The super-separability of the three-body inverse-square Calogero system

S. Benenti,[a] C. Chanu, and G. Rastelli
Department of Mathematics, University of Turin, Italy

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The geometrical theory of the variable separation for the Hamilton–Jacobi equation is applied to the classical three-body inverse-square Calogero system. It is proved that this system is separable in infinitely many inequivalent ways, related to five different kinds of separable webs in the Euclidean three-space, and the corresponding systems of independent first integrals in involution are computed. © *2000 American Institute of Physics.* [S0022-2488(00)05707-8]

I. INTRODUCTION

The three-body Calogero system consists of three identical particles moving on a line, with coordinates (x, y, z), respectively, under interactive forces with potential energy

$$V=\frac{g}{(x-z)^2}+\frac{g}{(y-x)^2}+\frac{g}{(z-y)^2},\quad g\in\mathbb{R}. \tag{1.1}$$

It is known that this dynamical system is super-integrable (see Refs. 1–4 and papers cited therein). It is also known that it is separable in the cylindrical coordinates associated with the reduction to a two-dimensional system.[5,6] In the present paper we show that the three-body Calogero system is in fact separable in infinitely many inequivalent ways thus, that it is super-separable. (A Hamiltonian system is ''*super-separable*'' if it is separable in at least two inequivalent ways; ''*inequivalent*'' means that the separation is related to distinct *separable webs*, i.e., to distinct algebras of first integrals in involution.) For this purpose, the three particles on the line will be interpreted as a single one moving in the Euclidean three-space $\mathbb{E}_3\simeq\mathbb{R}^3$, with rectangular Cartesian coordinates (x, y, z). Then, we shall apply to the Hamiltonian

$$H=\tfrac{1}{2}(p_x^2+p_y^2+p_z^2)+V, \tag{1.2}$$

the geometrical theory of the orthogonal separation based on the following theorem:[7,8]

Theorem 1.1: *A natural Hamiltonian $H=G+V$ on the cotangent bundle T^*Q of a Riemannian manifold Q_n is separable in orthogonal coordinates iff on Q there exists a Killing two-tensor* $\mathbf{K}$ *with simple eigenvalues and normal eigenvectors, such that*

$$d(\mathbf{K}dV)=0. \tag{1.3}$$

Here, G is the geodesic Hamiltonian and V is a function on Q, canonically lifted to T^*Q; $\mathbf{K}dV$ denotes the one-form image of dV by $\mathbf{K}$, interpreted as a linear endomorphism over one-forms, whose components are $g_{ih}K^{hj}\partial_jV$. Let $G=(g^{ij})=(g_{ij})^{-1}$ be the contravariant metric tensor. We recall that on a Riemannian manifold Q (with coordinates (q^i)) a contravariant symmetric tensor of any order, $\mathbf{K}=(K^{i\cdots j})$, is a **Killing tensor (K-tensor)** if the functions on T^*Q (with canonical coordinates (q^i,p_i))

$$P_{\mathbf{K}}=K^{i\cdots j}p_i\cdots p_j,\quad P_{\mathbf{G}}=g^{ij}p_ip_j, \tag{1.4}$$

[a]Electronic mail: benenti@dm.unito.it

are in involution in the canonical Lie–Poisson brackets

$$\{P_{\mathbf{K}},P_{\mathbf{G}}\}=0.$$

This means that $P_{\mathbf{K}}$ is a first integral of the geodesic flow ($G=\frac{1}{2}P_{\mathbf{G}}$ is the geodesic Hamiltonian). Two symmetric tensors of any order **A** and **B** are said to be **in involution** if $\{P_{\mathbf{A}},P_{\mathbf{B}}\}=0$.

In Theorem 1.1, ''normal'' means orthogonally integrable or surface forming: the eigenvectors of **K** are orthogonal to foliations of regular hypersurfaces. These foliations of submanifolds of codimension 1 form an **orthogonal web** and any coordinate system $\underline{q}=(q^i)$ adapted to this web (i.e., whose coordinate surfaces belong to the web) is orthogonal and **separable**, i.e., in these coordinates the metric tensor is diagonal and the corresponding Hamilton–Jacobi equation

$$\tfrac{1}{2}g^{ij}(\partial_i W)^2+V=h,$$

admits a complete solution of the form

$$W(\underline{q},\underline{c})=\sum_{i=1}^{n} W_i(q^i,\underline{c}),$$

($\underline{c}=(c_i)$ is a complete set of integration constants). For this reason, an orthogonal web is called **separable** if it is made of hypersurfaces orthogonal to the eigenvectors of a Killing two-tensor **K** with simple eigenvalues. The tensor **K** is said to be a **characteristic tensor** of the web (notice that it is not uniquely determined; for instance, $\mathbf{K}+a\mathbf{G}$ is still a characteristic tensor, $\forall a\in\mathbb{R}$). The existence of such a tensor characterizes the orthogonal separation of the pure geodesic Hamilton–Jacobi equation (case $V=0$), while Eq. (1.3), which we call **the characteristic equation**, characterizes the separability of a potential V (i.e., of the Hamiltonian $H=G+V$) in the web determined by **K**. A separable orthogonal web is the geometrical object representing an equivalence class of separable orthogonal coordinates: Two separable coordinate systems are equivalent if their coordinate hypersurfaces belong to the same web.

The meaning of the characteristic equation [Eq. (1.3)] is given by the following general property:[7,8] Let **K** be a contravariant symmetric two-tensor, $P_{\mathbf{K}}$ be defined as in (1.4) and $V_{\mathbf{K}}$ be a smooth function on Q (canonically extended to T^*Q). Then the function

$$H_{\mathbf{K}}=\tfrac{1}{2}P_{\mathbf{K}}+V_{\mathbf{K}} \tag{1.5}$$

is in involution with the natural Hamiltonian

$$H=H_{\mathbf{G}}=\tfrac{1}{2}P_{\mathbf{G}}+V_{\mathbf{G}}=G+V,$$

if and only if **K** is a Killing two-tensor and

$$dV_{\mathbf{K}}=\mathbf{K}dV. \tag{1.6}$$

Thus, the characteristic equation [Eq. (1.3)] is locally equivalent to the existence of a quadratic first integral of the kind (1.5).

It can be shown[7] that the existence of a characteristic K-tensor (i.e., of a K-tensor with simple eigenvalues and normal eigenvectors) **K** implies the existence of a n-dimensional linear space $\mathcal{K}$ of Killing two-tensors, including **K** and the metric tensor **G**, all in involution and with common eigenvectors. We shall call this space the **Killing–Stäckel involutive algebra** (briefly, **KS-algebra**) generated by (or associated with) the characteristic tensor **K**. It can be proved that if the characteristic equation [Eq. (1.3)] is satisfied by **K**, then it is satisfied by all elements of the KS-algebra $\mathcal{K}$ generated by **K**, and that the corresponding functions (1.5) form a n-dimensional space $\mathcal{H}$ of first integrals in involution. Actually, the orthogonal separation can be characterized by a system of n independent K-tensors in involution and commuting as linear operators, according to the celebrated Eisenhart theorem,[9] see also Kalnins and Miller[10] for a deeper discussion. However, in our approach to the separation of the Calogero system it turns out to be more convenient to relate the separation to a single characteristic K-tensor. In any separable coordinate system adapted to the web, all these tensors are diagonalized and for any basis $(\mathbf{K}_j)$ of $\mathcal{K}$, the diagonal components

$$K_j^{ii} = \varphi_{(j)}^i$$

form a regular $n \times n$ matrix $[\varphi_{(j)}^i]$ whose inverse $[\varphi_i^{(j)}]$ is a **Stäckel matrix:** Each element $\varphi_i^{(j)}$ is a function of the coordinate q^i corresponding to the lower index only. The functions V_j, such that

$$dV_j = \mathbf{K}_j dV,$$

have the form

$$V_j = \varphi_{(j)}^i \phi_i = K_j^{ii} \phi_i, \tag{1.7}$$

where ϕ_i is a function of the corresponding coordinate q^i only. If $\mathbf{G}$ is an element of the basis, say $\mathbf{G} = \mathbf{K}_n$, then

$$g^{ii} = \varphi_{(n)}^i, \quad V = V_n = g^{ii} \phi_i. \tag{1.8}$$

It follows that the quadratic first integrals in involution (H_j) generated by the basis have the form:

$$H_j = \tfrac{1}{2} \varphi_{(j)}^i (p_i^2 + 2\phi_i). \tag{1.9}$$

Furthermore, by setting $c_j = 2H_j$ and reversing the system of equations (1.9), we can see that the Lagrangian foliation of T^*Q generated by equations

$$p_i = \partial_i W$$

is locally represented by equations of the kind

$$p_i^2 = \Phi_i(q^i, \underset{\sim}{c}) = \varphi_i^{(j)} c_j - 2\phi_i. \tag{1.10}$$

This means that each p_i is a function of the corresponding coordinate q^i only, but in general of all the integration constants $\underset{\sim}{c}$. It follows that a separated solution W of the $H-J$ equation is the sum of the integrals:

$$W_i = \pm \int \sqrt{\Phi_i} dq^i,$$

for any suitable choice of the signs. Moreover, the inequalities

$$\Phi_i \geqslant 0, \tag{1.11}$$

following from (1.10) define regions of the space, depending on the constants of motion $\underset{\sim}{c}$, where the orbits are confined. By the Jacobi theorem, the orbits are locally determined by the $n-1$ equations

$$a_k = \frac{\partial W}{\partial c_k}, \quad k = 1,\dots,n-1, \tag{1.12}$$

if we choose (as it is customary) $c_n = h$ (the energy constant), while the time-dependence is given by the last equation

$$t-t_0=\frac{\partial W}{\partial h}, \tag{1.13}$$

representing a moving hypersurface.

In the present paper, we shall solve the characteristic equation (1.3) for the Calogero potential (1.1) and find all possible characteristic tensors **K**. They form a five-dimensional space and generates five different kinds of separable orthogonal webs. Furthermore, by solving Eqs. (1.6), we shall compute all the quadratic first integrals associated with the corresponding five KS-algebras, leaving to a further work the analysis of their expressions in separable coordinates and the discussion of the related equations (1.10)–(1.13).

