RENDICONTI DEL SEMINARIO MATEMATICO

Universita' e Politecnico
Torino

GEOMETRICAL STRUCTURES FOR
PHYSICAL THEORIES, I

VOLUME 54, N. 3, 1996

Geometrical aspects of the dynamics of non-holonomic systems

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Summary. *Dynamics of non-holonomic mechanical systems is interpreted as a submanifold of TT^*Q where Q is the configuration manifold. Integrability of dynamics is discussed for linear and non-linear constraints. The case of constrained geodesics of a Riemannian manifold studied by Synge is also considered. Local coordinate representations are used. An example of an ideal non-linear non-holonomic constraint is proposed.*

1. First order equations

A *first order (differential) equation* on a manifold M is a submanifold D of the tangent bundle TM. A first order equation is said to be *integrable* if for each $v \in D$ there exists a differentiable curve $\gamma\colon I \to M$ such that $\gamma(0) = v$ and $\gamma(t) \in D$ for each $t \in I$, where $\dot\gamma\colon I \to TM$ is the tangent curve to γ and I is an open real interval containing 0. Such a curve is called an *integral curve of D based on $v \in D$*. It is possible to extend this definition to the case in which D is a submanifold with boundary or a subset of TM. If D is a non-integrable first order equation, then the *integrable part* of D is the maximal subset of D which is integrable according to the definition above.

If (x^A) are local coordinates of M, then we denote by $(x^a, \dot x^B)$ the corresponding fibered coordinates on TM. A first order equation D is locally described by a system of equations

$$D^a(x^A, \dot x^B) = 0$$

$(A, B = 1, \dots, m;\ m = \dim(M);\ a = 1, \dots, l;\ l = \dim(D))$. An integral curve has a local representation $x^A = \gamma^A(t)$ such that for each t

$$D^a\big(\gamma^A(t), \dot\gamma^B(t)\big) = 0$$

where $D\gamma^A$ is the derivative of the real function γ^A.

Examples. (1) A subbundle D of TM, i.e. a *regular distribution* on M, is an integrable first order equation on M. For each $v \in D$ there exists an integral curve of D based on v, but it is not unique.

(2) Let $X\colon M \to TM$ be a differentiable section of the tangent bundle $\tau_M\colon TM \to M$, i.e. a differentiable *vector field* on M. The image $D = X(M)$ of X is an integrable

first order equation. In this case the *uniqueness property* holds (Cauchy theorem): if $\gamma: I \to M$ and $\gamma': I' \to M$ are integral curves based on $v \in D$, then they coincide in the intersection $I \cap I'$ of the intervals of definition.

2. Dynamics of holonomic systems

A fundamental example of first order equation is given by the dynamics of holonomic mechanical systems.

Let Q be the configuration manifold of a holonomic mechanical system with n degrees of freedom. Let (q^i) be local coordinates on Q (i.e. *Lagrangian coordinates* of the mechanical system). We denote by $(q^i, \dot{q}^j)$, (q^i, p_j) and $(q^i, p_j, \dot{q}^h, \dot{p}_k)$ the corresponding fibered coordinates on TQ, T^*Q and TT^*Q respectively. In the following discussion Latin indices $i, j, h, k \ldots$ run from 1 to $n = \dim(Q)$. The manifolds TQ and T^*Q represent the *velocity space* and the *phase space* of the mechanical system.

The *dynamics* of the mechanical system is the submanifold D of TT^*Q locally defined by equations

$$p_i - \frac{\partial L}{\partial \dot{q}^i} = 0, \qquad \dot{p}_i - \frac{\partial L}{\partial q^i} = 0, \tag{1}$$

or by equations

$$\dot{q}^i - \frac{\partial H}{\partial p_i} = 0, \qquad \dot{p}_i + \frac{\partial H}{\partial q^i} = 0, \tag{2}$$

where $L: TQ \to \mathbb{R}$ is the *Lagrangian function* locally represented by a function of the coordinates $(q^i, \dot{q}^j)$, and $H: T^*Q \to \mathbb{R}$ is the *Hamiltonian function* locally represented by a function of the coordinates (q^i, p_j). We call (1) and (2) the *Lagrangian representation* and the *Hamiltonian representation* of the dynamics D respectively. Equations (1) follow from the d'Alembert-Lagrange principle. Equations (2) follow from equations (1) under the regularity condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^i\, \partial \dot{q}^j}\right) \neq 0. \tag{3}$$

