Intrinsic characterization of the variable separation in the Hamilton–Jacobi equation

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The nonorthogonal separation of variables in the Hamilton–Jacobi equation corresponding to a natural Hamiltonian $H=\frac{1}{2}g^{ij}p_i p_j+V$, with a metric tensor of any signature, is intrinsically characterized by geometrical objects on the Riemannian configuration manifold: Killing vectors, Killing tensors, and Killing webs. Comparisons with previous characterizations and some illustrative examples are given. © *1997 American Institute of Physics.* [S0022-2488(97)01412-6]

I. INTRODUCTION

In this paper we investigate the intrinsic characterization in terms of Riemannian geometry of the additive separation of variables in the Hamilton–Jacobi equation. We are concerned with a Hamiltonian of the kind

$$H=\tfrac{1}{2}g^{ij}p_i p_j+V, \tag{1.1}$$

where $g^{ij}(\underline{q})$ are the contravariant components of a metric tensor $\mathbf{g}$ on a differentiable n-dimensional manifold Q and $V(\underline{q})$ is a smooth function on Q. We denote by $(\underline{q},\underline{p})=(q^i,p_i)$ canonical coordinates on T^*Q corresponding to coordinates $\underline{q}=(q^i)$ on Q (indices, $i,j,\ldots$ run from 1 to n). The coordinates $\underline{q}$ are called *separable* if the corresponding Hamilton–Jacobi equation

$$\frac{1}{2}\,g^{ij}\partial_i W\partial_j W+V=h \quad \left(\partial_i=\frac{\partial}{\partial q^i}\right) \tag{1.2}$$

has a complete solution of the form

$$W(\underline{q},\underline{c})=W_1(q^1,\underline{c})+\cdots+W_n(q^n,\underline{c}), \tag{1.3}$$

where $\underline{c}=(c_i)$ are integration constants. We say that the Hamiltonian $H=G+V$ is *separable* when such a coordinate system exists.

Hamiltonian systems of this kind form a large class of integrable systems and, moreover, the additive separation of the HJ-equation is related to the multiplicative separation of the corresponding Helmholtz (or Schrödinger) equation. It is known that the first integrals in involution arising from the additive separation of the HJ-equation (1.2) are linear or quadratic in the momenta $\underline{p}$. This is shown by the general procedure of the integration by separation of variables of the HJ-equation based on the general expressions of the functions g^{ij} and V in separable coordinates. Finding these expressions has been for a long time one of the main problems in the theory of separation of the HJ-equation, after the solution given by Stäckel[1] in 1893 for orthogonal coordinates. The general setting of this problem was clearly formulated by Levi-Civita,[2] who wrote the partial differential equations characterizing the separation of a Hamiltonian $H(\underline{q},\underline{p})$, and pointed out that the separation of the geodesic Hamiltonian,

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$$G=\tfrac{1}{2}g^{ij}p_ip_j \tag{1.4}$$

is a crucial problem, since it is a necessary condition for the separation of the Hamiltonian (1.1). He also suggested a method for discussing his equations, based on a division of the separable coordinates into two classes (as we shall see below). After various significant contributions (among which we mention those of Dall'Acqua,[3] Agostinelli,[4] Forbat,[5] Iarov-Iarovoi,[6] Havas[7]) Levi-Civita equations for a nonhomogenous quadratic time-dependent Hamiltonian have been completely and rigorously solved by Cantrijn[8] in the case of a positive-definite metric. For a nondefinite metric further technical difficulties arise from the occurrence of ''null coordinates'' (for which $g^{ii}=0$). A complete solution has been given in Ref. 9 (see also Refs. 10, 11).

Since quadratic first integrals correspond to Killing 2-tensors (K-tensors) and linear first integrals to Killing vectors (K-vectors), these objects can be used for an intrinsic characterization of the separation for both HJ-equation and Helmholtz equation. General theorems on the relationships between separation and Killing vectors and tensors have been proved by Eisenhart,[12] Kalnins and Miller,[13] and Shapovalov[14] for the separation in orthogonal coordinates, by Woodhouse[15] (see also Refs. 16, 17) for the separation of a single coordinate, and by Kalnins and Miller[18] for the general nonorthogonal separation in a metric of any signature. Since K-vectors and tensors are related to first and second order symmetry operators of the Laplace–Beltrami operator, also the group-theoretical aspect of separation on general or special manifolds has been widely explored (for a review see Ref. 19).

The intrinsic characterization of the additive separation of the HJ-equation proposed in this paper is focused over the following two points: (i) Since separable coordinates occur in equivalence classes (two systems separable coordinates are equivalent if they provide intrinsically the same complete integral of the HJ-equation) the separation phenomenon is related to the geometrical properties of the particular ''webs'' formed by the coordinate hypersurfaces. (ii) While in the previous characterizations (Refs. 12–18) a number $m\leq n$ of independent K-tensors ($m=n$ for the orthogonal case) and a complementary number $r=n-m$ of K-vectors are involved, here it is shown how the separation can be characterized by a single K-tensor with suitable properties, together with an Abelian subalgebra of K-vectors. The main statements are truly coordinate-independent (although for their proofs local coordinate representations are used). As usual, all objects are assumed to be smooth (C^∞). The classical approach to the separation of the HJ-equation (1.2) based on the Levi-Civita equations is revisited from the very beginning with valuable simplifications and in a way suitable for our purposes. The problem of relating the additive separation of the HJ-equation with the multiplicative separation of the Helmholtz equation, i.e., of extending the Robertson conditions[12,20] to the general nonorthogonal separation for a nonpositive metric, is not considered.

II. MAIN RESULTS

Our approach will be similar to that followed in a previous paper for the orthogonal separation,[21] based on the following simple remark: since the orthogonal separation is preserved under coordinate transformations with diagonal Jacobian (each coordinated is transformed separately) then the separation has to be considered as a geometrical property of the orthogonal web formed by the coordinate hypersurfaces. Let us consider the following definitions. An *orthogonal web* on a Riemannian manifold Q_n is a set $\mathscr{S}_n=(\mathscr{S}^i)=(\mathscr{S}^1,\ldots,\mathscr{S}^n)$ of n pairwise transversal and orthogonal foliations of connected submanifolds of codimension 1. (Two submanifolds of codimension 1 are orthogonal if their normal vectors are orthogonal; in a nondefinite metric orthogonality does not imply transversality.) A coordinate system $\underline{q}$ is *adapted* to a web $\mathscr{S}$ if its leaves are locally represented by equations $q^i=$constant. An orthogonal web is *separable* if in the adapted coordinates the geodesic Hamiltonian G is separable. A potential V is *separable* in an orthogonal web $\mathscr{S}$ if in the adapted coordinates the Hamiltonian $H=G+V$ is separable.

A simple example is the following (see also Sec. VI): on the Euclidean space $Q=\mathbb{R}^2=\{(x,y)\}$ a system of confocal conics form an orthogonal web—let us consider for instance the case of ellipses and hyperbola: a system of functions constant on these curves is clearly $q^1=|PF_1|+|PF_2$, $q^2=|PF_1|-|PF_2|$, where P is the generic point of the plane, (F_1,F_2) are the two focuses, and $|\cdot|$ denotes the distance between points. These functions form a coordinate system adapted to the web only locally, since two different points could have the same values of (q^1,q^2). Another system of functions constant on the conics is given by the roots (u^1,u^2) of the equation

$$\frac{x^2}{u-a}+\frac{y^2}{u-b}=1.$$

where $0<a<b$, $2(b-a)=|F_1F_2|^2$. They are called *elliptic coordinates* of the plane, although only locally they form coordinate systems in a strict sense (as in the previous case). As we know (see the comments in Sec. VI) such a web is separable and the separability can be characterized by the existence of a Killing tensor, according to the following statement, proved in Ref. 21 (see also Refs. 22, 23).

Theorem 1: An orthogonal web on a Riemannian manifold Q is separable if and only if there exists a K-tensor $\mathbf{K}$, (i) with pointwise simple real eigenvalues and (ii) with eigenvectors orthogonal to the leaves of the web. A potential V is separable in this web if and only if $d(\mathbf{K}\cdot dV)=0$.

Here $\mathbf{K}\cdot dV$ is the image of the 1-form dV by the linear endomorphism $\mathbf{K}$. We say that $\mathbf{K}$ is a *characteristic Killing tensor* of the separable web. Notice that it is not uniquely determined. We can restate this property in an equivalent form as follows:

Theorem 2: A Hamiltonian $H=G+V$ is separable in orthogonal coordinates if and only if on the manifold Q there exists a K-tensor $\mathbf{K}$ with pointwise simple real eigenvalues, orthogonally integrable eigenvectors (or closed eigenforms) and such that $d(\mathbf{K}\cdot dV)=0$.

We say that a vector field is *orthogonally integrable* if the orthogonal distribution is completely integrable (see below). A vector field with this property is also called *normal* or *normalizable*.[13] With respect the geometrical characterization given by Kalnins and Miller in Ref. 13 (Theorem 6)—for the geodesic case only—where n K-tensors are involved, here we have the advantage of dealing with only one K-tensor but the disadvantage of the practical difficulty of checking if a given K-tensor has normalizable eigenvectors. (In some cases it is possible to answer this question, without knowing the eigenvectors, by computing the Nijenhuis torsion of a related conformal K-tensor, see Ref. 21.)

Remark 1: (i) An orthogonal separable web as well as the corresponding characteristic K-tensor may be defined only on $Q-\Sigma$ where Σ is a suitable closed *singular* set (for instance, on the Euclidean plane the two focuses are the singular points of the web made of confocal conics). Similar remark will apply to the ''existence'' of the objects considered in the following statements. (ii) Starting from a characteristic K-tensor $\mathbf{K}$ it is possible to construct a n-dimensional space $\mathscr{K}$ of commuting K-tensors, including the metric tensor and $\mathbf{K}$ itself, having common eigenvectors with $\mathbf{K}$ and such that the condition $d(\mathbf{K}'\cdot dV)=0$ holds for all elements $\mathbf{K}'\in\mathscr{K}$. The space $\mathscr{K}$ can be constructed by using separable coordinates and the so-called *Stäckel matrices* associated with the metric (see for instance Refs. 9, 13, 21). In some cases it is possible to construct a basis of $\mathbf{K}$ by an intrinsic iterative process, which avoids the use of the separable coordinates (this is the case of the asymmetric separable webs on Euclidean spaces, see Ref. 22). (iii) If $(\mathbf{K}_a)$ $(a=1,...,n)$ is a basis of $\mathscr{K}$ and if the closed 1-forms $\mathbf{K}_a\cdot dV$ are exact i.e., $\mathbf{K}_a\cdot dV=dU_a$, then the n functions

$$F_a = \tfrac{1}{2} K_a^{ij} p_i p_j + U_a \tag{2.1}$$

are independent first integrals in involution. Notice that we can take the metric tensor **g** and the characteristic K-tensor **K** as elements of this basis.