II. KILLING VECTORS AND TENSORS IN THE EUCLIDEAN THREE-SPACE

It is known that in the Euclidean space $\mathbb{E}_3$, as in all manifolds of constant curvature, any *K*-tensor is **reducible**, i.e., a linear combination with constant coefficients of symmetric tensor products of *K*-vectors.[11] The **symmetric tensor product** $\odot$ of two vectors is defined by

$$\mathbf{A}\odot\mathbf{B}=\tfrac{1}{2}(\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A}). \tag{2.1}$$

With the rectangular Cartesian coordinates $(x_i)=(x_1,x_2,x_3)=(x,y,z)$ we associate the basic translational unit *K*-vectors $(\mathbf{X}_i)=(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)=(\mathbf{X},\mathbf{Y},\mathbf{Z})$

$$\mathbf{X}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \mathbf{Y}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \mathbf{Z}=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

and the basic rotational *K*-vectors $(\mathbf{R}_i)=(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_3)=(\mathbf{R}_x,\mathbf{R}_y,\mathbf{R}_z)$, defined by

$$\mathbf{R}_x=\mathbf{X}\times\mathbf{r}=\begin{bmatrix}0\\-z\\y\end{bmatrix},\quad \mathbf{R}_y=\mathbf{Y}\times\mathbf{r}=\begin{bmatrix}z\\0\\-x\end{bmatrix},\quad \mathbf{R}_z=\mathbf{Z}\times\mathbf{r}=\begin{bmatrix}-y\\x\\0\end{bmatrix}, \tag{2.2}$$

where

$$\mathbf{r}=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

is the position vector $\mathbf{r}=OP$ of the generic point $P\in\mathbb{E}_3$ with respect to the origin *O*. Here, we denote by $\mathbf{u}\times\mathbf{v}$ the skew-symmetric cross product of two vectors, whose Cartesian components are $\epsilon_{ijk}u^jv^k$, where ϵ_{ijk} is the Levi-Civita symbol. We denote by $\mathbf{u}\cdot\mathbf{v}$ the symmetric scalar product of two vectors. The following identities hold

$$\Sigma_i\mathbf{X}_i\odot\mathbf{X}_i=\mathbf{G},\quad \Sigma_i\mathbf{X}_i\odot\mathbf{R}_i=0,\quad \Sigma_i\mathbf{R}_i\odot\mathbf{R}_i=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}, \tag{2.3}$$

where

$$\mathbf{G}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

Let $\mathcal{K}_2(\mathbb{E}_3)$ be the space of Killing two-tensors on $\mathbb{E}_3$. Any element of $\mathcal{K}_2(\mathbb{E}_3)$ is represented as a linear combination of symmetric products of the basic *K*-vectors

$$\mathbf{K}=\mathbf{A}+\mathbf{B}+\mathbf{C}=a^{ij}\mathbf{X}_i\odot\mathbf{X}_j+b^{ij}\mathbf{X}_i\odot\mathbf{R}_j+c^{ij}\mathbf{R}_i\odot\mathbf{R}_j. \tag{2.4}$$

The components of the K-tensors

$$\mathbf{A}=[A^{ij}],\quad \mathbf{B}=[B^{ij}],\quad \mathbf{C}=[C^{ij}]$$

are constant, linear-homogeneous in the Cartesian coordinates and quadratic-homogeneous, respectively, and the constant coefficients appearing in (2.4) form matrices of the kind

$$[a^{ij}]=[A^{ij}]=\begin{bmatrix} a_1 & \alpha_3 & \alpha_2 \\ \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_2 & \alpha_1 & \alpha_3 \end{bmatrix},\quad [b^{ij}]=\begin{bmatrix} b^{11} & b^{12} & b^{13} \\ b^{21} & b^{22} & b^{23} \\ b^{31} & b^{32} & b^{33} \end{bmatrix},\quad [c^{ij}]=\begin{bmatrix} c_1 & \gamma_3 & \gamma_2 \\ \gamma_3 & c_2 & \gamma_1 \\ \gamma_2 & \gamma_1 & c_3 \end{bmatrix} \tag{2.5}$$

(the first and the third one are symmetric), for a total amount of 21 constant coefficients. However, the dimension of $\mathcal{K}_2(\mathbb{E}_3)$ is 20, since only the differences of the diagonal coefficients (b^{11},b^{22},b^{33}) are involved (see below)

$$\beta_1=b^{22}-b^{33},\quad \beta_2=b^{33}-b^{11},\quad \beta_3=b^{11}-b^{22},$$

and these three parameters β are constrained by the equation

$$\beta_1+\beta_2+\beta_3=0.$$

We can compute the components of the matrices **B** and **C** starting from the relations

$$\begin{gathered}
\mathbf{R}_i\cdot\mathbf{X}_j=\epsilon_{ikj}x_k=-\epsilon_{ijk}x_k \\
B^{ij}=b^{lm}\mathbf{X}_i\cdot(\mathbf{X}_l\odot\mathbf{R}_m)\mathbf{X}_j=\tfrac{1}{2}(b^{ih}\epsilon_{jhk}+b^{jh}\epsilon_{ihk})x_k \\
C^{ij}=c^{lm}\mathbf{X}_i\cdot(\mathbf{R}_l\odot\mathbf{R}_m)\mathbf{X}_j=c^{lm}\epsilon_{lih}\epsilon_{mjk}x_hx_k.
\end{gathered}$$

We obtain

$$\begin{gathered}
B^{11}=b^{12}x_3-b^{13}x_2=b^{12}z-b^{13}y \\
B^{22}=b^{23}x_1-b^{21}x_3=b^{23}x-b^{21}z \\
B^{33}=b^{31}x_2-b^{32}x_1=b^{31}y-b^{32}x \\
B^{12}=\tfrac{1}{2}(b^{22}-b^{11})x_3+\tfrac{1}{2}b^{13}x_1-\tfrac{1}{2}b^{23}x_2=\tfrac{1}{2}(b^{13}x-b^{23}y-\beta_3z) \\
B^{23}=\tfrac{1}{2}(b^{33}-b^{22})x_1+\tfrac{1}{2}b^{21}x_2-\tfrac{1}{2}b^{31}x_3=\tfrac{1}{2}(b^{21}y-b^{31}z-\beta_1x) \\
B^{31}=\tfrac{1}{2}(b^{11}-b^{33})x_2+\tfrac{1}{2}b^{32}x_3-\tfrac{1}{2}b^{12}x_1=\tfrac{1}{2}(b^{32}z-b^{12}x-\beta_2y) \\
C^{11}=c^{22}x_3^2+c^{33}x_2^2-2c^{23}x_2x_3=c_2z^2+c_3y^2-2\gamma_1yz \\
C^{22}=c^{33}x_1^2+c^{11}x_3^2-2c^{31}x_3x_1=c_3x^2+c_1z^2-2\gamma_2zx \\
C^{33}=c^{11}x_2^2+c^{22}x_1^2-2c^{12}x_1x_2=c_1y^2+c_2x^2-2\gamma_3xy \\
C^{12}=(c^{13}x_2+c^{23}x_1-c^{12}x_3)x_3-c^{33}x_1x_2=(\gamma_2y+\gamma_1x-\gamma_3z)z-c_3xy \\
C^{23}=(c^{21}x_3+c^{31}x_2-c^{23}x_1)x_1-c^{11}x_2x_3=(\gamma_3z+\gamma_2y-\gamma_1x)x-c_1yz \\
C^{31}=(c^{32}x_1+c^{12}x_3-c^{31}x_2)x_2-c^{22}x_3x_1=(\gamma_1x+\gamma_3z-\gamma_2y)y-c_2zx\,.
\end{gathered} \tag{2.6}$$

III. THE FIRST SEPARABILITY CONDITION

According to Theorem 1.1, a first condition for the separability is the existence of a K-tensor **K** solution of the characteristic equation (1.3) for the Calogero potential V (1.1) (there is no loss of generality in assuming $g=1$). By inserting into this equation the expression (2.4) of a general K-tensor of $\mathbb{E}_3$, we get a system of algebraic equations in the variables (x,y,z), to be identically satisfied. This provides a system of linear equations on the constant coefficients. With the help of a computer algebra system (we used Maple V®—an alternative method which avoids this calculation is illustrated in Ref. 12) it can be shown that these linear equations are

$$
\begin{gathered}
a_1=a_2=a_3=a\\
\alpha_1=\alpha_2=\alpha_3=\alpha\\
b^{13}=b^{21}=b^{32}=-b^{23}=-b^{31}=-b^{12}=b\\
\beta_1=\beta_2=\beta_3=0\\
c_1=c_2=c_3=c\\
\gamma_1=\gamma_2=\gamma_3=\gamma.
\end{gathered}
$$

It follows from (2.5) that

$$
[a^{ij}]=\begin{bmatrix} a & \alpha & \alpha \\ \alpha & a & \alpha \\ \alpha & \alpha & a \end{bmatrix},\quad [b^{ij}]=\begin{bmatrix} t & -b & b \\ b & t & -b \\ -b & b & t \end{bmatrix},\quad [c^{ij}]=\begin{bmatrix} c & \gamma & \gamma \\ \gamma & c & \gamma \\ \gamma & \gamma & c \end{bmatrix}, \tag{3.1}
$$

for any arbitrary $t\in\mathbb{R}$. However, the value of t is irrelevant, due to the second identity (2.3). We can choose $t=0$, so that the matrix $[b^{ij}]$ becomes skew-symmetric

$$
[b^{ij}]=\begin{bmatrix} 0 & -b & b \\ b & 0 & -b \\ -b & b & 0 \end{bmatrix}. \tag{3.2}
$$

The conclusion is

Proposition 3.1: The solutions of the characteristic equation $d(\mathbf{K}dV)=0$ for the Calogero potential (1.1) *form a five-dimensional linear space $\mathcal{C}$ of K-tensors of the kind* $\mathbf{K}=\mathbf{A}+\mathbf{B}+\mathbf{C}$, *where*

$$
\mathbf{A}=[a^{ij}]=\begin{bmatrix} a & \alpha & \alpha \\ '' & a & \alpha \\ '' & '' & a \end{bmatrix}, \tag{3.3}
$$

$$
\mathbf{B}=b\begin{bmatrix} -(y+z) & \frac{1}{2}(x+y) & \frac{1}{2}(x+z) \\ '' & -(x+z) & \frac{1}{2}(y+z) \\ '' & '' & -(x+y) \end{bmatrix}, \tag{3.4}
$$

$$
\mathbf{C}=\begin{bmatrix} c(y^2+z^2)-2\gamma yz & -cxy+\gamma z(x+y-z) & -czx+\gamma y(z+x-y) \\ '' & c(z^2+x^2)-2\gamma zx & -cyz+\gamma x(y+z-x) \\ '' & '' & c(x^2+y^2)-2\gamma xy \end{bmatrix}. \tag{3.5}
$$

The elements of $\mathcal{C}$ are determined by the values of the five parameters (a,α,b,c,γ). An equivalent decomposition of $\mathbf{K}\in\mathcal{C}$ is

$$
\mathbf{K}=(a-\alpha)\mathbf{G}+\alpha\mathbf{T}+b\mathbf{S}+c\mathbf{I}+\gamma\mathbf{J}, \tag{3.6}
$$

where

$$
\mathbf{T}=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \tag{3.7}
$$

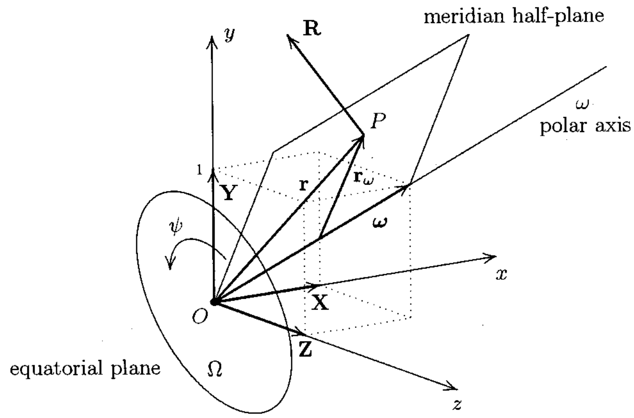

FIG. 1. The basic geometrical objects.

$$\mathbf{S}=\frac{1}{b}\mathbf{B}=\begin{bmatrix} -(y+z) & \frac{1}{2}(x+y) & \frac{1}{2}(x+z) \\ '' & -(x+z) & \frac{1}{2}(y+z) \\ '' & '' & -(x+y) \end{bmatrix}, \tag{3.8}$$

$$\mathbf{I}=\begin{bmatrix} y^2+z^2 & -xy & -zx \\ '' & z^2+x^2 & -yz \\ '' & '' & x^2+y^2 \end{bmatrix}, \quad \mathbf{J}=\begin{bmatrix} -2yz & z(x+y-z) & y(z+x-y) \\ '' & -2zx & x(y+z-x) \\ '' & '' & -2xy \end{bmatrix}. \tag{3.9}$$

However, as far as the separation is concerned, the first component $\mathbf{G}$ in the expression (3.6) is irrelevant, so that we can consider only K-tensors of the kind

$$\mathbf{K}=\alpha\mathbf{T}+b\mathbf{S}+c\mathbf{I}+\gamma\mathbf{J}. \tag{3.10}$$

Furthermore, as it will be shown in the next section, it is convenient to introduce the K-tensor

$$\mathbf{Q}=\mathbf{I}+\mathbf{J}=\begin{bmatrix} (z-y)^2 & (z-y)(x-z) & (z-y)(y-x) \\ '' & (x-z)^2 & (x-z)(y-x) \\ '' & '' & (y-x)^2 \end{bmatrix}, \tag{3.11}$$

so that (3.10) becomes equivalent to

$$\mathbf{K}=\alpha\mathbf{T}+b\mathbf{S}+(c-\gamma)\mathbf{I}+\gamma\mathbf{Q}. \tag{3.12}$$

IV. THE SECOND SEPARABILITY CONDITION

According to Theorem 1.1, we have to look for elements of $\mathcal{C}$ with normal eigenvectors and simple eigenvalues. In our analysis, the following objects will play a basic role (see Fig. 1): the constant vector

$$\boldsymbol{\omega}=\mathbf{X}+\mathbf{Y}+\mathbf{Z}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{4.1}$$

and its unit vector (the **director**)

$$\mathbf{d}=\frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}=\frac{1}{\sqrt{3}}\boldsymbol{\omega}, \tag{4.2}$$

the line ω passing through the origin O of the coordinates and parallel to $\boldsymbol{\omega}$ (called the **polar axis**); the rotational K-vector around ω

$$\mathbf{R}=\boldsymbol{\omega}\times\mathbf{r}=\mathbf{R}_x+\mathbf{R}_y+\mathbf{R}_z=\begin{bmatrix} z-y \\ x-z \\ y-x \end{bmatrix}; \tag{4.3}$$

the plane Ω through O and orthogonal to ω (the **equatorial plane**), and the half-planes issued from ω (the **meridian half-planes**).

