g^{ab})$ is the result of the projection of **G** by the $(n-m)$-dimensional group of isometries associated with D. Due to the D-invariance, the separable Killing algebra $(D,\mathcal{K})$ is projectable onto an m-dimensional *reduced Killing–Stäckel algebra* $\widetilde{\mathcal{K}}$, and a characteristic Killing tensor $\mathbf{K}=(K^{ij})$ onto a *reduced characteristic Killing tensor* $\widetilde{\mathbf{K}}=(K^{ab})$. Then, the *reduced Schrödinger equation* (5.25) is just the Schrödinger equation written on the reduced manifold, but with respect to the *reduced potential*

$$\widetilde{V}=V-g^{\alpha\beta}\kappa_\alpha\kappa_\beta.$$

For this reduced equation we have a free separation.

Remark 7.29: All the preceding discussion and statements hold for pseudo-Riemannian manifolds, by excluding the case of null (or isotropic) coordinates, i.e., under the assumption $g^{ii}\neq 0$ for the free separation and $g^{aa}\neq 0$ for the reduced separation.

VIII. INTEGRABILITY OF FRAMES

As mentioned in Sec. VII, n independent Killing tensors in involution and commuting as linear operators have common and normal eigenvectors. This remarkable fact, discovered by Kalnins and Miller,[27] reduces the number of sufficient conditions for the geodesic separation listed in the original version of the Eisenhart theorem,[3] as well as in the version given by Woodhouse.[28] We illustrate in this section a detailed proof of this property, showing that in fact it lies in basic properties of frames in pure differential manifolds, apart from any Riemannian structure, which plays a role only at the very end.

A *frame* (a "moving frame" in the classical literature) on a differentiable manifold Q_n is a set of n pointwise independent vector fields $(\mathbf{X}_i)$. In general, frames exist only locally. A manifold Q

admits global frames if and only if it is parallelizable, that is $TQ=Q\times\mathbb{R}^n$. Two frames $(\mathbf{X}_i)$ and $(\mathbf{X}'_i)$ are said to be *equivalent* if there are (nowhere vanishing) functions (f_i) such that $\mathbf{X}'_i=f_i\mathbf{X}_i$.

Definition 8.1: A frame is *integrable* if for each index $i=1,...,n$ the distribution $\Delta_{\hat{i}}$ of rank $n-1$ spanned by all vectors of the frame with the exception of $\mathbf{X}_i$ is completely integrable.

Remark 8.2: The integrability is an invariant property within a class of equivalent frames.

We have two equivalent definitions of integrability, illustrated in the two following propositions.

Proposition 8.3: A frame is integrable if and only if there exist local coordinate systems (q^i) such that the frame is equivalent to (∂_i),

$$\partial_i=f_i\mathbf{X}_i.$$

Proof: (i) Assume that the frame is integrable. Let us consider the distribution $\Delta_{\hat{1}}$ spanned by $(\mathbf{X}_2,\ldots,\mathbf{X}_n)$. The corresponding foliation can be locally parametrized by a regular function q^1; $\mathrm{d}q^1\neq 0$ is a characteristic one-form, so that $\langle\mathbf{X}_i,\mathrm{d}q^1\rangle=0$ for $i\neq 1$. Moreover, $\langle\mathbf{X}_1,\mathrm{d}q^1\rangle=g_1\neq 0$, otherwise $\mathrm{d}q^1=0$. Thus, $\langle f_1\mathbf{X}_1,\mathrm{d}q^1\rangle=1$, with $f_1=1/g_1$. By the same procedure applied to all distributions $\Delta_{\hat{i}}$ we get a system of functions (q^i) such that $\langle f_i\mathbf{X}_i,\mathrm{d}q^j\rangle=\delta_i^j$. Hence, (q^i) are independent functions forming a coordinate system such that $\partial_i=f_i\mathbf{X}_i$. (ii). The natural frame (∂_i) associated with coordinates is obviously integrable; hence the equivalent frame $(\mathbf{X}_i)$ is integrable (Remark 8.2). ■

Proposition 8.4: A frame is integrable if and only if for each pair of indices $i\neq j$ the distribution Δ_{ij} spanned by the two vectors $(\mathbf{X}_i,\mathbf{X}_j)$ is completely integrable.