In the general case of the nonorthogonal separation there are equivalent coordinate transformations with nondiagonal Jacobian.[9] Indeed, the separable coordinates are divided into two classes: a coordinate q^i is of *first class* if the ratio $(\partial H/\partial q^i)(\partial H/\partial p_i)^{-1}$ is linear (homogeneous) in the momenta, otherwise it is of *second class* (see Sec. III). The numbers (r,m) of first and second class coordinates are invariant within an equivalence class and moreover only the second class coordinates are related by separated transformations. In Sec. III, it will be shown how in the nonorthogonal separation, instead of an orthogonal web, we are led to consider a more general geometrical structure described in the following definition.

Definition 1: A *Killing web* (K-web) on a Riemannian manifold Q_n is a pair $(\mathscr{S}_m, D_r)$, where

(i) $\mathscr{S}_m = (\mathscr{S}^1, \ldots, \mathscr{S}^m) = (\mathscr{S}^a)$ $(a=1,...,m)$ is a set of $m \leqslant n$ pairwise transversal and orthogonal foliations of connected submanifolds of codimension 1.

(ii) D_r is an r-dimensional Abelian algebra of K-vectors tangent to the leaves of $\mathscr{S}_m$, $r=n-m$, spanning a distribution Δ with constant rank r and such that also the distribution $I=\Delta\cap\Delta^\perp$ has a constant rank m_0.

We shall omit the dimensional indices (m,r) when they are not needed. The rank of a distribution Δ at a point $q\in Q$ is the dimension of the space $\Delta_q=\Delta\cap T_qQ$. A distribution with constant rank will be called *regular*. The distribution orthogonal to Δ is denoted by $\Delta^\perp$. Notice that under the assumption that D is an Abelian subalgebra of K-vectors the dimension m_0 of the subspace I_q is constant on each orbit of D, but in general it could depend on the orbit. It follows from this definition that the leaves of $\mathscr{S}$ are D-invariant and that their complete intersections coincide with the orbits of D. Moreover, the distribution I is made of null vectors. Notice that $m_0=0$ for positive metrics and $m_0\leqslant 1$ for Lorentzian (hyperbolic) metrics. When $m_0=0$ each subspace Δ_q is metrically nondegenerate. A K-web with $m_0=0$ will be called *nondegederate*. The dimension r of D will be called the *degree of symmetry* of the web. We say that D is *orthogonally integrable* if $\Delta^\perp$ is completely integrable.

In Sec. III it will be proved that

Theorem 3: If the Hamiltonian $H=G+V$ is separable, then

(i) There exists a K-web $(\mathscr{S}_m, D_r)$ such that D is orthogonally integrable.

(ii) There exists a m-dimensional space $\mathscr{K}_m$ of commuting and D-invariant K-tensors, including the metric tensor, with m common eigenvectors orthogonal to the leaves of $\mathscr{S}$. In $\mathscr{K}$ there are elements with pairwise distinct real eigenvalues corresponding to the common eigenvectors.

(iii) For each K-tensor $\mathbf{K}\in\mathscr{K}$ and for any basis $(\mathbf{X}_\alpha)$ of D, $d(\mathbf{K}\cdot dg^{\alpha\beta})=0$, where $g^{\alpha\beta}$ are the inverse elements of the matrix $g_{\alpha\beta}=\mathbf{X}_\alpha\cdot\mathbf{X}_\beta$ $(\alpha,\beta=m+1,...,n)$.

(iv) The potential V is D-invariant and $d(\mathbf{K}\cdot dV)=0$ for each element $\mathbf{K}\in\mathscr{K}$.

The numbers (r,m,m_0) entering in this statement coincide, respectively, with the number of the first class, second class, and null second class coordinates. Moreover, the foliations $(\mathscr{S}^a)$ $(a=1,...,m)$ are coordinate hypersurfaces of the second class coordinates, while the derivations with respect to the first class coordinates (interpreted as vector fields) span the distribution Δ.

Remark 2: Point (iii) of Remark 1 also holds for this case: if $(\mathbf{K}_a)$ $(a=1,...,m)$ is a basis of the space $\mathscr{K}$ (the metric tensor can be included, as well as the characteristic tensor considered below) and if all the closed forms $\mathbf{K}_a\cdot dV$ are exact, i.e., $\mathbf{K}_a\cdot dV=dU_a$, then the m functions (2.1) are first integrals in involution. To these m quadratic first integrals we add the r linear first integrals

$$F_\alpha = X^i_\alpha p_i, \tag{2.2}$$

corresponding to a basis of D, and we get a complete system of n independent first integrals in involution.

The conditions listed in Theorem 3 are also sufficient for the separation, but as sufficient conditions they are redundant. A way of obtaining a minimal set of sufficient conditions will be investigated in Sec. IV. Assume that a K-web $(\mathscr{S},D)$ is given; then it generates local coordinates (q^a,q^α) $(a=1,...,m;\ \alpha=m+1,...,n)$ by choosing (i) a system of m independent functions (q^a) which are constant on the leaves of the foliations $(\mathscr{S}^a)$, (ii) a basis $(\mathbf{X}_\alpha)$ of D and (iii) a local section W of the orbits of D (W is a submanifold of dimension m transversal to the orbits). The coordinates (q^α) are the affine parameters of the integral curves of the vector fields $(\mathbf{X}_\alpha)$ starting from the points of W. Since these are K-vectors the coordinates (q^α) are ignorable. Coordinates of this kind will be called *adapted* to the web (or *generated* by the web). We say that the web is a *separable Killing web* if in adapted coordinates the geodesic Hamiltonian G is separable. Hence, as it has been done for the orthogonal separation, we characterize the separability of a K-web by means of a single *characteristic K-tensor* $\mathbf{K}$.

Theorem 4: A K-web $(\mathscr{S},D)$ is separable if and only if there exists a D-invariant K-tensor $\mathbf{K}$ with pairwise and pointwise distinct real eigenvalues corresponding to eigenvectors orthogonal to the leaves of $\mathscr{S}$ and moreover, for $m_0>1$, $d(\mathbf{K}\cdot dg^{\alpha\beta})=0$ for any basis $(\mathbf{X}_\alpha)$ of D.

We emphasize the fact that this last condition drops out when $m_0\leqslant 1$, for instance for positive-definite or Lorentzian metrics (further comments on this *additional condition* will be given at the end of the proof of this theorem). Notice that the ''only if'' part of this statement is included in Theorem 3. Theorem 4 is an extension of Theorem 1 to the general nonorthogonal separation (for a geodesic Hamiltonian).

The separability of a general Hamiltonian (1.1) can be characterized by the existence of a pair $(D_r,\mathbf{K})$ where D_r is an r-dimensional Abelian algebra of K-vectors and $\mathbf{K}$ is a K-tensor, according to the following:

Theorem 5: The Hamiltonian $H=G+V$ is separable if and only if on Q there exists a pair $(D_r,\mathbf{K})$ such that

(a) D_r is a r-dimensional Abelian algebra of K-vectors spanning a regular distribution Δ of rank r such that $I=\Delta\cap\Delta^\perp$ has a constant rank m_0.
(b) $\mathbf{K}$ is a D-invariant K-tensor with $m=n-r$ pointwise and pairwise distinct real eigenvalues with orthogonally integrable eigenvectors.
(c) The manifolds orthogonal to these eigenvectors are D-invariant.
(d) The potential V is D-invariant and $d(\mathbf{K}\cdot dV)=0$.
(e) For $m_0>1$, $d(\mathbf{K}\cdot dg^{\alpha\beta})=0$.

This is an extension of Theorem 2 to the general nonorthogonal separation. Notice that the orthogonal integrability of D does not appear in this statement, but it remains a crucial necessary condition.

Remark 3: The extreme cases $r=0$ and $r=n$ are included in Theorem 5. For $r=0$ the space D disappears and $\mathbf{K}$ plays the essential role. For $r=n$ no K-tensor is involved, V is constant and the manifold Q is locally flat. The separable coordinates determined by an orthogonal basis of D are rectangular Cartesian coordinates. Also the case $r=n-1$ is in some sense trivial, since as a characteristic K-tensor we can take the metric tensor $\mathbf{g}$ itself. In all these three cases the separation is orthogonal and $m_0=0$.

The nondegenerate separable systems $(m_0=0)$ are of particular interest. Among them we find all the separable systems in positive definite metrics and all the orthogonal separable systems. They are examined in Sec. V.

III. NECESSARY CONDITIONS FOR THE SEPARATION

We base our discussion on the following known properties concerning separable coordinates.

(a) A Hamiltonian $H(\underline{q},\underline{p})$ is separable in a coordinate system $\underline{q}$ if and only if equations

$$\partial^i\partial^j H\partial_i H\partial_j H - \partial_i\partial^j H\partial^i H\partial_j H + \partial_i\partial_j H\partial^i H\partial^j H - \partial^i\partial_j H\partial_i H\partial^j H = 0 \tag{3.1}$$

are identically satisfied, for $i \neq j$ (no summation over these indices). Here the notation $\partial_i = \partial/\partial q^i$ and $\partial^i = \partial/\partial p_i$ is used. These are the *separability conditions of Levi-Civita* mentioned in the Introduction.[2]

(b) If a Hamiltonian H is separable in two separable and overlapping coordinate systems then these two systems are called *equivalent* if in the intersection of their domains they yield the same complete solution of the H–J equation.

(c) Let a Hamiltonian H be separable in a coordinate system $\underline{q}$. A coordinate q^i is of *first class* if the ratio $\partial_i H/\partial^i H$ is a linear (homogeneous) function in the momenta $\underline{p}$, otherwise it is of *second class*.[2,3,8,9,18] When $\partial_i H = 0$ the coordinate q^i is *ignorable*. An ignorable coordinate is obviously of first class. It is convenient to denote by (q^α) (with Greek indices running from $m+1$ to n) the first class coordinates and by (q^a) (with first Latin indices running from 1 to m) the second class coordinates. Working on the separability conditions of Levi-Civita (3.1) it can be proved that[9,10] (1) the numbers (r,m) of first and second class coordinates are invariant; they are the same in two equivalent systems of separable coordinates; (2) by a coordinate transformation preserving the separation all first class coordinates are reducible to ignorable coordinates, $\partial_\alpha H = 0$.

(d) Let G be the geodesic Hamiltonian of a Riemannian manifold and V a potential function. If the Hamiltonian $H = G + V$ is separable then also G is separable in the same coordinates.[2] Thus the separation of the geodesic Hamiltonian G is a necessary condition for the separation of the complete Hamiltonian $H = G + V$.

(e) For a geodesic Hamiltonian G the partial derivatives (∂_α) with respect to the ignorable coordinates (q^α), interpreted as vector fields, are independent and commuting K-vectors. We recall that a K-vector is a vector field $\mathbf{X}$ on Q such that the function $E_X = X^i p_i$ is a (linear) first integral of the geodesic flow: $\{G, E_X\} = 0$. A K-vector generates a local one-parameter group of isometries. We recall that two vector fields commute, i.e., their Lie-brackets are zero, if and only if the corresponding linear functions on T^*Q are in involution.

