JOURNAL OF MATHEMATICAL PHYSICS VOLUME 42, NUMBER 5 MAY 2001

Variable separation for natural Hamiltonians with scalar and vector potentials on Riemannian manifolds

S. Benenti,[a)] C. Chanu, and G. Rastelli
Department of Mathematics, University of Turin, 10123 Torino, Italy

(Received 9 October 2000; accepted for publication 6 November 2000)

The additive variable separation in the Hamilton–Jacobi equation is studied for a natural Hamiltonian with scalar and vector potentials on a Riemannian manifold with positive–definite metric. The separation of this Hamiltonian is related to the separation of a suitable geodesic Hamiltonian over an extended Riemannian manifold. Thus the geometrical theory of the geodesic separation is applied and the geometrical characterization of the separation is given in terms of Killing webs, Killing tensors, and Killing vectors. The results are applicable to the case of a nondegenarate separation on a manifold with indefinite metric, where no null essential separable coordinates occur.
[DOI: 10.1063/1.1340868]

I. INTRODUCTION

A smooth real function V and a smooth vector field $\mathbf{A}$ on a Riemannian n-manifold $(Q,\mathbf{g})$ define a Hamiltonian function on the cotangent bundle T^*Q,

$$H=\tfrac{1}{2}g^{ij}(p_i+A_i)(p_j+A_j)+V=\tfrac{1}{2}g^{ij}p_ip_j+A^ip_i+U, \tag{1.1}$$

where the function on Q

$$U=V+\tfrac{1}{2}A^iA_i=V+\tfrac{1}{2}\mathbf{A}\cdot\mathbf{A} \tag{1.2}$$

is extended to T^*Q as a function constant on the fibers. Hamiltonians of this kind appear in many classical problems of analytical mechanics and physics, and for this reason they are called *natural*. The Hamiltonian (1.1) corresponds to a Lagrangian $L:TQ\to\mathbb{R}$ of the form

$$L=\tfrac{1}{2}g_{ij}\dot q^i\dot q^j-A_i\dot q^i-V, \tag{1.3}$$

where $\frac{1}{2}g_{ij}\dot q^i\dot q^j$ is the *kinetic energy* and V and $\mathbf{A}$ play the role of *scalar* and *vector potentials*, respectively, generating Lagrangian forces

$$F_i=(\partial_jA_i-\partial_iA_j)\dot q^j-\partial_iV. \tag{1.4}$$

Here we denote by $(\underline{q},\underline{p})=(q^i,p_i)$ and by $(\underline{q},\underline{\dot q})=(q^i,\dot q^i)$ the coordinate systems on T^*Q and TQ, associated with a coordinate system $\underline{q}=(q^i)$ on Q. We denote by ∂_i the partial derivative with respect to the variable q^i. In the following we shall use the symbol ∂^i for the partial derivative with respect to p_i.

A natural Hamiltonian is called separable if there are coordinates $\underline{q}$ on Q such that the Hamilton–Jacobi equation

$$H(\underline{q},\underline{p})=h, \quad p_i=\partial_iW \tag{1.5}$$

a)Electronic mail: benenti@dm.unito.it

admits a *separated complete solution* of the form

$$W(\underline{q},\underline{c}) = W_1(q^1,\underline{c}) + \cdots W_n(q^n,\underline{c}), \tag{1.6}$$

where $\underline{c} = (c_i)$ is a set of n constants satisfying the *completeness condition*

$$\det\left[\frac{\partial^2 W}{\partial q^i \partial c_j}\right] \neq 0. \tag{1.7}$$

The interest of separable Hamiltonians lies essentially on two facts: (1) in separable coordinates $\underline{q}$ the integration of the Hamilton–Jacobi equation is reduced to (at most n) simple integrals (i.e., involving single variables); (2) the separation of variables is characterized by the existence of n first integrals in involution, quadratic or linear in the conjugate momenta $\underline{p}$. Hence, separable Hamiltonians give rise to a particular but wide class of completely integrable Hamiltonian systems. In the theory of separation of variables a basic role is played by the geodesic Hamiltonian

$$G = \tfrac{1}{2} g^{ij} p_i p_j \,. \tag{1.8}$$

Indeed, as pointed out by Levi-Civita,[1] a necessary condition for the separability of a natural Hamiltonian (1.1) is the separability of the corresponding geodesic Hamiltonian (1.8). Moreover, it is known that the separability of G is characterized by the existence of Killing vectors and Killing 2-tensors on the Riemannian manifold Q (which generate quadratic and linear first integrals in involution) satisfying suitable properties.[2–9] This shows that the separability is not simply a local property concerning with coordinates but it is in fact related to the existence of intrinsic objects satisfying coordinate-independent properties. As a consequence, the intrinsic characterization of the separability (by means of algebraic objects like Killing vectors and tensors[4–8] and geometrical objects like ''Killing webs''[9,10]) provide a useful and effective tool for finding and constructing separable Hamiltonian systems. While the theory of the geodesic separability can be easily extended to natural Hamiltonians of the kind

$$G = \tfrac{1}{2} g^{ij} p_i p_j + V, \tag{1.9}$$

involving a scalar potential only, the extension to the general Hamiltonian (1.1) with a vector potential meets some difficulties, as explained below. However, several important results are already present in the literature, but all concerning the general form of the functions (g^{ij},A^i,V) in separable coordinates[11–14] (also in the time-dependent case). The aim of the present paper is to revisit all this matter at the light of the more recent progress in the geometrical characterization of the separation.[10] As it has been done for a pure geodesic Hamiltonian G, for investigating on the intrinsic properties of the objects $(\mathbf{g},\mathbf{A},V)$ underlying the separation, a starting point could be the fundamental *Levi-Civita separability conditions*[1]

$$\partial^i \partial^j H \partial_i H \partial_j H + \partial_i \partial_j H \partial^i H \partial^j H - \partial^i \partial_j H \partial_i H \partial^j H - \partial_i \partial^j H \partial^i H \partial_j H = 0 \tag{1.10}$$

(no sum over the indices $i \neq j$) which yield second-order differential equations on the functions (g^{ij},A^i,V). But these equations turn out to be of such a complexity that this way seems to be hopeless. An alternative method could be the analysis of the known expressions[11,12] of the functions (g^{ij},A^i,V) in separable coordinates (as done for instance in Ref. 15, for the orthogonal separation, on the basis of previous results by Steigeberger[13]). But also this method appears to be rather difficult and, moreover, it does not provide a good and complete understanding of the intrinsic meaning of the separation, where a basic and simplifying role is played by particular classes of coordinates, called *normal separable coordinates*.[7–10] Instead, we propose here a direct and geometrical method which makes the problem clear and easily solvable from the very beginning. The basic (and very simple) idea of this method is the following: we replace the original Hamiltonian (1.1) by an ''equivalent'' geodesic Hamiltonian on the ''extended manifold'' Q

$\times\mathbb{R}$ endowed with a suitable ''extended metric'' (Sec. IV); then, we apply to this new Hamiltonian the well-known methods of the theory of the geodesic separability.[7,8,10]

In the present paper we consider, for simplicity, only the case of a positive–definite metric. This makes the discussion considerably easier, since we avoid the cases of degenerate separation where the so-called second-class null coordinates occur. However, all results hold for the nondegenerate separation in a metric of any signature. The case of a Lorentzian metric will be considered in detail in a further paper.

II. NOTATION

We denote by $\langle \mathbf{X},\varphi\rangle = X^i\varphi_i$ the evaluation between a vector field $\mathbf{X}$ and a 1-form φ. In particular, $\langle \mathbf{X},dV\rangle = X^i\partial_i V$ is the derivative of the function V with respect to the vector $\mathbf{X}$. We denote by $\mathbf{u}\cdot\mathbf{v}$ the scalar product of two vectors, $\mathbf{u}\cdot\mathbf{v}=\mathbf{g}(\mathbf{u},\mathbf{v})=g_{ij}u^iv^j$. The canonical Poisson–Lie brackets of functions over a cotangent bundle are defined by

$$\{f,g\}=\partial^i f\,\partial_i g-\partial^i g\,\partial_i f. \tag{2.1}$$

We consider the natural identification between contravariant symmetric tensors $\mathbf{K}=(K^{i_1\cdots i_k})$ on Q and the homogeneous polynomial functions on the cotangent bundle T^*Q, defined by

$$P(\mathbf{K})=P_{\mathbf{K}}=K^{i_1\cdots i_k}p_{i_1}\cdots p_{i_k}. \tag{2.2}$$

For a function f on Q (tensor of order 0) P_f is its natural extension to T^*Q constant on the fibers. Then the Poisson brackets induce Nijenhuis–Lie brackets between contravariant symmetric tensors on Q by setting

$$\{P_{\mathbf{K}},P_{\mathbf{L}}\}=P_{[\mathbf{K},\mathbf{L}]}. \tag{2.3}$$

If $\mathbf{K}$ and $\mathbf{L}$ are of order k and l, respectively, then $[\mathbf{K},\mathbf{L}]$ is of order $k+l-1$. In particular, for two vector fields, $[\mathbf{X},\mathbf{Y}]$ are the ordinary Lie brackets, and $[\mathbf{X},\mathbf{K}]$ is the Lie derivative of the tensor field $\mathbf{K}$ with respect to the vector field $\mathbf{X}$. We say that two (symmetric) tensors are *in involution* (or that they *commute*) if $[\mathbf{K},\mathbf{L}]=0$. This means that the corresponding polynomial functions are in involution: $\{P_{\mathbf{K}},P_{\mathbf{L}}\}=0$. Killing vectors and Killing tensors are defined by the *Killing equations*

$$[\mathbf{X},\mathbf{G}]=0,\quad [\mathbf{K},\mathbf{G}]=0, \tag{2.4}$$

where

$$\mathbf{G}=(g^{ij})$$

is the contravariant metric tensor. This means that the corresponding functions $P_{\mathbf{X}}$ and $P_{\mathbf{K}}$ are first integrals of the geodesic flow. As for any symmetric 2-tensor on a Riemannian manifold, a Killing tensor $\mathbf{K}$ can be interpreted as a linear operator over 1-forms or vector fields; we shall denote by $\mathbf{K}\varphi$ and by $\mathbf{KX}$, respectively, the images by $\mathbf{K}$ of a 1-form φ and of a vector $\mathbf{X}$, whose local representations, in any coordinate system $\underline{q}$, are

$$\mathbf{K}\varphi=g_{ij}K^{jh}\varphi_h\,dq^i,\quad \mathbf{KX}=K^{ih}g_{hj}X^j\partial_i. \tag{2.5}$$

The contravariant metric tensor $\mathbf{G}$ corresponds to the identity mapping,

$$\mathbf{G}\varphi=\varphi,\quad \mathbf{GX}=\mathbf{X}.$$

We denote by $\flat$ the bijective mapping from vector fields to 1-forms on Q, defined by the equivalent equations

$$\langle \mathbf{Y},\mathbf{X}^{\flat}\rangle = \mathbf{Y}\cdot\mathbf{X}, \quad \mathbf{G}(\mathbf{X}^{\flat},\varphi) = \langle \mathbf{X},\varphi\rangle. \tag{2.6}$$

III. AN OUTLINE ON THE GEODESIC SEPARATION

In order to make this paper self-contained we recall in this section, with suitable adaptations, the basic definitions and results of the geometrical theory of the separation of the geodesic Hamilton–Jacobi equation.

(A) An *orthogonal web* on a Riemannian manifold Q_n is a set $(\mathcal{S}^a)$ $(a=1,...,m)$ of $m \leqslant n$ pairwise transversal and orthogonal foliations of leaves of codimension 1. In a positive–definite metric the orthogonality implies the transversality, and moreover, the intersections of all the leaves of $(\mathcal{S}^a)$ form a foliation $\mathcal{O}$ of submanifolds of dimension $r=n-m$. If these submanifolds are the orbits of a r-dimensional space D of commuting Killing vectors, then we say that the set

$$(\mathcal{S}^a,D) = (\mathcal{S}^1,\ldots,\mathcal{S}^m,D) \tag{3.1}$$

is a *Killing web*. The orbits of D are locally flat submanifolds.

(B) If the foliations $(\mathcal{S}^a)$ are, respectively, orthogonal to m eigenvectors $(\mathbf{X}_a)$ of a Killing 2-tensor $\mathbf{K}$ associated with m pointwise distinct eigenvalues (λ^a), and if $\mathbf{K}$ is D-invariant $([\mathbf{X},\mathbf{K}]=0,\ \forall \mathbf{X}\in D)$ then we say that the set

$$(\mathcal{S}^1,\ldots,\mathcal{S}^m,D,\mathbf{K}) \tag{3.2}$$

is a *separable Killing web* and that $\mathbf{K}$ is a *characteristic Killing tensor* of the web. Since only the eigenvectors (or eigenforms) orthogonal to the foliations $(\mathcal{S}^a)$ are relevant for the separation, we call them *main eigenvectors* (or *main eigenforms*) of $\mathbf{K}$. Points of Q where these objects are not defined or do not satisfy the above requirements are called *singular points* of the web. They form the *singular set* of the web.