The Hamiltonian representation D shows that the dynamics D is the image of a vector field X on the phase space T^*Q. Hence, D is integrable with the uniqueness property. The local expression of the vector field X is

$$X = X^i \frac{\partial}{\partial q^i} + X_j \frac{\partial}{\partial p_j},$$

where

$$X^i = \frac{\partial H}{\partial p_i}, \qquad X_j = -\frac{\partial H}{\partial q^j}.$$

The vector field is globally defined by equation

$$i_X\, d\theta = -\, dH,$$

where $\theta = p_i\, dq^i$ is the fundamental 1-form of T^*Q (the *Liouville form*).

3. Dynamics with non-holonomic linear constraints

We assume that further constraints are imposed on the holonomic system. The possible kinematical states of the system are represented by vectors $v \in TQ$ which belong to a subset K of T. In most of the applications K is a subbundle of TQ, i.e. a regular distribution on Q. If K is not completely integrable, then the constraints are called *non-holonomic linear constraints.*

The distribution K can be represented by local equations

(1) $$K_i^a\, \dot{q}^i = 0 \qquad (a = 1, \ldots, l),$$

where K_i^a are functions on the domain of the coordinates (q^i) forming a matrix of maximal rank:

(2) $$\operatorname{rank}(K_i^a) = l.$$

It follows from the D'Alembert-Lagrange principle that the dynamics D is the subset of TT^*Q locally defined by equations

(3) $$p_i - \frac{\partial L}{\partial \dot{q}^i} = 0, \qquad \dot{p}_i - \frac{\partial L}{\partial q^i} = \lambda_a\, K_i^a, \qquad K_i^a\, \dot{q}^i = 0,$$

or by equations

(4) $$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0, \qquad \dot{p}_i + \frac{\partial H}{\partial q^i} = \lambda_a\, K_i^a, \qquad K_i^a\, \dot{q}^i = 0,$$

where (λ_a) are the *Lagrange multipliers.* A point of TT^*Q belongs to D if and only if its coordinates satisfy equations (3) or (4) with some values of the parameters (λ_a). The terms

(5) $$R_i = \lambda_a\, K_i^a$$

represent the possible *reaction forces* of the constraints.

The *Lagrangian representation* (3) of the dynamics D is suitable for proving that D is a submanifold. The *Hamiltonian representation* (4) is suitable for discussing the integrability of D. The integrability will be discussed in the next section.

The distribution K can be represented by parametric equations of the kind

(6) $$q^i = X^i, \qquad \dot{q}^i = F_\alpha^i(x^j)\, \omega^\alpha \qquad (\alpha = l+1, \ldots, n),$$

where the functions F^i_α form a matrix of maximal rank:

(7) $$\operatorname{rank}\big(F^i_\alpha\big) = n - l.$$

The parameters $(x^i = q^i, \omega^\alpha)$ can be interpreted as coordinates on K and equations (6) as representing a local immersion of K into TQ. Coordinates (ω^α) are known in the classical literature as "pseudo-velocities". The equivalence of the representations (1) and (6) implies that

(8) $$F^i_\alpha\, K^a_i = 0.$$

The substitution of (6) into the first two sets of the Lagrange equations (3) yields equations of the kind

(9) $$p_i = f_i(x^j, \omega^\alpha); \qquad \dot{p}_i = g_i(x^j, \omega^\alpha) + \lambda_a\, K^a_i(x^j).$$

The system of equations (6) and (9) gives the *parametric representation* of the dynamics D. The $2n$ parameters $(x^i, \lambda_a, \omega^\alpha)$ can be interpreted as coordinates on D and equations (6) and (9) as representing a local immersion of D into TT^*Q. Indeed, a straightforward calculation shows that the Jacobian matrix of the functions at the left sides of (6) and (9), with respect to the variables $(x^i, \lambda_a, \omega^\alpha)$, has maximal rank. This proves

Proposition 1. *Under the regularity conditions (3,§2) and (2) the subset $D \subset TT^*Q$ defined by equations (3) is a submanifold of dimension $2n$.*

4. The integrability theorem and the elimination of the Lagrangian multipliers

For the sake of simplicity we shall use the following notation:

$$H_i = \frac{\partial H}{\partial q^i}, \qquad H^i = \frac{\partial H}{\partial p_i}, \qquad \text{etc.}$$

The image by the tangent fibration $\tau_{T^*Q}\colon TT^*Q \to T^*Q$ of the first order equation D considered in Section 3 is the subset $C \subset T^*Q$ locally defined by equations

(1) $$C^a = K^a_i\, H^i = 0$$

which are obtained by combining the first and the third set of equations (4.§3).