Remark 4.1: The Calogero potential (1.1) is not defined on the meridian planes containing the axes, whose equations are $z=y$, $x=z$ and $x=y$. ●

Remark 4.2: With any smooth real function $f(u)$ we associate a potential energy

$$V_f=f(x-z)+f(y-x)+f(z-y). \tag{4.4}$$

For $f(u)=gu^{-2}$ we find the Calogero potential (1.1). For all f, the function V_f is $\boldsymbol{\omega}$-invariant, $\boldsymbol{\omega}\cdot\nabla V_f=0$. This means that the function

$$p_\omega=\boldsymbol{\omega}\cdot\mathbf{p}=p_x+p_y+p_z \tag{4.5}$$

is a (linear) first integral of the Hamiltonian $H=G+V_f$; for the three-body system on a line, this is precisely the mass-center first integral (or the linear momentum integral). Indeed, from

$$\begin{aligned} \partial_x V_f&=f'(x-z)-f'(y-x)\\ \partial_y V_f&=f'(y-x)-f'(z-y)\\ \partial_z V_f&=f'(z-y)-f'(x-z), \end{aligned} \tag{4.6}$$

it follows that:

$$\boldsymbol{\omega}\cdot\nabla V_f=(\partial_x+\partial_y+\partial_z)V_f=0. \quad ●$$

Proposition 4.3: The K-vector $\mathbf{R}$ *is an eigenvector of all elements of* $\mathcal{C}$.

Proof: We observe that the following intrinsic expressions hold for the **basic K-tensors** of $\mathcal{C}$ defined in (3.7)–(3.9) and (3.11):

$$\begin{aligned} \mathbf{Q}&=\mathbf{R}\otimes\mathbf{R}\\ \mathbf{T}&=\boldsymbol{\omega}\otimes\boldsymbol{\omega}\\ \mathbf{I}&=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}\\ \mathbf{S}&=\boldsymbol{\omega}\odot\mathbf{r}-s\mathbf{G}, \end{aligned} \tag{4.7}$$

where

$$r^2=\mathbf{r}\cdot\mathbf{r}=x^2+y^2+z^2, \quad s=\boldsymbol{\omega}\cdot\mathbf{r}=x+y+z. \tag{4.8}$$

Moreover,

$$R^2=\mathbf{R}\cdot\mathbf{R}=3r^2-s^2=3\mathbf{r}_\omega^2, \quad \mathbf{R}\cdot\boldsymbol{\omega}=\mathbf{R}\cdot\mathbf{r}=0, \tag{4.9}$$

where r_ω is the distance from the axis ω. From these expressions it follows that

$$\begin{aligned}\mathbf{QR}&=R^2\mathbf{R}\\ \mathbf{TR}&=0\\ \mathbf{IR}&=r^2\mathbf{R}\\ \mathbf{SB}&=-s\mathbf{R}.\end{aligned} \tag{4.10}$$

■

Remark 4.4: We recall that a Lie algebra structure is defined on the contravariant symmetric tensor fields of a manifold, by equation [see (1.4)]

$$\{P_{\mathbf{A}},P_{\mathbf{B}}\}=P_{[\mathbf{A},\mathbf{B}]}, \tag{4.11}$$

where $\{\cdot,\cdot\}$ are the Poisson–Lie brackets on functions over T^*Q

$$P_{\mathbf{A}}=P(\mathbf{A})=A^{i\cdots j}p_i\cdots p_j,$$

and for a function (symmetric tensor of rank 0) f over Q,

$$P_f=f,$$

where we denote by the same symbol f its natural extension to T^*Q (constant on the fibers). The corresponding Lie-brackets $[\cdot,\cdot]$ are known as Schouten–Nijenhuis brackets (for symmetric tensors), [13,14] whose expression in any local coordinate system is

$$[\mathbf{A},\mathbf{B}]^{i_1\cdots i_{p+q-1}}=pA^{i(i_1\cdots i_{p-1}}\partial_i B^{i_p\cdots i_{p+q-1})}-qB^{i(i_1\cdots i_{q-1}}\partial_i A^{i_q\cdots i_{q+p-1}}, \tag{4.12}$$

where (p,q) are the ranks of $\mathbf{A}$ and $\mathbf{B}$, respectively, and round brackets ( ) around indices denote symmetrization over those indices. We remark that the rank of $[\mathbf{A},\mathbf{B}]$ is $p+q-1$. In particular, for vector fields $\mathbf{X}$ and $\mathbf{Y}$, $[\mathbf{X},\mathbf{Y}]$ are the ordinary Lie-brackets

$$[\mathbf{X},f]=\langle\mathbf{X},df\rangle,$$

is the derivative of the function f with respect to the vector $\mathbf{X}$, and

$$d_{\mathbf{X}}\mathbf{A}=[\mathbf{X},\mathbf{A}],$$

is the Lie-derivative of the tensor $\mathbf{A}$ with respect to the vector $\mathbf{X}$. Thus, the tensor $\mathbf{A}$ is invariant with respect to (the flow generated by) the vector field $\mathbf{X}$ iff $[\mathbf{X},\mathbf{A}]=0$. In particular, $\mathbf{X}$ is a Killing vector iff $[\mathbf{X},\mathbf{G}]=0$. The Leibniz rule holds

$$[\mathbf{A},\mathbf{B}\odot\mathbf{C}]=[\mathbf{A},\mathbf{B}]\odot\mathbf{C}+[\mathbf{A},\mathbf{C}]\odot\mathbf{B}, \tag{4.13}$$

where the symmetric tensor product $\odot$ is defined by

$$P_{\mathbf{A}\odot\mathbf{B}}=P_{\mathbf{A}}P_{\mathbf{B}}.$$

For two vectors we find (2.1). ●

Proposition 4.5: $\mathcal{C}$ is invariant with respect to $\mathbf{R}$*:* $[\mathbf{R},\mathbf{K}]=0$, $\forall\mathbf{K}\in\mathcal{C}$.

Proof: By the Leibniz rule

$$[\mathbf{R},\mathbf{T}]=[\mathbf{R},\boldsymbol{\omega}\odot\boldsymbol{\omega}]=2[\mathbf{R},\boldsymbol{\omega}]\odot\boldsymbol{\omega}=0,$$

$$[\mathbf{R},\mathbf{Q}]=[\mathbf{R},\mathbf{R}\odot\mathbf{R}]=2[\mathbf{R},\mathbf{R}]\odot\mathbf{R}=0,$$

$$[\mathbf{R},\mathbf{I}]=[\mathbf{R},r^2]\mathbf{G}+r^2[\mathbf{R},\mathbf{G}]-2[\mathbf{R},\mathbf{r}]\odot\mathbf{r}=0,$$

$$[\mathbf{R},\mathbf{S}]=[\mathbf{R},\boldsymbol{\omega}]\odot\mathbf{r}+[\mathbf{R},\mathbf{r}]\odot\boldsymbol{\omega}-[\mathbf{R},s]\mathbf{G}-s[\mathbf{R},\mathbf{G}]=0,$$

since $\mathbf{R}$ is a K-vector and $\boldsymbol{\omega}$ and $\mathbf{r}$ are invariant with respect to $\mathbf{R}$ (i.e., under any rotation around ω):

$$[\mathbf{R},\mathbf{G}]=0, \quad [\mathbf{R},\boldsymbol{\omega}]=[\mathbf{R},\mathbf{r}]=0, \quad [\mathbf{R},r^2]=[\mathbf{R},s]=0. \qquad \blacksquare$$

From this invariance it follows that:

Proposition 4.6: All elements of $\mathcal{C}$ have normal eigenvectors.

Proof: The common eigenvector $\mathbf{R}$ is normal: The orthogonal surfaces are the meridian half-planes. Let $(\mathbf{E}_1,\mathbf{E}_2)$ be other two orthogonal eigenvectors of an element $\mathbf{K}\in\mathcal{C}$. These vectors are tangent to the half-planes. Since $\mathbf{K}$ is $\mathbf{R}$-invariant, we can choose these vectors to be $\mathbf{R}$-invariant. Let us consider the integral curves of $\mathbf{E}_1$ on a half-plane. They are orthogonal to $\mathbf{E}_2$. By rotating the half-plane we get surfaces of revolution orthogonal to $\mathbf{E}_2$. Thus $\mathbf{E}_2$ is normal. The same for $\mathbf{E}_1$. ■

It remains to look for the elements of $\mathcal{C}$ with simple eigenvalues: these elements will be characteristic K-tensors of separable orthogonal webs. Since $\mathbf{R}$ is a common eigenvector of $\mathcal{C}$, all these webs will be of revolution around ω and include the foliation of the meridian half-planes. We know five possible separable webs of this kind:

$$\mathcal{W}_{\rm cyl}=\text{circular cylindrical web},$$

$$\mathcal{W}_{\rm par}=\text{circular parabolic web},$$

$$\mathcal{W}_{\rm sph}=\text{spherical web},$$

$$\mathcal{W}_{\rm pro}=\text{prolate spheroidal web},$$

$$\mathcal{W}_{\rm obl}=\text{oblate spheroidal web}.$$

We shall show that in fact there exists in $\mathcal{C}$ a characteristic K-tensor $\mathbf{K}_*$ for any web $\mathcal{W}_*$ of this kind. For a graphical and coordinate representation of these webs see Ref. 15.