(C) From a purely algebraic point of view a separable Killing web is then completely determined by a pair

$$(D,\mathbf{K}) \tag{3.3}$$

which we call *characteristic Killing pair*, made of a r-dimensional linear space $(r\leqslant n)$ D of commuting Killing vectors and of a Killing 2-tensor $\mathbf{K}$ satisfying the following requirements: (i) the vectors of D span a regular distribution Δ of rank r (i.e., a subbundle $\Delta \subseteq TQ$ such that $\dim(\Delta)=n+r$); (ii) $\mathbf{K}$ is D-invariant; (iii) $\mathbf{K}$ has $m=n-r$ *normal* (i.e., orthogonally integrable) eigenvectors $(\mathbf{X}_a)$ $(a=1,...,m)$ (the *main eigenvectors*) orthogonal to D and associated with m pointwise distinct eigenvalues (λ^a).

(D) In a neighborhood of a nonsingular point a Killing web $(\mathcal{S}^a,D)$ generates coordinate systems (q^a,q^α) $(a=1,...,m;\ \alpha=m+1,...,n)$ such that each dq^a is a characteristic 1-form of the corresponding foliation $\mathcal{S}^a$ (q^a is constant on the leaves of $\mathcal{S}^a$) and (q^α) are the affine parameters of the integral curves of r vector fields $(\mathbf{X}_\alpha)$ forming a basis of D, with zero values on a chosen submanifold $\mathcal{Z}$ of codimension r, transversal to the orbits of D. It follows that the coordinates (q^a) are orthogonal, $g^{ab}=\mathbf{G}(dq^a,dq^b)=0$ for $a\neq b$, and their coordinate hypersurfaces are open submanifolds of the leaves of the web. Moreover, the coordinates (q^α) are ignorable, $\partial_\alpha g^{ij}=0$, since they are generated by Killing vectors. We say that such a coordinate system is *adapted* to or *generated* by the Killing web, and *based on the section* $\mathcal{Z}$ (Fig. 1).

(E) It can be shown that[10] the coordinates adapted to a Killing web are separable for the geodesic Hamiltonian G if and only if there exists a Killing 2-tensor $\mathbf{K}$ satisfying conditions of item (B), i.e., if and only if $(\mathcal{S}^a,D,\mathbf{K})$ is a separable Killing web. This is equivalent to say that the geodesic Hamiltonian G is separable if and only if there exists a characteristic Killing pair $(D,\mathbf{K})$ [see item (C)].

(F) It can be proved that[9,10] in a separable Killing web the distribution $\Delta^\perp$ orthogonal to D is completely integrable, so that there exists a foliation of m-dimensional manifolds orthogonal to the

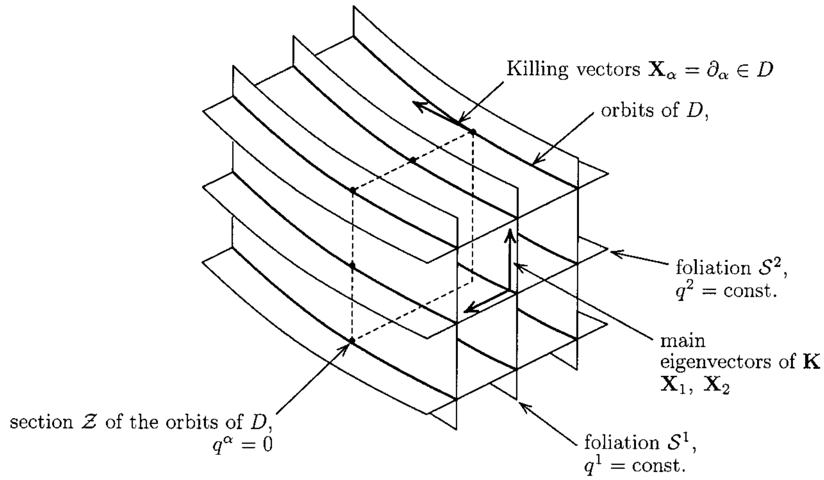

FIG. 1. Illustration of the elements of a separable Killing web $(\mathcal{S}^a, D, \mathbf{K})$ (for $a=1,2$).

orbits of D. The separable coordinates adapted to a separable Killing web and based on a section $\mathcal{Z}$ orthogonal to D are called *normal separable coordinates*. In these coordinates the contravariant metric assumes the semidiagonal *standard form*

$$[g^{ij}] = \left[\begin{array}{ccccc|c} g^{11} & & & & & \\ & \ddots & & 0 & & \\ & & g^{aa} & & & 0 \\ & 0 & & \ddots & & \\ & & & & g^{mm} & \\ \hline & & 0 & & & g^{\alpha\beta} \end{array}\right]. \tag{3.4}$$

(G) There are two extreme cases of the above description: (i) $m=n$, $r=0$; in this case the space of Killing vectors D vanishes, the Killing web is simply an orthogonal web of n foliations of codimension 1; (ii) $m=0$, $r=n$; in this case the foliations $\mathcal{S}^a$ disappear, and only the n-dimensional space D of commuting K-vectors is present; such a K-web is always separable, with $\mathbf{K}=0$. There is a further particular case: (iii) $m=1$, $r=n-1$; in this case we have a single foliation of codimension 1 made of the orbits of $n-1$ commuting K-vectors; such a K-web is always separable, with $\mathbf{K}=\mathbf{G}$.

(H) Separable coordinate systems occur in equivalence classes: two separable systems are equivalent if the corresponding complete integrals generate the same Lagrangian foliation in T^*Q. A separable Killing web is the geometrical counterpart of an equivalence class of separable coordinates for the geodesic Hamiltonian. According to Levi-Civita,[1] the coordinates (q^i) of a separable system are divided into two classes: a coordinate q^i is of *first class* if the fraction $\partial_i H / \partial^i H$ is linear (homogeneous) in the momenta (p_j). Otherwise, it is of *second class*. Second-class coordinates are also called *essential separable coordinates*. They are usually labeled by indices $a,b,\ldots$ running from 1 to $m \leq n$. The first-class coordinates are labeled by indices α, β,... running from $m+1$ to n. The numbers (r, m) of coordinates of first and second class, respectively, are the same for two equivalent separable systems and moreover, a separable systems is always equivalent to a normal separable system, see item (F), in which the first-class coordinates are ignorable and the metric tensor has the standard form (2.1).[7,8] In the transformation from a generic separable coordinate system to a normal one, the second-class coordinates remain essentially unchanged (they are related by a separated transformation, whose Jacobian is diagonal) so that their coordinate surfaces are invariant; these surfaces span the foliations $\mathcal{S}^a$ of the underlying separable Killing web. Moreover, the partial derivatives (∂_α) with respect to the first-class coordinates (ignorable or not), interpreted as vector fields, span the space D of the underlying separable Killing web.

(I) The nonvanishing metric components (3.4) in normal separable coordinates have the form

$$g^{aa}=\varphi^a_{(m)}, \quad g^{\alpha\beta}=g^{aa}\phi_a^{\alpha\beta}, \tag{3.5}$$

where $\phi_a^{\alpha\beta}$ are functions of q^a only and $\varphi^a_{(m)}$ is the mth row of the inverse of a $m\times m$ *Stäckel matrix* $[\varphi_a^{(b)}]$: this is a matrix of functions depending only on the coordinate q^a corresponding to the lower index.

(J) A characteristic Killing pair $(D,\mathbf{K})$ generates an m-dimensional space $\mathcal{K}$ of Killing 2-tensors which are (i) D-invariant, (ii) in involution, and (iii) with m eigenvectors in common orthogonal to D (the main eigenvectors of the characteristic tensor $\mathbf{K}$). The components of an element of $\mathcal{K}$ in normal separable coordinates form a matrix similar to that of the metric (3.4). By formulas similar to (3.5), the rows of the inverse Stäckel matrix generate the components of a basis $(\mathbf{K}_b)$ of $\mathcal{K}$, $b=1,...,m$, with $\mathbf{K}_m=\mathbf{G}$,

$$K_b^{aa}=\varphi^a_{(b)}, \quad K_b^{\alpha\beta}=\varphi^a_{(b)}\phi_a^{\alpha\beta}. \tag{3.6}$$

This space includes $\mathbf{K}$ and the contravariant metric tensor $\mathbf{G}$. We call $\mathcal{K}$ the separable Killing algebra generated by $(D,\mathbf{K})$. If $\mathbf{K}^0$ is an element of $\mathcal{K}$ with distinct eigenvalues corresponding to the main eigenvectors, then the pairs $(D,\mathbf{K}^0)$ and $(D,\mathbf{K})$ are said to be equivalent (the define the same $\mathcal{K}$). The m quadratic functions

$$G_b=\tfrac{1}{2}P(\mathbf{K}_b)=\tfrac{1}{2}K_b^{ij}p_ip_j=\tfrac{1}{2}\varphi^a_{(b)}(p_a^2+\phi_a^{\alpha\beta}p_\alpha p_\beta) \tag{3.7}$$

together with the r linear functions

$$G_\alpha=P(\mathbf{X}_\alpha)=p_\alpha$$

associated with a basis $(\mathbf{X}_\alpha)$ of D, form a system of n independent first integrals in involution of the geodesic flow. Moreover, from the eigenform equations

$$\mathbf{K}_b\,dq^a=\lambda_b^a\,dq^a, \tag{3.8}$$

we derive the following relation between the main eigenvalues of $\mathbf{K}_b$ (corresponding to the main eigenvectors of the characteristic tensor $\mathbf{K}$) and the inverse Stäckel matrix,

$$\lambda_b^a=\frac{\varphi^a_{(b)}}{\varphi^a_{(m)}}, \tag{3.9}$$

so that

$$G_b=\tfrac{1}{2}\lambda_b^a g^{aa}(p_a^2+\phi_a^{\alpha\beta}p_\alpha p_\beta). \tag{3.10}$$

In this last formula, the quadratic first integrals in involution are expressed in terms of the main eigenvalues of the Killing tensors forming a basis of $\mathcal{K}$, without any reference to the Stäckel matrix.

(K) It can be shown that[10] a natural Hamiltonian $H=G+V$ is separable if and only if there exists a characteristic Killing pair $(D,\mathbf{K})$ such that

$$D(V)=0, \quad d(\mathbf{K}\,dV)=0. \tag{3.11}$$

The first of these two conditions means that V is D-invariant, $\langle \mathbf{X}, dV \rangle = 0$, $\forall \mathbf{X} \in D$, the second one that the 1-form $\mathbf{K}\, dV$ (image of dV by $\mathbf{K}$) is closed, hence locally exact. We call the second equation (3.11) the *characteristic equation of a separable potential*. Moreover,[7,8] a function V satisfies conditions (3.11) if and only if in a normal separable coordinate system is of the form

$$V = g^{aa}\phi_a = \varphi^a_{(m)}\phi_a, \tag{3.12}$$

where each ϕ_a is a function of q^a only. Functions of this kind are called *Stäckel multipliers*.[6] We observe that the first-class metric components $g^{\alpha\beta}$ (3.5) are Stäckel multipliers. It is remarkable that if V satisfies equations (3.11) then the characteristic equation holds for all elements of the algebra $\mathcal{K}$ generated by the characteristic Killing pair $(D, \mathbf{K})$. Hence, with a basis $(\mathbf{K}_b)$ of $\mathcal{K}$, we can associate (at least locally) m D-invariant functions (V_b) such that

$$\mathbf{K}_b\, dV = dV_b. \tag{3.13}$$

These *associated potentials* have a form similar to (3.12),

$$V_b = K_b^{aa}\phi_a = \varphi^a_{(b)}\phi_a. \tag{3.14}$$

The n functions

$$\begin{gathered} H_b = G_b + V_b = \tfrac{1}{2}\varphi^a_{(b)}(p_a^2 + \phi_a^{\alpha\beta}p_\alpha p_\beta + 2\phi_a) = \tfrac{1}{2}\lambda^a_b g^{aa}(p_a^2 + \phi_a^{\alpha\beta}p_\alpha p_\beta + 2\phi_a), \\ H_\alpha = P(\mathbf{X}_\alpha) = p_\alpha \end{gathered} \tag{3.15}$$

are independent first integrals in involution.

(L) It is useful to remark that, from an intrinsic point of view, a Stäckel multiplier is always the sum of scalar products of gradients of functions constant on the leaves of the web.

IV. THE EXTENDED METRIC

Let Q be a differentiable manifold with local coordinates (q^i) and let $\hat{Q} = Q \times \mathbb{R}$ be the *extended manifold* with local coordinates $(q^A) = (q^i, q^0)$ (q^0 is the natural coordinate over the real line). Let us consider on Q a positive–definite contravariant metric tensor $\mathbf{G} = (g^{ij})$, a vector field $\mathbf{A} = (A^i)$, and a function U. The triple

$$\hat{\mathbf{G}} = (\mathbf{G}, \mathbf{A}, U)$$

generates a contravariant symmetric 2-tensor $\hat{\mathbf{G}}$ on $\hat{Q}$ by setting

$$\hat{G}^{AB} = \begin{bmatrix} \hat{G}^{ij} & \hat{G}^{i0} \\ \hat{G}^{0j} & \hat{G}^{00} \end{bmatrix} = \begin{bmatrix} g^{ij} & A^i \\ A^j & 2U \end{bmatrix}. \tag{4.1}$$

In matrix notation,

$$\hat{\mathbf{G}} = \begin{bmatrix} \mathbf{G} & \mathbf{A} \\ \mathbf{A}^\top & 2U \end{bmatrix}. \tag{4.2}$$

If $\det \hat{\mathbf{G}} > 0$, then $\hat{\mathbf{G}}$ is a positive–definite metric tensor, which we call *extended metric tensor*. Since $\det \mathbf{G} > 0$ and the determinant of the matrix (4.2) is a sum containing the term $2U \cdot \det \mathbf{G}$, the regularity condition $\det \hat{\mathbf{G}} > 0$ can be locally satisfied by adding to the function U a suitable positive constant. Because of the physical meaning of the function U (1.2) any additional constant is inessential. If the function U has a lower bound, then this process of regularization is global. However, the local definition of the extended metric when U has no lower bound is not an obstruction to our purposes, since we shall use it as a local device.