(f) For a geodesic Hamiltonian G it can be shown that[9,10] (1) in two equivalent separable systems the second class coordinates are related by separated transformations i.e. by a transformations with diagonal Jacobian matrix; (2) the second class coordinates are orthogonal, i.e., $g^{ab} = 0$ for $a \neq b$ (see Ref. 3 for a positive-definite metric and Ref. 10 for a nondefinite metric).

(g) Among the second class coordinates we consider a further classification. A second class coordinate q^a is *null* if $g^{aa} = 0$. We label the second class null coordinates by $q^{\bar{a}}$ and the non-null coordinates by $q^{\hat{a}}$, with $\hat{a} = 1, \ldots. m_1$, $\bar{a} = m_1 + 1, \ldots, m = m_1 + m_0$. Due to (f)-1 also the number m_0 of the null coordinates is invariant. It will be shown that

$$m_0 \leqslant \min(p,q), \quad m_0 \leqslant r, \tag{3.2}$$

where (p,q) is the signature of the metric.

(h) There exist equivalent coordinate systems $(q^{\hat{a}}, q^{\bar{a}}, q^\alpha)$ such that all first class coordinates (q^α) are ignorable and $g^{\hat{a}\alpha} = 0$ for any non-null second class index $\hat{a}$.[9] These coordinates are called *normal separable coordinates*. In these coordinates the matrix of the contravariant components of the metric tensor assumes the *standard form*

$$
(g^{ij}) = \begin{array}{c c} & \begin{array}{ccc} \overbrace{\quad}^{m_1} & \overbrace{\quad}^{m_0} & \overbrace{\quad}^{r} \end{array} \\ \begin{array}{c} m_1 \{ \\ m_0 \{ \\ r \{ \end{array} & \begin{pmatrix} g^{\hat{a}\hat{a}} & 0 & 0 \\ 0 & 0 & g^{\bar{b}\beta} \\ 0 & g^{\alpha\bar{a}} & g^{\alpha\beta} \end{pmatrix} \end{array} \tag{3.3}
$$

In the proof of this property, which is based on the separability conditions of Levi-Civita (3.1), we find the equations[9] $\partial_b \partial^a G = f^a_b \partial^a G$, for $a \neq b$ indices of second class, where f^a_b are functions of the coordinates q only. These equations are equivalent to

$$\partial_b g^{ai} = f^a_b g^{ai} \tag{3.4}$$

and are used for generating an equivalent separable coordinate system such that $g^{\hat{a}\alpha} = 0$. From the standard form (3.3) we can see that (3.2) holds since $m_0 > r$ would imply $\det(g^{ij}) = 0$.

Let us examine the geometrical implications of these results for a geodesic Hamiltonian G. Assume that on $Q - \Sigma$, where Σ is a closed singular set, there is an atlas of equivalent separable charts. Since second class coordinates remain essentially unchanged in an equivalent coordinate transformation [points (c)-1 and (f)-1], the corresponding coordinate hypersurfaces build up m transversal foliations $(\mathscr{S}^a)$. Due to (f)-2; these foliations are pairwise orthogonal. Let us consider a subatlas of separable charts with ignorable first class coordinates [this subatlas exists because of (c)-2]. The K-vectors corresponding to the ignorable coordinates are tangent to the leaves of the foliations S^a, since $\partial_\alpha q^a = 0$. They commute and span a regular distribution Δ, which is completely integrable and whose integral submanifolds (of dimension r) coincide with the complete intersections of the leaves of $(\mathscr{S}^a)$. Let $(\mathbf{X}^a)$ be the vector fields corresponding to the differentials (dq^a) of the second class coordinates. Since the vector fields (∂_α) span the distribution Δ and $\mathbf{X}^a \cdot \partial_\alpha = \partial_\alpha q^a = 0$, it follows that the independent vector fields $(\mathbf{X}^a)$ span the orthogonal distribution $\Delta^\perp$. Moreover, from $\mathbf{X}^a = g^{ai}\partial_i = g^{a\alpha}\partial_\alpha + g^{aa}\partial_a$ it follows in particular that $\mathbf{X}^{\bar{a}} = g^{\bar{a}\alpha}\partial_\alpha$. This shows that these vector fields also belong to the distribution Δ, thus they belong to the distribution $I = \Delta \cap \Delta^\perp$. Furthermore, no linear combination $f_{\hat{a}}\mathbf{X}^{\hat{a}}$ belongs to Δ, since from $f_{\hat{a}}\mathbf{X}^{\hat{a}} = f^\alpha \partial_\alpha$ by scalar multiplication by $\mathbf{X}^{\hat{b}}$ it follows $0 = f_{\hat{a}}\mathbf{X}^{\hat{a}} \cdot \mathbf{X}^{\hat{b}} = f_{\hat{a}} g^{\hat{a}\hat{b}} = f_{\hat{b}} g^{\hat{b}\hat{b}}$ (no summation on $\hat{b}$), that is $f_{\hat{b}} = 0$. This shows that the $m = m_1 + m_0$ vector fields $(\mathbf{X}^{\hat{a}} \cdot \mathbf{X}^{\bar{a}})$ span $\Delta^\perp$ and the m_0 vector fields $(\mathbf{X}^{\bar{a}})$ span the intersection distribution I. As a consequence, $m_0 = \dim(I_q)$. Since this number is invariant [point (g)] the distribution I is regular. Since the vector fields $(\mathbf{X}^a)$ span the orthogonal distribution $\Delta^\perp$ and in normal separable coordinates $\mathbf{X}^{\hat{a}} = g^{\hat{a}\alpha}\partial_\alpha + g^{\hat{a}\hat{a}}\partial_{\hat{a}}$ reduces to $\mathbf{X}^{\hat{a}} = g^{\hat{a}\hat{a}}\partial_{\hat{a}}$, this distribution is also spanned by the vector fields $(\partial_{\hat{a}}, \mathbf{X}^{\bar{a}})$. Since (q^α) are ignorable coordinates, it follows that $[\mathbf{X}^{\bar{a}}, \mathbf{X}^{\bar{b}}] = [g^{\bar{a}\alpha}\partial_\alpha, g^{\bar{b}\beta}\partial_\beta] = 0$. Moreover, due to (3.4), $[\partial_{\hat{a}}, \mathbf{X}^{\bar{a}}] = \partial_{\hat{a}} g^{\bar{a}\alpha}\partial_\alpha = f^{\bar{a}}_{\hat{a}} g^{\bar{a}\alpha}\partial_\alpha = f^{\bar{a}}_{\hat{a}}\mathbf{X}^{\bar{a}}$. This proves that both $\Delta^\perp$ and I are involutive, thus completely integrable (it is a general property that if both Δ and $\Delta^\perp$ are involutive then $I = \Delta \cap \Delta^\perp$ is involutive). Thus we have proved that

Proposition 1: An atlas of equivalent separable coordinate systems with r first class coordinates and m_0 null second class coordinates generates $m = n - r$ pairwise transversal and orthogonal foliations $(\mathscr{S}^a) = (\mathscr{S}^1, \ldots, \mathscr{S}^m)$ of submanifolds of codimension 1, whose complete intersections form a foliation $\mathscr{E}$ of submanifolds of dimension r which are the orbits of the action of an Abelian group of isometries. If Δ is the distribution of vectors tangent to the foliation $\mathscr{E}$, then the orthogonal distribution $\Delta^{\perp}$ and the intersection $I=\Delta\cap\Delta^{\perp}$ are completely integrable. The rank of the distribution I is m_0.

Notice that the distribution I is made of null vectors (this is equivalent to $I\subseteq I^{\perp}$). This implies the bound $(3.2)_1$. Possibly by enlarging the critical set Σ on Q, we can assume that there is a global r-dimensional space D of commuting K-vectors generating Δ, then we have a K-web according to Definition 2 of Sec. II. Hence, Proposition 1 leads to point (i) of Theorem 3.

Now we consider the connection between separation and K-tensors. A Killing 2-tensor is a contravariant 2-tensor $\mathbf{K}=(K^{ij})$ such that the function

$$E_{\mathbf{K}}=\tfrac{1}{2}\,K^{ij}p_ip_j$$

is a (quadratic) first integral of the geodesic flow: $\{G,E_{\mathbf{K}}\}=0$. In the theory of separation the role played by these tensors is essentially algebraic; interpreted as linear endomorphisms on vectors and 1-forms they produce eigenvalues, eigenvectors, and eigenforms, and the properties of such objects are used for characterizing the separation. For our purposes, it is convenient to summarize this crucial topic in the following three propositions.

Proposition 2: For a metric tensor of the standard form (3.3) with (q^{α}) ignorable coordinates, the H–J equation is separable if and only if[9]

$$g^{\bar{a}\alpha}=\theta^{\alpha}_{\bar{a}}\varphi^{\bar{a}}\quad(\bar{a}\ n.s.),\quad g^{\hat{a}\hat{a}}=\varphi^{\hat{a}},\tag{3.5}$$

where each $\theta^{\alpha}_{\bar{a}}$ is a function of $x^{\bar{a}}$ only, and the functions $(\varphi^{a})=(\varphi^{\hat{a}},\varphi^{\bar{a}})$ and $g^{\alpha\beta}$ satisfy the following differential equations:

$$\begin{cases}\partial_a\partial_b\varphi^c-\partial_a\ \ln\ \varphi^b\partial_b\varphi^c-\partial_b\ \ln\ \varphi^a\partial_a\varphi^c=0,\\ \partial_a\partial_b g^{\alpha\beta}-\partial_a\ \ln\ \varphi^b\partial_b g^{\alpha\beta}-\partial_b\ \ln\ \varphi^a\partial_a g^{\alpha\beta}=0.\end{cases}\quad(a\neq b)\tag{3.6}$$

Proof: For a metric of the form (3.3) the geodesic Hamiltonian is

$$G=\tfrac{1}{2}g^{\hat{a}\hat{a}}p^2_{\hat{a}}+g^{\bar{a}\alpha}p_{\bar{a}}p_{\alpha}+\tfrac{1}{2}g^{\alpha\beta}p_{\alpha}p_{\beta},$$

and since the coordinates (q^{α}) are ignorable the separability conditions of Levi-Civita are not trivial only for pairs of indices of second class $(i,j)=(a,b)$. By these conditions it is possible to prove that $g^{\bar{a}\alpha}$ has the form $(3.5)_1$. (see the proof of Theorem 5.4 in Ref. 10). Thus a straightforward calculation shows that the separability conditions, which are polynomial equations of fourth degree in the momenta, are equivalent to (3.6). ■

In (3.6) and in the following discussion by $\partial_a\ \ln\ \varphi^c$ we actually mean $\partial_a\ \ln|\varphi^c|=(\varphi^c)^{-1}\partial_a\varphi^c$ when $\varphi^c<0$. Now we examine Eqs. (3.6) from a different point of view.