Remark 4.7: For each web $\mathcal{W}_*$, a basis of the corresponding KS-algebra $\mathcal{K}_*$ is given by the triple $(\mathbf{K}_*,\mathbf{Q},\mathbf{G})$, where $\mathbf{K}_*$ is a characteristic tensor. Indeed, the tensor $\mathbf{Q}$ is a common element of all these subalgebras, since any vector orthogonal to $\mathbf{R}$ is an eigenvector of $\mathbf{Q}$ (with zero eigenvalues) and, moreover, $\mathbf{Q}$ is in involution with all elements of $\mathcal{C}$, due to the $\mathbf{R}$-invariance

$$[\mathbf{K},\mathbf{Q}]=[\mathbf{K},\mathbf{R}\odot\mathbf{R}]=2[\mathbf{K},\mathbf{R}]\odot\mathbf{R}=0. \qquad \bullet$$

Remark 4.8: The axially symmetric orthogonal webs listed above, with the exception of the first one, are centered: They refer to a distinguished point on the polar axis ω, which in our case is the origin O of the coordinates. However, due to the invariance with respect to $\boldsymbol{\omega}$, any arbitrary translation along the line ω (which produces a translation of the center) leads to webs which are still separable for the Calogero system. ●

In the following discussion, among the eigevectors we shall find the vector fields

$$\mathbf{r}\times\mathbf{R}=\begin{bmatrix} y^2+z^2-x(y+z) \\ z^2+x^2-y(z+x) \\ x^2+y^2-z(x+y)\end{bmatrix}=r^2\boldsymbol{\omega}-s\mathbf{r}, \quad \boldsymbol{\omega}\times\mathbf{R}=\begin{bmatrix} y+z-2x \\ z+x-2y \\ x+y-2z\end{bmatrix}=s\boldsymbol{\omega}-3\mathbf{r}. \tag{4.14}$$

Besides (4.10), we shall use the following formulas, derived from (4.7):

$$\begin{gathered} \mathbf{Qr}=0 \\ \mathbf{Tr}=s\boldsymbol{\omega} \\ \mathbf{Ir}=0 \\ \mathbf{Sr}=\tfrac{1}{2}r^2\boldsymbol{\omega}-\tfrac{1}{2}s\mathbf{r}, \end{gathered} \tag{4.15}$$

$$\begin{aligned}\mathbf{Q}\boldsymbol{\omega}&=0\\ \mathbf{T}\boldsymbol{\omega}&=3\boldsymbol{\omega}\\ \mathbf{I}\boldsymbol{\omega}&=r^2\boldsymbol{\omega}-s\mathbf{r}\\ \mathbf{S}\boldsymbol{\omega}&=\tfrac{3}{2}\mathbf{r}-\tfrac{1}{2}s\boldsymbol{\omega},\end{aligned}\tag{4.16}$$

$$\begin{aligned}\mathbf{Q}(\mathbf{r}\times\mathbf{R})&=0\\ \mathbf{T}(\mathbf{r}\times\mathbf{R})&=R^2\boldsymbol{\omega}\\ \mathbf{I}(\mathbf{r}\times\mathbf{R})&=r^2\mathbf{r}\times\mathbf{R}\\ \mathbf{S}(\mathbf{r}\times\mathbf{R})&=\tfrac{1}{2}R^2\mathbf{r}-s\mathbf{r}\times\mathbf{R},\end{aligned}\tag{4.17}$$

$$\begin{aligned}\mathbf{Q}(\boldsymbol{\omega}\times\mathbf{R})&=0\\ \mathbf{T}(\boldsymbol{\omega}\times\mathbf{R})&=0\\ \mathbf{I}(\boldsymbol{\omega}\times\mathbf{R})&=r^2\boldsymbol{\omega}\times\mathbf{R}+R^2\mathbf{r}\\ \mathbf{S}(\boldsymbol{\omega}\times\mathbf{R})&=-\tfrac{1}{2}R^2\boldsymbol{\omega}-s\boldsymbol{\omega}\times\mathbf{R}.\end{aligned}\tag{4.18}$$

V. THE CIRCULAR CYLINDRICAL WEB $\mathcal{W}_{\mathrm{cyl}}$

This web is determined by a single geometrical element: A line ω (the polar axis). It is made of the half-planes issued from ω (the meridian half-planes), the planes orthogonal to ω, and the circular cylinders with axis ω. The singular set (where the web is not defined) is ω. These surfaces are, respectively, orthogonal to the vector fields

$$(\mathbf{R},\boldsymbol{\omega},\boldsymbol{\omega}\times\mathbf{R}),\tag{5.1}$$

where: $\boldsymbol{\omega}$ is a (constant) vector parallel to ω, $\mathbf{R}=\boldsymbol{\omega}\times\mathbf{r}$, and $\mathbf{r}=OP$ is the position vector of the generic point $P\in\mathbb{E}_3$ with respect a point $O\in\omega$.

We can see from (4.10), (4.16), and (4.18) that the vectors (5.1), where $\boldsymbol{\omega}$ and $\mathbf{R}$ are just the vectors defined in (4.1) and (4.3), are eigenvectors of the K-tensor $\mathbf{Q}-\mathbf{T}\in\mathcal{C}$, with eigenvalues

$$(R^2,-3,0).\tag{5.2}$$

These eigeivalues are simple on $\mathbb{E}_3\backslash\omega$, since $R^2=0$ on ω. Thus, we have proved

Proposition 5.1: The Calogero system is separable in the circular cylindrical web $\mathcal{W}_{\mathrm{cyl}}$ with axis ω and characteristic tensor

$$\mathbf{K}_{\mathrm{cyl}}=\mathbf{Q}-\mathbf{T}=\mathbf{R}\otimes\mathbf{R}-\boldsymbol{\omega}\otimes\boldsymbol{\omega}.\tag{5.3}$$

We observe that also $\mathbf{Q}+\mathbf{T}$ is a characteristic tensor everywhere, with the exclusion of the polar axis and of the circular cylinder $R^2=3$. Indeed, the eigenvalues of this tensor are $(R^2,3,0)$.

VI. THE PARABOLIC WEB $\mathcal{W}_{\mathrm{par}}$

This web is determined by two geometrical elements (ω,O): A line ω (the axis) and a point $O\in\omega$ (the focus or center). It is made of the meridian half-planes and of the two families of paraboloids of revolution around ω with focus O. The singular set is ω. It can be shown that a triple of vector fields orthogonal to these surfaces is

$$(\mathbf{R},\mathbf{u}+\mathbf{d},\mathbf{u}-\mathbf{d}),\tag{6.1}$$

where $\mathbf{d}$ is a director of the axis ω, $\mathbf{u}$ is the unit vector determined by the radius vector $\mathbf{r}$ referred to the center O

$$\mathbf{u}=\frac{\mathbf{r}}{r},\tag{6.2}$$

and $\mathbf{R}$ is any rotation vector around ω. From (4.7), (4.10), (4.15), and (4.16) we can see that the vectors (6.1), where $\mathbf{d}$ and $\mathbf{R}$ are defined in (4.2) and (4.3), are eigenvectors of

$$\mathbf{K}_O=\mathbf{S}=\boldsymbol{\omega}\odot\mathbf{r}-\boldsymbol{\omega}\cdot\mathbf{r}\mathbf{G}\in\mathcal{C}, \tag{6.3}$$

with eigenvalues

$$(-s,\tfrac{1}{2}(\sqrt{3}r-s),-\tfrac{1}{2}(\sqrt{3}r+s)). \tag{6.4}$$

These eigenvalues are simple on $\mathbb{E}_3\backslash\omega$. Indeed, on ω they assume the values $(-s,0,-s)$, since $\mathbf{r}=r\mathbf{d}$ and $s=\boldsymbol{\omega}\cdot\mathbf{r}=\sqrt{3}\mathbf{d}\cdot\mathbf{r}=\sqrt{3}r$.

Let us consider the position vector $\mathbf{r}_C$ with respect to a point $C\in\omega$

$$\mathbf{r}_C=\mathbf{r}-t\mathbf{d}=\mathbf{r}-\frac{1}{\sqrt{3}}t\boldsymbol{\omega}, \tag{6.5}$$

where

$$|OC|^2=t^2. \tag{6.6}$$

Then

$$\mathbf{K}_C=\boldsymbol{\omega}\odot\mathbf{r}_C-\boldsymbol{\omega}\cdot\mathbf{r}_C\mathbf{G} \tag{6.7}$$

is a characteristic tensor of the parabolic web centered at $C\in\omega$. By inserting (6.5) in (6.7), we can see that also this tensor is an element of C, in agreement with Remark 4.8

$$\mathbf{K}_C=\mathbf{S}-\frac{1}{\sqrt{3}}t\mathbf{T}+\sqrt{3}t\mathbf{G}. \tag{6.8}$$

Hence, we have proved [the last term in (6.8) can be disregarded]

Proposition 6.1: The Calogero system is separable in any parabolic web with axis ω *and focus* $C\in\omega$, *and with characteristic tensor*

$$\mathbf{K}_{\text{par}}=\mathbf{S}-\frac{1}{\sqrt{3}}t\mathbf{T}=\boldsymbol{\omega}\odot\mathbf{r}-\boldsymbol{\omega}\cdot\mathbf{r}\mathbf{G}-\frac{1}{\sqrt{3}}t\boldsymbol{\omega}\otimes\boldsymbol{\omega}\quad(t^2=|OC|^2). \tag{6.9}$$

VII. THE SPHERICAL WEB $\mathcal{W}_{\text{sph}}$

This web is determined by two geometrical elements (ω, O): A line ω (the polar axis) and a point $O\in\omega$ (the center). It is made of the half-planes issued from ω (the meridian half-planes), the spheres centered at O and the circular cones with axis ω and vertex O. The singular set is the axis ω. A triple of vectors orthogonal to these surfaces is

$$(\mathbf{R},\mathbf{r},\mathbf{r}\times\mathbf{R}). \tag{7.1}$$

From (4.7), (4.15), and (4.17) we can see that these vectors are eigenvectors of the K-tensor

$$\mathbf{K}_O=\mathbf{I}+\mathbf{Q}=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}+\mathbf{R}\otimes\mathbf{R}\in\mathcal{C}, \tag{7.2}$$

with eigenvalues

$$(R^2+r^2,0,r^2). \tag{7.3}$$

These eigenvalues are simple on $\mathbb{E}_3 \backslash \omega$, since $R^2=0$ on ω. By applying again the transformation (6.5), we can see that also the tensor

$$\mathbf{K}_C = r_C^2 \mathbf{G} - \mathbf{r}_C \otimes \mathbf{r}_C + \mathbf{R} \otimes \mathbf{R} = \mathbf{I} + \mathbf{Q} + \frac{2}{\sqrt{3}} t\mathbf{S} - \frac{1}{3} t^2 \mathbf{T} + t^2 \mathbf{G} \tag{7.4}$$

is an element of C (Remark 4.8). Thus we have proved

Proposition 7.1: The Calogero system is separable in any spherical web with polar axis ω *and center* $C \in \omega$, *and with characteristic tensor*

$$\mathbf{K}_{\mathrm{sph}} = \mathbf{I} + \mathbf{Q} + \frac{2}{\sqrt{3}} t\mathbf{S} - \frac{1}{3} t^2 \mathbf{T} = r^2 \mathbf{G} - \mathbf{r} \otimes \mathbf{r} + \mathbf{R} \otimes \mathbf{R} + \frac{2}{\sqrt{3}} t \boldsymbol{\omega} \odot \mathbf{r} - \boldsymbol{\omega} \cdot \mathbf{r} \mathbf{G} - \frac{1}{3} t^2 \boldsymbol{\omega} \otimes \boldsymbol{\omega} \quad (t^2 = |OC|^2). \tag{7.5}$$