Remark 4.1: In order to get a globally regular metric we could extend the manifold Q by two real axes, $\hat{Q}=Q\times\mathbb{R}\times\mathbb{R}$, and consider the contravariant metric

$$\hat{\mathbf{G}}=\begin{bmatrix} \mathbf{G} & \mathbf{A} & 0 \\ \mathbf{A}^{\top} & 2U & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4.3}$$

for which $\det\hat{\mathbf{G}}=-\det\mathbf{G}$. However, this metric is Lorentzian. Both the extensions (4.2) and (4.3) are contravariant. The metric (4.2) is a sort of Kaluza–Klein metric. Metrics similar to (4.2) and (4.3), with $\mathbf{A}=0$, have been considered by Eisenhart[16] in his interpretation of the dynamical trajectories of a holonomic system, with time-dependent constraints and potentials, as geodesics on a Riemannian manifold.

Remark 4.2: Any real function f on Q has a natural extension to $\hat{Q}=Q\times\mathbb{R}$ (constant along the fiber $\mathbb{R}$). For the sake of simplicity we denote this extension by the same symbol f. From the definition (4.2) it follows that the extended metric is characterized by the following equations, where $(f,\ g)$ are arbitrary functions on Q:

$$\begin{aligned} &\hat{\mathbf{G}}(df,dg)=\mathbf{G}(df,dg),\\ &\hat{\mathbf{G}}(df,dq^0)=\langle\mathbf{A},df\rangle,\\ &\hat{\mathbf{G}}(dq^0,dq^0)=2U. \end{aligned} \tag{4.4}$$

Remark 4.3: The extended geodesic Hamiltonian is

$$\hat{G}=\tfrac{1}{2}P(\hat{\mathbf{G}})=\tfrac{1}{2}\hat{G}^{AB}p_Ap_B=\tfrac{1}{2}g^{ij}p_ip_j+A^ip_ip_0+Up_0^2 \tag{4.5}$$

(with indices $A=0,1,...,n;\ i,\ j=1,...,n$). Since q^0 is ignorable, the corresponding momentum p_0 is a first integral. As a consequence, the integral curves with $p_0=1$ of the Hamilton equations of $\hat{G}$ reduce to the integral curves of the Hamilton equations of H (1.1). In other words, the geodesic flow of the extended metric is projectable onto the Hamiltonian flow of H.

Remark 4.4: Let $W(\underline{q},\underline{c})$ be a complete solution of the Hamilton–Jacobi equation (1.5). Then the function

$$\hat{W}(\underline{q},q^0,\underline{c},c_0)=c_0(W(\underline{q},\underline{c})+q^0) \tag{4.6}$$

is a complete solution of the Hamilton–Jacobi equation associated with $\hat{G}$:

$$\tfrac{1}{2}g^{ij}\partial_i\hat{W}\partial_j\hat{W}+A^i\partial_i\hat{W}\partial_0\hat{W}+U(\partial_0\hat{W})^2=k.$$

Indeed, this equation reduces to

$$c_0^2(\tfrac{1}{2}g^{ij}\partial_iW\partial_jW+A^i\partial_iW+U)=k,$$

i.e., to Eq. (1.5) with $h=k/c_0^2$. Furthermore,

$$\begin{bmatrix} \dfrac{\partial^2 \hat{W}}{\partial q^i \partial c_j} & \dfrac{\partial^2 \hat{W}}{\partial q^i \partial c_0} \\ \dfrac{\partial^2 \hat{W}}{\partial q^0 \partial c_j} & \dfrac{\partial^2 \hat{W}}{\partial q^0 \partial c_0} \end{bmatrix} = \begin{bmatrix} c_0 \dfrac{\partial^2 W}{\partial q^i \partial c_j} & \dfrac{\partial W}{\partial q^i} \\ 0 & 1 \end{bmatrix}$$

and the completeness condition, i.e., the regularity of this matrix, is satisfied for $c_0 \neq 0$. By (4.6) we observe that if W is a separated complete solution of the form (1.6), then also $\hat{W}$ is separated. In other words, if (q^i) are separable coordinates for the Hamiltonian H, then (q^i, q^0) are also separable for the geodesic extended Hamiltonian $\hat{G}$. This shows that the separation of $\hat{G}$ is a necessary condition for the separation of H, and this is the reason why we shall analyze the separation in the extended space (Sec. V). However, as we shall see, the converse is not always true: the separation of $\hat{G}$ does not imply the separation of H, unless we consider a more general kind of separation, the *gauge separation* (see Definition 5.9 below).

Let us look at some properties of the fundamental objects defined on the extended manifold: vectors, 1-forms and 2-tensors. A vector field on $\hat{Q}$ is represented by a pair

$$\hat{\mathbf{X}} = (\mathbf{X}, \xi), \tag{4.7}$$

where $\mathbf{X}$ is a q^0-dependent vector field on Q and ξ a function on $\hat{Q}$. Its components are

$$(\hat{X}^A) = (X^i, \xi), \quad \hat{X}^i = X^i, \quad \hat{X}^0 = \xi,$$

so that, as a derivation,

$$\hat{\mathbf{X}} = X^i \frac{\partial}{\partial q^i} + \xi \frac{\partial}{\partial q^0} = X^i \partial_i + \xi \partial_0 .$$

A vector field $\hat{\mathbf{X}}$ is *horizontal* if $\xi = 0$, *vertical* if $\mathbf{X} = 0$. If we introduce the *fundamental vertical* vector field

$$\hat{\mathbf{X}}_0 = (0,1) = \partial_0 \tag{4.8}$$

then the expression (4.7) can be replaced with

$$\hat{\mathbf{X}} = \mathbf{X} + \xi \hat{\mathbf{X}}_0 . \tag{4.9}$$

We say that a vector field $\hat{\mathbf{X}}$ on $\hat{Q}$ is *vertically invariant* if $[\hat{\mathbf{X}}, \hat{\mathbf{X}}_0] = 0$. A vector field is vertically invariant iff both components $(\mathbf{X}, \xi)$ are q^0-independent. In this case, $\mathbf{X}$ is a vector field on Q and ξ is a function on Q. We call $\mathbf{X}$ the *basic component* of $\hat{\mathbf{X}}$ and ξ the *vertical component*.

Proposition 4.5: Two vertically invariant vector fields $\hat{\mathbf{X}} = (\mathbf{X}, \xi)$ and $\hat{\mathbf{Y}} = (\mathbf{Y}, \eta)$ commute, $[\hat{\mathbf{X}}, \hat{\mathbf{Y}}] = 0$, iff

$$\begin{gathered} \langle \mathbf{X}, d\eta \rangle = \langle \mathbf{Y}, d\xi \rangle, \\ [\mathbf{X}, \mathbf{Y}] = 0. \end{gathered} \tag{4.10}$$

Proof: Since all components do not depend on q^0, we have

$$[\hat{\mathbf{X}}, \hat{\mathbf{Y}}]^i = \hat{X}^A \partial_A Y^i - \hat{Y}^A \partial_A X^i = \hat{X}^j \partial_j Y^i - \hat{Y}^j \partial_j X^i = [\mathbf{X}, \mathbf{Y}]^i .$$

$$[\hat{\mathbf{X}}, \hat{\mathbf{Y}}]^0 = \hat{X}^A \partial_A \hat{Y}^0 - \hat{Y}^A \partial_A \hat{X}^0 = X^j \partial_j \hat{Y}^0 - Y^j \partial_j \hat{X}^0 = \langle \mathbf{X}, d\eta \rangle - \langle \mathbf{Y}, d\xi \rangle .$$

■

Proposition 4.6: A vertically invariant vector field $\hat{\mathbf{X}}=(\mathbf{X},\xi)$ is a Killing vector iff

$$\langle \mathbf{A}, d\xi\rangle = \langle \mathbf{X}, dU\rangle,$$
$$[\mathbf{X},\mathbf{A}] = \nabla\xi, \tag{4.11}$$
$$[\mathbf{X},\mathbf{G}] = 0.$$

Proof: The Killing equation $[\hat{\mathbf{X}},\hat{\mathbf{G}}]=0$ is equivalent to

$$\{\xi p_0 + X^i p_i, \tfrac{1}{2} g^{ij} p_i p_j + A^i p_i p_0 + U p_0^2\} = 0,$$

that is to

$$X^i(\tfrac{1}{2}\partial_i g^{hk} p_h p_k + \partial_i A^h p_h p_0 + \partial_i U p_0^2) - (\partial_i \xi p_0 + \partial_i X^h p_h)(g^{ik} p_k + A^i p_0) = 0.$$

The coefficients of p_0^2, $p_0 p_k$, and $p_h p_k$ generate equations

$$X^i \partial_i U - A^i \partial_i \xi = 0,$$
$$X^i \partial_i A^k - g^{ik}\partial_i \xi - A^i \partial_i X^k = 0,$$
$$(\tfrac{1}{2} X^i \partial_i g^{hk} - g^{ik}\partial_i X^h) p_h p_k = 0,$$

which are the coordinate representations of Eqs. (4.11).

■

The last equation (4.11) means that the basic component $\mathbf{X}$ of $\hat{\mathbf{X}}$ is a Killing vector. We notice that the fundamental vertical vector $\hat{\mathbf{X}}_0$ is a Killing vector. As for the contravariant metric, a contravariant symmetric 2-tensor on $\hat{Q}$ is represented by a triple

$$\hat{\mathbf{K}} = (\mathbf{K}, \mathbf{C}, F), \tag{4.12}$$

where $\mathbf{K}=(K^{ij})$ is a contravariant symmetric 2-tensor, $\mathbf{C}=(C^i)$ is a vector field, and F is a function on Q (all these objects may be q^0-dependent). In components,

$$\hat{K}^{AB} = \begin{bmatrix} K^{ij} & K^{i0} \\ K^{0j} & K^{00} \end{bmatrix} = \begin{bmatrix} K^{ij} & C^i \\ C^j & 2F \end{bmatrix}. \tag{4.13}$$

In matrix notation,

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{C}^{\top} & 2F \end{bmatrix}. \tag{4.14}$$

With this tensor we associate the Hamiltonian

$$\tfrac{1}{2} P(\hat{\mathbf{K}}) = \tfrac{1}{2}\hat{K}^{AB} p_A p_B = \tfrac{1}{2} K^{ij} p_i p_j + C^i p_i p_0 + F p_0^2. \tag{4.15}$$

This tensor is vertically invariant, $[\hat{\mathbf{X}}_0, \hat{\mathbf{K}}]=0$, iff all components are q^0-independent. In this case $\mathbf{K}$, $\mathbf{C}$, and F are objects on Q.

Proposition 4.7: A vertically invariant 2-tensor $\hat{\mathbf{K}}=(\mathbf{K},\mathbf{C},F)$ is a Killing tensor iff

$$\begin{aligned}&[\mathbf{G},\mathbf{K}]=0,\\&[\mathbf{C},\mathbf{G}]=[\mathbf{A},\mathbf{K}],\\&[\mathbf{C},\mathbf{A}]=\nabla F-\mathbf{K}\nabla U,\\&\langle \mathbf{C},dU\rangle=\langle \mathbf{A},dF\rangle.\end{aligned}\tag{4.16}$$

Proof: The Killing equation $[\hat{\mathbf{K}},\hat{\mathbf{G}}]=0$ is equivalent to $\{P_{\hat{\mathbf{G}}},P_{\hat{\mathbf{K}}}\}=0$,

$$\begin{aligned}&(g^{il}p_l+A^ip_0)(\tfrac{1}{2}\partial_iK^{hk}p_hp_k+\partial_iC^kp_kp_0+\partial_iFp_0^2)\\&\quad-(K^{il}p_l+C^ip_0)(\tfrac{1}{2}\partial_ig^{hk}p_hp_k+\partial_iA^kp_kp_0+\partial_iUp_0^2)=0.\end{aligned}$$

The first equation (4.16) is determined by the coefficient of $(p_hp_kp_l)$. The coefficients of $p_0p_hp_k$, $p_kp_0^2$, and p_0^3 give rise, respectively, to equations

$$\begin{aligned}&(g^{ih}\partial_iC^k+\tfrac{1}{2}A^i\partial_iK^{hk}-K^{ih}\partial_iA^k-\tfrac{1}{2}C^i\partial_ig^{hk})p_hp_k=0,\\&g^{ik}\partial_iF+A^i\partial_iC^k-K^{ik}\partial_iU-C^i\partial_iA^k=0,\\&A^i\partial_iF-C^i\partial_iU=0,\end{aligned}$$

which are the coordinate representations of the last three equations (4.16). ■

We notice that the first equation (4.16) means that the basic component $\mathbf{K}$ is a Killing tensor.