Proof: With each frame $(\mathbf{X}_i)$ we can always associate a *co-frame* (ξ^i) made of independent one-forms such that $\langle\mathbf{X}_i,\xi^j\rangle=\delta_i^j$. Then ξ^i is a characteristic form of the distribution $\Delta_{\hat{i}}$, that is $\langle\Delta_{\hat{i}},\xi^i\rangle=0$. As we know, this distribution is completely integrable if and only if $\mathrm{d}\xi^i\wedge\xi^i=0$. Moreover, each distribution Δ_{ij} is characterized by the $n-2$ forms (ξ^a), $a\neq i,j$, and for instance, Δ_{12} is completely integrable if and only if $(\dagger)\mathrm{d}\xi^a\wedge\xi^3\wedge\cdots\wedge\xi^n=0$ for $a=3,...,n$. (i) Assume that the frame $(\mathbf{X}_i)$ is integrable: all $\Delta_{\hat{i}}$ are completely integrable, thus $\mathrm{d}\xi^i\wedge\xi^i=0$ for all i. Then $(\dagger)$ is satisfied, as well as the similar equations associated with all pairs of distinct indices. (ii) Assume that all Δ_{ij} are completely integrable. Then, by the Frobenius theorem each Lie bracket $[\mathbf{X}_i,\mathbf{X}_j]$ is a linear combination of the same vectors $(\mathbf{X}_i,\mathbf{X}_j)$. This is enough for the complete integrability of any distribution (of any rank) spanned by any choice of the vectors of the frame. ■

Remark 8.5: There is a geometrical (and intuitive) proof of this proposition. (i) If the frame is integrable, let us consider the integral foliations $\mathcal{S}_{\hat{i}}$ of $\Delta_{\hat{i}}$. They are made of submanifolds of codimension 1. Submanifolds belonging to different foliations are transversal. For any choice of $i\neq j$ the foliation given by the intersection of all $\mathcal{S}_{\hat{k}}$ with $k\neq i,j$ is made of submanifolds of dimension 2 which are tangent to $(\mathbf{X}_i,\mathbf{X}_j)$. Hence, Δ_{ij} is integrable. (ii) Conversely, assume that all Δ_{ij} are integrable and let us consider the corresponding foliations $\mathcal{S}_{ij}$ made of submanifolds of dimension 2. For any fixed i, let us consider the foliation $\mathcal{S}_{\hat{i}}$ given by the union of the foliations $\mathcal{S}_{jk}$ with $j,k\neq i$. This is a foliation of submanifolds of codimension 1 tangent to $\Delta_{\hat{i}}$; then this distribution is integrable.

We base our discussion on the following general considerations. As we have seen in Sec. VII, we have a one-to-one correspondence between contravariant symmetric tensors of any order $\mathbf{K}=(K^{i\cdots j})$ on Q and homogeneous polynomial functions $P(\mathbf{K})=K^{i\cdots j}p_i\cdots p_j$ on T^*Q. For a function f (zero-order tensor) on Q, $P(f)$ is by definition the canonical lift to T^*Q (constant along the fibers), which we denote by the same symbol f, so that

$$P(f)=f.$$

By this correspondence we define two operations over symmetric tensors. (i) The symmetric tensor product $\odot$,

$$P(\mathbf{K}\odot\mathbf{L})=P(\mathbf{K})P(\mathbf{L}).$$

This product is comutative and associative. In particular, for vector fields $\mathbf{X}$ and $\mathbf{Y}$,

$$\mathbf{X}\odot\mathbf{Y}=\tfrac{1}{2}(\mathbf{X}\otimes\mathbf{Y}+\mathbf{Y}\otimes\mathbf{X}).$$

(ii) The Lie–Nijenhuis bracket $[\cdot,\cdot]$,

$$P([\mathbf{K},\mathbf{L}])=\{P(\mathbf{K}),P(\mathbf{L})\},$$

where $\{,\}$ is the canonical Poisson–Lie bracket of functions on T^*Q, defined in local canonical coordinates (q^i,p_i) by

$$\{E,F\}=\partial^i E\partial_i F-\partial^i F\partial_i E,\quad \partial_i=\frac{\partial}{\partial q^i},\quad \partial^i=\frac{\partial}{\partial p_i}.$$