VIII. THE PROLATE SPHEROIDAL WEB $\mathcal{W}_{\mathrm{pro}}$

This web is determined by three elements (ω, O, c): A line ω (the polar axis), a point $O \in \omega$ (the center) and a positive constant c (the parameter). The parameter c defines on ω two points (foci) (F_1, F_2), whose distance from O is c and whose position vectors are

$$\mathbf{r}_1 = \mathbf{r} - c\mathbf{d}, \quad \mathbf{r}_2 = \mathbf{r} + c\mathbf{d}, \tag{8.1}$$

where $\mathbf{d}$ is a unit vector parallel to ω (the director). This web is made of the meridian half-planes and of the quadrics of revolution, ellipsoids and two-folded hyperboloids, obtained by rotating around ω the confocal conics (ellipses and hyperbolae) with foci (F_1, F_2) over the meridian planes. The singular set is ω. It can be shown (see for details Ref. 12) that the vectors

$$(\mathbf{R}_{\mathbf{d}}, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2), \tag{8.2}$$

where

$$\mathbf{R}_{\mathbf{d}} = \mathbf{d} \times \mathbf{r}, \quad \mathbf{u}_1 = \mathbf{r}_1 / r_1, \quad \mathbf{u}_2 = \mathbf{r}_2 / r_2 \tag{8.3}$$

are orthogonal to these three families of surfaces, and that they are eigenvectors of the tensor

$$\mathbf{K}_O = r^2 \mathbf{G} - \mathbf{r} \otimes \mathbf{r} + c^2 \mathbf{d} \otimes \mathbf{d}, \tag{8.4}$$

with eigenvalues

$$(r^2, \tfrac{1}{4}(r_1 - r_2)^2, \tfrac{1}{4}(r_1 + r_2)^2). \tag{8.5}$$

These eigevalues are simple on $\mathbb{E}_3 \backslash \omega$. We recognize the tensor $\mathbf{K}_O$ as an element of $\mathcal{C}$

$$\mathbf{K}_O = \mathbf{I} + \frac{c^2}{3} \mathbf{T} = r^2 \mathbf{G} - \mathbf{r} \otimes \mathbf{r} + \frac{c^2}{3} \boldsymbol{\omega} \otimes \boldsymbol{\omega}. \tag{8.6}$$

By applying the transformation (6.5), we can see that

$$\mathbf{K}_C = r_C^2 \mathbf{G} - \mathbf{r}_C \otimes \mathbf{r}_C + c^2 \mathbf{d} \otimes \mathbf{d} = I + \frac{2}{\sqrt{3}} t\mathbf{S} + \frac{1}{3}(c^2 - t^2)\mathbf{T} + t^2 \mathbf{G}. \tag{8.7}$$

Thus,

Proposition 8.1: The Calogero system is separable in any prolate spheroidal web with polar axis ω *and center* $C \in \omega$, *with any value of the parameter c, and with characteristic tensor*

$$\mathbf{K}_{\text{pro}}=\mathbf{I}+\frac{2}{\sqrt{3}}t\mathbf{S}+\frac{1}{3}(c^2-t^2)\mathbf{T}$$
$$=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}+\frac{2}{\sqrt{3}}t(\boldsymbol{\omega}\odot\mathbf{r}-\boldsymbol{\omega}\cdot\mathbf{r}\mathbf{G})+\frac{1}{3}(c^2-t^2)\boldsymbol{\omega}\otimes\boldsymbol{\omega}\quad(t^2=|OC|^2). \tag{8.8}$$

IX. THE OBLATE SPHEROIDAL WEB $\mathcal{W}_{\text{obl}}$

This web is again determined by three elements (ω,O,c). The positive constant c defines on the equatorial plane Ω a circle Γ of radius c and center O. On each meridian plane we consider the confocal conics with foci (F_1,F_2) belonging to Γ. The web is made by the ellipsoids and one-folded hyperboloids of revolution generated by these confocal conics, and by the meridian half-planes. The singular set of this web is $\Gamma\cup\omega$. With each position vector $\mathbf{r}$ (for points $\notin\omega$) we associate its projection $\mathbf{c}$ onto the equatorial plane, renormalized in such a way that $\mathbf{c}\cdot\mathbf{c}=c^2$. This vector is defined by

$$\mathbf{c}=\frac{c}{r_\omega}(\mathbf{r}-\mathbf{r}\cdot\mathbf{d}\mathbf{d}).$$

Then we define the position vectors with respect to the foci F_1 and F_2

$$\mathbf{r}_1=\mathbf{r}-\mathbf{c},\quad \mathbf{r}_2=\mathbf{r}+\mathbf{c},$$

and the corresponding unit vectors

$$\mathbf{u}_1=\mathbf{r}_1/r_1,\quad \mathbf{u}_2=\mathbf{r}_2/r_2.$$

It can be shown[12] that the vectors

$$(\mathbf{R}_\mathbf{d},\mathbf{u}_1+\mathbf{u}_2,\mathbf{u}_1-\mathbf{u}_2)$$

are orthogonal to the surfaces of the web, and that they are eigevectors of the tensor

$$\mathbf{K}_O=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}-c^2\mathbf{d}\otimes\mathbf{d}, \tag{9.1}$$

with eigenvalues

$$(r^2,\tfrac{1}{2}(\alpha-1)r_1r_2,\tfrac{1}{2}(\alpha+1)r_1r_2),\quad \alpha=\mathbf{u}_1\cdot\mathbf{u}_2. \tag{9.2}$$

These eigenvalues are simple on $\mathbb{E}_3\backslash(\Gamma\cup\omega)$, so that the tensor (9.1) is a characteristic tensor of the oblate spheroidal web with axis ω and center O, determined by the triple $(O,\mathbf{d},c)$. We recognize the tensor $\mathbf{K}$ as an element of C

$$\mathbf{K}_O=\mathbf{I}-\frac{c^2}{3}\mathbf{T}=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}-\frac{c^2}{3}\boldsymbol{\omega}\otimes\boldsymbol{\omega}. \tag{9.3}$$

By the transformation (6.5) we can see that

$$\mathbf{K}_C=r_C^2\mathbf{G}-\mathbf{r}_C\otimes\mathbf{r}_C-c^2\mathbf{d}\otimes\mathbf{d}=\mathbf{I}+\frac{2}{\sqrt{3}}t\mathbf{S}-\frac{1}{3}(c^2+t^2)\mathbf{T}+t^2\mathbf{G}. \tag{9.4}$$

Hence, we have proved

Proposition 9.1: The Calogero system is separable in any oblate spheroidal web with polar axis ω and center $C\in\omega$, with any value of the parameter c, and with characteristic tensor

$$\mathbf{K}_{\mathrm{obl}}=\mathbf{I}+\frac{2}{\sqrt{3}}t\mathbf{S}-\frac{1}{3}(c^2+t^2)\mathbf{T}$$

$$=r^2\mathbf{G}-\mathbf{r}\otimes\mathbf{r}+\frac{2}{\sqrt{3}}t(\boldsymbol{\omega}\odot\mathbf{r}-\boldsymbol{\omega}\cdot\mathbf{r}\mathbf{G})-\frac{1}{3}(c^2+t^2)\boldsymbol{\omega}\otimes\boldsymbol{\omega}\quad (t^2=|OC|^2). \tag{9.5}$$

X. THE FIRST INTEGRALS ASSOCIATED WITH THE SEPARATION

As it was mentioned in in the Introduction and in Sec. IV, each separable web $\mathcal{W}_*$ generates a three-dimensional KS-algebra $\mathcal{K}_*\subset C$, including the characteristic tensor $\mathbf{K}_*$, the tensor $\mathbf{Q}$ and the metric tensor $\mathbf{G}$. This subalgebra is also commutative, interpreting the K-tensors as linear operators. Indeed, all tensors of $\mathcal{K}_*$ have common eigenvectors and at least one of them (the characteristic tensor) has simple eigenvalues. Furthermore, $\mathcal{K}_*$ generates a three-dimensional involutive algebra $\mathcal{H}_*$ of first integrals defined by Eq. (1.5) and (1.6). For computing all these first integrals, one should integrate Eq. (1.6) (i.e., the closed one-form $\mathbf{K}dV$) for each one of the basic elements (4.7) of $\mathcal{C}$. However, this cumbersome process of integration can be avoided, due to a remarkable property of the inverse-square Calogero potential. To show this, we recall a general property of the orthogonal separable systems, expressed by formula (1.8) in the Introduction:

Proposition 10.1: Let (q^i) be orthogonal separable coordinates and let

$$V=g^{ii}\phi_i,\quad g^{ii}=\mathbf{G}(dq^i,dq^i),\quad \partial_j\phi_i=0\ \ (i\neq j) \tag{10.1}$$

be the expression of a separable potential in these coordinates (each ϕ_i is a function of the corresponding coordinate q^i only). Then, for each element $\mathbf{K}$ of the KS-algebra $\mathcal{K}$ corresponding to these coordinates, a solution $V_{\mathbf{K}}$ of Eq. (1.6) is

$$V_{\mathbf{K}}=\lambda_i g^{ii}\phi_i=K^{ii}\phi_i, \tag{10.2}$$

where λ_i are the eigenvalues of $\mathbf{K}$ corresponding to the eigenforms dq^i.

Proof: Equation (1.6) is equivalent to

$$\partial_j V_{\mathbf{k}}=\lambda_j\partial_j V, \tag{10.3}$$

while the Killing equation $[\mathbf{K},\mathbf{G}]=0$ is equivalent to

$$K^{jj}\partial_j g^{ii}=g^{jj}\partial_j K^{ii}$$

(no sum over the repeated indices) that is to

$$\lambda_j\partial_j g^{ii}=\partial_j K^{ii}.$$

It follows that:

$$\partial_j(K^{ii}\phi_i)=\partial_j K^{ii}\phi_i+K^{ii}\partial_j\phi_i=\lambda_j\partial_j g^{ii}\phi_i+\lambda_j g^{ii}\partial_j\phi_i=\lambda_j\partial_j(g^{ii}\phi_i)=\lambda_j\partial_j V.$$

This shows that (10.2) is a solution of Eq. (10.3). ■

For the Calogero system, all separable webs are of revolution around ω and the rotation angle ψ, mesured from a fixed meridian half-plane, can be chosen as a coordinate (say $q^3=\psi$) in any separable coordinate system (q^i) (see Fig. 1). Its gradient $\nabla\psi$ is proportional to the rotational vector $\mathbf{R}$, and the orientation of ψ can be chosen in such a way that

$$\nabla\psi=\frac{\mathbf{d}\times\mathbf{r}}{r_\omega^2}=\frac{\mathbf{R}}{\sqrt{3}r_\omega^2}. \tag{10.4}$$

Moreover,

$$g^{33}=\nabla\psi\cdot\nabla\psi=\frac{1}{r_\omega^2}, \tag{10.5}$$

where r_ω is the distance from the axis ω.

Proposition 10.2: In any separable coordinate system (q^i) *with* $q^3=\psi$, *the Calogero potential has the form*

$$V=g^{33}\phi_3=\frac{1}{r_\omega^2}\phi(\psi), \tag{10.6}$$

where $\phi_3=\phi(\psi)$ *is a function of the rotation angle only.*

This means that in the expression (10.1) of the Calogero potential, $\phi_1=\phi_2=0$.