Remark 4.8: As for any Riemannian manifold, the bijective mapping $\flat$ from 1-forms to vector fields on $\hat{Q}$ is defined by, see (2.6),

$$\langle \hat{\mathbf{X}},df\rangle=\hat{\mathbf{G}}(df,\hat{\mathbf{X}}^\flat),\tag{4.17}$$

where f is a function on $\hat{Q}$. Since

$$df=\frac{\partial f}{\partial q^i}dq^i+\frac{\partial f}{\partial q^0}dq^0,$$

it follows that

$$\hat{\mathbf{G}}(df,dq^0)=\frac{\partial f}{\partial q^i}\hat{G}^{i0}+\frac{\partial f}{\partial q^0}\hat{G}^{00}=A^i\frac{\partial f}{\partial q^i}+2U\frac{\partial f}{\partial q^0}.$$

This shows that

$$dq^0=(\mathbf{A},2U)^\flat.\tag{4.18}$$

Remark 4.9: A 1-form $\hat{\varphi}$ on $\hat{Q}$ is represented by a pair (φ,φ_0), where φ is a q^0-dependent 1-form on Q and φ_0 is a function on $\hat{Q}$. In local coordinates (q^i,q^0) we have $\hat{\varphi}=\hat{\varphi}_A\,dq^A=\varphi_i\,dq^i+\varphi_0\,dq^0$, where $\varphi=\varphi_i\,dq^i$. For any vector field $\hat{\mathbf{X}}=(\mathbf{X},\xi)$,

$$\langle \hat{\mathbf{X}},\hat{\varphi}\rangle=\langle \mathbf{X},\varphi\rangle+\xi\varphi_0.\tag{4.19}$$

We say that $\hat{\varphi}$ is a *basic 1-form* if

$$\varphi_0=\langle \hat{\mathbf{X}}_0,\hat{\varphi}\rangle=0.\tag{4.20}$$

The contravariant components of the image of a 1-form $\hat{\varphi}$ by a symmetric 2-tensor $\hat{\mathbf{K}}$ are

$$\begin{aligned}\hat{K}^{AB}\hat{\varphi}_B &= (\hat{K}^{ij}\varphi_j + \hat{K}^{i0}\varphi_0, \hat{K}^{0i}\varphi_i + \hat{K}^{00}\varphi_0)\\ &= (\hat{K}^{ij}\varphi_j + C^i\varphi_0, C^i\varphi_i + 2F\varphi_0).\end{aligned} \tag{4.21}$$

This shows that the eigenform equation $\hat{\mathbf{K}}\hat{\varphi} = \lambda\hat{\varphi}$ is equivalent to equations

$$\begin{aligned}\mathbf{K}\varphi + \varphi_0\mathbf{C}^\flat &= \lambda(\varphi + \varphi_0\mathbf{A}^\flat),\\ \langle\mathbf{C},\varphi\rangle + 2F\varphi_0 &= \lambda(\langle\mathbf{A},\varphi\rangle + 2U\varphi_0).\end{aligned} \tag{4.22}$$

For a basic eigenform these equations become

$$\begin{aligned}\mathbf{K}\varphi &= \lambda\varphi,\\ \langle\mathbf{C},\varphi\rangle &= \lambda\langle\mathbf{A},\varphi\rangle.\end{aligned} \tag{4.23}$$

V. SEPARABLE KILLING WEBS IN THE EXTENDED METRIC

Assume that the extended geodesic Hamiltonian $\hat{G}$ is separable. According to the general theory of the geodesic separability, this fact is characterized by the existence of a separable Killing web,

$$(\hat{\mathcal{S}}^a, \hat{D}, \hat{\mathbf{K}}), \tag{5.1}$$

where (I) $\hat{\mathcal{S}}^a$ is a set of m orthogonal foliations of submanifolds of codimension 1 ($a=1,...,m$); (II) $\hat{D}$ is a $r+1$-dimensional linear space of commuting Killing vectors ($m+r=n$). These Killing vectors are tangent to the orbits of $\hat{D}$, and these orbits coincide with the complete intersections of the leaves of the foliations $\hat{\mathcal{S}}^a$; (III) $\hat{\mathbf{K}}$ is a Killing tensor of order 2. (III.a) $\hat{\mathbf{K}}$ is $\hat{D}$-invariant (it commutes with all elements of $\hat{D}$); (III.b) $\hat{\mathbf{K}}$ has m main eigenvectors orthogonal to the leaves of $\hat{\mathcal{S}}^a$, corresponding to distinct eigenvalues. It follows that locally on $\hat{Q}$ there are m independent functions $(\hat{q}^a)$ such that $(d\hat{q}^a)$ are characteristic 1-forms of the web, so that

$$\hat{\mathbf{K}}\,dq^a = \lambda^a\,dq^a, \quad \langle\hat{\mathbf{X}}, d\hat{q}^a\rangle = 0, \quad \forall\hat{\mathbf{X}} \in \hat{D}, \tag{5.2}$$

and

$$\hat{\mathbf{G}}(d\hat{q}^a, d\hat{q}^b) = 0, \quad a \neq b. \tag{5.3}$$

As it will be justified below, it is interesting to consider the particular case in which the fundamental vertical vector field $\hat{\mathbf{X}}_0$ is an element of $\hat{D}$.

Proposition 5.1: A separable Killing web $(\hat{\mathcal{S}}^a, \hat{D}, \hat{\mathbf{K}})$ on the extended manifold $\hat{Q}$, such that $\hat{\mathbf{X}}_0 \in \hat{D}$, is reducible to a separable Killing web on Q,

$$(\mathcal{S}^a, D, \mathbf{K}). \tag{5.4}$$

The meaning of the term ''reducible'' is explained in the following proof.

Proof: Since the second equation (5.2) implies in particular

$$\langle\hat{\mathbf{X}}_0, d\hat{q}^a\rangle = 0, \tag{5.5}$$

the functions $(\hat{q}^a)$ are vertically invariant and reduce to functions (q^a) on Q, so that, according to Remark 4.2, we can use the simplified notation $\hat{q}^a = q^a$. As a consequence, the web $(\hat{\mathcal{S}}^a)$ reduces to a web $(\mathcal{S}^a)$ with characteristic 1-forms (dq^a). Because of (4.4) and (5.3),

$$\mathbf{G}(dq^a, dq^b) = \hat{\mathbf{G}}(dq^a, dq^b) = 0 \quad (a \neq b) \tag{5.6}$$

and the reduced web is orthogonal. According to Propositions 4.5 and 4.6, the Killing vectors $\hat{\mathbf{X}} = (\mathbf{X}, \xi) \in \hat{D}$ reduce to commuting Killing vectors $\mathbf{X}$ on Q and form a space D of dimension $r = n - m$ (one dimension is lost by the vertical vector $\hat{\mathbf{X}}_0 \in \hat{D}$, which projects onto the zero vector field of Q). Since (dq^a) are basic 1-forms, from (4.19) it follows that

$$\langle \mathbf{X}, dq^a \rangle = \langle \hat{\mathbf{X}}, dq^a \rangle = 0. \tag{5.7}$$

Thus, the reduced Killing vectors are tangent to the leaves of the reduced web. The Killing tensor $\hat{\mathbf{K}}$ reduces to a Killing tensor $\mathbf{K}$ on Q (Proposition 4.7). The reduced Killing tensor commutes with all the reduced Killing vectors of D; the proof that $[\hat{\mathbf{X}}, \hat{\mathbf{K}}] = 0$ implies $[\mathbf{X}, \mathbf{K}] = 0$ is similar to that in the proof of Proposition 4.6. Finally, because of (4.23), the eigenform equation (5.2) reduces to equation

$$\mathbf{K}\, dq^a = \lambda^a\, dq^a,$$

and this shows that the reduced characteristic 1-forms (dq^a) are eigenforms of $\mathbf{K}$ corresponding to the distinct eigenvalues (λ^a). Since these eigenvalues are vertically invariant, they reduce to functions on Q.

■

Remark 5.2: If we choose a local basis $(\hat{\mathbf{X}}_\alpha, \hat{\mathbf{X}}_0)$ of $\hat{D}$ including the fundamental vertical vector field and a local section $\hat{\mathcal{Z}}$ orthogonal to the orbits of $\hat{D}$, then normal separable coordinates $(\hat{q}^A) = (\hat{q}^a, \hat{q}^\alpha, \hat{q}^0)$ are defined on $\hat{Q}$ such that $\hat{q}^a = q^a$,

$$\frac{\partial}{\partial \hat{q}^\alpha} = \hat{\mathbf{X}}_\alpha, \quad \frac{\partial}{\partial \hat{q}^0} = \hat{\mathbf{X}}_0, \tag{5.8}$$

and

$$\langle \hat{\mathbf{X}}_0, d\hat{q}^0 \rangle = 1, \quad \langle \hat{\mathbf{X}}_0, d\hat{q}^\alpha \rangle = 0, \quad \langle \hat{\mathbf{X}}_\alpha, d\hat{q}^0 \rangle = 0, \quad \langle \hat{\mathbf{X}}_\alpha, d\hat{q}^\beta \rangle = \delta_\alpha^\beta. \tag{5.9}$$

According to the general theory of the geodesic separation, the m separable coordinates (q^a) are essential, the $r+1$ coordinates $(\hat{q}^\alpha, \hat{q}^0)$ are ignorable, and the contravariant components of the extended metric

$$\hat{G}^{AB} = \hat{\mathbf{G}}(d\hat{q}^A, d\hat{q}^B), \tag{5.10}$$

have a form similar to (3.4) and (3.5), with one additional line and row with index 0 (index of first-class),

$$\begin{gathered} \hat{G}^{ab} = 0 \quad (a \neq b), \quad \hat{G}^{a\alpha} = 0, \quad \hat{G}^{a0} = 0, \\ \hat{G}^{aa} = \varphi^a_{(m)}, \quad \hat{G}^{00} = \phi_a \hat{G}^{aa}, \quad \hat{G}^{\alpha 0} = \phi_a^\alpha \hat{G}^{aa}, \quad \hat{G}^{\alpha\beta} = \phi_a^{\alpha\beta} \hat{G}^{aa}, \end{gathered} \tag{5.11}$$

where ϕ_a, ϕ_a^α, $\phi_a^{\alpha\beta}$ are functions of the coordinate corresponding to the lower index only. Furthermore, since all the elements of $\hat{D}$ commute, we have in particular $[\hat{\mathbf{X}}_0, \hat{\mathbf{X}}_\alpha] = 0$, and, due to (5.9), also the coordinates $(\hat{q}^\alpha)$ reduce to coordinates (q^α) on Q, so that we can use the simpler notation q^α instead of $\hat{q}^\alpha$. It follows that (q^a, q^α) is a normal separable coordinate system associated with the reduced separable Killing web (5.4). However, as we shall see below, these coordinates are not separable with respect to the complete Hamiltonian H (1.1). For the separability of H further conditions are required. From the first characteristic equation of the extended metric (4.4) it follows that

$$\begin{aligned} \hat{G}^{ab} &= \hat{\mathbf{G}}(dq^a, dq^b) = \mathbf{G}(dq^a, dq^b) = g^{ab}, \\ \hat{G}^{a\alpha} &= \hat{\mathbf{G}}(dq^a, dq^\alpha) = \mathbf{G}(dq^a, dq^\alpha) = g^{a\alpha}, \\ \hat{G}^{\alpha\beta} &= \hat{\mathbf{G}}(dq^\alpha, dq^\beta) = \mathbf{G}(dq^\alpha, dq^\beta) = g^{\alpha\beta}. \end{aligned} \tag{5.12}$$

Hence, the comparison with (5.11) shows that the metric components (g^{ij}) maintain the same expressions (3.4) and (3.5),

$$\begin{aligned} & g^{ab} = 0 \quad (a \neq b), \quad g^{aa} = \varphi^a_{(m)}, \\ & g^{a\alpha} = 0, \quad g^{\alpha\beta} = g^{aa} \phi_a^{\alpha\beta} = \varphi^a_{(m)} \phi_a^{\alpha\beta}. \end{aligned} \tag{5.13}$$