Proposition 3: The differential equations (3.6) are the necessary and sufficient conditions for the complete integrability of the linear differential system

$$\begin{cases}\partial_a\varrho_b=(\varrho_a-\varrho_b)\partial_a\ \ln\ \varphi^b,\\ \partial_a K^{\alpha\beta}=\varrho_a\partial_a g^{\alpha\beta},\end{cases}\tag{3.7}$$

in the unknown functions $(\varrho_a,K^{\alpha\beta})$ of the variables (q^a).

Proof: A straightforward calculation shows that the integrability conditions $\partial_a\partial_b\varrho_c=\partial_b\partial_a\varrho_c$ of the system (3.7) are (a, b n.s.)

$$(\varrho_a-\varrho_b)(\partial_a\partial_b\varphi^c-\partial_a\ \ln\ \varphi^b\partial_b\varphi^c-\partial_b\ \ln\ \varphi^a\partial_a\varphi^c)=0.\tag{3.8}$$

If Eqs. $(3.6)_1$. hold, then (3.8) are identically satisfied. Conversely, if the linear system $(3.7)_1$ is completely integrable then it has local solutions such that $\varrho_a\neq\varrho_b$ for $a\neq b$ (indeed in a vector space there are vectors with distinct components with respect to any fixed basis) so that (3.8) implies $(3.6)_1$. Under the assumption that (3.7) is integrable the integrability condition of (3.7) becomes

$$(\varrho_a-\varrho_b)(\partial_a\partial_b g^{\alpha\beta}-\partial_b \ln \varphi^a \partial_a g^{\alpha\beta}-\partial_a \ln \varphi^b \partial_b g^{\alpha\beta})=0. \tag{3.9}$$

Following the same reasoning as before, we conclude that Eqs. (3.9) are equivalent to $(3.6)_2$. ∎

Remark 1: This proof shows that the complete integrability of the subsystem (3.7) is equivalent to the existence of a solution such that $\varrho_a \neq \varrho_b$ for $a\neq b$.

Proposition 4. Let (g^{ij}) be a metric tensor of the form (3.3) with (q^α) ignorable coordinates and such that (3.5) hold. Then the 2-tensor $\mathbf{K}=(K^{ij})$ defined by

$$(K^{ij})=\begin{pmatrix} \varrho_{\hat a} g^{\hat a\hat a} & 0 & 0 \\ 0 & 0 & \varrho_{\bar b} g^{\bar b\beta} \\ 0 & \varrho_{\bar a} g^{\alpha\bar a} & K^{\alpha\beta} \end{pmatrix} \tag{3.10}$$

is a K-tensor if and only if Eqs. (3.7) are satisfied. All K-tensors of this kind commute (i.e., the corresponding functions $E_{\mathbf{K}}$ are all in involution).

Proof: The left-hand side of the Killing equation $\{G,E_{\mathbf{K}}\}=0$ is a third degree polynomial in the momenta $\underline{p}$, so that all the coefficients must vanish. In the present case

$$\begin{cases} G=\frac{1}{2}\,\varphi^{\hat a}p_{\hat a}^2+\theta_{\bar a}^{\alpha}\varphi^{\bar a}p_{\bar a}p_\alpha+\frac{1}{2}\,g^{\alpha\beta}p_\alpha p_\beta, \\ E_{\mathbf{K}}=\frac{1}{2}\,\varrho_{\hat a}\varphi^{\hat a}p_{\hat a}^2+\varrho_{\bar a}\theta_{\bar a}^{\alpha}\varphi^{\bar a}p_{\bar a}p_\alpha+\frac{1}{2}\,K^{\alpha\beta}p_\alpha p_\beta. \end{cases}$$

A straightforward calculation shows that $\{G,E_{\mathbf{K}}\}=0$ is equivalent to (3.7) and $\{E_K,E_{K'}\}=0$ for two K-tensors $\mathbf{K}$ and $\mathbf{K}'$ determined by two solutions of (3.7). ∎

The functions (ϱ_a) entering in this discussion are eigenvalues of $\mathbf{K}$ corresponding to eigenforms (dq^a). The metric tensor $\mathbf{g}$ $(\varrho_a=1)$ is always a solution of (3.7). The first-class (ignorable) coordinates (q_α) do not appear at all, so that $\partial_\alpha K^{ij}=0$. This means that the K-tensors are D-invariant. Thus from Propositions 2, 3, 4 and Remark 1 we derive point (ii) of Theorem 3.

Remark 2: Equations $(3.7)_2$ are equivalent to

$$dK^{\alpha\beta}=\mathbf{K}\cdot dg^{\alpha\beta}. \tag{3.11}$$

Indeed, for a function F the components of the 1-form $\eta=\mathbf{K}\cdot dF$ are $\eta_i=g_{ij}K^{jh}\partial_h F$. If $\partial_\alpha F=0$ by (3.10) we find

$$\begin{cases} \eta_\alpha=g_{\alpha j}\varrho_a g^{ja}\partial_a F=\delta_\alpha^a\varrho_a\partial_a F=0, \\ \eta_a=g_{aj}\varrho_b g^{jb}\partial_b F=\delta_a^b\varrho_b\partial_b F=\varrho_a\partial_a F. \end{cases}$$

so that (3.11) is equivalent to $(3.7)_2$. The integrability condition of (3.11) is

$$d(\mathbf{K}\cdot dg^{\alpha\beta})=0. \tag{3.12}$$

This proves point (iii) of Theorem 3.

So far we considered a geodesic Hamiltonian G. Now we examine a separable Hamiltonian $H=G+V$. All the preceding procedure should be repeated from the very beginning, by dividing the coordinates into two classes and so on. However, this long way can be avoided since we can reduce again to a geodesic case by considering the so-called *Eisenhart metric*.[24] Let us consider the manifold $\bar{Q}=\mathbb{R}\times Q$ with local coordinates (q^0,q^i) (q^0 is the natural coordinate over $\mathbb{R}$) and the cotangent bundle $T^*\bar{Q}=\mathbb{R}^2\times T^*Q$ with momenta (p_0,p_i). Since at this level we deal with local objects we can assume that $V\neq 0$, so that we can consider locally on $\bar{Q}$ a contravariant metric tensor $\bar{g}$ whose contravariant components are g^{ij}, $g^{0i}=0$ and $g^{00}=2V$. Then the geodesic Hamiltonian is

$$\bar{G}=G+Vp_0^2=\tfrac{1}{2}g^{ij}p_ip_j+\tfrac{1}{2}g^{00}p_0^2.$$

Since q^0 is ignorable, p_0 is a constant of motion. If we consider on $T^*\bar{Q}$ the geodesic flow of $\bar{G}$ then we can see that the integral curves with $p_0=1$ are projected onto T^*Q in the integral curves of the Hamiltonian $H=G+V$. Moreover, the separation of the H–J equation corresponding to H is equivalent to the separation of the H–J equation corresponding to $\bar{G}$, which is a geodesic Hamiltonian. Indeed, if $W(\underset{\sim}{q},\underset{\sim}{c})=W(q^1,\underset{\sim}{c})+\cdots+W_n(q^n,\underset{\sim}{c})$ is a separated complete integral of $H=G+V$, then $\bar{W}=W(q^1,\underset{\sim}{c})+\cdots+W_n(q^n,\underset{\sim}{c})+c_0q^0$ is a separated complete integral of $\bar{G}$, where c_0 is a further constant.

By applying to $\bar{G}$ the results concerning the separation of the geodesic H–J equation we find on $\bar{Q}$ a K-web $(\bar{\mathscr{S}},\bar{D})$, where the space $\bar{D}$ contains the K-vector ∂_0, and a K-tensor $\bar{\mathbf{K}}$. Since they are ∂_0-invariant, they project onto a K-web $(\mathscr{S},D)$ and a K-tensor $\mathbf{K}$ of Q, satisfying the properties considered above. Moreover, since the metric $\bar{\mathbf{g}}$ is represented by the pair $(\mathbf{g},V)$ and $\bar{\mathbf{K}}$ by a pair $(\mathbf{K},U)$, it follows that $\bar{\mathbf{K}}$ is a K-tensor if and only if $\mathbf{K}$ is a K-tensor and

$$dU=\mathbf{K}\cdot dV. \tag{3.13}$$

This implies

$$d(\mathbf{K}\cdot dV)=0. \tag{3.14}$$

Moreover, since $\bar{\mathbf{g}}$ is $\bar{D}$-invariant, it follows that V is D-invariant. This completes the proof of Theorem 3 of Sec. II.

Remark 3: Equations (3.6) and (3.7) hold for the extended metric, thus also for $g^{00}=2V$ (notice that $\partial_\alpha V=0$ since V is D-invariant). Hence, the separability conditions $(3.6)_2$ as well equations $(3.7)_2$ are implemented by the analogous equations corresponding to V, namely,

$$\partial_a\partial_bV-\partial_a\ \ln\ \varphi^b\partial_bV-\partial_b\ \ln\ \varphi^a\partial_aV=0. \tag{3.15}$$

and

$$\partial_aU=\varrho_a\partial_aV, \tag{3.16}$$

where $K^{00}=2U$. This last equation is equivalent to (3.13) and the separability conditions (3.15) are equivalent to the integrability condition (3.14) (since the eigenvalues are distinct).

Remark 4: We say that a K-web $(\mathscr{S}_m,D_r)$ is *reducible* if there exists a K-web $(\mathscr{S}'_{m'},D'_{r'})$ such that $m'<m$, $D\subset D'$ and $\mathscr{S}'$ is a subweb of $\mathscr{S}$. The necessary conditions listed in Theorem 3 do not exclude the existence of a reduced K-web $(\mathscr{S}',D')$ satisfying the same conditions, but such that the potential V is no more D'-invariant. A simple concrete example is the following (see also Sec. VI). Let $(\mathscr{S}_2,D_1)$ be the K-web in $Q=\mathbb{R}^3=\{(x,y,z)\}$ (the Euclidean space) where D_1 are the rotations around the z-axis and $\mathscr{S}=(\mathscr{S}^1,\mathscr{S}^2)$ are the cylinders around the z-axis and the planes orthogonal to the z-axis, respectively. This web is reducible to $(\mathscr{S}'_1,D'_2)$ where D'_2 contains D_1 and the translations along the z-axis, and $\mathscr{S}'_1$ is the subweb made of the cylinders. Then we observe that the Hamiltonian $H=G+V$ with a potential of the kind $V=A(z)+B(r)$ is separable in the cylindrical coordinates $(q^1,q^2,q^3)=(z,r,\theta)$ and the K-web produced by the proof of Theorem 3 is just $(\mathscr{S}_2,D_1)$. However the cylindrical coordinates by themselves generate the reduced web $(\mathscr{S}'_1,D'_2)$, where V is not D'-invariant.