Proof: Let us consider the function

$$\phi=r_\omega^2 V.$$

This function is obviously invariant with respect to $\boldsymbol{\omega}$, i.e., $\boldsymbol{\omega}\cdot\nabla\phi=0$, since both functions r_ω and V are invariant. The function ϕ is also invariant with respect to $\mathbf{r}$. To show this we introduce the vector

$$\mathbf{r}_\omega=\mathbf{r}-\tfrac{1}{3}\boldsymbol{\omega}\cdot\mathbf{r}\boldsymbol{\omega}, \tag{10.7}$$

representing the component of $\mathbf{r}$ orthogonal to the axis ω, and such that $\mathbf{r}_\omega\cdot\mathbf{r}_\omega=r_\omega^2$. Because of the meaning of r_ω (the distance from the axis ω), $\nabla r_\omega=(r_\omega)^{-1}\mathbf{r}_\omega$, so that

$$\mathbf{r}\cdot\nabla r_\omega^2=2r_\omega\mathbf{r}\cdot\nabla r_\omega=2r_\omega^2.$$

For the Calogero potential (see Remark 10.3 below)

$$\mathbf{r}\cdot\nabla V=-2V, \tag{10.8}$$

so that $\mathbf{r}\cdot\nabla\phi=\mathbf{r}\cdot\nabla(r_\omega^2V)=\mathbf{r}\cdot\nabla r_\omega^2V+\mathbf{r}\cdot\nabla Vr_\omega^2=0$. A function which is invariant with respect to $\boldsymbol{\omega}$ and $\mathbf{r}$ is invariant with respect to $\boldsymbol{\omega}$ and $\mathbf{r}_\omega$, thus it is a function of the angle ψ only. ∎

Remark 10.3: For a potential V_f (4.4),

$$\mathbf{r}\cdot\nabla V_f=(x\partial_x+y\partial_y+z\partial_z)V_f=(x-z)f'(x-z)+(y-x)f'(y-x)+(z-y)f'(z-y).$$

If $f(u)=u^p$, then $\mathbf{r}\cdot\nabla V_f=pV_f$. Therefore, the crucial condition (10.8) holds only for the inverse square potential $(p=-2)$. ●

From Propositions 10.1 and 10.2 it follows that:

Proposition 10.4: For all $\mathbf{K}\in C$, a solution of Eq. (1.6) is

$$V_{\mathbf{K}}=\lambda_{\mathbf{K}}V=\lambda_{\mathbf{K}}\frac{\phi}{r_\omega^2}, \tag{10.9}$$

where $\lambda_{\mathbf{K}}$ is the eigenvalue of $\mathbf{K}$ associated with the eigenvector $\mathbf{R}$.

For the basic elements of C, Eqs. (4.10) show that

$$\begin{gathered}\lambda_{\mathbf{Q}}=R^2=3r_\omega^2=(x-z)^2+(y-x)^2+(z-y)^2\\ \lambda_{\mathbf{T}}=0\\ \lambda_{\mathbf{I}}=r^2=x^2+y^2+z^2\\ \lambda_{\mathbf{S}}=-s=-(x+y+z),\end{gathered} \tag{10.10}$$

hence, by (10.9)

$$
\begin{aligned}
V_{\mathbf{Q}} &= 3\phi \\
V_{\mathbf{T}} &= 0 \\
V_{\mathbf{I}} &= r^2 V = \frac{r^2}{r_\omega^2}\phi \\
V_{\mathbf{S}} &= -sV = -\frac{s}{r_\omega^2}\phi.
\end{aligned} \tag{10.11}
$$

Using (10.9) and (10.10) we could write the expressions of these functions in Cartesian coordinates. Furthermore, if we introduce the momentum vector

$$
\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \tag{10.12}
$$

and the associated quantities

$$
\begin{aligned}
p_\omega &= P_{\boldsymbol{\omega}} = \boldsymbol{\omega}\cdot\mathbf{p} = p_x + p_y + p_z \\
p_r &= \mathbf{r}\cdot\mathbf{p} = xp_x + yp_y + zp_z \\
p_\psi &= \mathbf{M}\cdot\mathbf{d} = \mathbf{r}\times\mathbf{p}\cdot\mathbf{d} = \frac{1}{\sqrt{3}}((y-x)p_z + (z-y)p_x + (x-z)p_y),
\end{aligned} \tag{10.13}
$$

$$
\mathbf{M} = \mathbf{r}\times\mathbf{p} = \begin{bmatrix} yp_z - zp_y \\ zp_x - xp_z \\ xp_y - yp_x \end{bmatrix}, \tag{10.14}
$$

the **mass-center momentum**, the **radial momentum**, the **axial angular momentum** (with respect to the polar axis ω), and the **angular momentum vector**, respectively, then we derive from (4.7)

$$
\begin{aligned}
P_{\mathbf{Q}} &= 3p_\psi^2 \\
P_{\mathbf{T}} &= p_\omega^2 \\
P_{\mathbf{I}} &= r^2 p^2 - p_r^2 = M^2 \\
P_{\mathbf{S}} &= p_\omega p_r - sp^2.
\end{aligned} \tag{10.15}
$$

The resulting **basic first integrals** (1.5) are

$$
\begin{aligned}
H_{\mathbf{Q}} &= 3(\tfrac{1}{2}p_\psi^2 + \phi) \\
H_{\mathbf{T}} &= \tfrac{1}{2}p_\omega^2 \\
H_{\mathbf{I}} &= \frac{1}{2}M^2 + \frac{r^2}{r_\omega^2}\phi \\
H_{\mathbf{S}} &= \frac{1}{2}(p_\omega p_r - sp^2) - \frac{s}{r_\omega^2}\phi.
\end{aligned} \tag{10.16}
$$

Using (10.9), (10.10), (10.13), and (10.14), we could express these first integrals in Cartesian coordinates and see that they are rational in (x,y,z) (and, of course, quadratic in the momenta). From (10.9) it follows that all these first integrals (except p_ω) are not defined on the meridian planes passing through the Cartesian axes (see Remark 4.1).

Remark 10.5: The existence of rational first integrals for the n-body Calogero system has been proved in Ref. 16. Actually, $H_{\mathbf{T}}$ is equivalent to the linear first integral p_ω, which appears in the KS-algebra of the cylindrical web (see below). The existence of four independent first integrals, besides the Hamiltonian itself, shows that the Calogero system is ''*maximally super-integrable*,''

TABLE I. Basic elements of the separable webs for the calogero system.

Web $\mathcal{W}_*$	Basis of $\mathcal{K}_*$	Basis of $\mathcal{H}_*$
$\mathcal{W}_{\rm cyl}$	**G, Q, T**	$H, H_{\mathbf{Q}}, H_{\mathbf{T}}$
$\mathcal{W}_{\rm par}$	**G, Q, S**	$H, H_{\mathbf{Q}}, H_{\mathbf{S}}$
$\mathcal{W}_{\rm sph}$	**G, Q, I**	$H, H_{\mathbf{Q}}, H_{\mathbf{I}}$
$\mathcal{W}_{\rm pro}$	$\mathbf{G},\mathbf{Q},\mathbf{I}+(c^2/3)\mathbf{T}$	$H, H_{\mathbf{Q}}, H_{\mathbf{I}}+(c^2/3)H_{\mathbf{T}}$
$\mathcal{W}_{\rm obl}$	$\mathbf{G},\mathbf{Q},\mathbf{I}-(c^2/3)\mathbf{T}$	$H, H_{\mathbf{Q}}, H_{\mathbf{I}}-(c^2/3)H_{\mathbf{T}}$

in the sense that it has the maximal number of first integrals allowed by its dimension. (The maximal number of independent first integrals of a N-dimensional dynamical system is $N-1$; for a Hamiltonian system $N=2n$, and in our case $N=6$. About the definition of super-integrability see, for instance, Refs. 3, 4, and 17.) However, if one deserves the term ''*Arnold–Liouville super-integrable*'' to those Hamiltonian systems which admit at least two inequivalent systems of n independent first integrals in involution, generating two distict Lagrangian foliations in the cotangent bundle T^*Q, then the above discussion shows that the Calogero system is in fact, as any super-separable system, also AL-super-integrable. ●

As a conclusion of the above discussion, in Table I we list for each separable web $\mathcal{W}_*$ of the Calogero system (we consider only the webs centered at the origin O of the coordinates) the basic elements of the KS-algebra $\mathcal{K}_*$ and of the involutive function algebra $\mathcal{H}_*$.

Remark 10.6: The first integral $H_{\mathbf{Q}}$ is in involution with all the basic first integrals (10.16). Therefore, it belongs to every involutive function algebra $\mathcal{H}_*$ associated with the separation. Since

$$p_\psi = r_\omega^2 \dot{\psi}, \tag{10.17}$$

it yields the constant of motion[6]

$$\tfrac{1}{2} r_\omega^4 \dot{\psi}^2 + \phi = \text{constant}. \tag{10.18}$$

The explicit expression of the function $\phi(\psi)$ can be found in Ref. 5

$$\phi(\psi) = \frac{9g}{2\sin^2(3\psi)}. \tag{10.19}$$

The following proof of this formula exhibits some interesting properties of the inverse-square potential. Let us consider the equatorial plane Ω and the projection operator which associates with any vector $\mathbf{v}$ the orthogonal projection $\mathbf{v}_\omega$ over Ω

$$\mathbf{v}_\omega = \mathbf{v} - \tfrac{1}{3}\boldsymbol{\omega}\cdot\mathbf{v}\boldsymbol{\omega}, \tag{10.20}$$

so that $\mathbf{v}_\omega\cdot\boldsymbol{\omega}=0$ [see (10.7) and Fig. 1 for the case of the position vector $\mathbf{r}$]. Let us consider the projections of the basic coordinate vectors (**X, Y, Z**)

$$\begin{aligned} \mathbf{X}_\omega &= \mathbf{X} - \tfrac{1}{3}\boldsymbol{\omega}\cdot X\boldsymbol{\omega} \\ \mathbf{Y}_\omega &= \mathbf{Y} - \tfrac{1}{3}\boldsymbol{\omega}\cdot \mathbf{Y}\boldsymbol{\omega} \\ \mathbf{Z}_\omega &= \mathbf{Z} - \tfrac{1}{3}\boldsymbol{\omega}\cdot \mathbf{Z}\boldsymbol{\omega}. \end{aligned} \tag{10.21}$$

For these vectors

$$\mathbf{X}_\omega\cdot\mathbf{X}_\omega = \tfrac{2}{3},\ldots,\quad \mathbf{X}_\omega\cdot\mathbf{Y}_\omega = -\tfrac{1}{3},\ldots\ . \tag{10.22}$$

For any position vector $\mathbf{r}$, we consider the angles (ψ_x, ψ_y, ψ_z) formed by $\mathbf{r}_\omega$ and $(\mathbf{X}_\omega, \mathbf{Y}_\omega, \mathbf{Z}_\omega)$, respectively (see Fig. 2), and oriented in such a way that

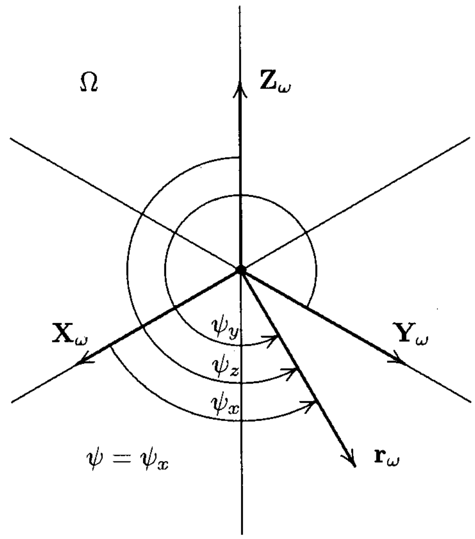

FIG. 2. Orthogonal projection onto the equatorial plane Ω.

$$\mathbf{X}_\omega \times \mathbf{r}_\omega = |\mathbf{X}_\omega||\mathbf{r}_\omega| \sin \psi_x \mathbf{d} = \sqrt{\tfrac{2}{3}}\, r_\omega \sin \psi_x \mathbf{d}, \quad \ldots \quad (\boldsymbol{\omega} = \sqrt{3}\mathbf{d}). \tag{10.23}$$

We choose $\psi = \psi_x$ as the fundamental angle. Thus,

$$\psi_y = \psi + \tfrac{2}{3}\pi, \quad \psi_z = \psi + \tfrac{4}{3}\pi \simeq \psi - \tfrac{2}{3}\pi$$

and

$$\sin(3\psi_y) = \sin(3\psi), \quad \sin(3\psi_z) = \sin(3\psi). \tag{10.24}$$

From the definitions (10.21) it follows that:

$$\mathbf{r}_\omega \times \mathbf{X}_\omega = \mathbf{r} \times \mathbf{X} - \tfrac{1}{3} s\, \boldsymbol{\omega} \times \mathbf{X} - \tfrac{1}{3} \mathbf{r} \times \boldsymbol{\omega} = \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix} - \frac{s}{3} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} z-y \\ x-z \\ y-x \end{bmatrix} = \frac{1}{3}(z-y)\boldsymbol{\omega} = \frac{1}{3} \mathbf{R} \cdot \mathbf{X} \boldsymbol{\omega}. \tag{10.25}$$