Remark 5.3: The natural coordinate q^0 of $\hat{Q}$ does not coincide with the separable coordinate $\hat{q}^0$ determined by $\hat{\mathbf{X}}_0$ in the basis of $\hat{D}$. As for any function of $\hat{Q}$, we can consider the differential of q^0 in the coordinates $(\hat{q}^A) = (q^a, q^\alpha, \hat{q}^0)$, written in the form

$$dq^0 = f\, d\hat{q}^0 + f_\alpha\, dq^\alpha + \xi_a\, dq^a.$$

Since we have $\langle \hat{\mathbf{X}}_0, dq^0 \rangle = 1$ because of the definition of $\hat{\mathbf{X}}_0$, from (5.9) and (5.5) (where $\hat{q}^\alpha = q^\alpha$, $\hat{q}^a = q^a$) it follows that $f = 1$. Moreover, by applying to both sides of this equation the Killing vector $\hat{\mathbf{X}}_\alpha = (\mathbf{X}_\alpha, \xi_\alpha)$, due again to (5.9) and to (5.2) we get

$$\langle \hat{\mathbf{X}}_\alpha, d\hat{q}^0 + f_\alpha\, dq^\alpha + \xi_a\, dq^a \rangle = f_\alpha, \quad \langle \hat{\mathbf{X}}_\alpha, dq^0 \rangle = \langle \mathbf{X}_\alpha + \xi_\alpha \hat{\mathbf{X}}_0, dq^0 \rangle = \xi_\alpha,$$

so that $f_\alpha = \xi_\alpha$. Hence,

$$dq^0 = d\hat{q}^0 + \xi_\alpha\, dq^\alpha + \xi_a\, dq^a, \tag{5.14}$$

where (ξ_α) are just the vertical components of the Killing vectors $(\hat{\mathbf{X}}_\alpha)$. Since the Killing vectors commute with $\hat{\mathbf{X}}_0$, these components reduce to functions on Q. By developing the commutation relations

$$\begin{aligned} 0 &= \left[\frac{\partial}{\partial \hat{q}^a}, \frac{\partial}{\partial \hat{q}^\alpha} \right] = \left[\frac{\partial}{\partial \hat{q}^a}, \hat{\mathbf{X}}_\alpha \right] \\ &= \left[\frac{\partial}{\partial \hat{q}^a}, \mathbf{X}_\alpha + \xi_\alpha \hat{\mathbf{X}}_0 \right] = \left[\frac{\partial}{\partial q^a}, \mathbf{X}_\alpha \right] + \frac{\partial \xi_\alpha}{\partial q^a} \hat{\mathbf{X}}_0 \\ &= \partial_a \xi_\alpha \hat{\mathbf{X}}_0, \end{aligned}$$

we find that

$$\partial_a \xi_\alpha = 0. \tag{5.15}$$

By differentiating Eq. (5.14), we find equation

$$d\xi_\alpha \wedge dq^\alpha + d\xi_a \wedge dq^a = 0.$$

Due to (5.15) and the q^0-independence of ξ_α, we obtain

$$\partial_\beta \xi_\alpha\, dq^\beta \wedge dq^\alpha + \partial_\alpha \xi_a\, dq^\alpha \wedge dq^a + \partial_b \xi_a\, dq^b \wedge dq^a + \partial_0 \xi_a\, dq^0 \wedge dq^a = 0.$$

It follows that

$$\partial_\alpha \xi_\beta = \partial_\beta \xi_\alpha, \quad \partial_\alpha \xi_a = 0, \quad \partial_a \xi_b = \partial_b \xi_a, \quad \partial_0 \xi_a = 0. \tag{5.16}$$

The last equation (5.16) shows that also the functions (ξ_a) appearing in (5.14) are q^0-independent. The remaining equations show that on Q there exist local functions $S_1(q^a)$ and $S_2(q^\alpha)$ depending on the essential coordinates (q^a) and on the ignorable coordinates (q^α), respectively, such that

$$\xi_a = \partial_a S_1, \quad \xi_\alpha = \partial_\alpha S_2. \tag{5.17}$$

Thus, the link (5.14) between q^0 and $\hat{q}^0$ takes the form

$$d\hat{q}^0 = dq^0 - d(S_1 + S_2), \quad S_1 = S_1(q^a), \quad S_2 = S_2(q^\alpha). \tag{5.18}$$

Remark 5.4: From Eqs. (4.4) it follows that

$$\begin{aligned} \hat{\mathbf{G}}(dq^a, dq^0) &= \langle \mathbf{A}, dq^a \rangle = A^a, \\ \hat{\mathbf{G}}(dq^\alpha, dq^0) &= \langle \mathbf{A}, dq^\alpha \rangle = A^\alpha, \\ \hat{\mathbf{G}}(dq^0, dq^0) &= 2U. \end{aligned} \tag{5.19}$$

On the other hand, from (5.14), using (5.11) and (5.12), and recalling that $\hat{q}^a = q^a$, $\hat{q}^\alpha = q^\alpha$, we derive

$$\begin{aligned} \hat{\mathbf{G}}(dq^a, dq^0) &= \hat{\mathbf{G}}(dq^a, d\hat{q}^0) + \hat{\mathbf{G}}(dq^a, dq^\alpha)\xi_\alpha + \hat{\mathbf{G}}(dq^a, dq^b)\xi_b \\ &= \hat{G}^{a0} + \hat{G}^{a\alpha}\xi_\alpha + \hat{G}^{ab}\xi_b = \hat{G}^{aa}\xi_a = g^{aa}\xi_a, \\ \hat{\mathbf{G}}(dq^\alpha, dq^0) &= \hat{\mathbf{G}}(dq^\alpha, d\hat{q}^0) + \hat{\mathbf{G}}(dq^\alpha, dq^\beta)\xi_\beta + \hat{\mathbf{G}}(dq^\alpha, dq^b)\xi_b \\ &= \hat{G}^{\alpha 0} + \hat{G}^{\alpha\beta}\xi_\beta + \hat{G}^{\alpha b}\xi_b = \hat{G}^{aa}\phi_a^\alpha + \hat{G}^{\alpha\beta}\xi_\beta = g^{aa}\phi_a^\alpha + g^{\alpha\beta}\xi_\beta, \\ \hat{\mathbf{G}}(dq^0, dq^0) &= \hat{\mathbf{G}}(d\hat{q}^0, d\hat{q}^0) + \hat{\mathbf{G}}(dq^\alpha, dq^\beta)\xi_\alpha\xi_\beta + \hat{\mathbf{G}}(dq^a, dq^b)\xi_a\xi_b \\ &\quad + 2\hat{G}(d\hat{q}^0, dq^\alpha)\xi_\alpha + 2\hat{\mathbf{G}}(d\hat{q}^0, dq^a)\xi_a + 2\hat{\mathbf{G}}(dq^\alpha, dq^a)\xi_\alpha\xi_a \\ &= \hat{G}^{00} + \hat{G}^{\alpha\beta}\xi_\alpha\xi_\beta + \hat{G}^{ab}\xi_a\xi_b + 2\hat{G}^{\alpha 0}\xi_\alpha = g^{aa}(\phi_a + \phi_a^{\alpha\beta}\xi_\alpha\xi_\beta + \xi_a^2 + 2\phi_a^\alpha\xi_\alpha). \end{aligned} \tag{5.20}$$

The comparison of Eqs. (5.19) and (5.20) shows that

$$\begin{aligned} A^a &= g^{aa}\xi_a, \\ A^\alpha &= g^{aa}\phi_a^\alpha + g^{\alpha\beta}\xi_\beta = g^{aa}(\phi_a^\alpha + \phi_a^{\alpha\beta}\xi_\beta), \\ 2U &= g^{aa}(\phi_a + \phi_a^{\alpha\beta}\xi_\alpha\xi_\beta + \xi_a^2 + 2\phi_a^\alpha\xi_\alpha). \end{aligned} \tag{5.21}$$

We can summarize the preceding remarks in the following.

Proposition 5.5: If the extended metric admits a separable Killing web $(\hat{\mathcal{S}}^a, \hat{D}, \hat{\mathbf{K}})$ with $\hat{\mathbf{X}}_0 \in \hat{D}$, then on Q there exists a coordinate system (q^a, q^α) such that the components of $\mathbf{G}$ and $\mathbf{A}$ and the function U assume the form (5.13), (5.21), with $(\phi_a^\alpha, \phi_a^{\alpha\beta}, \phi_a)$ functions of the coordinate corresponding to the lower index only, and $\xi_i = \partial_i(S_1 + S_2)$, with $S_1(q^a)$ and $S_2(q^\alpha)$ functions of the essential coordinates (q^a) and of the ignorable coordinates (q^α), respectively.

Remark 5.6: From Eqs. (5.21) we observe that the vector field $\mathbf{A}$ is a sum of three vectors:

$$\mathbf{A}=\mathbf{A}_{(1)}+\mathbf{A}_{(2)}+\mathbf{A}_{(3)},$$

$$\mathbf{A}_{(1)}=g^{aa}\xi_a\mathbf{X}_a=\nabla S_1, \qquad \mathbf{A}_{(2)}=g^{\alpha\beta}\xi_\beta\mathbf{X}_\alpha=\nabla S_2=g^{aa}\phi_a^{\alpha\beta}\xi_\beta\mathbf{X}_\alpha, \tag{5.22}$$

$$\mathbf{A}_{(3)}=g^{aa}\phi_a^{\alpha}\mathbf{X}_\alpha,$$

where

$$\mathbf{X}_a=\partial_a=\frac{\partial}{\partial q^a}, \quad \mathbf{X}_\alpha=\partial_\alpha=\frac{\partial}{\partial q^\alpha}. \tag{5.23}$$

Since $\mathbf{X}_a\cdot\mathbf{X}_\alpha=0$, both vectors $\mathbf{A}_{(2)}$ and $\mathbf{A}_{(3)}$ are orthogonal to $\mathbf{A}_{(1)}$:

$$\mathbf{A}_{(1)}\cdot\mathbf{A}_{(2)}=0, \quad \mathbf{A}_{(1)}\cdot\mathbf{A}_{(3)}=0. \tag{5.24}$$

From the last equation (5.21) we get the following decomposition for the function U:

$$U=\tfrac{1}{2}\mathbf{A}_{(1)}\cdot\mathbf{A}_{(1)}+\tfrac{1}{2}\mathbf{A}_{(2)}\cdot\mathbf{A}_{(2)}+\mathbf{A}_{(2)}\cdot\mathbf{A}_{(3)}+V^0, \tag{5.25}$$

where

$$V^0=g^{aa}\phi_a \tag{5.26}$$

is a Stäckel multiplier. From (1.2), (5.22), (5.24), and (5.25) we derive the following expression for the (physical) scalar potential:

$$V=U-\tfrac{1}{2}\mathbf{A}\cdot\mathbf{A}=V^0-\tfrac{1}{2}\mathbf{A}_{(3)}\cdot\mathbf{A}_{(3)}. \tag{5.27}$$

Remark 5.7: Let us consider the reduced separable Killing web $(\mathcal{S}^a,D,\mathbf{K})$ of Proposition 5.1. Each foliation $\mathcal{S}^a$ is locally represented by equation $q^a=\text{const}$, the vectors $\mathbf{X}_\alpha=\partial_\alpha$ form a local basis of D, and the vectors $\mathbf{X}_a=\partial_a$ are eigenvectors of $\mathbf{K}$ orthogonal to D. Then the function S_1 is constant on the orbits of D, since it depends on the coordinates (q^a) only, while the function S_2 is constant on the submanifolds orthogonal to the orbits of D, since $\langle\mathbf{X}_a,dS_2\rangle=\partial_a S_2=0$. Hence, the vectors of the decomposition (5.22) are completely characterized by the following properties:

$$\begin{gathered}\mathbf{A}_{(1)}\ \text{is a gradient of the orbits of}\ D,\\ \mathbf{A}_{(2)}\ \text{is a gradient of the foliation orthogonal to the orbits of}\ D,\\ \mathbf{A}_{(3)}\ \text{is tangent to the orbits of}\ D\ \text{and its components in any}\\ \text{basis of}\ D\ \text{are Stäckel multipliers.}\end{gathered} \tag{5.28}$$

Here, by *gradient of a foliation* we mean a vector field which is the gradient of a function constant on the leaves of the foliation (i.e., the corresponding 1-form is the differential of a function constant on the leaves). It follows in particular that $\mathbf{A}_{(1)}$ and $\mathbf{A}_{(3)}$ are D-invariant.

Remark 5.8: As a consequence of the expressions (5.13) and (5.21), the Hamilton–Jacobi equation (1.5) can be written in the form

$$\tfrac{1}{2}\varphi_{(m)}^a(\bar{p}_a^2+\phi_a^{\alpha\beta}\bar{p}_\alpha\bar{p}_\beta+2\phi_a^\alpha\bar{p}_\alpha+\phi_a)=h, \tag{5.29}$$

by setting

$$\bar{p}_i = p_i + \xi_i = p_i + \partial_i (S_1 + S_2) \Leftrightarrow \begin{cases} \bar{p}_a = p_a + \xi_a = p_a + \partial_a S_1, \\ \bar{p}_\alpha = p_\alpha + \xi_\alpha = p_\alpha + \partial_\alpha S_2. \end{cases} \tag{5.30}$$

We can consider this equation as the last one of the following system of m equations:

$$\varphi^a_{(b)} (\bar{p}_a^2 + \phi_a^{\alpha\beta} \bar{p}_\alpha \bar{p}_\beta + 2\phi_a^\alpha \bar{p}_\alpha + \phi_a) = c_b, \tag{5.31}$$

where (c_b) are m arbitrary constants, and $c_m = 2h$. By applying the Stäckel matrix $[\varphi_a^{(b)}]$ we get the equivalent system

$$\bar{p}_a^2 + \phi_a^{\alpha\beta} \bar{p}_\alpha \bar{p}_\beta + 2\phi_a^\alpha \bar{p}_\alpha + \phi_a = \varphi_a^{(b)} c_b. \tag{5.32}$$

By setting $\bar{p}_\alpha = c_\alpha = \text{const}$, this system splits into m separated equations:

$$(p_a + \partial_a S_1)^2 = \Phi_a(q^a, \underline{c}), \quad p_\alpha = c_\alpha - \partial_\alpha S_2, \tag{5.33}$$

where

$$\Phi_a(q^a, \underline{c}) = \varphi_a^{(b)} c_b - \phi_a^{\alpha\beta} c_\alpha c_\beta - 2\phi_a^\alpha c_\alpha - \phi_a \tag{5.34}$$

are functions of the coordinate corresponding to the index only, and (in general) of all the n constants $\underline{c} = (c_b, c_\alpha)$. If we consider the integrals (with any choice of the signs)

$$W_a(q^a, \underline{c}) = \pm \int \sqrt{\Phi_a(q^a, \underline{c})}\, dq^a, \tag{5.35}$$

then we build a complete solution of the Hamilton–Jacobi equation of the form

$$W = c_\alpha q^\alpha + \sum_a W_a - S, \quad S = S_1 + S_2. \tag{5.36}$$

We observe that this is not a separated complete solution, due to the presence of the function S, which is not in general a sum of functions of single coordinates. However, this function does not contain the constants $\underline{c}$.