In particular, for vector fields $\mathbf{X}$ and $\mathbf{Y}$,

$$\{P(\mathbf{X}),P(\mathbf{Y})\}=P([\mathbf{X},\mathbf{Y}]),$$

where $[,]$ is the ordinary Lie bracket, and

$$\{P(\mathbf{X}),P(f)\}=[\mathbf{X},f]=\langle\mathbf{X},\mathrm{d}f\rangle.$$

This bracket is anticommutative, bilinear, and obeys the Jacobi rule. Since the Poisson bracket is a bi-derivation, the Leibnitz rule holds,

$$[\mathbf{K},\mathbf{L}\odot\mathbf{M}]=[\mathbf{K},\mathbf{L}]\odot\mathbf{M}+[\mathbf{K},\mathbf{M}]\odot\mathbf{L}.$$

Moreover, two tensors are said to be in involution if $[\mathbf{K},\mathbf{L}]=0$, i.e., $\{P(\mathbf{K}),P(\mathbf{L})\}=0$.

Let $(\mathbf{X}_i)$ be a frame on Q. Let us set

$$[\mathbf{X}_i,\mathbf{X}_j]=\Omega_{ij}{}^h\mathbf{X}_h,\quad \Omega_{ij}{}^h=-\Omega_{ji}{}^h,$$

and use the notation

$$P(\mathbf{X}_i)=x_i, \tag{8.1}$$

so that

$$\{x_i,f\}=\langle\mathbf{X}_i,\mathrm{d}f\rangle,\quad \{x_i,x_j\}=P([\mathbf{X}_i,\mathbf{X}_j])=\Omega_{ij}{}^h x_h.$$

Let us consider two contravariant symmetric two-tensors $\mathbf{K}$ and $\mathbf{L}$. Suppose that they are diagonalized in the frame, i.e.,

$$\mathbf{K}=K^i\mathbf{X}_i\odot\mathbf{X}_i,\quad \mathbf{L}=L^i\mathbf{X}_i\odot\mathbf{X}_i.$$

By using the previous formulas, it is straightforward to compute the Poisson bracket of the corresponding quadratic functions:

$$\begin{aligned}\{P(\mathbf{K}),P(\mathbf{L})\}&=\{P(K^i\mathbf{X}_i\odot\mathbf{X}_i),P(L^h\mathbf{X}_h\odot\mathbf{X}_h)\}\\&=\{K^i x_i^2,L^h x_h^2\}\\&=K^iL^h\{x_i^2,x_h^2\}+K^i\{x_i^2,L^h\}x_h^2+L^h\{K^i,x_h^2\}x_i^2\\&=4K^iL^h\Omega_{ih}{}^j x_i x_h x_j+2(K^i\langle\mathbf{X}_i,\mathrm{d}L^h\rangle-L^i\langle\mathbf{X}_i,\mathrm{d}K^h\rangle)x_i x_h^2\\&=2(2K^iL^h\Omega_{ih}{}^j+(K^i\langle\mathbf{X}_i,\mathrm{d}L^k\rangle-L^i\langle\mathbf{X}_i,\mathrm{d}K^k\rangle)\,\delta_k^h\delta_k^j)x_i x_h x_j.\end{aligned} \tag{8.2}$$

By using this formula we can prove

Theorem 8.6: *Suppose that n contravariant two-tensors* $(\mathbf{K}_a)$ *are (i) pointwise independent, (ii) in involution, and (iii) simultaneously diagonalized in a frame* $(\mathbf{X}_i)$. *Then the distributions* Δ_{ij} *spanned by pairs of vectors* $(\mathbf{X}_i,\mathbf{X}_j)$ *are completely integrable*.

Proof: By assumption (iii), $\mathbf{K}_a=K_a^i\mathbf{X}_i\odot\mathbf{X}_i$, and the independence of the tensors is equivalent to

$$\det[K_a^i]\neq 0.$$

Due to (8.2), equation $\{P(\mathbf{K}_a),P(\mathbf{K}_b)\}=0$ is equivalent to

$$(2K_a^iK_b^h\Omega_{ih}{}^j+(K_a^i\langle\mathbf{X}_i,\mathrm{d}K_b^k\rangle-K_b^i\langle\mathbf{X}_i,\mathrm{d}K_a^k\rangle)\,\delta_k^h\delta_k^j)x_ix_hx_j=0.$$