The regularity condition (3.§2) is equivalent to

(2) $$\det\big(H^{ij}\big) \neq 0.$$

The regularity conditions (2.§3) and (2) imply that the l functions at the left side of (1) are independent. Hence, the subset C is a submanifold of T^*Q of dimension $2n - l$. The first set of the Lagrange equations (3.§3) can be interpreted as the local definition of a fiber bundle isomorphism $\Lambda: THQ \to T^*Q$, i.e. of the Legendre transformation. The first set of the Hamilton equations can be interpreted as the local definition of Λ^{-1}. Hence, $C = \Lambda(K)$.

Let $\gamma: I \to T^*Q$ be integral curve of D. The image $\gamma(I)$ is contained in C. Hence, the image $\dot{\gamma}(I)$ of the tangent curve $\dot{\gamma}: I \to TT^*Q$ is contained in TC. It follows that the integrable part of D is contained in the intersection $D \cap TC$. The submanifold TC is defined by equations (1) and equations

(3) $$\dot{q}^j \left(K^a_{ij}\, H^i + K^a_i\, H^i_j\right) + \dot{p}_j\, K^a_i\, H^{ij} = 0,$$

obtained from (1) by formal derivation. Since the submanifold D is defined by equations (4.§3), the intersection $D \cap TC$ is characterized by equations

(4) $$K^a_{ij}\, H^i\, H^j + K^a_i\, \left(H^i_j\, H^j - H^{ij}\, H_j\right) + K^a_i\, K^b_j\, H^{ij}\, \lambda_b = 0.$$

Under the condition

(5) $$\det\left(H^{ij}\, K^a_i\, K^b_j\right) \neq 0,$$

we can solve equations (4) with respect to the multipliers (λ_a). We obtain

(6) $$\lambda_a = G_{ab}\, L^b,$$

where

(7) $$\begin{cases} L^b = \{H, C^b\} = K^b_i\, \left(H^{ij}\, H_j - H^i_j\, H^j\right) - K^b_{ij}\, H^i\, H^j, \\ \left(G_{ab}\right) = \left(G^{ab}\right)^{-1}, \qquad G^{ab} = H^{ij}\, K^a_i\, K^b_j. \end{cases}$$

In the first equation $\{\cdot,\cdot\}$ denotes the canonical Poisson bracket of functions on the cotangent bundle T^*Q. Hence, for each point $p \in C$ there exists one and only one element of D belonging to the intersection $D \cap T_pC$. This means that $D \cap TC$ is the image of a section $X: C \to TC$ of the tangent fibration $\tau_C: TC \to C$ i.e., the image of a vector field X on C. Hence, the intersection $D \cap TC$ coincides with the integrable part of D. This proves:

Proposition 2. *If the regularity conditions (2.§3), (2) and (5) are satisfied, then the integrable part of the first order differential equation D defined by equations (4.§3) is the image of a vector field X in the submanifold $C \subset T^*Q$ defined by equations (1).*

Remark 1. (i) If the quadratic form defined by the matrix (H^{ij}) is positive-definite, then the regularity condition (5) is a consequence of condition (2.§3). This is the case of the ordinary holonomic mechanical systems. (ii) The explicit form (6)–(7) of the Lagrangian multipliers has been derived by Eden [7] by a different method.

The intersection $D \cap TC$ is defined by equations

(8) $$\dot{q}^i - H^i = 0, \qquad \dot{p}_i + H_i = G_{ab}\, L^a\, K^b_i, \qquad K^a_i\, \dot{q}^i = 0.$$

Hence, the vector field X can be interpreted as the restriction to C of the vector field $\overline{X}$ on T^*Q whose components are

$$\overline{X}^i = H^i, \qquad \overline{X}_i = -H_i + G_{ab}\, L^a\, K_i^b. \tag{9}$$

This proves

Proposition 3. *The integral curves of the first order equation D are the integral curves of the of the vector field $\overline{X}$ based on the points of C, i.e. the solutions of the following system of differential equations*

$$\frac{dq^i}{dt} = H^i, \qquad \frac{dp_i}{dt} = -H_i + G_{ab}\, L^a\, K_i^b \tag{10}$$

whose initial conditions satisfy equations (1).