Remark 5: Let us consider on T^*Q the Hamiltonian $H=G+V$ and the functions

$$E_{\mathbf{K}}=\tfrac{1}{2}K^{ij}p_i p_j, \quad F_{\mathbf{K}}=E_{\mathbf{K}}+U,$$

where $\mathbf{K}$ is a symmetric 2-tensor. Then $F_{\mathbf{K}}$ is a first integral, that is $\{H,F_{\mathbf{K}}\}=0$, if and only if

$$\{G,E_{\mathbf{K}}\}=0, \quad dU=\mathbf{K}\cdot dV.$$

This first equation means that $\mathbf{K}$ is a K-tensor. The second one is just Eq. (3.13). Thus Eq. (3.14) where $\mathbf{K}$ is a Killing tensor is a necessary condition for the existence of a first integral of the kind $F_{\mathbf{K}}=E_{\mathbf{K}}+U$. This also shows that if we know a basis $(\mathbf{K}_a)$ of $\mathscr{K}$ then locally we can construct m quadratic first integrals in involution of this kind by integrating the closed 1-forms $\mathbf{K}_a\cdot dV$ (Remarks 1, 2, Sec. II).

IV. SUFFICIENT CONDITIONS FOR SEPARATION

Let $(\mathscr{S}_m, D_r)$ be a K-web and let (q^a,q^α) be adapted coordinates defined as in Sec. II; (q^a) are independent functions such that the differentials (dq^a) are characteristic 1-forms of the foliations $(\mathscr{S}^a)$ and the coordinates (q^α) are the affine parameters of the integral curves of a basis $(\mathbf{X}_\alpha)$ of D, based on the points of an arbitrary chosen local section W of these foliations. Since these vectors are tangent to the foliations $(\mathscr{S}^a)$ we have $\langle \mathbf{X}_\alpha, dq^a\rangle=0$. Since they commute, a local coordinate system (q^a,q^α) is defined such that $\mathbf{X}_\alpha=\partial_\alpha$. Since they are K-vectors, the coordinates (q^α) are ignorable, i.e., $\partial_\alpha g^{ij}=0$. The vector field $\mathbf{X}^a$ corresponding to the 1-form dq^a is orthogonal to the manifolds of the foliation $\mathscr{S}^a$. Since these foliations are orthogonal it follows that $g^{ab}=\mathbf{X}^a\cdot\mathbf{X}^b=0$ (for $a\neq b$). Moreover, by the same reasoning used in Sec. III, it follows that the m independent vector fields $(\mathbf{X}^a)$ span the orthogonal distribution $\Delta^\perp$ and that $m_0=\dim(\Delta_p\cap\Delta_p^\perp)$ is the number of those coordinates $(q^{\bar a})$ for which $g^{\bar a\bar a}=0$ (null coordinates). Hence, in the coordinates (q^a,q^α) the matrix (g^{ij}) has the form

$$(g^{ij}) = \begin{array}{c} \\ m_1\,\{ \\ m_0\,\{ \\ r\,\{ \end{array} \!\!\begin{array}{c} \begin{array}{ccc} \overbrace{\quad}^{m_1} & \overbrace{\quad}^{m_0} & \overbrace{\quad}^{r} \end{array} \\ \begin{pmatrix} g^{\hat a\hat a} & 0 & g^{\hat b\beta} \\ 0 & 0 & g^{\bar b\beta} \\ g^{\alpha\hat a} & g^{\alpha\bar a} & g^{\alpha\beta} \end{pmatrix} \end{array}. \tag{4.1}$$

Let us assume that there exists a D-invariant K-tensor $\mathbf{K}$ such the vector fields orthogonal to the foliations $(\mathscr{S}^a)$ are eigenvectors corresponding to distinct eigenvalues (ϱ_a). Since $q^a=\text{constant}$ on $\mathscr{S}^a$, it follows that the 1-forms (dq^a) satisfy the eigenform equation

$$\mathbf{K}\cdot dq^a=\varrho_a dq^a, \tag{4.2}$$

equivalent to

$$K^{ai}=\varrho_a g^{ai}.$$

We have in particular

$$K^{a\alpha}=\varrho_a g^{a\alpha}, \quad K^{\hat a\hat a}=\varrho_{\hat a} g^{\hat a\hat a}, \quad K^{\bar a\bar a}=0, \quad K^{ab}=0 \quad (a\neq b). \tag{4.3}$$

Thus the matrix (K^{ij}) has a form similar to (4.1). Since $\mathbf{K}$ is a K-tensor, the Killing equation holds $\{G,E_{\mathbf{K}}\}=0$, where

$$\begin{cases} G = \frac{1}{2} g^{ij} p_i p_j = \frac{1}{2} g^{\hat{a}\hat{a}} p_{p\hat{a}}^2 + g^{a\alpha} p_a p_\alpha + \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, \\ E_{\mathbf{K}} = \frac{1}{2} K^{ij} p_i p_j = \frac{1}{2} K^{\hat{a}\hat{a}} p_{\hat{a}}^2 + K^{a\alpha} p_a p_\alpha + \frac{1}{2} K^{\alpha\beta} p_\alpha p_\beta. \end{cases}$$

Since $\mathbf{K}$ is D-invariant, the components (K^{ij}) do not depend on the ignorable coordinates (q^α); $\partial_\alpha K^{ij} = 0$. Thus we find the equation

$$g^{\hat{a}\hat{a}} p_{\hat{a}} \partial_{\hat{a}} E_K + g^{a\beta} p_\beta \partial_a E_{\mathbf{K}} - K^{\hat{a}\hat{a}} p_{\hat{a}} \partial_{\hat{a}} G - K^{a\beta} p_\beta \partial_a G = 0. \tag{4.4}$$

The left-hand side is a homogeneous polynomial of third degree in the momenta p whose coefficients must vanish. This leads to seven equations, corresponding to seven different kinds of coefficients, which we analyze separately.

(1) The equation corresponding to the coefficient of $p_{\hat{a}} p_{\hat{b}}^2$ is

$$g^{\hat{a}\hat{a}} \partial_{\hat{a}} K^{\hat{b}\hat{b}} - K^{\hat{a}\hat{a}} \partial_{\hat{a}} g^{\hat{b}\hat{b}} = 0.$$

Due to (4.3) this is equivalent to

$$\partial_{\hat{a}} \varrho_{\hat{b}} = (\varrho_{\hat{a}} - \rho_{\hat{b}}) \partial_{\hat{a}} \ln g^{\hat{b}\hat{b}}. \tag{4.5}$$

For $\hat{a} = \hat{b}$ we have in particular $\partial_{\hat{a}} \varrho_{\hat{a}} = 0$.

(2) The equation corresponding to the coefficient of $p_{\hat{a}} p_{\hat{b}} p_\alpha$ for $\hat{a} \neq \hat{b}$ is

$$g^{\hat{a}\hat{a}} \partial_{\hat{a}} K^{\hat{b}\alpha} + g^{\hat{b}\hat{b}} \partial_{\hat{b}} K^{\hat{a}\alpha} - K^{\hat{a}\hat{a}} \partial_{\hat{a}} g^{\hat{b}\alpha} - K^{\hat{b}\hat{b}} \partial_{\hat{b}} g^{\hat{a}\alpha} = 0.$$

This is equivalent to

$$g^{\hat{a}\hat{a}} (g^{\hat{b}\alpha} \partial_{\hat{a}} \varrho_{\hat{b}} - (\varrho_{\hat{a}} - \varrho_{\hat{b}}) \partial_{\hat{a}} g^{\hat{b}\alpha}) + g^{\hat{b}\hat{b}} (g^{\hat{a}\alpha} \partial_{\hat{b}} \varrho_{\hat{a}} - (\varrho_{\hat{b}} - \varrho_{\hat{a}}) \partial_{\hat{b}} g^{\hat{a}\alpha}) = 0.$$

By (4.5) and the assumption $\varrho_{\hat{a}} \neq \varrho_{\hat{b}}$, this last equation reduces to

$$g^{\hat{a}\hat{a}} (g^{\hat{b}\alpha} \partial_{\hat{a}} \ln g^{\hat{b}\hat{b}} - \partial_a g^{\hat{b}\alpha}) = g^{\hat{b}\hat{b}} (g^{\hat{a}\alpha} \partial_{\hat{b}} \ln g^{\hat{a}\hat{a}} - \partial_{\hat{b}} g^{\hat{a}\alpha}),$$

that is

$$\partial_{\hat{a}} \left(\frac{g^{\hat{b}\alpha}}{g^{\hat{b}\hat{b}}} \right) = \partial_{\hat{b}} \left(\frac{g^{\hat{a}\alpha}}{g^{\hat{a}\hat{a}}} \right) \quad (\hat{a} \neq \hat{b}).$$

It is a remarkable fact that these equations can be interpreted as the integrability conditions of the following linear differential system in the r unknown functions $\omega^\alpha(q^{\hat{a}})$,

$$\partial_{\hat{a}} \omega^\alpha = - \frac{g^{\hat{a}\alpha}}{g^{\hat{a}\hat{a}}}.$$

Indeed, a solution of this system provides a coordinate transformation

$$q^{\alpha'} = q^\alpha + \omega^\alpha(q^{\hat{a}}),$$

such that $g^{\hat{a}\alpha'} = 0$. From a geometrical point of view this coordinate transformation corresponds to a change of the local section W considered in the definition of the coordinates adapted to the web. Hence, from now on we can assume that $g^{\hat{a}\alpha} = 0$, so that the matrix (g^{ij}) [as well as the matrix

(K^{ij})] assumes the standard form (3.3). After this point we are in the situation discussed by Kalnins and Miller in Ref. 18—proof of Theorem 3, which we shall follow with suitable modifications.