Hence, we are led to consider a more general kind of separation.

Definition 5.9: A Hamiltonian is *gauge-separable* if the corresponding Hamilton–Jacobi equation admits a complete solution of the form

$$W(\underline{q}, \underline{c}) = \sum_{i=1}^n W_i(q^i, \underline{c}) - S(\underline{q}). \tag{5.37}$$

The gauge-separation is also called *R-separation* in connection with the multiplicative separation of the Helmholtz equation.[17]

Thus, we have proved

Proposition 5.10: If the extended metric admits a separable Killing web with $\hat{\mathbf{X}}_0 \in \hat{D}$, then the Hamiltonian H (1.1) is gauge separable.

Remark 5.11: We have the ordinary separation of the Hamiltonian H if and only if (ξ_a) and (ξ_α) are functions of the coordinate corresponding to the index only, i.e.,

$$\partial_b \xi_a = 0 \quad (a \neq b), \quad \partial_\beta \xi_\alpha = 0 \quad (\alpha \neq \beta). \tag{5.38}$$

Since these functions are the covariant components of the vectors $\mathbf{A}_{(1)}$ and $\mathbf{A}_{(2)}$, it follows that the first equation (5.38) and the second equation (5.38) are, respectively, equivalent to the following two conditions:

$$\begin{aligned} &(1)\ \mathbf{A}_{(1)} \text{ is the sum of gradients of the foliations } S^a, \\ &(2)\ \text{there is a basis } (\mathbf{X}_\alpha) \text{ of } D \text{ such that } \langle \mathbf{X}_\alpha, d(\mathbf{A}_{(2)}\cdot\mathbf{X}_\beta)\rangle = 0 \text{ for } \alpha\neq\beta. \end{aligned} \tag{5.39}$$

Furthermore, going back to (5.14), we remark that conditions (5.38) are necessary and sufficient for the separability of the coordinate system (q^a,q^α,q^0), which in this case is equivalent to the coordinate system $(q^a,q^\alpha,\hat{q}^0)$ associated with the separable Killing web on the extended manifold.

VI. FINAL STATEMENTS AND REMARKS

From the discussion in the preceding section we can derive the following theorem on the intrinsic characterization of the separation of a natural Hamiltonian with scalar and vector potential.

Theorem 6.1: The Hamiltonian (1.1) is separable if and only if (i) on Q there exists a separable Killing web $(\mathcal{S}^a, D, \mathbf{K})$; (ii) the vector field $\mathbf{A}$ is a sum of three vectors,

$$\mathbf{A} = \mathbf{A}_{(1)} + \mathbf{A}_{(2)} + \mathbf{A}_{(3)},$$

where (ii.1) $\mathbf{A}_{(1)}$ is locally the sum of gradients of the foliations $\mathcal{S}^a$, (ii.2) $\mathbf{A}_{(2)}$ is locally a gradient of the foliation orthogonal to the orbits of D, and there exists a basis $(\mathbf{X}_\alpha)$ of D $(\alpha = m+1,\ldots,n)$ such that

$$\langle \mathbf{X}_\alpha, d(\mathbf{A}_{(2)}\cdot\mathbf{X}_\beta)\rangle = 0 \quad \text{for } \alpha\neq\beta;$$

(ii.3) $\mathbf{A}_{(3)}$ is tangent to the orbits of D and its components $A^\alpha_{(3)}$ with respect to any basis $(\mathbf{X}_\alpha)$ of D are Stäckel multipliers,

$$\langle \mathbf{X}, dA^\alpha_{(3)}\rangle = 0, \ \forall \mathbf{X}\in D, \quad d(\mathbf{K}\, dA^\alpha_{(3)}) = 0;$$

(iii) the function V is a sum

$$V = V^0 - \tfrac{1}{2}\mathbf{A}_{(3)}\cdot\mathbf{A}_{(3)},$$

where V^0 is a Stäckel multiplier,

$$\langle \mathbf{X}, dV^0\rangle = 0, \quad \forall \mathbf{X}\in D, \quad d(\mathbf{K}\, dV^0) = 0.$$

Proof: Assume that the Hamiltonian (1.1) is separable in a coordinate system (q^i). Then (Remark 4.4) the extended metric is separable in the coordinate system (q^i,q^0), with q^0 ignorable. As a consequence, on $\hat{Q}$ there exists a separable Killing web (5.1). Since q^0 is ignorable, the vector field ∂_0 belongs to $\hat{D}$. But this vector coincides with the fundamental vector field $\hat{\mathbf{X}}_0$. Thus, we are in the situation considered in Sec. V, and because of Proposition 5.5, Remarks 5.6–5.7, Proposition 5.10, and Remark 5.11, the conditions (i)–(iii) are fulfilled. Conversely, assume that these conditions are satisfied. Then, because of Remarks 5.7, 5.8, and 5.11, the Hamilton–Jacobi equation admits a separated solution.

■

Remark 6.2: The separable coordinates (q^α) are ignorable (hence, of first class) with respect to both the geodesic Hamiltonians $\hat{G}$ and G but in general they could be nonignorable and of second class for the Hamiltonian H, due to the presence of the functions ξ_α in the components of the vector potential. More precisely, an ignorable coordinate q^α is also of first class and ignorable in the whole Hamiltonian H if and only if the corresponding function ξ_α is constant. To see this, we consider the Hamiltonian written in the form

$$H = \tfrac{1}{2} g^{\alpha\beta}(p_\alpha + A_\alpha)(p_\beta + A_\beta) + \tfrac{1}{2} g^{aa}(p_a + A_a)^2 + V.$$

The coordinates (q^α) appear only in the components (A_α). Because of (5.21),

$$A_\alpha = \xi_\alpha + g^{aa} g_{\alpha\beta} \phi_a^\beta,$$

so that $\partial_\alpha A_\beta = \partial_\alpha \xi_\beta = \delta_{\alpha\beta} \xi_\alpha^0$, where $\xi_\alpha^0 = \partial_\alpha \xi_\alpha$. Thus,

$$\frac{\partial_\alpha H}{\partial^\alpha H} = \frac{g^{\beta\gamma}(p_\beta + A_\beta)\partial_\alpha \xi_\gamma}{g^{\alpha\beta}(p_\beta + A_\beta)} = \xi_\alpha^0,$$

and this fraction becomes a linear (homogeneous) function in the momenta if and only if $\xi_\alpha^0 = 0$; in this case, $\partial_\alpha H = 0$.

Remark 6.3: The only physically interesting component of the vector potential $\mathbf{A}$ is $\mathbf{A}_{(3)}$, since the other two components are gradients and do not influence the motion of the system in the configuration space. Since the orbits in the configuration space are determined, via the Jacobi method, by the partial derivatives of W with respect to the constants $\underset{\sim}{c}$, the independence of the motions from the gradient components can also be observed by the expressions of the separated solution (5.34)–(5.36), where the covariant components (ξ_a) and (ξ_α) of $\mathbf{A}_{(1)}$ and $\mathbf{A}_{(2)}$ do not appear explicitly. It follows, in particular, that there are no physically interesting separable systems with a vector potential $\mathbf{A}$, without the occurrence of symmetries (Killing vectors), since in this case $\mathbf{A}_{(3)}$ vanishes.

After this last remark we can confine our interest to the case $\mathbf{A}_{(1)} = \mathbf{A}_{(2)} = 0$, that is $\mathbf{A} = \mathbf{A}_{(3)}$, and consider the following simplified version of Theorem 6.1.

Theorem 6.4: The Hamiltonian (1.1) is separable if and only if (i) on Q there exists a characteristic Killing pair $(D, \mathbf{K})$, (ii) up to a gauge transformation the vector potential $\mathbf{A}$ is D-invariant, tangent to the orbits of D and its components (A^α) with respect to any basis $(\mathbf{X}_\alpha)$ of D are Stäckel multipliers,

$$\langle \mathbf{X}, dA^\alpha \rangle = 0, \quad \forall \mathbf{X} \in D, \quad d(\mathbf{K}\, dA^\alpha) = 0; \tag{6.1}$$

(iii) the scalar potential V is a sum

$$V = U - \tfrac{1}{2} \mathbf{A} \cdot \mathbf{A} \tag{6.2}$$

where U is a Stäckel multiplier,

$$\langle \mathbf{X}, dU \rangle = 0, \quad \forall \mathbf{X} \in D, \quad d(\mathbf{K}\, dU) = 0. \tag{6.3}$$

Remark 6.5: In (ii) the condition that $\mathbf{A}$ is D-invariant is redundant, since it follows from the other requirements. However, in view of the applications, it is convenient to mention it explicitly in the statement. We also observe that the Stäckel multiplier U in (6.2) is just the scalar part of the Hamiltonian (1.1). The expression (6.2) exhibits a relation of the ''physical'' potential energy V with the vector potential $\mathbf{A}$. This represents a very strong restriction for the separability of a physical system with $\mathbf{A} \neq 0$.

Remark 6.6: According to Theorem 6.4, the separation of a Hamilton–Jacobi equation always occurs in coordinates (q^a, q^α) for which (i) the metric tensor components assume the form (3.4)–(3.5); (ii) up to a gauge transformation the components of the vector potential have the form

$$A^a = 0, \quad A^\alpha = g^{aa} \phi_a^\alpha; \tag{6.4}$$

(iii) the scalar potentials have the form

$$U = g^{aa} \phi_a, \quad V = U - \tfrac{1}{2} g^{aa} g^{bb} \phi_a^\alpha \phi_b^\beta g_{\alpha\beta}, \tag{6.5}$$

where ϕ_a^α and ϕ_a are functions depending on the coordinate corresponding to the lower index only. All these expressions are derived from (5.21), with $\xi_a = 0$ and $\xi_\alpha = 0$. From (1.4) it follows that the Lagrangian forces are

$$F_a = -\partial_a V - \partial_a A_\alpha \dot{q}^\alpha, \quad F_\alpha = \partial_a A_\alpha \dot{q}^a. \tag{6.6}$$

In the case of a vanishing scalar potential, $V=0$, also the scalar product

$$\mathbf{A}\cdot\mathbf{A} = g^{aa} g^{bb} \phi_a^\alpha \phi_b^\beta g_{\alpha\beta} \tag{6.7}$$

must be a Stäckel multiplier. This is a further very strong restriction for the separability, which, however, disappears in the case $m=1$.

Remark 6.7: Theorem 6.4 has another interesting consequence. Let $(\mathbf{K}_b)$ $(b=1,...,m)$ be a basis of the Killing algebra $\mathcal{K}$ generated by the characteristic Killing pair $(D,\mathbf{K})$, with $\mathbf{K}_1=\mathbf{K}$ and $\mathbf{K}_m=\mathbf{G}$ [see item (J) of Sec. III]. Then, besides the $r=n-m$ linear first integrals $H_\alpha = P(\mathbf{X}_\alpha)$ associated with a basis of D, we have m independent quadratic (nonhomogeneous) first integrals in involution of the form

$$H_b = \tfrac{1}{2} P_{\mathbf{K}_b} + P_{\mathbf{A}_b} + U_b, \tag{6.8}$$

where $\mathbf{A}_b = A_b^\alpha \mathbf{X}_\alpha$ are m vector fields and U_b are m functions such that

$$\mathbf{K}_b\, dA^\alpha = dA_b^\alpha, \quad \mathbf{K}_b\, dU = dU_b. \tag{6.9}$$