This is a homogeneous polynomial equation which must be identically satisfied for all values of the variables (p_k), thus for all values of the variables (x_i), since $x_i=P(\mathbf{X}_i)=X_i^kp_k$, and $\det[X_i^k]\neq 0$. Thus, all coefficients vanish. In particular, the coefficient of $x_1x_2x_3$ (as well as for all possible choice of three distinct indices) gives rise to

$$K_a^1K_b^2\Omega_{12}{}^3+K_a^1K_b^3\Omega_{13}{}^2+K_a^2K_b^3\Omega_{23}{}^1+K_a^2K_b^1\Omega_{21}{}^3+K_a^3K_b^1\Omega_{31}{}^2+K_a^3K_b^2\Omega_{32}{}^1=0,$$

i.e., due to the skew-symmetry of Ω,

$$\Omega_{12}{}^3(K_a^1K_b^2-K_a^2K_b^1)+\Omega_{23}{}^1(K_a^2K_b^3-K_a^3K_b^2)+\Omega_{31}{}^2(K_a^3K_b^1-K_a^1K_b^3)=0.$$

This equation can be represented in the form

$$\det\begin{bmatrix}\Omega_{23}{}^1 & \Omega_{31}{}^2 & \Omega_{12}{}^3\\ K_a^1 & K_a^2 & K_a^3\\ K_b^1 & K_b^2 & K_b^3\end{bmatrix}=0.$$

This means that the three vectors of $\mathbb{R}^3$,

$$\boldsymbol{\omega}=(\Omega_{23}{}^1,\Omega_{31}{}^2,\Omega_{12}{}^3),\quad \boldsymbol{\kappa}_a=(K_a^1,K_a^2,K_a^3),\quad \boldsymbol{\kappa}_b=(K_b^1,K_b^2,K_b^3)$$

are linearly dependent (i.e., coplanar). Assume that $\boldsymbol{\omega}\neq 0$. It follows that all vectors $(\boldsymbol{\kappa}_1,\ldots,\boldsymbol{\kappa}_n)$ belong to a same three-plane (containing $\boldsymbol{\omega}$). This means that, for any choice of three distinct indices (a,b,c), we have

$$\det\begin{bmatrix}K_a^1 & K_a^2 & K_a^3\\ K_b^1 & K_b^2 & K_b^3\\ K_c^1 & K_c^2 & K_c^3\end{bmatrix}=0.$$

As a consequence, by applying the Laplace rule to the lines (1,2,3) of the matrix $[K_a^i]$ for the calculus of its determinant, we get $\det[K_a^i]=0$: absurd. Thus $\boldsymbol{\omega}=0$. This means that $\Omega_{23}{}^1=\Omega_{31}{}^2=\Omega_{12}{}^3=0$. So, we have proved that $\Omega_{ij}{}^h=0$ for any choice of distinct indices. This means that $[\mathbf{X}_i,\mathbf{X}_j]$ is a linear combination of $\mathbf{X}_i$ and $\mathbf{X}_j$ only. Due to Frobenius' theorem, the statement is proved. ■

Due to Propositions 8.3 and 8.4 it follows that

Theorem 8.7: *Under the same assumptions of Theorem 8.6, the frame* $(\mathbf{X}_i)$ *is integrable and there are coordinates* (q^i) *in which all tensors are simultaneously diagonalized:* $K_a^{ij}=0$ *for* $i\neq j$.

We remark that the preceding statements hold in a pure differential framework; they do not involve a metric at all. When a metric tensor $\mathbf{G}$ is present, we can consider the case in which the frame is orthogonal and made of common eigenvectors of n independent symmetric two-tensors (not necessarily Killing tensors) $(\mathbf{K}_a)$. Then we can write

$$\mathbf{K}_a=\lambda_a^i\varepsilon^i\mathbf{X}_i\odot\mathbf{X}_i,\quad \mathbf{X}_i\cdot\mathbf{X}_j=0,\quad i\neq j,$$

where λ_a^i is the eigenvalue of $\mathbf{K}_a$ corresponding to the eigenvector $\mathbf{X}_i$,

$$\mathbf{K}_a\mathbf{X}_i=\lambda_a^i\mathbf{X}_i,$$

and

$$\varepsilon^i=\varepsilon_i^{-1},\quad \varepsilon_i=\mathbf{X}_i\cdot\mathbf{X}_i.$$

Thus, we can assert that

Theorem 8.8: *An orthogonal frame made of common eigenvectors of n independent symmetric two-tensors in involution is integrable and the tensors are simultaneously diagonalzed in an orthogonal coordinate system.*