(3) The coefficient of $p_{\hat{a}}^2 p_\alpha$ yields the equation (sum over b)

$$g^{b\alpha}\partial_b(\varrho_{\hat{a}} g^{\hat{a}\hat{a}}) = \varrho_b g^{b\alpha}\partial_b g^{\hat{a}\hat{a}},$$

which reduces to (sum over $\bar{a}$)

$$g^{\bar{a}\alpha}[\partial_{\bar{a}}\varrho_{\hat{a}} - (\varrho_{\bar{a}} - \varrho_{\hat{a}})\partial_{\bar{a}} \ln g^{\hat{a}\hat{a}}] = 0.$$

Since the submatrix $(g^{\bar{a}\alpha})$ has maximal rank and $m_0 \leqslant r$, it follows that

$$\partial_{\bar{a}}\varrho_{\hat{a}} = (\varrho_{\bar{a}} - \varrho_{\hat{a}})\partial_{\bar{a}} \ln g^{\hat{a}\hat{a}}. \tag{4.6}$$

(4) The equation corresponding to the coefficient of $p_{\hat{a}} p_{\bar{a}} p_\alpha$ is $\partial_{\hat{a}}(\varrho_{\bar{a}} g^{\bar{a}\alpha}) = \varrho_{\hat{a}}\partial_{\bar{a}} g^{\bar{a}\alpha}$. For $g^{\bar{a}\alpha} \neq 0$ this is equivalent to

$$\partial_{\hat{a}}\varrho_{\bar{a}} = (\varrho_{\hat{a}} - \varrho_{\bar{a}})\partial_{\hat{a}} \ln g^{\bar{a}\alpha}. \tag{4.7}$$

However, for any fixed index $\bar{a}$ there is at least one index β for which $g^{\bar{a}\beta} \neq 0$ [otherwise $\det(g^{ij}) = 0$]. Let us set

$$\varphi^{\bar{a}} = g^{\beta\bar{a}} \neq 0 \tag{4.8}$$

and write for any index α

$$g^{\alpha\bar{a}} = \theta^\alpha_{\bar{a}}\varphi^{\bar{a}}. \tag{4.9}$$

By subtracting term by term Eq. (4.7) and the same equation written for $\alpha = \beta$ we find

$$\partial_{\hat{a}}\theta^\alpha_{\bar{a}} = 0. \tag{4.10}$$

Thus Eq. (4.7) can be written

$$\partial_{\hat{a}}\varrho_{\bar{a}} = (\varrho_{\hat{a}} - \varrho_{\bar{a}})\partial_{\hat{a}} \ln \varphi^{\bar{b}}. \tag{4.11}$$

We remark that at this point we can prove that Δ is orthogonally integrable. Indeed, as we have seen in Sec. III [the metric tensor has the standard form (3.3)], the orthogonal distribution is spanned by the vector fields $(\partial_{\hat{a}}, \mathbf{X}^{\bar{a}})$ where $\mathbf{X}^{\bar{a}} = g^{\bar{a}\alpha}\partial_\alpha$ and $[\mathbf{X}^{\bar{a}}, \mathbf{X}^{\bar{b}}] = 0$. Moreover $[\partial_{\hat{a}}, \mathbf{X}^{\bar{a}}] = g^{\bar{a}\alpha}\partial_\alpha = \theta^\alpha_{\bar{a}}\partial_{\hat{a}}\varphi^{\bar{a}}\partial_\alpha = (\varphi^{\bar{a}})^{-1} g^{\bar{a}\alpha}\partial_\alpha = (\varphi^{\bar{a}})^{-1}\mathbf{X}^{\bar{a}}$, and this shows that $\Delta^\perp$ is involutive.

(5) The equation corresponding to the coefficient $p_{\bar{b}} p_\alpha p_\beta$ is (sum over a)

$$g^{a(\alpha}\partial_a K^{\beta)\bar{b}} = K^{a(\alpha}\partial_a g^{\beta)\bar{b}},$$

and it is equivalent to

$$g^{\bar{a}(\alpha}\left(g^{\beta)\bar{b}}\partial_{\bar{a}}\varrho_{\bar{b}} - (\varrho_{\bar{a}} - \varrho_{\bar{b}})\partial_{\bar{a}} g^{\beta)\bar{b}}\right) = 0, \tag{4.12}$$

with no summation over $\bar{b}$. As shown in Ref. 18, the discussion of this equation leads to the following conclusion: Eq. (4.8) hold with

$$\partial_{\bar{b}}\theta^\alpha_{\bar{a}} = 0 \quad (\bar{b} \neq \bar{a}). \tag{4.13}$$

and

$$\partial_{\bar{a}}\varrho_{\bar{b}}=(\varrho_{\bar{a}}-\varrho_{\bar{b}})\partial_{\bar{a}}\ln\varphi^{\bar{b}}. \tag{4.14}$$

At this point by setting

$$\varphi^{\hat{a}}=g^{\hat{a}\hat{a}}$$

we can see that Eqs. (4.5), (4.6), (4.11), and (4.14) form a system of the kind $(3.7)_1$,

$$\partial_a\varrho_b=(\varrho_a-\varrho_b)\partial_a\ln\varphi^b,$$

including $\partial_a\varrho_a=0$ for $a=b$. This linear system has a solution such that $\varrho_a\neq\varrho_b$ for $a\neq b$, so it is completely integrable (Remark 1 of Sec. III). On the other hand, Eqs. (4.10) and (4.13) show that each function $\theta^{\alpha}_{\bar{a}}$ entering in the representation (4.9) depends on the variable $q^{\bar{a}}$ only. Thus by Propositions 2 and 3 of Sec. III we conclude that the separability conditions $(3.6)_1$ are satisfied. It remains to show that also $(3.6)_2$ are fulfilled, and for this we can use the remaining two equations following from the Killing Eq. (4.4).

(6) The equation corresponding to the coefficient $p_{\hat{a}}p_{\alpha}p_{\beta}$ is

$$\partial_{\hat{a}}K^{\alpha\beta}=\varrho_{\hat{a}}\partial_{\hat{a}}\partial_{\hat{a}}g^{\alpha\beta}. \tag{4.15}$$

(7) The part of Eq. (4.4) corresponding to the monomials $p_{\alpha}p_{\beta}p_{\gamma}$ can be written

$$g^{\bar{a}\alpha}(\partial_{\bar{a}}K^{\beta\gamma}-\varrho_{\bar{a}}\partial_{\bar{a}}g^{\beta\gamma})p_{\alpha}p_{\beta}p_{\gamma}=0, \tag{4.16}$$

with summation on $\bar{a}$. If $m_0=0$ (no null coordinates) this equation is meaningless. If $m_0=1$, this equation implies

$$(\partial_{\bar{a}}K^{\alpha\beta}-\varrho_{\bar{a}}\partial_{\bar{a}}g^{\alpha\beta})p_{\alpha}p_{\beta}=0,$$

i.e.,

$$\partial_{\bar{a}}K^{\alpha\beta}=\varrho_{\bar{a}}\partial_{\bar{a}}g^{\alpha\beta}. \tag{4.17}$$

We can put together (4.15) and (4.17) by writing

$$\partial_aK^{\alpha\beta}=\varrho_a\partial_ag^{\alpha\beta}. \tag{4.18}$$

These are the second equations in (3.7). Thus the separability conditions are all satisfied and Theorem 4 in Sec. II is proved for $m_0=1$.

Remark 1: For $m_0>1$ Eqs. (4.17) do not follow from (4.16) so that they must be considered as further conditions to be imposed on $\mathbf{K}$ for the separability of the web. They involve the null second class coordinates and the corresponding eigenvalues. Unfortunately their intrinsic meaning remains obscure. Thus we return to the whole system (4.18), which includes (4.17); as we remarked in Sec. III this system is equivalent to $dK^{\alpha\beta}=\mathbf{K}\cdot dg^{\alpha\beta}$ and we can take the integrability condition

$$d(\mathbf{K}\cdot dg^{\alpha\beta})=0 \tag{4.19}$$

as an additional condition on $\mathbf{K}$ in order to get the separation of the K-web in the case $m_0>1$.

Finally, we prove Theorem 5 of Sec. II. The conditions listed in the statement are necessary for the separation because of Theorem 3. They are also sufficient: if a pair $(D,\mathbf{K})$ satisfying these conditions is given, then the foliations $(\mathscr{S}^1,\ldots,\mathscr{S}^m)$ orthogonal to the m eigenvectors of $\mathbf{K}$ corresponding to the distinct eigenvalues and the algebra D form a K-web, because of conditions (a), (b), (c). Moreover, conditions (b) and (e) imply that this K-web is separable, due to Theorem 4. Finally, the D-invariance of V means that $\partial_\alpha V=0$ for the ignorable coordinates adapted to the web, so that condition $d(\mathbf{K}\cdot dV)=0$ is equivalent to the separability conditions (3.15), since the eigenvalues are distinct. Thus Theorem 5 is proved.

V. NONDEGENERATE SEPARATION

We say that the separation is *nondegenerate* or *regular* if all second class coordinates are non-null; $g^{aa}\neq 0$ $(a=1\ldots,m)$, i.e., $m_0=0$. Intrinsically this means that $\Delta_q\cap\Delta_q^\perp=0$ at each point q, so that the subspace Δ_q spanned by D is metrically nondegenerate. This is always the case for a definite metric. Since $\Delta^\perp$ is completely integrable, there is a foliation $\mathscr{W}$ of m-dimensional submanifolds orthogonal to the orbits of D. These submanifolds are isometric under the action of D. Let Q' be the quotient set of the orbits of D. Locally Q' can be identified with one of the leaves of $\mathscr{W}$. Moreover, the potential V reduces to a function on Q'. The second class coordinates (q^a) can be interpreted as orthogonal coordinates on Q'. When $m_0=0$ the standard form of the metric is

$$(g^{ij})=\begin{pmatrix} g^{aa} & 0 \\ 0 & g^{\alpha\beta}\end{pmatrix}$$

and the separability conditions of Levi-Civita are equivalent to the following equations:

$$\begin{cases} \partial_a\partial_b g^{cc}-\partial_a \ln g^{bb}\partial_b g^{cc}-\partial_b \ln g^{aa}\partial_a g^{cc}=0, \\ \partial_a\partial_b g^{\alpha\beta}-\partial_a \ln g^{bb}\partial_b g^{\alpha\beta}-\partial_b \ln g^{aa}\partial_a g^{\alpha\beta}=0. \\ \partial_a\partial_b V-\partial_a \ln g^{bb}\partial_b V-\partial_b \ln g^{aa}\partial_a V=0, \end{cases} \tag{5.1}$$

with $a\neq b$ (not summed). Thus the coordinates (q^a) on Q' are separable and the manifold Q' has an orthogonal separable web.

If D has an orthogonal basis, then also the submatrix $(g^{\alpha\beta})$ can be diagonalized and the separation occurs in orthogonal coordinates. In Ref. 25 it is proved that on a manifold with positive metric and constant curvature an Abelian algebra of K-vectors D which is orthogonally integrable has an orthogonal basis (this property also holds for an hyperbolic metric with constant positive curvature, when D is metrically nondegenerate), so that in these manifolds the separation is orthogonal. This property was previously proved by Kalnins and Miller (see Refs. 26–28) by another method.

If $(\mathbf{X}_\alpha)$ is an orthogonal basis of D, then

$$\mathbf{K}_*=\mathbf{K}+c^\alpha\mathbf{X}_\alpha\otimes\mathbf{X}_\alpha \quad (c^\alpha\in\mathbb{R}) \tag{5.2}$$

is a K-tensor with eigenvectors $(\mathbf{X}^a,\mathbf{X}_\alpha)$. We can choose the constants (c^α) in order to get all distinct eigenvalues for $\mathbf{K}_*$. Thus the tensor $\mathbf{K}_*$ characterizes the orthogonal separation according to Theorem 1 of Sec. II.