Similar results hold for $\mathbf{Y}_\omega$ and $\mathbf{Z}_\omega$. The comparison with (10.23) shows that

$$(z-y)^2 = 2 r_\omega^2 \sin^2 \psi_x = \tfrac{2}{3} R^2 \sin^2 \psi_x. \tag{10.26}$$

Thus,

$$\phi = r_\omega^2 V = \frac{g}{2} \left(\frac{1}{\sin^2 \psi_x} + \frac{1}{\sin^2 \psi_y} + \frac{1}{\sin^2 \psi_z} \right). \tag{10.27}$$

This proves once more that ϕ is a function of the angle ψ alone. We can transform this expression by using the identity

$$\sin 3\psi = \sin \psi (3 - 4 \sin^2 \psi),$$

so that, due to (10.24)

$$\phi=\frac{g}{2\sin^2(3\psi)}[(3-4\sin^2\psi_x)^2+(3-4\sin^2\psi_y)^2+(3-4\sin^2\psi_z)^2]$$
$$=\frac{g}{2\sin^2(3\psi)}[27+16(\sin^4\psi_x+\sin^4\psi_y+\sin^4\psi_z)-24(\sin^2\psi_x+\sin^2\psi_y+\sin^2\psi_z)]. \tag{10.28}$$

From (10.23)

$$\sin^2\psi_x=\frac{3}{2}\frac{(z-y)^2}{R^2},\quad \ldots .$$

Thus

$$\sin^2\psi_x+\sin^2\psi_y+\sin^2\psi_z=\tfrac{3}{2}. \tag{10.29}$$

Actually, this last formula is a general identity which holds for any triple (ϕ_x,ψ_y,ψ_z) of angles differing by $\frac{2}{3}\pi$ (modulo 2π). It is remarkable that a similar formula holds for the fourth powers

$$\sin^4\psi_x+\sin^4\psi_y+\sin^4\psi_z=\tfrac{9}{8}. \tag{10.30}$$

Indeed,

$$\sin^4\psi_x=\sin^2\psi_x-\tfrac{1}{4}\sin^2(2\psi_x),\quad \ldots .$$

Summing these three expressions, we can apply formula (10.28) to the angles $(2\psi_x,2\psi_y,2\psi_z)$ and get (10.30). Thus, (10.28) leads to (10.19). ●

Remark 10.7: For the basic K-tensors (4.7), Eqs. (1.3) are, respectively, equivalent to

$$\begin{gathered}d(\mathbf{R}\cdot\nabla V\mathbf{R}^b)=0\\ d(\boldsymbol{\omega}\cdot\nabla V\boldsymbol{\omega}^b)=0\\ d(\mathbf{r}\cdot\nabla V\mathbf{r}^b-r^2dV)=0\\ d(\boldsymbol{\omega}\cdot\nabla V\mathbf{r}^b+\mathbf{r}\cdot\nabla V\boldsymbol{\omega}^b-2\boldsymbol{\omega}\cdot\mathbf{r}dV)=0,\end{gathered} \tag{10.31}$$

where

$$\begin{gathered}\mathbf{R}^b=(z-y)dx+(x-z)dy+(y-x)dz=\sqrt{3}r_\omega^2d\psi\\ \mathbf{r}^b=xdx+ydy+zdz=rdr\\ \boldsymbol{\omega}^b=dx+dy+dz=ds,\end{gathered} \tag{10.32}$$

while Eqs. (1.6) are equivalent to

$$\begin{gathered}\mathbf{R}\cdot\nabla V\mathbf{R}^b=dV_{\mathbf{Q}},\\ \boldsymbol{\omega}\cdot\nabla V\boldsymbol{\omega}^b=0,\\ \mathbf{r}\cdot\nabla V\mathbf{r}^b-r^2dV=dV_{\mathbf{I}},\\ \boldsymbol{\omega}\cdot\nabla V\mathbf{r}^b+\mathbf{r}\cdot\nabla V\boldsymbol{\omega}^b-2\boldsymbol{\omega}\cdot\mathbf{r}dV=-dV_{\mathbf{S}}.\end{gathered} \tag{10.33}$$

●

XI. COMMUTATION RELATIONS

For functions on T^*Q of the kind (1.5)

$$H_{\mathbf{A}}=\tfrac{1}{2}P_{\mathbf{A}}+V_{\mathbf{A}}, \tag{11.1}$$

with $V_{\mathbf{A}}$ satisfying (1.6), $dV_{\mathbf{A}}=\mathbf{A}dV$, the following general commutation relation holds [see also (4.11), we replace here, when convenient, notation $P_{\mathbf{A}}$ by $P(\mathbf{A})$]

$$\{H_{\mathbf{A}},H_{\mathbf{B}}\}=\tfrac{1}{4}\{P_{\mathbf{A}},P_{\mathbf{B}}\}+P([\mathbf{A},\mathbf{B}]_a\nabla V)=\tfrac{1}{4}P([\mathbf{A},\mathbf{B}])+P([\mathbf{A},\mathbf{B}]_a\nabla V), \tag{11.2}$$

where

$$[\mathbf{A},\mathbf{B}]_a=\mathbf{AB}-\mathbf{BA} \tag{11.3}$$

denotes the *algebraic commutator* of the linear operators $\mathbf{A}$ and $\mathbf{B}$. The term $P([\mathbf{A},\mathbf{B}])$ in (11.2) is cubic in (p_i), while $P([\mathbf{A},\mathbf{B}]_a\nabla V)$ is linear. Notice that $[\mathbf{A},\mathbf{B}]_a\nabla V$ is the vector field image of the gradient ∇V by the linear (skew-symmetric) operator $[\mathbf{A},\mathbf{B}]_a$.

In order to express the commutation relations for all the basic first integrals (10.16), it is convenient to introduce the adjoint skew-symmetric two-tensor of the vector field $\mathbf{R}$

$$\mathbf{\Omega}=*\mathbf{R}=\begin{bmatrix} 0 & y-x & z-x \\ x-y & 0 & z-y \\ x-z & y-z & 0 \end{bmatrix}, \tag{11.4}$$

such that, for any vector $\mathbf{v}$

$$\mathbf{\Omega v}=\mathbf{R}\times\mathbf{v}. \tag{11.5}$$

Indeed, since the basic K-tensors (4.7) have the common eigenvector $\mathbf{R}$, their algebraic commutators are all of the kind $f\mathbf{\Omega}$, where f is a function on $\mathbb{E}_3$. This follows from the general identity:

$$[\mathbf{A},\mathbf{B}]_a\mathbf{R}=*[\mathbf{A},\mathbf{B}]_a\times\mathbf{R},$$

and from the condition

$$[\mathbf{A},\mathbf{B}]_a\mathbf{R}=0,$$

which imply

$$*[\mathbf{A},\mathbf{B}]_a=f\mathbf{R},$$

i.e.,

$$[\mathbf{A},\mathbf{B}]_a=f\mathbf{\Omega}.$$

We find

$$\begin{gathered}
[\mathbf{T},\mathbf{S}]_a=\tfrac{3}{2}\mathbf{\Omega}=\tfrac{1}{2}(3\lambda_{\mathbf{G}}-\lambda_{\mathbf{T}})\mathbf{\Omega}\\
[\mathbf{T},\mathbf{I}]_a=-(x+y+z)\mathbf{\Omega}=\lambda_S\mathbf{\Omega}\\
[\mathbf{S},\mathbf{I}]_a=-\tfrac{1}{2}(x^2+y^2+z^2)\mathbf{\Omega}=-\tfrac{1}{2}\lambda_{\mathbf{I}}\mathbf{\Omega}\\
[\mathbf{Q},\mathbf{I}]_a=[\mathbf{Q},\mathbf{T}]_a=[\mathbf{Q},\mathbf{S}]_a=0.
\end{gathered} \tag{11.6}$$

Moreover, a straightforward calculation shows that the Poisson brackets of the five functions $P_{\mathbf{K}}$ for these K-tensors have a similar behavior

$$
\begin{gathered}
\{P_{\mathbf{T}},P_{\mathbf{S}}\}=-2p_{\omega}(3P_{\mathbf{G}}-P_{\mathbf{T}})\\
\{P_{\mathbf{T}},P_{\mathbf{I}}\}=-4p_{\omega}P_{\mathbf{S}}\\
\{P_{\mathbf{S}},P_{\mathbf{I}}\}=2p_{\omega}P_{\mathbf{I}}\\
\{P_{\mathbf{Q}},P_{\mathbf{I}}\}=\{P_{\mathbf{Q}},P_{\mathbf{T}}\}=\{P_{\mathbf{Q}},P_{\mathbf{S}}\}=0,
\end{gathered}
\tag{11.7}
$$

as well as the Poisson brackets of the five first integrals, derived from (11.2)

$$
\begin{gathered}
\{H_{\mathbf{T}},H_{\mathbf{S}}\}=-p_{\omega}(3H-H_{\mathbf{T}})\\
\{H_{\mathbf{T}},H_{\mathbf{I}}\}=-2p_{\omega}H_{\mathbf{S}}\\
\{H_{\mathbf{S}},H_{\mathbf{I}}\}=p_{\omega}H_{\mathbf{I}}\\
\{H_{\mathbf{Q}},H_{\mathbf{I}}\}=\{H_{\mathbf{Q}},H_{\mathbf{T}}\}=\{H_{\mathbf{Q}},H_{S}\}=0.
\end{gathered}
\tag{11.8}
$$

Remark 11.1: The commutation relations (11.8) show that the nonvanishing Poisson brackets yield cubic first integrals which factorize in the product of $p_{\omega}=\mathbf{p}\cdot\boldsymbol{\omega}$ (10.13) (which is linear) and the basic quadratic first integrals themselves [or a linear combination of them, as in $(11.8)_1$]. Since $p_{\omega}=\sqrt{H_{\mathbf{T}}}$, it follows that no new independent first integrals are generated by the Poisson brackets (see Remark 10.5) and that the algebra of first integrals generated in this way is quadratically closed. ●

Remark 11.2: About the independence of the basic elements (H_i) of each involutive function algebra $\mathcal{H}_*$ (see Table I), we observe that they are in particular **vertically independent**, i.e.,

$$
\det\left[\frac{\partial H_i}{\partial p_j}\right]\neq 0,
$$

for all $\mathbf{p}$ not tangent to a hypersurface of the web. This is in fact a general property of the orthogonal separation. Indeed, from (1.9) it follows that:

$$
\det\left[\frac{\partial H_i}{\partial p_j}\right]=\det[\varphi_{(i)}^{j}p_j]=p_1\cdots p_n\det[\varphi_{(i)}^{j}].
$$

Since $\det[\varphi_{(i)}^{j}]\neq 0$, the result is zero iff at least one p_i vanishes. On the other hand, $p_i=0$ means that $\mathbf{p}$ is tangent to the foliation $q^i=\text{constant}$. ●

XII. FINAL REMARKS

This paper leaves open interesting questions concerning, for example, (i) the case $n>3$ and, (ii) the case of a ''multiparametrized'' Calogero system (in the sense explained below) at least for $n=3$. These cases are currently under investigation, and we have at the moment only partial results. About case (i) we mention for instance Ref. 18 where separability is stated for the n-particle elliptic Calogero–Moser system up to general (complex) canonical transformations.