Note that the metric tensor **G** may be one of the tensors considered in this statement (as it happens in a Killing–Stäckel algebra). Then all the two-tensors are Killing tensors. This proves Theorem 7.12, by remarking that item (i) is equivalent to the existence of common eigenvectors, according to the following

Proposition 8.9: Let $(\mathbf{K}_a)$ *be a set of n pointwise independent symmetric two-tensors on a Riemannian manifold. If they commute as linear operators then they generate locally a unique orthogonal frame (up to an equivalence) made of common eigenvectors.*

This is a pure algebraic property, which follows from the spectral theorem of self-adjoint linear operators (see for instance Ref. 29, Secs. 79, 84). However, for the sake of completeness, here we give direct proof based on the following

Proposition 8.10: Let **K** *and* **L** *be two symmetric linear operators over an n-dimensional real Euclidean vector space* $\mathbb{E}_n$. *If they commute, then they have n common eigenvectors.*

Proof: Let us denote by $\mathbf{u}\cdot\mathbf{v}=\mathbf{g}(\mathbf{u},\mathbf{v})$ the scalar product of vectors of $\mathbb{E}_n$, defined by a positive-definite metric tensor **g**. A linear operator $\mathbf{K}:\mathbb{E}_n\to\mathbb{E}_n$ is, by definition, symmetric if $\mathbf{v}\cdot\mathbf{K}\mathbf{u}=\mathbf{u}\cdot\mathbf{K}\mathbf{v}$ for each pair of vectors. It is well known that, for a metric tensor of any signature, (i) eigenvectors corresponding to distinct eigenvalues are orthogonal:

$$\mathbf{K}\mathbf{u}=\lambda\mathbf{u},\quad \mathbf{K}\mathbf{u}'=\lambda'\mathbf{u}',\quad \lambda\neq\lambda'\Leftrightarrow\mathbf{u}\cdot\mathbf{u}'=0.$$

(ii) An eigenvalue λ determines a maximal invariant linear subspace $S(\lambda)\subseteq\mathbb{E}_n$ of eigenvectors; if λ is complex, then also the conjugate λ^* is an eigenvalue (with the same multiplicity) and the dimension of $S(\lambda)$ is even. (iii) $\mathrm{am}(\lambda)\geqslant\mathrm{gm}(\lambda)$, where $\mathrm{am}(\lambda)$ is the *algebraic multiplicity* of λ as root of the characteristic equation, and $\mathrm{gm}(\lambda)$ is its *geometric multiplicity*, which is the dimension of $S(\lambda)$, if λ is real, or half of this dimension if complex. However, in a positive definite metric the eigenvalues are real and $\mathrm{am}(\lambda)=\mathrm{gm}(\lambda)=\dim S(\lambda)$. It follows that if all the eigenvalues are simple, $\dim S(\lambda)=1$ and the eigenvectors form (up to factors) a canonical basis. Let **L** be a linear operator commuting with **K**. Then

$$\mathbf{v}\in S(\lambda)\Leftrightarrow\mathbf{K}\mathbf{v}=\lambda\mathbf{v}\Rightarrow\mathbf{L}\mathbf{K}\mathbf{v}=\lambda\mathbf{L}\mathbf{v}\Rightarrow\mathbf{K}\mathbf{L}\mathbf{v}=\lambda\mathbf{L}\mathbf{v}\Rightarrow\mathbf{L}\mathbf{v}\in S(\lambda).$$

This means that $S(\lambda)$ is an invariant space of **L**. If λ is a simple root of **K**, then **v** is also an eigenvector of **L**. It follows that the space $\mathbb{E}_n$ is decomposed into a direct sum

$$\mathbb{E}_n=V_1\oplus\cdots\oplus V_k\oplus W_1\oplus\cdots\oplus W_l,$$

of mutually orthogonal subspaces, where $(V_1,\ldots,V_k)$ are one-dimensional subspaces of common eigenvectors generated by simple eigenvalues of **K** and **L**, and $(W_1,\ldots,W_k)$ are subspaces of dimension $\geqslant 2$ made of simultaneous eigevectors of **K** and **L**. By choosing an orthogonal basis on each of these subspaces we find a basis made of orthogonal common eigenvectors. ■

If, instead of two operators, we consider a set of *n* commuting symmetric operators $(\mathbf{K}_a)$ we find again a direct-sum decomposition as previously and an orthogonal basis made of common

eigenvectors. But if these operators are independent, the spaces W are not present and the basis is uniquely determined up to constant factors. Indeed, if $(\mathbf{X}_i)$ is a basis of common eigenvectors, then $\mathbf{K}_a\mathbf{X}_i=\lambda_a^i\mathbf{X}_i$ and the independence of $(\mathbf{K}_a)$ is equivalent to $\det[\lambda_a^i]\neq 0$. It follows that we can always find a linear combination of $(\mathbf{K}_a)$ which has all distinct eigenvalues. This proves Proposition 8.9.