These remarks suggest the following inverse problem. Assume that on a Riemannian manifold Q there is a linear r-dimensional space D of commuting K-vectors such that (i) the distribution Δ spanned by D is regular and metrically nondegenerate and (ii) $\Delta^\perp$ is completely integrable. Furthermore, assume that on an integral manifold Q' of $\Delta^\perp$ there is an orthogonal separable web $\mathscr{S}'$. Then by the action of D we can extend this web to a K-web $(\mathscr{S},D)$ on Q. When is this K-web separable? An answer is given by the following:

Proposition 1: Assume that the orthogonal sepable web $\mathscr{S}'$ on Q' is characterized by a K-tensor $\mathbf{K}'_*$ with pointwise simple eigenvalues and orthogonally integrable eigenvectors. Then the orthogonal K-web $(\mathscr{S},D)$ is separable if and only if on Q'

$$d(\mathbf{K}'_{*}\cdot dg^{\alpha\beta})=0, \tag{5.3}$$

where $(g^{\alpha\beta})$ is the inverse matrix of $(g_{\alpha\beta})$, $g_{\alpha\beta}=\mathbf{X}_\alpha\cdot\mathbf{X}_\beta$ and $(\mathbf{X}_\alpha)$ is a local basis of D. In this case a characteristic K-tensor of the separable web $(\mathscr{S},D)$ is

$$\mathbf{K}=\bar{\mathbf{K}}'_{*}+\tfrac{1}{2}K^{\alpha\beta}\mathbf{X}_\alpha\cap\mathbf{X}_\beta, \tag{5.4}$$

where $\bar{\mathbf{K}}'_{*}$ is the extension of $\mathbf{K}'_{*}$ to Q by the action of D, $\cap$ is the symmetric tensor product, and

$$dK^{\alpha\beta}=\mathbf{K}'_{*}\cdot dg^{\alpha\beta}. \tag{5.5}$$

Proof: Notice that the functions $g_{\alpha\beta}$ as well as $g^{\alpha\beta}$ are D-invariant, so that they reduce to functions on Q'. As we already observed Eqs. (5.3) are the integrability conditions of Eqs. $(3.7)_2$, which coincide with the separability conditions $(5.1)_2$. Furthermore, let us consider local coordinates (q^a,q^α) adapted to the splitting (Q',D); (q^a) are coordinates on Q' and the coordinates (q^α) are such that $\partial_\alpha=\mathbf{X}_\alpha$ form a basis of D (thus they are ignorable). The geodesic Hamiltonians G' and G on Q' and Q, respectively, are

$$G'=\tfrac{1}{2}g^{ab}p_ap_b,\quad G=\tfrac{1}{2}g^{ab}p_ap_b+\tfrac{1}{2}g^{\alpha\beta}p_\alpha p_\beta=G'+R.$$

Moreover, if

$$E_{\mathbf{K}'_{*}}=\tfrac{1}{2}K^{ab}p_ap_b, \tag{5.6}$$

then

$$E_{\mathbf{K}}=\tfrac{1}{2}K^{ab}p_ap_b+\tfrac{1}{2}K^{\alpha\beta}p_\alpha p_\beta=E_{\mathbf{K}'_{*}}+S. \tag{5.7}$$

Since $\{G',E_{\mathbf{K}'_{*}}\}=0$ ($\mathbf{K}'_{*}$ is a K-tensor on Q') and $\{R,S\}=0$, equation $\{G,E_{\mathbf{K}}\}=0$ ($\mathbf{K}$ is a K-tensor) is equivalent to $\{G',S\}+\{R,E_{\mathbf{K}'_{*}}\}=0$, that is to $dK^{\alpha\beta}=\mathbf{K}'_{*}\cdot dg^{\alpha\beta}=0$. Finally, we observe that the characteristic conditions of Theorem 5 of Sec. II are satisfied by the pair $(D,\mathbf{K})$. ■

A longer discussion is needed when the orthogonal separation on Q' is determined by a pair $(D',\mathbf{K}')$ where D' is a r'-dimensional space of commuting K-vectors on Q' (with an orthogonal basis) and $\mathbf{K}'$ is a K-tensor characterizing the separation. However, we can always reduce the problem to the previous case by considering on Q' a characteristic K-tensor of the form (5.2),

$$\mathbf{K}'_{*}=\mathbf{K}'+c^{a'}\mathbf{X}_{a'}\otimes\mathbf{X}_{a'},\quad (c^{a'}\in\mathbb{R}), \tag{5.8}$$

where $(\mathbf{X}_{\alpha'})$ is an orthogonal basis of D'. With this choice Eqs. (5.3) split in the two subsystems

$$d(\langle\mathbf{X}_{a'},dg^{\alpha\beta}\rangle\xi_{a'})=0,\quad d(\mathbf{K}'\cdot dg^{\alpha\beta})=0, \tag{5.9}$$

where $\xi_{a'}$ is the 1-form corresponding to $\mathbf{X}_{a'}$.

In this situation a further question arises. By the action of D an element $\mathbf{X}'\in D'$ is extended to a vector field $\bar{\mathbf{X}}'$ on Q; when is this vector a K-vector? [Notice that when this happens then the K-web $(\mathscr{S}^a,D)$ is reducible.] The answer is given by the following:

Proposition 2: The vector field $\bar{\mathbf{X}}'$ is a K-vector if and only if

$$\langle\mathbf{X}',dg^{\alpha\beta}\rangle=0. \tag{5.10}$$

Proof: In the local coordinates used in the preceding proof we have $E_{\mathbf{X}'}=E_{\bar{\mathbf{X}}'}=X^ap_a$. Since $\{G',E_{\bar{\mathbf{X}}'}\}=0$, equation $\{G,E_{\mathbf{X}'}\}=0$ is equivalent to $X^a\partial_ag^{\alpha\beta}=0$, i.e., to (5.8). ■

VI. ILLUSTRATIVE EXAMPLES

In order to illustrate how the intrinsic method works we consider some examples concerning the general theory and the separation in the Euclidean spaces of dimension 2 and 3. Only rectangular Cartesian coordinates and ordinary vector calculus will be used for representing and dealing with the intrinsic objects. For the sake of brevity we shall not write the coordinate transformations relating the Cartesian coordinates with the separable coordinates adapted to the separable webs encountered in the following examples.

A. The Bertrand–Darboux–Whittaker theorem

In a two-dimensional manifold a separable web is characterized by a single K-tensor $\mathbf{K}$ with simple eigenvalues, since the orthogonal integrability of the eigenvectors is obviously satisfied. The points where the two eigenvalues coincide (or are not real, in the case of a hyperbolic metric[23]) are *singular points*. A potential V is separable in the separable web characterized by $\mathbf{K}$ if and only if the 1-form $\varphi = \mathbf{K} \cdot dV$ is closed. If $\varphi = dU$, then the function

$$F = E_{\mathbf{K}} + U = \tfrac{1}{2} K^{ij} p_i p_j + U \tag{6.1}$$

is a first integral. Thus for $n=2$ the separation is equivalent to the existence of a quadratic first integral (different from the energy). This is essentially the content of the so-called Bertrand–Darboux–Whittaker theorem for the Euclidean plane (see Ref. 29, Secs. 152, 153, and also Refs. 30, 31), which however holds for any two-dimensional manifold. This theorem can be extended to a manifold of any dimension, provided the K-tensor corresponding to the quadratic first integral satisfies the conditions of Theorem 2 of Sec. II (for the orthogonal separation) or of Theorem 3 (for the nonorthogonal separation). For instance, since the separation on a Riemannian manifold with positive metric and constant curvature is orthogonal, we can affirm that

Proposition 1: On a Riemannian manifold with positive metric and constant curvature the separation of $H = G + V$ occurs if and only if there exists a quadratic first integral such that the corresponding K-tensor has pointwise single eigenvectors and orthogonally integrable eigenvectors (or closed eigenforms).

In Ref. 30 we find an analogus statement, but restricted to a flat metric and involving n first integrals.

B. Separation in the Euclidean plane $\mathbb{R}^2$

Let us consider a Hamiltonian of the form

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + V(x,y). \tag{6.2}$$

We can interpret (x,y) as rectangular Cartesian coordinates. For establishing that it is separable, in these coordinates or in some other system of coordinates, we can check if equation $d\varphi = d(\mathbf{K} \cdot dV) = 0$ is satisfied for a generic K-tensor in the Euclidean plane. Since the components of such a tensor are

$$\begin{cases} K^{xx} = A + 2\alpha y + \gamma y^2, \\ K^{yy} = B + 2\beta x + \gamma x^2, \\ K^{xy} = C - \alpha x - \beta y - \gamma x y, \end{cases} \tag{6.3}$$

where $(A,B,C,\alpha,\beta,\gamma)$ are constant, we can translate the separability condition $d(\mathbf{K} \cdot dV) = 0$ into a differential equation on $V(x,y)$ (see Ref. 30, where such a differential equation is obtained by another method). However, it is known that on $\mathbb{R}^2$ there are four kinds of separable orthogonal webs, made of confocal conics, including degenerate cases. They can be characterized by the K-tensors of the form[21]

$$\mathbf{K}=\mathbf{R}_{F_1}\cap\mathbf{R}_{F_2}, \tag{6.4}$$

where $\cap$ is the symmetric tensor product ($\mathbf{A}\cap\mathbf{B}=\mathbf{A}\otimes\mathbf{B}+\mathbf{B}\otimes\mathbf{A}$) and $\mathbf{R}_F$ denotes the unitary rotation vector field around the point F; $\mathbf{R}_F(\mathbf{r}_F)=\boldsymbol{\omega}\times\mathbf{r}_F$, where $\mathbf{r}_F$ is the position vector with respect to the point F and $\boldsymbol{\omega}$ is a unitary vector orthogonal to the plane. The center F can go to infinity; in this case $\mathbf{R}_F$ is a unitary constant vector field orthogonal to the direction of F (a translation field). The centers (F_1,F_2) are the focuses of the conics. When the two focuses are distinct we have the elliptic–hyperbolic web; when the focuses coincide we have the polar web; when one focus goes to infinity we have the parabolic web and finally, when both focuses go to infinity, we have the Cartesian web. The focuses are the singular points of the K-tensors and form the singular set of the web. Thus it is possible to check if the Hamiltonian (6.2) is separable or not by trying if equation $d\varphi=d(\mathbf{K}\cdot dV)=0$ is satisfied for one of the four kinds of K-tensors (6.4). For the elliptic web the K-tensor has the form $\mathbf{K}=\boldsymbol{\omega}\times(\mathbf{r}-\mathbf{r}_1)\cap\boldsymbol{\omega}\times(\mathbf{r}-\mathbf{r}_2)$, where $\mathbf{r}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are the radius vectors, with respect to the origin O of the coordinates, of the generic point $P=(x,y)$ and of the two focuses $F_1=(x_1,y_1)$, $F_2=(x_2,y_2)$, respectively. A straightforward calculation shows that the separability condition $d\varphi=0$ is equivalent to the differential equation

$$\begin{aligned}&[(y-y_1)(x-x_2)+(y-y_2)(x-x_1)](V_{xx}-V_{yy})+2[(y-y_1)(y-y_2)-(x-x_1)(x-x_2)]V_{xy}\\&\quad+3(2y-y_1-y_2)V_x-3(2x-x_1-x_2)V_y=0.\end{aligned} \tag{6.5}$$