Remark 2. According to the classical terminology, equations (1) are *invariant relations* of the differential system (10). Analogous results, but within the Lagrangian formalism, have been obtained by Synge [13] (see also Agostinelli [1]) for quadratic Hamiltonians, as we shall see in the next section. By the Legendre transformation $\Lambda^{-1}: T^*Q \to TQ$ the vector field $X|: C \to TC$ is transformed into a vector field $Y: K \to TK$ over the subbundle $K \subset TQ$. This vector field is the geometric representation of the Gibbs-Appel equations or the Maggi-Volterra equations. The image $Y(K)$ of the vector field Y, which is a submanifold of TK, is locally represented by equations involving the coordinates $(q^i, \omega^\alpha, \dot{q}^i, \dot{\omega}^\alpha)$ of TK.

Remark 3. *(Let us introduce on T^*Q the vertical 1-form*

$$\phi = G_{ab}\, L^a\, K_i^b\, dq^i. \tag{11}$$

Then the vector field $\overline{X}$ is intrinsically defined by equation

$$i_{\overline{X}}\, d\theta = -dH + \phi, \tag{12}$$

where θ is the Liouville 1-form.)

Remark 4. Any *other* extension can be chosen for *finding the integral curves of D*. To the vector field $\overline{X}$ defined above we can add any arbitrary *(sooth)* vector field vanishing on the submanifold C.

Remark 5. The Hamilton-Jacobi method for integrating first order equations can be applied only to Hamiltonian vector fields on cotangent bundles, i.e. to vector fields Z such that $i_Z\, d\theta$ is an exact form. It follows from Remarks 3 and 4 that if we know a submanifold $J \subset C$ such that (i) X is tangent to J, (ii) the pull-back of ϕ to J is closed, then we can find a local integral $F :: J \to \mathbb{R}$ of $\phi|J$ and a local extension $\overline{F}$ of this function on T^*Q. The Hamiltonian vector field Z generated by $H' = H - \overline{F}$ is such that $Z|J = X|J$. Hence, we can apply the Hamilton-Jacobi method to the vector field

Z for finding the integral curves of $X|J$, i.e. the motions of the non-holonomic system lying on J.

5. The constrained geodesics

Let us consider the fundamental case of the quadratic Hamiltonian

$$H = \tfrac{1}{2}\, g^{ij}\, p_i\, p_j. \tag{1}$$

of the geodesics of a Riemannian (or pseudo-Riemannian) manifold (Q, g). The distribution $K \subset TQ$ can be represented by l independent vector fields

$$K^a = K^{ai}\, \frac{\partial}{\partial q^i}, \qquad K^{ai} = g^{ij}\, K^a_j \qquad (a = 1, \ldots, l) \tag{2}$$

which span the orthogonal distribution $K^\perp$ of K. In this case we have in (7.§4):

$$\begin{cases} H^{ij} = g^{ij}, \\ G^{ab} = g^{ij}\, K^a_i\, K^b_j = K^a \cdot K^b, \\ L^a = -\, g^{ih}\, g^{jk}\, \nabla_h K^a_k\, p_i\, p_j = -\, D^{aij}\, p_i\, p_j, \end{cases} \tag{3}$$

where

$$D^{aij} = g^{ih}\, g^{jk}\, D^a_{hk}, \qquad D^a_{hk} = \tfrac{1}{2}(\nabla_h K^a_k + \nabla_k K^a_h). \tag{4}$$

Here ∇ is the covariant derivative with respect to the Levi-Civita connection of (Q, g). The functions D^{aij} are the contravariant components of the *deformation tensor* of the vector field K^a. We have $D^{aij} = 0$ if and only if K^a is a *Killing vector*. These deformation tensors give the explicit expression of the reaction forces of the non-holonomic constraints (or the components of the 1-form ϕ) along a motion:

$$R_i = \phi_i = -\, G_{ab}\, D^{ahk}\, K^b_i\, p_h\, p_k. \tag{5}$$

These components are quadratic forms in the momenta (p_i).