Finally, we prove Theorem 7.24.

Proof: Conditions (i)–(ii)–(iii) are obviously necessary. To prove that they are also sufficient, let us apply Proposition 8.9 to the restrictions $\mathcal{K}^\perp$ of $\mathcal{K}$ to the orthogonal distribution $\Delta^\perp$; it follows that they define, in a unique way up to an equivalence, a *subframe* $(\mathbf{X}_a)$ $(a=1,...,m)$ of eigenvectors of $\mathcal{K}$. Since $\mathcal{K}$ is D-invariant, this subframe can be chosen to be D-invariant. If $(\mathbf{X}_\alpha)$ $(\alpha=m+1,...,n)$ is a (local) basis of D, then $(\mathbf{X}_a,\mathbf{X}_\alpha)$ form a frame such that $[\mathbf{X}_a,\mathbf{X}_\alpha]=0$, and two elements of $\mathcal{K}$ can be written in the form

$$\mathbf{K}=K^a\mathbf{X}_a\odot\mathbf{X}_a+K^{\alpha\beta}\mathbf{X}_\alpha\odot\mathbf{X}_\beta,\quad \mathbf{L}=L^a\mathbf{X}_a\odot\mathbf{X}_a+L^{\alpha\beta}\mathbf{X}_\alpha\odot\mathbf{X}_\beta,$$

where all the components are D-invariant. Recalling notation (8.1) we have

$$\{x_a,x_\alpha\}=0,\quad \{x_\alpha,x_\beta\}=0,\quad \{x_a,x_b\}=\Omega_{ab}^c x_c+\Omega_{ab}^\alpha,x_\alpha,$$

where Ω's are D-invariant. A calculation similar to that in the proof of Theorem 8.6, shows that the involutivity of a basis $(\mathbf{K}_a)$ of $\mathcal{K}$ implies that all Ω's with distinct indices vanish. Then we apply Proposition 8.4. ■

The second part of this proof can be replaced by the following geometrical (and intuitive) reasoning. We can consider (at least locally) the reduced manifold $\tilde{Q}$ (see Remark 7.28). Due to the D-invariance of $\mathcal{K}$, the vector fields $\mathbf{X}_a$ can be chosen (up to factors) to be projectable onto an orthogonal frame $\tilde{\mathbf{X}}_a$ of $\tilde{Q}$ and the orthogonal components of the elements of $\mathcal{K}$ onto the m-dimensional reduced Killing–Stäckel algebra $\tilde{\mathcal{K}}$. Then we can apply to this algebra the above-given results related to Killing–Stäckel algebras, remarking that the involutivity of $\mathcal{K}$ implies the involutivity of $\tilde{\mathcal{K}}$. It follows that the frame $(\tilde{\mathbf{X}}_a)$ is integrable. Pulling back the integral orthogonal manifolds, we get submanifolds of dimension $m-1$ whose unions with the orbits of D are manifolds of dimension $n-1$ orthogonal to the $(\mathbf{X}_a)$. Thus, these vectors are normal.

IX. AN ILLUSTRATIVE EXAMPLE: THE COMPARISON BETWEEN THE FREE AND THE REDUCED SEPARATION

Let us consider in the Euclidean three-space $Q=\mathbb{E}_3$ the Hamilton–Jacobi equation of the Kepler problem, $V=-k/r$,

$$\frac{1}{2}|\mathbf{p}|^2-\frac{k}{r}-E=0,$$

where r is the distance from the origin, and the corresponding Schrödinger equation of the "hydrogen atom" (as usual, see Sec. IV, we replace $(2/\hbar^2)V$ and $(2/\hbar^2)E$ with V and E)

$$\Delta\psi+\left(E+\frac{k}{r}\right)\psi=0.$$

It is known that the Kepler problem is separable with respect to four separable webs: spherical, spherical conical, parabolic, all centered at the origin, and prolate spheroidal centered at any point with focus at the origin (see Ref. 30 for a group-theoretical approach and Ref. 31 for a geometrical and tensorial approach). Since $\mathbf{R}=0$, the Schrödinger equation is separable in all possible coordinate systems associated with these webs. In order to analyze the different features between the free and the reduced separation, let us consider the spherical web, which is orthogonal (thus

allowing the free separation) and rotational (thus allowing the reduced separation). This web can be parametrized by the usual spherical coordinates $(q^1,q^2,q^3)=(r,\theta,\varphi)$.