If we want to extend the geometrical intrinsic method presented here to these more general cases (as well as to any dynamical system in a Euclidean space) what we need is a complete ''list'' or ''dictionary'' relating all possible separable orthogonal webs in a Euclidean space with the intrinsic expressions (in terms of products of vectors with a clear geometrical meaning) of all possible characteristic K-tensors and of a basis of the associated KS-algebras. This dictionary, besides that one which assigns separable systems of coordinates to symmetry operators, should give a further help in the application of the separability theory to the Hamilton–Jacobi equations as well as to the Schrödinger or Helmholtz equations. Indeed, for testing if a potential V is separable we should simply check if the characteristic equation (1.3) $d(\mathbf{K}\,dV)=0$ is satisfied for at least one of the characteristics tensors $\mathbf{K}$ of the list, and this can be done in any coordinate system we like (not necessarily separable) since the characteristic equation (1.3) has an intrinsic meaning. When a characteristic K-tensor satisfying (1.3) is found then, as a second step, we proceed to integrate equations (1.6) for a basis of the associated KS-algebra; this will produce n first integrals in involution (1.5) $H_{\mathbf{K}}$.

Actually, in the present paper we did not follow precisely this procedure, since we solved the characteristic equation (1.3) with respect to $\mathbf{K}$, and not with respect to the potential V. The reason is that, even in the case $n=3$, the kind of list we are looking for is at the moment incomplete (a paper is in preparation, Ref. 12). Indeed, our analysis of the characteristic equation of the Calogero potential leads to consider only rotational KS-algebras, around the particular axis ω determined by the ''diagonal vector'' $\boldsymbol{\omega}=(1,1,1)$. All the remaining separable webs in the Euclidean three-space are not involved. The necessity and the effectiveness of such a list becomes evident in investigating case (ii). We give here only an outline, leaving a complete discussion to a next paper. Let us consider the following two generalized versions of the Calogero potential:

$$\begin{aligned} V_1 &= \frac{1}{(\alpha x-\gamma z)^2}+\frac{1}{(\beta y-\alpha x)^2}+\frac{1}{(\gamma z-\beta y)^2} \\ V_2 &= \frac{\alpha}{(x-z)^2}+\frac{\beta}{(y-x)^2}+\frac{\gamma}{(z-y)^2}, \end{aligned} \tag{12.1}$$

where $\alpha,\beta,\gamma\in\mathbb{R}$, and $\neq 0$ in V_1. Up to a rescaling of the coordinates and for $\alpha,\beta,\gamma>0$, the potential V_1 represents the case of three particles with different masses. Let us consider the five rotationsl characteristic K-tensors introduced in this paper, but referred to an arbitrary axis ω (through the origin O of the Cartesian coordinates) and to an arbitrary point $C\in\omega$. In this general situation, expressions (5.3), (6.7), (7.4), (8.7), and (9.4) become

$$\begin{aligned} \mathbf{K}_{\mathrm{cyl}} &= \mathbf{R}\otimes\mathbf{R}-\mathbf{d}\otimes\mathbf{d} \\ \mathbf{K}_{\mathrm{par}} &= \mathbf{d}\odot\mathbf{r}_C-\mathbf{d}\cdot\mathbf{r}_C\mathbf{G} \\ \mathbf{K}_{\mathrm{sph}} &= r_C^2\mathbf{G}-\mathbf{r}_C\otimes\mathbf{r}_C+\mathbf{R}\otimes\mathbf{R} \\ \mathbf{K}_{\mathrm{pro}} &= r_C^2\mathbf{G}-\mathbf{r}_C\otimes\mathbf{r}_C+c^2\mathbf{d}\otimes\mathbf{d} \\ \mathbf{K}_{\mathrm{obl}} &= r_C^2\mathbf{G}-\mathbf{r}_C\otimes\mathbf{r}_C-c^2\mathbf{d}\otimes\mathbf{d}, \end{aligned} \tag{12.2}$$

where $\mathbf{d}$ is a unit vector parallel to the axis ω, $\mathbf{R}=\mathbf{d}\times\mathbf{r}_C$, $\mathbf{r}_C$ is the radius vector referred to the point $C\in\omega$, and $c\in\mathbb{R}$. Now, it is easy to see that the characteristic equation (1.3) for $V=V_1$ is satisfied by all K-tensors in the list (12.2), provided both vectors $\mathbf{d}$ and OC be parallel to the vector $(1/\alpha,1/\beta,1/\gamma)$, while for $V=V_2$, $\mathbf{d}$ and OC must be parallel to (1,1,1), as for the original Calogero potential (1.1). Thus, also V_1 and V_2 are super-separable. Furthermore, for a quadratic potential $V_k=k(x^2+y^2+z^2)(k\in\mathbb{R})$ the characteristic equation (1.3) can be satisfied for all tensors (12.2), with the exception of $\mathbf{K}=\mathbf{K}_{\mathrm{par}}$, for $C=O$ and for any arbitrary unit vector $\mathbf{d}$. Thus, also V_k is super-separable.

As we said in the Introduction, the aim of this paper is to prove the super-separability of the Calogero three-system, and to construct the corresponding first integrals in involution. The further work of writing and discussing Eqs. (1.10)–(1.13) for all possible choice of separable coordinates is in progress.

Other questions arise about the separability of the Schrödinger or Helmholtz equation and the associated symmetries. A general remark concerning this topic is the following. On a Riemannian manifold $(Q_n, \mathbf{G})$, with a function $H_{\mathbf{A}}$ of the kind (11.1) we associate the second-order differential operator (on functions ψ over Q)

$$\hat{H}_{\mathbf{A}}\psi=-\frac{\hbar}{2}\Delta_{\mathbf{A}}\psi+V_{\mathbf{A}}\psi, \tag{12.3}$$

where (∇_i is the covariant derivative with respect to the Levi-Civita connection)

$$\Delta_{\mathbf{A}}\psi=\delta(\mathbf{A}d\psi)=\nabla_i(A^{ij}\partial_j\psi) \tag{12.4}$$

is the **pseudo-Laplacian** associated with the symmetric two-tensor $\mathbf{A}=(A^{ij})$. Note that $\Delta_{\mathbf{G}}=\Delta$ is the ordinary Laplace–Beltrami operator and $\hat{H}=\hat{H}_{\mathbf{G}}$. It can be shown[19] that the commutation relation

$$\hat{H}_{\mathbf{A}}\hat{H}_{\mathbf{B}}-\hat{H}_{\mathbf{B}}\hat{H}_{\mathbf{A}}=0$$

holds for all $\mathbf{A}$ and $\mathbf{B}$ belonging to a KS-algebra $\mathcal{K}$, provided the commutation condition

$$\mathbf{A}\mathbf{Ric}-\mathbf{Ric}\mathbf{A}=0 \tag{12.5}$$

holds for all $\mathbf{A}\in\mathcal{K}$, or at least for a characteristic tensor of $\mathcal{K}$, where **Ric** is the Ricci tensor (here interpreted as linear operator on one-forms or vectors). Equation (12.5) is an intrinsic version of the well-known **Robertson condition**.[9,20] On manifolds of constant curvature (for instance on Euclidean spaces) it is obviously satisfied. All this shows that on these manifolds an involutive function algebra $\mathcal{H}$ associated with the orthogonal additive separation of the Hamilton–Jacobi equation corresponds to a system of second-order symmetry operators of the Schrödinger equation $\hat{H}\psi=\lambda\psi$, which are related to its multiplicative separation.[21] For this reason, the analysis of the separability in terms of symmetry operators of the Schrödinger equations in Euclidean or on constant curvature spaces, as done in Refs. 17, 22, 23 for $n=2,3$, is equivalent to the analysis of the separability of the corresponding Hamilton–Jacobi equations in terms of first integrals and Killing tensors.

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1. M. A. Olshanestky and A. M. Perelomov, ''Classical integrable finite-dimensional systems related to Lie algebras,'' Phys. Rep. **71**, 313–400 (1981).
2. S. Wojciechowski, ''Superintegrability of the Calogero-Moser system,'' Phys. Lett. A **95**, 279–281 (1983).
3. N. W. Evans, ''Superintegrability in classical mechanics,'' Phys. Rev. A **41**, 5666–5676 (1990).
4. M. F. Rañada, ''Superintegrability of the Calogero-Moser system: Costants of motion, master symmetries, and time-dependent symmetries,'' J. Math. Phys. **40**, 236–247 (1999).
5. F. Calogero, ''Solution of a three-body problem in one dimension,'' J. Math. Phys. **10**, 2191–2196 (1969).
6. C. Marchioro, ''Solution of a three-body scattering problem in one dimension,'' J. Math. Phys. **11**, 2193–2196 (1970).
7. S. Benenti, ''Orthogonal separable dynamical systems,'' in *Differential Geometry and Its Applications, Vol. I*, Proceedings of the 5th International Conference on Differential Geometry and Its Applications, Silesian University at Opava, August 24–28, 1992, edited by O. Kowalsky and D. Krupka, pp. 163–184 (1993). Electronic edition ELibEMS, htpp://www.emis.de/proceedings/.
8. S. Benenti, ''Intrinsic characterization of the variable separation in the Hamilon-Jacobi equation,'' J. Math. Phys. **38**, 6578–6602 (1997).
9. L. P. Eisenhart, ''Separable systems of Stäckel,'' Ann. Math. **35**, 284–305 (1934).
10. E. G. Kalnins and W. Miller, Jr., ''Killing tensors and variable separation for Hamilton–Jacobi and Helmholtz equations,'' SIAM (Soc. Ind. Appl. Math.) J. Math. Anal. **11**, 1011–1026 (1980).
11. H. Katzin and J. Levine, ''Quadratic first integrals of the geodesic in spaces of constant curvature,'' Tensor **10**, 97–104 (1965).
12. S. Benenti, C. Chanu, and G. Rastelli, ''Killing tensors and separable webs in the Euclidean three-space'' (in preparation).
13. A. Nijenhuis, ''Jacobi-type identities for bilinear differential concomitants of certain tensor fields,'' Indagationes Mathematicae **17**, 390–403 (1955).
14. L. K. Norris, ''Schouten-Nijenhuis brackets,'' J. Math. Phys. **38**, 2694–2709 (1977).
15. P. Moon and D. E. Spencer, *Field Theory Handbook* (Springer-Verlag, Berlin, 1961).
16. J. Moser, ''Three integrable Hamiltonian systems connected with isospectral deformations,'' Adv. Math. **16**, 197–220 (1975).
17. E. G. Kalnins, G. C. Williams, W. Miller, Jr., and G. S. Pogosyan, ''Superintegrability in three-dimensional Euclidean spaces,'' J. Math. Phys. **40**, 708–725 (1999).

[18] V. B. Kuznetsov, F. W. Nijhoff, and E. K. Sklyanin, ''Separation of variables for the Ruijsenaars system,'' Commun. Math. Phys. **189**, 855–877 (1997).
[19] S. Benenti, ''Remarks on the connection between the additive separation of the Hamilton–Jacobi equation and the multiplicative separation of the Schrödinger equation on a Riemannian manifold'' (in preparation).
[20] E. G. Kalnins, *Separation of variables for Riemannian spaces of constant curvature*, Pitman Monographs and Surveys in P. A. Math. **28**, (1986).
[21] W. Miller, Jr., *Symmetry and Separation of Variables* (Addison-Wesley, Reading, MA, 1977).
[22] E. G. Kalnins, W. Miller, Jr., and G. S. Pogosyan, ''Superintegrability and associated polynomial solutions: Euclidean space and the sphere in two dimensions,'' J. Math. Phys. **37**, 6439–6467 (1996).
[23] E. G. Kalnins, W. Miller, Jr., and G. S. Pogosyan, ''Superintegrability on the two-dimensional hyperboloid,'' J. Math. Phys. **38**, 5416–5433 (1997).