The functions (G^{ab}) are the contravariant components of the metric tensor g restricted to the orthogonal distribution $K^\perp$ with respect to the non-holonomic frame (K^a). The regularity condition (5.§4) is equivalent to the following one: the metric induced in each subspace of the distribution K (or of the orthogonal distribution $K^\perp$) is non-degenerate. If g is positive-definite this condition is always satisfied (see Remark 1.§4).

It follows from the preceding formulae that the components of the vector field $\overline{X}$ defined in (9.§4) are

$$\overline{X}^i = g^{ij}\, p_j, \qquad \overline{X}_i = -\left(\tfrac{1}{2}\, \partial_i g^{hk} + G_{ab}\, D^{ahk}\, K^b_i\right) p_h\, p_k. \tag{6}$$

The projections onto Q of the integral curves of $\overline{X}$ satisfy the second order equations

$$\frac{d^2q^i}{dt^2} + \left(\Gamma^i_{hj} + G_{ab}\, K^{ai}\, \nabla_h K^b_j\right) \frac{dq^h}{dt}\, \frac{dq^j}{dt} = 0, \tag{7}$$

which are equivalent to

$$\frac{d^2q^i}{dt^2} + \left(\Gamma^i_{hj} + G_{ab}\, K^{ai}\, D^b_{hj}\right) \frac{dq^h}{dt}\, \frac{dq^j}{dt} = 0, \tag{8}$$

where Γ^i_{hj} a the coefficients of the Levi-Civita connection. The projections of the integral curves of the vector field X satisfy the constraint equations

$$K^a_i\, \frac{dq^i}{dt} = 0. \tag{9}$$

We recognize in system (7) the differential equations *of the first kind* of the *constrained geodesics* derived by Synge from a variational principle [13] (see also Prange [11], Agostinelli [1]). Equations (7) are the geodesic equations of the connection

$$\overset{0}{\Gamma}{}^i_{hj} = \Gamma^i_{hj} + G_{ab}\, K^{ai}\, \nabla_h K^b_j, \tag{10}$$

whose symmetrical part is

$$\overset{1}{\Gamma}{}^i_{hj} = \Gamma^i_{hj} + G_{ab}\, K^{ai}\, D^b_{hj}. \tag{11}$$

These connections depend on the choice of the frame (K^a). As pointed out by Synge, in agreement with Remark 4.§4, it is convenient to introduce the connection

$$\overset{2}{\Gamma}{}^i_{hj} = \Gamma^i_{jh} + \nabla_h G^i_j, \tag{12}$$

where

$$G_{ij} = G_{ab}\, K^a_j\, K^b_j, \qquad G^i_j = g^{ih}\, G_{hj}. \tag{13}$$

Since G_{ij} are the components of the metric tensor reduced to the orthogonal distribution $K^\perp$, this connection does not depend on the choice of the representation of the distribution K. The corresponding geodesic equations

$$\frac{d^2q^i}{dt^2} + \left(\Gamma^i_{hj} + \nabla_h G^i_j\right) \frac{dq^h}{dt}\, \frac{dq^j}{dt} = 0, \tag{14}$$

called *equations of the second kind* by Synge, can be obtained in a direct way from the Lagrange equations

$$\frac{d^2q^i}{dt^2} + \Gamma^i_{hj}\, \frac{dq^h}{dt}\, \frac{dq^j}{dt} = \lambda_a\, K^{ai}, \tag{15}$$

by applying the orthogonal projection operator $P\colon TQ \to K$ defined by

$$P^i_j = \delta^i_j - G^i_j. \tag{16}$$

Since $P \cdot K^a = 0$, the right side vanishes, so that the Lagrange multipliers are eliminated and the left side becomes the left side of (15) plus the term

$$\frac{dq^h}{dt}\, \nabla_h \left(\frac{dq^j}{dt}\, G^i_j \right),$$

which vanishes when the constraint equations (9) are imposed.

The procedure of eliminating the Lagrange multipliers by the projection operators $P\colon TQ \to K$ and $G\colon TQ \to K^\perp$ was considered by Cattaneo [4] for a single constraint equation, i.e. for a distribution represented by a single vector field (see also Ferrarese [8]). The so-called *constrained covariant derivative* of Cattaneo-Gasparini [5] was used. This technique was previously employed by the same authors for a *relative decomposition* of the equations of motion of a test particle in a Lorentzian space-time with respect to a physical frame of reference, represented by a unitary time-like vector field.