Note that $\mathbf{A}_m = \mathbf{A}$ and $U_m = U$. In the separable coordinates (q^a, q^α), these objects have the following expressions, involving the inverse Stäckel matrix $[\varphi_{(b)}^a]$:

$$A_b^\alpha = \varphi_{(b)}^a \phi_a^\alpha, \quad U_b = \varphi_{(b)}^a \phi_a, \tag{6.10}$$

so that the final coordinate expressions of the first integrals are

$$\begin{aligned} H_\alpha &= p_\alpha, \\ H_b &= \tfrac{1}{2} \varphi_{(b)}^a (p_a^2 + \phi_a^{\alpha\beta} p_\alpha p_\beta + 2\phi_a^\alpha p_\alpha + 2\phi_a) \\ &= \tfrac{1}{2} \lambda_{(b)}^a g^{aa} (p_a^2 + \phi_a^{\alpha\beta} p_\alpha p_\beta + 2\phi_a^\alpha p_\alpha + 2\phi_a), \end{aligned} \tag{6.11}$$

where λ_b^a are the eigenvalues of the Killing tensors [see (3.9)]. For the case $\mathbf{A}=0$ they reduce to the expressions (3.15). These first integrals correspond to the constants of integration (c_b, c_α) of the separated Hamilton–Jacobi equations of the kind (5.31) (in the present case $\bar{p}_\alpha = p_\alpha = c_\alpha$). Thus, due to the Jacobi theorem, they are certainly first integrals in involution. However, it is interesting to prove that functions (6.8) are first integrals in involution, in a direct and intrinsic way, from their defining equations (6.8) and (6.9) and from the D-invariance, by analyzing their Poisson brackets with the Hamiltonian

$$H = H_m = \tfrac{1}{2} P_{\mathbf{G}} + P_{\mathbf{A}} + U.$$

We get

$$\begin{aligned} \{H_b, H\} = {} & \tfrac{1}{4}\{P_{\mathbf{K}_b}, P_{\mathbf{G}}\} + \tfrac{1}{2}\{P_{\mathbf{K}_b}, P_{\mathbf{A}}\} + \tfrac{1}{2}\{P_{\mathbf{A}_b}, P_{\mathbf{G}}\} + \tfrac{1}{2}\{P_{\mathbf{K}_b}, U\} \\ & + \{P_{\mathbf{A}_b}, P_{\mathbf{A}}\} + \tfrac{1}{2}\{U_b, P_{\mathbf{G}}\} + \{P_{\mathbf{A}_b}, U\} + \{U_b, P_{\mathbf{A}}\}. \end{aligned} \tag{6.12}$$

The terms in (6.12) are, in the order, polynomials of third, second, first, and 0th degree in the momenta. Thus, the Poisson brackets vanish iff these polynomials vanish separately. This gives rise to equations similar to (4.16),

$$
\begin{aligned}
&[\mathbf{G},\mathbf{K}_b]=0,\\
&[\mathbf{A}_b,\mathbf{G}]=[\mathbf{A},\mathbf{K}_b],\\
&[\mathbf{A}_b,\mathbf{A}]=\nabla U_b-\mathbf{K}_b\nabla U,\\
&\langle \mathbf{A}_b,dU\rangle=\langle \mathbf{A},dU_b\rangle.
\end{aligned}
\tag{6.13}
$$

This shows, in other words, that the fact that (H_b) are first integrals in involution is equivalent to the fact that the 2-tensors $\hat{\mathbf{K}}_b=(\mathbf{K}_b,\mathbf{A}_b,U_b)$ in the extended manifold are Killing tensors in involution and form the Killing algebra $\hat{\mathcal{K}}$ associated with the characteristic Killing pair $(\hat{D},\hat{\mathbf{K}})$, where $\hat{D}$ is spanned by the vectors $\hat{\mathbf{X}}_\alpha=(\mathbf{X}_\alpha,0)$ and by $\hat{\mathbf{X}}_0$. The first equation (6.13) is just the Killing equation for $\mathbf{K}_b$. If we assume that all these objects, in particular the functions U_b (including $U_m=U$), are D-invariant (which is equivalent to assume that H_b and H_α are in involution) and that the vector fields $\mathbf{A}_b$ are tangent to D, then both terms on the right-hand side of the third equation (6.13) are orthogonal to D, while the Lie bracket at the left one is tangent to D. Hence, both sides vanish identically and we get the second equation (6.9) together with $[\mathbf{A}_b,\mathbf{A}]=0$. Under the same assumptions, both sides of the fourth equation (6.13) vanish identically. The second remaining equation (6.13) is equivalent to the first equation (6.9). The fact that all (H_b) are in involution can be proved in a similar way. We remark that all the vector potentials commute, $[\mathbf{A}_b,\mathbf{A}_a]=0$.

VII. ILLUSTRATIVE EXAMPLES

Let us apply the above results to the Euclidean three-space $Q=\mathbb{E}_3$. In the following examples we give only the expressions of separable scalar and vectors potentials, without entering in the details of the integration of the corresponding Hamilton–Jacobi equations. In $\mathbb{E}_3$ the Lagrangian forces (1.4) are the components of the Lorentz force

$$\mathbf{F}=\mathbf{B}\times\mathbf{v}-\nabla V,\quad \mathbf{B}=\nabla\times\mathbf{A},$$

where $\mathbf{v}$ is the velocity of the particle, ∇ is the gradient operator, $\nabla\times$ is the curl operator, and $\times$ is the cross product of vectors. We shall use the well-known formula

$$\nabla\times(f\mathbf{V})=\nabla f\times\mathbf{V}+f\nabla\times\mathbf{V},$$

for any smooth function f and vector field $\mathbf{V}$. We consider on $\mathbb{E}_3$ Cartesian rectangular coordinates $(x,\ y,\ z)$ with origin at a point O and denote by $(\mathbf{X},\ \mathbf{Y},\ \mathbf{Z})$ the corresponding unit vectors. Due to Remark 6.3, only the cases of separable webs with symmetries (rotational or translational) are interesting for the separation of a vector potential. We consider for brevity and simplicity the cylindrical and the spherical web only (although the remaining two rotational webs, the prolate and oblate spheroidal ones, could be of some interest for the applications).

Example 1. The cylindrical web. In this first example we consider the cylindrical web around the z axis, made of cylinders around the axis, half-planes issued from the axis (the *meridian planes*) and planes orthogonal to the axis (the *equatorial planes*). These surfaces are, respectively, orthogonal to the vectors

$$(\mathbf{u}_z,\mathbf{R}_z,\mathbf{Z}),$$

where

$$\mathbf{u}_z=\frac{\mathbf{r}_z}{|\mathbf{r}_z|}$$

is the unit vector determined by the radius vector orthogonal to the z axis,

$$\mathbf{r}_z = \mathbf{r} - z\mathbf{Z}, \quad \mathbf{r} = x\mathbf{X} + y\mathbf{Y} + z\mathbf{Z} = r\mathbf{u},$$

and

$$\mathbf{R}_z = \mathbf{Z} \times \mathbf{r}$$

is the rotational vector around the z axis. The standard cylindrical coordinates are (ρ, θ, z), where ρ is the distance from the z axis, $\rho = |\mathbf{r}_z|$, and θ is the rotation angle around it, oriented as $\mathbf{R}_z$ and starting (for instance) from the (x, z) plane. Thus we have

$$\nabla \rho = \mathbf{u}_z, \quad |\mathbf{u}_z| = 1,$$

$$\nabla \theta = \rho^{-2}\mathbf{R}_z, \quad |\mathbf{R}_z| = \rho = |\mathbf{r}_z|,$$

$$\nabla z = \mathbf{Z}, \quad |\mathbf{Z}| = 1,$$

and from

$$\mathbf{p} = \mathbf{v} = p_\rho \nabla \rho + p_\theta \nabla \theta + p_z \nabla z = p_\rho \mathbf{u}_z + p_\theta \rho^{-2}\mathbf{R}_z + p_z \mathbf{Z},$$

we get the well-known expression of the geodesic Hamiltonian

$$G = \tfrac{1}{2}\mathbf{p}\cdot\mathbf{p} = \tfrac{1}{2}(p_\rho^2 + \rho^{-2}p_\theta^2 + p_z^2).$$

The curls of all vectors above are zero, with the exception of

$$\nabla \times \mathbf{R}_z = 2\mathbf{Z}.$$

We have three inequivalent characteristic Killing pairs $(D, \mathbf{K})$ associated with this web.

Case 1. $r = \dim(D) = 2$:

$$D = \mathrm{span}(\mathbf{Z}, \mathbf{R}_z), \quad \mathbf{K} = \mathbf{G}.$$

With respect to this Killing pair, (θ, z) are first-class (ignorable), and ρ is the essential (second-class) coordinate, so that a Stäckel multiplier is any function $U(\rho)$. Thus in this case the most general separable vector potential has the form

$$\mathbf{A} = \phi(\rho)\mathbf{Z} + \psi(\rho)\mathbf{R}_z.$$

It follows that

$$\begin{aligned}\mathbf{B} = \nabla \times (\phi\mathbf{Z} + \psi\mathbf{R}_z) &= \nabla\phi \times \mathbf{Z} + \nabla\psi \times \mathbf{R}_z + 2\psi\mathbf{Z}\\ &= \phi'\mathbf{u}_z \times \mathbf{Z} + \psi'\mathbf{u}_z \times \mathbf{R}_z + 2\psi\mathbf{Z}\\ &= \phi'\rho^{-1}\mathbf{r} \times \mathbf{Z} + \psi'\rho\mathbf{Z} + 2\psi\mathbf{Z},\end{aligned}$$

that is

$$\mathbf{B} = -\rho^{-1}\phi'\mathbf{R}_z + (\rho\psi' + 2\psi)\mathbf{Z} = f(\rho)\mathbf{R}_z + h(\rho)\mathbf{Z},$$

where $f = -\rho^{-1}\phi'$, $h = \rho\psi' + 2\psi$ are two independent functions. Since

$$\mathbf{A}\cdot\mathbf{A} = \phi^2(\rho) + \rho^2\psi(\rho)$$

is a function of ρ only, the most general separable scalar potential (6.2) is any function $V(\rho)$. Thus the Hamiltonian is

$$H=\tfrac{1}{2}\mathbf{p}^2+\mathbf{A}\cdot\mathbf{p}+U=\tfrac{1}{2}(p_\rho^2+\rho^{-2}p_\theta^2+p_z^2)+\phi(\rho)p_z+\rho^{-2}\psi(\rho)p_\theta+U(\rho).$$

Note that the Coriolis and centrifugal forces, appearing in a frame rotating around the z axis with constant angular velocity $\boldsymbol{\omega}=\omega\mathbf{Z}$ with respect to an inertial one, fits with this scheme, being

$$V=\tfrac{1}{2}\omega^2\rho^2,\quad \mathbf{A}=-\omega\mathbf{R}_z,\quad \omega\in\mathbb{R},$$

so that

$$\mathbf{F}=-2\omega\mathbf{Z}\times\mathbf{v}+\omega^2\rho\mathbf{u}_z.$$

Case 2. $r=1$:

$$D=\mathrm{span}(\mathbf{Z}),\quad \mathbf{K}=\mathbf{R}_z\otimes\mathbf{R}_z.$$

Note that $\mathbf{K}$ has eigenvectors $(\mathbf{u}_z,\mathbf{R}_z)$ orthogonal to D, with distinct eigenvalues $(0,\rho^2)$ (they coincide on the z axis, which is the singular set of the web). In this case only z is ignorable, while $(\rho,\ \theta)$ are essential coordinates, so that any Stäckel multiplier is of the kind

$$U=f(\rho)+\rho^{-2}h(\theta),$$

where $f(\rho)$ is any smooth function and $h(\theta)$ is any periodic smooth function (the same is understood for any function of θ considered below). Thus, a separable vector potential has the form

$$\mathbf{A}=(\phi(\rho)+\rho^{-2}\psi(\theta))\mathbf{Z},$$

and consequently

$$\mathbf{B}=\rho^{-1}(2\psi(\theta)\rho^{-3}-\phi'(\rho))\mathbf{R}_z+\psi'(\theta)\rho^{-3}\mathbf{u}_z.$$

The corresponding Hamiltonian is

$$H=\tfrac{1}{2}\mathbf{p}^2+\mathbf{A}\cdot\mathbf{p}+U=\tfrac{1}{2}(p_\rho^2+\rho^{-2}p_\theta^2+p_z^2)+(\phi(\rho)+\rho^{-2}\psi(\theta))p_z+f(\rho)+\rho^{-2}\psi(\theta).$$

Since

$$\mathbf{A}\cdot\mathbf{A}=\phi^2(\rho)+\rho^{-4}\psi^2(\theta)-2\rho^{-2}\psi(\rho)\psi(\theta),$$

the separable scalar potential (6.2) has the form

$$V=f(\rho)+\rho^{-2}h(\theta)-\tfrac{1}{2}\phi^2(\rho)-\tfrac{1}{2}\rho^{-4}\psi^2(\theta)-\rho^{-2}\phi(\rho)\psi(\theta),$$

i.e.,

$$V=f(\rho)+\rho^{-2}(h(\theta)-\phi(\rho)\psi(\theta))-\tfrac{1}{2}\rho^{-4}\psi^2(\theta),$$

where $f(\rho)$ and $h(\theta)$ are arbitrary functions, while $\phi(\rho)$ and $\psi(\theta)$ are the functions entering in the expressions of $\mathbf{A}$ and $\mathbf{B}$. In this case we have a quadratic first integral

$$H_1=\tfrac{1}{2}P_{\mathbf{K}_1}+P_{\mathbf{A}_1}+U_1,\quad \mathbf{K}_1=\mathbf{K}.$$

We compute its elements U_1 and $\mathbf{A}_1$ as follows: for any Stäckel multiplier U

$$\nabla U=(f'(\rho)-2\rho^{-3}h(\theta))\mathbf{u}_z+\rho^{-4}h'(\theta)\mathbf{R}_z$$

and

$$\mathbf{K}\nabla U=\rho^{-2}h'(\theta)\mathbf{R}_z,$$

since $\mathbf{R}_z \cdot \mathbf{u}_z = 0$ and $\mathbf{R}_z^2 = \rho^2$. It follows that

$$U_1 = h(\theta),$$

since $\nabla U_1 = h'(\theta)\nabla\theta = h'(\theta)\rho^{-2}\mathbf{R}_z$. By applying the same method to the component of $\mathbf{A}$ (which is a Stäckel multiplier) we find

$$\mathbf{A}_1 = \psi(\theta)\mathbf{Z}.$$

Thus the quadratic first integral is

$$H_1 = \tfrac{1}{2}(\mathbf{R}_z \cdot \mathbf{p})^2 + \mathbf{A}_1 \cdot \mathbf{p} + U_1 = \tfrac{1}{2} p_\theta^2 + \psi(\theta) p_z + h(\theta).$$