Free separation. The Killing–Stäckel algebra $\mathcal{K}_3$ associated with the spherical web has the following basis:

$$\mathbf{K}_1=\partial_\theta\otimes\partial_\theta+\frac{1}{\sin^2\theta}\partial_\varphi\otimes\partial_\varphi,$$

$$\mathbf{K}_2=\partial_\varphi\otimes\partial_\varphi, \tag{9.1}$$

$$\mathbf{G}=\partial_r\otimes\partial_r+\frac{1}{r^2}\partial_\theta\otimes\partial_\theta+\frac{1}{r^2\sin^2\theta}\partial_\varphi\otimes\partial_\varphi.$$

The (diagonal) components of these tensors form the inverse matrix

$$[\varphi^i_{(j)}]=\begin{bmatrix}0 & 1 & \frac{1}{\sin^2\theta}\\ 0 & 0 & 1\\ 1 & \frac{1}{r^2} & \frac{1}{r^2\sin^2\theta}\end{bmatrix} \tag{9.2}$$

of the Stäckel matrix

$$[\varphi^{(j)}_i]=\begin{bmatrix}-\frac{1}{r^2} & 1 & 0\\ 0 & -\frac{1}{\sin^2\theta} & 1\\ 1 & 0 & 0\end{bmatrix}.$$

The last line of matrix (9.2) represents the diagonal components g^{ii} of the metric tensor. The contracted Christoffel symbols (4.3) are

$$\Gamma_1=-\frac{2}{r},\quad \Gamma_2=-\frac{1}{\tan\theta},\quad \Gamma_3=0. \tag{9.3}$$

Looking at the expression of the Newtonian potential energy, written as a Stäckel multiplier (1.6),

$$V=-\frac{k}{r}=\phi_i g^{ii}=\phi_i\varphi^i_{(3)}=\phi_1(r)+\phi_2(\theta)\frac{1}{r^2}+\phi_3(\varphi)\frac{1}{r^2\sin^2\theta},$$

we get (at least) two solutions,

$$\phi_1=-\frac{k}{r}-\frac{c}{r^2},\quad \phi_2=c\in\mathbb{R},\quad \phi_3=0,$$

or

$$\phi_1=-\frac{k}{r},\quad \phi_2=-\frac{c}{\sin^2\theta},\quad \phi_3=c\in\mathbb{R}.$$

However, for our purposes, there is no loss of generality in choosing $c=0$, so that they coincide:

$$\phi_1 = -\frac{k}{r}, \quad \phi_2 = 0, \quad \phi_3 = 0. \tag{9.4}$$

In the free separation the Killing tensors of the spherical Killing–Stäckel algebra correspond to the constants of motion $a = (a_j)$ entering Eqs. (4.32),

$$\varphi^i_{(j)} \left(\frac{\psi_i''}{\psi_i} - \Gamma_i \frac{\psi_i'}{\psi_i} - \phi_i \right) + a_j = 0.$$

Due to (9.3) and (9.4), these equations are equivalent to the well-known separated ordinary differential equations of the kind (4.29),

$$\psi_1'' + \frac{2}{r}\psi_1' + \left(a_3 - \frac{1}{r^2} a_1 + \frac{k}{r} \right) \psi_1 = 0 \quad (a_3 = E),$$

$$\psi_2'' + \cot\theta \psi_2' + \left(a_1 - \frac{1}{\sin^2\theta} a_2 \right) \psi_2 = 0, \tag{9.5}$$

$$\psi_3'' + a_2 \psi_3 = 0,$$

in the unknown functions $\psi_1(r)$, $\psi_2(\theta)$, $\psi_3(\varphi)$. The general solution of the first equation (9.5) can be written