The general setting of the projection procedure is based on the fact (see also Vershik [14]) that if ∇ is a connection on a manifold Q, K is a subbundle of TQ and $P\colon TQ \to K$ is a projection operator, then the equation

$$\tilde{\nabla} X = p \cdot \nabla Z,$$

where Z is a vector field compatible with the distribution K, i.e. such that $Z(Q) \subset K$, defines a connection $\tilde{\nabla}$ on the subbundle K, which is called the *reduced connection* of ∇ to K by P. The connection $\overset{2}{\nabla}$, whose coefficients are defined in (12), provide a "canonical" extension to Q of the reduced connection $P \cdot \nabla$, where ∇ is the Levi-Civita connection and P is the orthogonal projection over K. This fact is shown by the following calculation:

$$\begin{aligned} \tilde{\nabla}_i Z^i &= \nabla_i Z^h\, P^j_h = \nabla_i Z^h - \nabla_i Z^h\, G^j_h \\ &= \nabla_i Z^h - \nabla_i \big(G^j_h\, Z^h \big) + Z^h\, \nabla G^j_h = \overset{2}{\nabla}_i Z^h, \end{aligned}$$

since $Z^h\, G^j_h = 0$.

6. Dynamics with non-linear non-holonomic constraints

When the kinematical constraints are represented by a submanifold K of TQ defined by local equations

$$(1) \qquad K^a(q^i, \dot{q}^j) = 0 \qquad (a = 1, \ldots, l),$$

it is assumed that the dynamics is the first order equation $D \subset TT^*Q$ locally defined by equations

$$(2) \qquad p_i - \frac{\partial L}{\partial \dot{q}^i} = 0, \qquad \dot{p}_i - \frac{\partial L}{\partial q^i} = \lambda_a \frac{\partial K^a}{\partial \dot{q}^i} \qquad K^a = 0.$$

or by equations

$$(3) \qquad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0, \qquad \dot{p}_i + \frac{\partial H}{\partial q^i} = \lambda_a \frac{\partial K^a}{\partial \dot{q}^i}, \qquad K^a = 0,$$

where (λ_a) are the Lagrange multipliers. Equations (2) follow from the Gauss principle (see Prange [11]). If the constraints are linear, then equations (2) and (3) reduce to equations (3) and (4) of §3 respectively. We mention he the recent articles by Vershik [14] and Weber [15] on geometrical approaches to non-linear constraints which generalize this assumption.

Let U be the domain of the coordinates (q^i). Then in the open subset $T^*U \subset T^*Q$ we can define the following functions:

$$(4) \qquad A_i^a = \left.\frac{\partial K^a}{\partial \dot{q}^i}\right|_*, \qquad B_i^a = \left.\frac{\partial K^a}{\partial q^i}\right|_*, \qquad C^a = K^a|_*, \qquad G^{ab} = A_i^a\, A_j^b\, H^{ij},$$

$$(5) \qquad L^a = A_i^a \left(H^{ij}\, H_j - H_j^i\, H^j\right) - B_i^a\, H^i = \{H, C^a\},$$

where the symbol $|_*$ denotes the substitution $\dot{q}^i = H^i$. By a procedure analogous to that of §3 and §4, it can be shown that

Proposition 4. *If the regularity conditions*

$$(6) \qquad \det\left(H^{ij}\right) \neq 0, \qquad \operatorname{rank}\left(A_i^a\right) = l, \qquad \det\left(G^{ab}\right) \neq 0,$$

are satisfied, then: (i) the subset $D \subset TT^*Q$ defined by equations (2) is a submanifold of dimension $2n$; (ii) the subset $C = \tau_{T^*Q}(D) \subset T^*Q$ locally defined by equations

$$(7) \qquad C^a = 0,$$

is a submanifold of dimension $2n - l$; (iii) the integrable part of D is the image of a vector field X on C; (iv) the vector field X is the restriction of a vector field $\overline{X}$ on T^*U with components

$$(8) \qquad \overline{X}^i = H^i, \qquad \overline{X}_i = -H_i + G_{ab}\, L^a\, A_i^b,$$