Case 3. $r=1$:

$$D = \mathrm{span}(\mathbf{R}_z), \quad \mathbf{K} = \mathbf{Z} \otimes \mathbf{Z}.$$

The Killing tensor $\mathbf{K} = \mathbf{Z} \otimes \mathbf{Z}$ has eigenvectors $(\mathbf{Z}, \mathbf{u_z})$ orthogonal to D, with distinct eigenvalues (1, 0). In this case θ is ignorable, while (ρ, z) are essential coordinates. Thus any Stäckel multiplier has the form

$$U = f(\rho) + h(z),$$

and the most general separable vector potential is

$$\mathbf{A} = (\phi(\rho) + \psi(z))\mathbf{R}_z.$$

As a consequence,

$$\mathbf{B} = (\rho\phi'(\rho) + 2\phi(\rho) + 2\psi(z))\mathbf{Z} - \psi'(z)\rho\mathbf{u}_z.$$

The corresponding Hamiltonian is

$$H = \tfrac{1}{2}\mathbf{p}^2 + \mathbf{A} \cdot \mathbf{p} + U = \tfrac{1}{2}(p_\rho^2 + \rho^{-2} p_\theta^2 + p_z^2) + (\phi(\rho) + \psi(z)) p_\theta + f(\rho) + h(z).$$

Since

$$\mathbf{A} \cdot \mathbf{A} = \rho^2(\phi^2(\rho) + \psi^2(z) + 2\phi(\rho)\psi(z)),$$

the separable scalar potential (6.2) has the form

$$V = f(\rho) + h(z) - \tfrac{1}{2}\rho^2(\phi^2(\rho) + \psi^2(z) + 2\phi(\rho)\psi(z)),$$

i.e.,

$$V = f(\rho) + h(z) - \rho^2(\tfrac{1}{2}\psi^2(z) + \phi(\rho)\psi(z)),$$

where $f(\rho)$, $h(z)$ are arbitrary functions, while $\psi(z)$, $\phi(\rho)$ are the functions entering in the expressions of $\mathbf{A}$ and $\mathbf{B}$. Also in this case we have a quadratic first integral H_1. Since

$$\nabla U = f'(\rho)\mathbf{u}_z + h'(z)\mathbf{Z}, \quad \mathbf{K}\nabla U = h'(z)\mathbf{Z},$$

we find $U_1 = h(z)$, and in a similar way, $\mathbf{A}_1 = \psi(z)\mathbf{R}_z$. Thus the quadratic first integral is

$$H_1 = \tfrac{1}{2}(\mathbf{Z} \cdot \mathbf{p})^2 + \mathbf{A}_1 \cdot \mathbf{p} + U_1 = \tfrac{1}{2} p_z^2 + \psi(z) p_\theta + h(z).$$

Example 2. The spherical web. This web is made of spheres around the origin, meridian half-planes (issued from the z axis), and circular cones around the axis with vertex at the origin. These surfaces are, respectively, orthogonal to the vectors

$$(\mathbf{r},\mathbf{R}_z,\mathbf{l}),$$

where $\mathbf{l}$ is the unit vector

$$\mathbf{l}=\frac{\mathbf{r}\times\mathbf{R}_z}{|\mathbf{r}\times\mathbf{R}_z|}=\frac{\mathbf{r}\times\mathbf{R}_z}{r\rho}$$

tangent to the meridian planes and to the spheres. The standard spherical coordinates are (r,θ,ϕ), where ϕ is the latitude, so that

$$\nabla r=\mathbf{u},\quad \nabla\theta=\rho^{-2}\mathbf{R}_z,\quad \mathbf{p}=p_r\mathbf{u}+p_\theta\rho^{-2}\mathbf{R}_z+r^{-1}p_\phi\mathbf{l},\quad \nabla\phi=r^{-1}\mathbf{l},$$

with $\rho=r\cos\phi$, and the geodesic Hamiltonian assumes the standard form

$$G=\tfrac{1}{2}(p_r^2+\rho^{-2}p_\theta^2+r^{-2}p_\phi^2).$$

Up to equivalences, there is only one characteristic Killing pair characterizing this web, with $r=1$:

$$D=\mathrm{span}(\mathbf{R}_z),\quad \mathbf{K}=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}.$$

The vectors $(\mathbf{r},\,\mathbf{l})$ are eigenvectors of $\mathbf{K}$ orthogonal to $\mathbf{R}_z$, with distinct eigenvalues $(0,r^2)$. The coordinates (r,ψ) are essential, θ is ignorable, and a Stäckel multiplier is a function

$$U=f(r)+r^{-2}h(\phi).$$

Thus the separable vector potential is

$$\mathbf{A}=(\alpha(r)+r^{-2}\beta(\phi))\mathbf{R}_z,$$

and

$$\mathbf{B}=(\alpha'-2\beta r^{-3})\rho\mathbf{l}-\beta' r^{-3}\rho\mathbf{u}+2(\alpha+\beta r^{-2})\mathbf{Z}.$$

The Hamiltonian is

$$H=\tfrac{1}{2}(p_r^2+\rho^{-2}p_\phi^2)+(\alpha(r)+r^{-2}\beta(\phi))p_\theta+f(r)+r^{-2}h(\phi).$$

Since

$$\nabla U=(f'-2r^{-3}h)\mathbf{u}+r^{-3}h'\mathbf{l},\quad \mathbf{K}\nabla U=h'(\phi)\nabla\phi=\nabla h,$$

we find $U_1=h(\phi)$ and, in a similar way, $\mathbf{A}_1=\beta(\phi)\mathbf{R}_z$. It follows that the associated quadratic first integral is

$$H_1=r^2G-\tfrac{1}{2}(\mathbf{p}\cdot\mathbf{r})^2+\mathbf{A}_1\cdot\mathbf{p}+U_1=\tfrac{1}{2}r^2(\rho^{-2}p_\theta^2+r^{-2}p_\phi^2)+\beta(\phi)p_\theta+h(\phi).$$

Example 3. Rotational surfaces in $\mathbb{E}_3$. For a particle moving on a regular surface S in $\mathbb{E}_3$ only the restriction of the scalar potential V to S and the tangent component of the vector potential $\mathbf{A}$ have influence on the motion, as well as the orthogonal part of $\mathbf{B}$. Only the case of a surface with symmetry (translational or rotational) is relevant for the separation of a vector potential. Let us consider the case of a rotational surface around the z axis. Then the dynamics of the point on this surface is separable for any scalar and vector potential in $\mathbb{E}_3$ invariant under the rotation $\mathbf{R}_z$. Indeed, let us consider the cylindrical web of Example 1 and the cylindrical coordinates (ρ,θ,z). Let us consider the decomposition of the vector potential,

$$\mathbf{A}=\alpha\mathbf{u}_z+\beta\mathbf{R}_z+\gamma\mathbf{Z},$$

where, due to the rotational invariance, the functions (α, β, γ) do not depend on the rotation angle θ. It follows that

$$\mathbf{B}=\nabla\alpha\times\mathbf{u}_z+\nabla\beta\times\mathbf{R}_z+2\beta\mathbf{Z}+\nabla\gamma\times\mathbf{Z}.$$

But the first and the last terms are vectors parallel to $\mathbf{R}_z$, since the gradients of θ-invariant functions are tangent to the meridian half-planes, thus they are tangent to the surface and can be disregarded. The relevant potential is then

$$\mathbf{A}=\beta(\rho,z)\mathbf{R}_z,$$

which is tangent to the surface and orthogonal to the meridian planes. On the surface (ρ,z) can be represented as functions of a parameter u, so that the scalar and vector potentials are

$$V=f(u), \quad \mathbf{A}=\phi(u)\mathbf{R}_z.$$

The coordinates on the surfaces are then (θ,u), with θ ignorable and u essential coordinate.

Example 4. The Euclidean plane $\mathbb{E}_2$. We consider $\mathbb{E}_2$ as the (x,y) plane in the three-dimensional Euclidean space $\mathbb{E}_3$. In the rectangular Cartesian web the only interesting case is $m=1$, $D=\mathrm{span}(\mathbf{X})$, $\mathbf{K}=\mathbf{G}$, so that

$$\mathbf{A}=A(y)\mathbf{X}, \quad V=V(y).$$

It follows that

$$\mathbf{B}=A'(y)\mathbf{Y}\times\mathbf{X}=-A'(y)\mathbf{Z}, \quad \mathbf{F}=-A'(y)\mathbf{Z}\times\mathbf{v}-V'(y)\mathbf{Y}.$$

For the polar web we have $r=1$, $D=\mathrm{span}(\mathbf{R}_z)$, $\mathbf{K}=\mathbf{G}$, and

$$\mathbf{A}=A(r)\mathbf{R}_z, \quad V=V(r).$$

It follows that

$$\begin{aligned}\mathbf{B}=\nabla\times(A\mathbf{R}_z)&=\nabla A\times\mathbf{R}+2A\mathbf{Z}\\&=A'(r)\boldsymbol{\nabla} r\times\mathbf{R}+2A\mathbf{Z}\\&=A'(r)r^{-1}\mathbf{r}\times(\mathbf{Z}\times\mathbf{r})+2A\mathbf{Z}\\&=(rA'+2A)\mathbf{Z},\end{aligned}$$

and the corresponding separable force is

$$\mathbf{F}=B(r)\mathbf{Z}\times\mathbf{v}-V'(r)\mathbf{u}, \quad B(r)=rA'+2A, \quad \mathbf{r}=r\mathbf{u}.$$

ACKNOWLEDGMENTS

Research supported by Dipartimento di Matematica, Università di Torino and by M.U.R.S.T., Ministero dell'Università e della Ricerca Scientifica e Tecnologica under the research project ''Geometry of integrable systems.''

[1] T. Levi-Civita, ''Sulla integrazione della equazione di Hamilton–Jacobi per separazione di variabili,'' Math. Ann. **59**, 3383–3397 (1904).
[2] P. Stäckel, ''Uber die Bewegung eines Punktes in einer n-fachen Mannigfaltigkeit,'' Math. Ann. **42**, 537–563 (1893); ''Uber quadratizche Integrale der Differentialgleichungen der Dynamik,'' Ann. Mat. Pura **26**, 55–60 (1897).
[3] L. P. Eisenhart, ''Separable systems of Stäckel,'' Ann. Math. **35**, 284–305 (1934).
[4] N. M. J. Woodhouse, ''Killing tensors and the separation of the Hamilton–Jacobi equation,'' Commun. Math. Phys. **44**, 9–38 (1975).
[5] E. G. Kalnins and W. Miller, Jr., ''Killing tensors and variable separation for Hamilton–Jacobi and Helmholtz equations,'' SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **11**, 1011–1026 (1980).
[6] E. G. Kalnins and W. Miller, Jr., ''Killing tensors and nonorthogonal variable separation for Hamilton–Jacobi equations,'' SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **12**, 617–629 (1981).
[7] S. Benenti, ''Separability structures on Riemannian manifolds,'' Lect. Notes Math. **863**, 512–538 (1980).
[8] S. Benenti, ''Separation of variables in the geodesic Hamilton–Jacobi equation,'' Progress in Mathematics, Birkhäuser **99**, 1–36 (1991).
[9] S. Benenti, ''Orthogonal separable dynamical systems,'' *Differential Geometry and Its Applications*, Vol. I, Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992 (Silesian University, Opava, 1993), pp. 163–184. Electronic edition: ELibEMS, http://www.emis.de/proceedings/.
[10] S. Benenti, ''Intrinsic characterization of the variable separation in the Hamilton–Jacobi equation,'' J. Math. Phys. **38**, 6578–6602 (1997).
[11] M. S. Iarov-Iarovoi, ''Integration of the Hamilton–Jacobi equation by the methods of separation of variables,'' J. Appl. Math. Mech. **27**, 1499–1520 (1963).
[12] F. Cantrijn, ''Separation of variables in the Hamilton–Jacobi equation for non-conservative systems,'' J. Phys. A **10**, 491–505 (1977).
[13] J. Steigeberger, ''On separation of variables in mechanical systems,'' Bull. Cl. Sci., Acad. R. Belg. **49**, 990–1014 (1963); *ibid.* **51**, 1200–1204 (1965).
[14] S. Benenti and G. Pidello, ''Separability structures corresponding to conservative dynamical systems,'' Meccanica **19**, 275–281 (1984).
[15] G. Rastelli, ''Orthogonal separation of variables for natural Hamiltonians with vector potentials,'' Atti Ac. Sci. Torino (in press).
[16] L. P. Eisenhart, ''Dynamical trajectories and geodesics,'' Ann. Math. **30**, 591–606 (1929).
[17] E. G. Kalnins and W. Miller, Jr., ''The general theory of R-separation for Helmholtz equations,'' J. Math. Phys. **24**, 1047–1053 (1983).