$$\psi_1 = c_1 F_1 + c_2 F_2$$

where $F_1(a_1, a_3; r)$ and $F_2(a_1, a_3; r)$ are two independent confluent hypergeometric functions[32] and (c_1, c_2) arbitrary constants. Since any ψ_i can be determined up to an inessential multiplicative constant, we can choose the solution

$$\psi_1 = F_1 + b_1 F_2, \quad b_1 \in \mathbb{R}.$$

In a similar way, for the second equation (9.5) we consider the solution

$$\psi_2 = S_1(a_1, a_2; \theta) + b_2 S_2(a_1, a_2; \theta),$$

where S_1 and S_2 are independent spherical harmonics.[32] For the last equation (9.5) we can consider the solution

$$\psi_3 = e^{-\sqrt{-a_2}\varphi} + b_3 e^{\sqrt{-a_2}\varphi}.$$

Thus, the resulting separated solution $\psi = \psi_1 \psi_2 \psi_3$ depends on $2n = 6$ constants (a_i, b_i). However, in order to get solutions $\psi = \psi_1 \psi_2 \psi_3$ with a physical meaning, the constant parameters (a_1, a_2, a_3), corresponding to the constants of motion, must verify further well known (quantization) conditions, assuring for instance, the summability of ψ_1 and ψ_2, and the periodicity of ψ_3. These are the results of the free separation, as explained in Remarks 4.7, 4.8, and 4.9.

Reduced separation. Since the spherical web is rotational, so that φ is ignorable, we can consider the reduced separation according to Definition 5.1 and Theorem 5.2, by setting (being $\alpha = 3$)

$$\psi_3 = e^{\kappa_3 \varphi}. \tag{9.6}$$

Now the Stäckel matrix and its inverse are the 2×2 matrices

$$[\varphi_b^{(a)}]=\begin{bmatrix} -\frac{1}{r^2} & 1 \\ 1 & 0 \end{bmatrix}, \quad [\varphi_{(a)}^b]=\begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{r^2} \end{bmatrix},$$

respectively. The other two functions $\psi_1(r)$ and $\psi_2(\theta)$ must satisfy the separated equations (5.27). In the present case, from the general expressions (3.9), where $\alpha=\beta=3$, we derive

$$g^{33}=\frac{1}{r^2\sin^2\theta}=\phi_1^{33}+\phi_2^{33}\frac{1}{r^2},$$

thus,

$$\phi_1^{33}=\frac{c}{r^2}, \quad \phi_2^{33}=\frac{1}{\sin^2\theta}-c, \quad c\in\mathbb{R},$$

and ϕ_1, ϕ_2, Γ_1, and Γ_2 are as above. For $c=0$ (5.27) become

$$\psi_1''+\frac{2}{r}\psi_1'+\left(\tilde{a}_2-\frac{1}{r^2}\tilde{a}_1+\frac{k}{r}\right)\psi_1=0 \quad (\tilde{a}_2=E),$$

$$\psi_2''+\cot\theta\psi_2'+\left(\tilde{a}_1+\frac{1}{\sin^2\theta}\kappa_3^2\right)\psi_2=0.$$

Up to the transformation of the constant parameters,

$$\tilde{a}_1=a_1, \quad \tilde{a}_2=a_3, \quad \kappa_3^2=-a_2,$$

these equations coincide with the first two equations (9.5), and ψ_3 given by the constraint (9.1) is a solution of the third equation (9.5). Finally, we observe that the Killing–Stäckel algebra $\mathcal{K}_3$ associated with the free separation in the spherical web is determined by the Killing tensors $(\mathbf{G},\mathbf{K}_1,\mathbf{K}_2)$ given in (9.1). With the same web we associate the separable Killing algebra $(D_1,\mathcal{K}_2)$, with $\mathcal{K}_2=\mathrm{span}(\mathbf{G},\mathbf{K}_1)$ and $D=\mathrm{span}(\partial_\varphi)$. This geometrical structure can be interpreted in two different ways. (i) We can use ∂_φ for constructing $\mathbf{K}_2=\partial_\varphi\otimes\partial_\varphi$, by interpreting φ as an essential coordinate (this is possible since the web is orthogonal). In this way we reconstruct the Killing–Stäckel algebra $\mathcal{K}_3$, associated with the free separation. (ii) We can associate with D the solution (9.6), by interpreting φ as an ignorable coordinate, and we reduce the problem of the separation to the remaining two essential coordinates: this is the reduced separation.

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