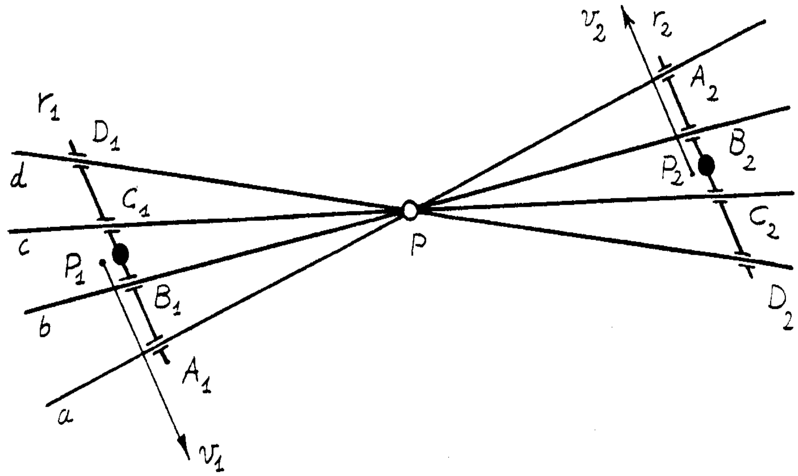
where (G_{ab}) *is the inverse matrix of* (G^{ab})*.*

Remarks analogous to those of §4 hold for non-linear constraints.

A non-holonomic constraint is called *homogeneous* if $v \in K$ implies $rv \in K$ for each real number r. The equations (1) can be chosen to be homogeneous in the coordinates $(\dot{q}^i)$. Caratheodory [2] pointed out that if the constraints are homogeneous then no work is done by the reaction forces (see also Saletan and Cromer [12]). Then the constraints are said to be *ideal. (For homogeneous (non-linear) constraints the regularity conditions* $(6)_2$ *are not fulfilled for* $\dot{q}^i = 0$*, i.e. when the mechanical system is at rest (see the example below). The corresponding singular points of* D *should be analyzed more closely: in general the integrability is preserved but not the uniqueness.)*

It seems that no mechanical system with ideal non-linear non-holonomic constraints is known other than that of Appell (see Fufaev and Neimark [9]). This example, however, suffers of some defects (for criticisms and discussions we refer to Delassus [6], Castoldi [3], Fufaev-Neimark [9] and Pironnau [10]). Castoldi proposed a different example, but its construction seems to be rather complicated. In fact, Hertz pointed out that non-linear constraints can be realized as a limit of linear constraints, when certain masses and distances become negligible. We propose here a simple example, leaving to further investigation the question as to whether it is realistic or not and whether it confirms the theory or not.

Example. Two identical rods r_1 and r_2 move on a plane in such a way that the rods and the velocities v_1 and v_2 of the midpoints P_1 and P_2 remain parallel. This constraint can be produced by installing a sharp wheel or a sharp blade (as in an ice skate) at the center of each rod. To guarantee that the two *"skates"* r_1 and r_2 remain parallel we constrain four points (A_1, B_1, C_1, D_1) of r_1 and corresponding four points (A_2, B_2, C_2, D_2) of r_2 to slide without friction along four rigid bars (a, b, c, d) respectively. These four bars can pivot without friction around a common point P which moves freely in the plane. At each configuration the two skates r_1 and r_2 are in a symmetrical position with respect to the point P. The use of four bars instead of three avoids a certain singularity in the construction, which arises when one of the bars is orthogonal to the skates. If we consider *(on each skate an heavy small body, whose centers of masses* P'_1 *and* P'_2 *can move along the skates slightly from the midpoints* P_1 *and* P_2 *respectively, in order that their velocities a nd those of the skates remain parallel, and we disregard the masses of all the components of the device)*, then we have constructed a system of two material points P_1 and P_2 which move in a plane and are constrained to have parallel velocities. This constraint is non-linear and homogenous. It is represented by equation $v_1 \times v_2 = 0$, i.e. by a *(single)* scalar *homogeneous quadratic* equation in the components of the velocities. Unfortunately, the regularity condition $(6)_2$ is not satisfied for $_1 = v_2 = 0$. Hence, either the current theory or the example, or both, are unsatisfactory. In fact, this singularity is first of all due to the construction. If we leave the two points at rest in a configuration, then we do not know the behavior of the system without specifying the initial directions of the skates. But this information must be "a priori" ignored because of our assumption of disregarding *(all the remaining parts)* of the device.



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