For $r_1=r_2$ we have the polar web and the separability Eq. (6.5) reduces to

$$(y-y_1)(x-x_1)(V_{xx}-V_{yy})+[(y-y_1)^2-(x-x_1)^2]V_{xy}+3(y-y_1)V_x-6(x-x_1)V_y=0. \tag{6.6}$$

For the parabolic web the K-tensor has the form $\mathbf{K}=\boldsymbol{\omega}\times(\mathbf{r}-\mathbf{r}_1)\cap\mathbf{X}$, where $\mathbf{X}=a\mathbf{i}+b\mathbf{j}$ is a constant vector [we denote by $(\mathbf{i},\mathbf{j})$ the unitary vectors of the coordinates (x,y)]. In this case the separability condition is

$$[a(x-x_1)-b(y-y_1)](V_{xx}-V_{yy})+2[b(x-x_1)+a(y-y_1)]V_{xy}+3aV_x+3bV_y=0. \tag{6.7}$$

Finally the Cartesian web is characterized by a K-tensor of the kind $\mathbf{K}=\mathbf{X}\otimes\mathbf{X}$ so that the separability condition is

$$(a^2-b^2)V_{xy}+ab(V_{yy}-V_{xx})=0. \tag{6.8}$$

Thus we have proved

Proposition 2: The Hamiltonian (6.2) is separable if and only if one of the Eqs. (6.5), (6.6), (6.7), and (6.8) is satisfied with some values of the constant parameters (x_1,y_1), (x_2,y_2), (a,b).

The values of these parameters locate the focuses of the web and the directions of the relevant axes, so that the corresponding separable coordinates can be immediately determined [this is an advantage with respect to the use of the generic K-tensor (6.3)].

By applying Proposition 2 we can find as particular cases some separable systems known in the literature (see for instance Refs. 30, 32). Let us consider for instance a cubic potential

$$V=\alpha x+\beta y+\gamma x^2+\delta y^2+\epsilon xy+\lambda x^3+\mu y^3+\varrho x^2y+\sigma xy^2. \tag{6.9}$$

By applying Eq. (6.7) it can be seen that this potential is separable in a parabolic web if and only if the following six equations are satisfied [they correspond to the six coefficients of the second-degree polynomial resulting from (6.7)],

$$\begin{cases} 7b\varrho+5a\lambda-2a\sigma=0, \\ 7a\sigma+15b\mu-2b\varrho=0, \\ 2a\varrho+2b\sigma-a\mu-b\lambda=0, \\ 8a\gamma-2a\delta+5b\epsilon+2(3\lambda-\sigma)(by_1-ax_1)-4\varrho(bx_1+ay_1)=0, \\ 8b\delta-2b\gamma+5a\epsilon+2(\varrho-3\mu)(by_1-ax_1)-4\sigma(bx_1+ay_1)=0, \\ 2(\gamma-\delta)(by_1-ax_1)-2\epsilon(bx_1+ay_1)+3a\alpha+3b\beta=0. \end{cases} \tag{6.10}$$

The known integrable cases of the so-called Henon–Heiles potential can be found. For instance, the case $\epsilon=\mu=\varrho=0$, that is the potential

$$V=\alpha x+\beta y+\gamma x^2+\delta y^2+\lambda x^3+\sigma xy^2, \tag{6.11}$$

leads to equations

$$\begin{cases} a(15\lambda-2\sigma)=0, \\ a\sigma=0. \\ b(2\sigma-\lambda)=0, \\ 4a\gamma-a\delta+(3\lambda-\sigma)(by_1-ax_1)=0, \\ 4b\delta-b\gamma-2\sigma(bx_1+ay_1)=0, \\ 2(\gamma-\delta)(by_1-ax_1)+3a\alpha+3b\beta=0. \end{cases} \tag{6.12}$$

The second equation is satisfied for $a=0$ or $\sigma=0$. In the case $a=0$ (so that $b\neq 0$) and $\sigma\neq 0$ we find

$$\lambda=2\sigma, \quad y_1=0, \quad x_1=\frac{4\delta-\gamma}{2\sigma}, \quad \beta=0.$$

This means that the potential

$$V=\alpha x+\gamma x^2+\delta y^2+\sigma(2x^3+xy^2) \tag{6.13}$$

is separable in the parabolic web with focus at the point $[(+\delta-\gamma)/2\sigma,0]$ and axis the x-axis. Assuming $a=\sigma=0$, Eq. (6.12) imply

$$\lambda=0, \quad \gamma=4\delta, \quad y_1=0, \quad \beta=0.$$

We conclude that the potential

$$V=\alpha x+\delta(4x^2+y^2) \tag{6.14}$$

is separable in all the parabolic webs with axis the x-axis. The case $a\neq 0$ and $\sigma=0$ leads to the trivial potential $V=0$. It is interesting to remark that Eq. (6.8) is satisfied by the potential (6.11) for $a=b$, $\delta=\gamma$ and $\sigma=3\lambda$. This means that the potential

$$V=\alpha x+\beta y+\gamma(x^2+y^2)+\lambda x(x^2+3y^2) \tag{6.15}$$

is separable in the Cartesian web obtained from that corresponding to the coordinates (x,y) by a rotation of $\pi/4$.

When a potential is known to be separable in a web characterized by a K-tensor $\mathbf{K}$, then by using an integral function U of the form φ we can construct the first integral F (6.1). Let us consider for example the separable potential (6.13) with $\gamma=4\delta$,

$$V=\alpha x+\delta(4x^2+y^2)+\sigma(2x^3+xy^2).$$

The parabolic web is now centered at the origin of the coordinates (x,y). In this web the components of the K-tensor and of the form φ (6.11) are

$$K^{xx}=0, \quad K^{xy}=-y, \quad K^{yy}=x,$$

and

$$\varphi_x=-2\,\delta y^2-2\,\sigma xy^2, \quad \varphi_y=-4\,\delta xy-2\,\sigma x^2 y-\alpha y-\sigma y^3,$$

so that $\varphi=dU$ with

$$U=-2\,\delta xy^2-\sigma x^2 y^2-\tfrac{1}{2}\alpha y^2-\tfrac{1}{4}\sigma y^4.$$

Thus we find the first integral

$$F=\tfrac{1}{2}xp_y^2-yp_xp_y-2\,\delta xy^2-\tfrac{1}{2}\alpha y^2-\sigma(x^2y^2+\tfrac{1}{4}y^4).$$

C. Separation in the Euclidean space $\mathbb{R}^3$

By discussing the differential equations arising from the separability conditions and the vanishing of the Riemann tensor Eisenhart[12] proved that in $\mathbb{R}^3$ there are 11 kinds of inequivalent orthogonal separable coordinates and that the corresponding coordinate surfaces are confocal quadrics, including planes. (The separation in the Euclidean 3-space was previously investigated by Weinacht;[33] from his analysis, based on results of Dall'Acqua[34] concerning the separation on 3-manifolds, it follows that any possible separable system in $\mathbb{R}^3$ have an orthogonal equivalent. As we already mentioned, the orthogonal separation on manifolds of constant curvature can be proved by an intrinsic method.) However, all the separable webs in $\mathbb{R}^3$ can be determined and classified by means of Proposition 1 of Sec. V. We have to consider all possible r-dimensional orthogonally integrable spaces D_r of commuting K-vectors (with $r=0,1,2,3$) and all the separable orthogonal webs on a transversal manifold Q' of the orbits of D_r characterized by a K-tensor $\mathbf{K}_*$.

Case $r=0$: This case corresponds to asymmetric orthogonal separable webs. It is known that there are exactly three kinds of such webs, generated by three different K-tensors with simple eigenvalues and orthogonally integrable eigenvectors. How to find these Killing tensors is explained in Refs. 21 and 22 (for any dimension n). Two of them can be interpreted as the inertia tensors of an asymmetric body of massive points with total mass $m\neq 0$ or $m=0$ (the masses are assumed to be eighter positive or negative numbers). Both tensors have the form

$$\mathbf{K}=\mathrm{tr}(\mathbf{L})\mathbf{g}-\mathbf{L}. \tag{6.16}$$

For the case $m\neq 0$ the tensor $\mathbf{L}$ is defined by

$$\mathbf{L}=\mathbf{A}+m\mathbf{r}\otimes\mathbf{r}, \tag{6.17}$$

where $\mathbf{r}$ is the radius vector with respect to the center of mass and $\mathbf{A}$ is a constant linear operator with simple eigenvalues (a_α) $(\alpha=1,...,n)$ (the inertia tensor at the center of mass). For $m=0$ the tensor $\mathbf{L}$ is defined by

$$\mathbf{L}=\mathbf{A}+\mathbf{r}\otimes\mathbf{w}+\mathbf{w}\otimes\mathbf{r}, \tag{6.18}$$

where $\mathbf{r}$ is the radius vector with respect to a certain focal point O, $\mathbf{w}$ is a constant vector such that $\mathbf{A}\cdot\mathbf{w}=0$, and again $\mathbf{A}$ is a constant linear operator with simple eigenvalues (a_α) $(\alpha=1,...,n)$ (in this case one of them is zero). The explicit expressions of the corresponding K-tensors are

$$\begin{cases} \mathbf{K}_e=(\mathrm{tr}(\mathbf{A})+mr^2)\mathbf{g}-\mathbf{A}-m\mathbf{r}\otimes\mathbf{r}, \\ \mathbf{K}_p=(\mathrm{tr}(\mathbf{A})+2\mathbf{r}\cdot\mathbf{w})\mathbf{g}-\mathbf{A}-\mathbf{r}\otimes\mathbf{w}-\mathbf{w}\otimes\mathbf{r} \quad (\mathbf{A}\cdot\mathbf{w}=0). \end{cases} \tag{6.19}$$

The K-tensors determined in this way have simple eigenvalues everywhere and their eigenvectors generate the *elliptic-hyperbolic web* and the *parabolic web*, respectively. Both webs are made of confocal quadrics. Each one of these two tensors generates a whole space $\mathscr{K}$ of commuting K-tensors by an iterative process described in Ref. 21. In the case $m\neq 0$ all these tensors are linear nonhomogeneous functions of m. Taking the coefficients of m (which in a sense corresponds to consider the limit for $m\to\infty$) we get a new space of commuting K-tensors generating a new separable web, the *spherical-conical web*. A K-tensor with simple eigenvalues corresponding to this web is

$$\mathbf{K}_s=(r^2\ \mathrm{tr}(\mathbf{A})-\mathbf{r}\cdot\mathbf{A}\cdot\mathbf{r})\mathbf{g}-r^2\mathbf{A}-\mathrm{tr}(\mathbf{A})\mathbf{r}\otimes\mathbf{r}+\mathbf{A}\cdot\mathbf{r}\otimes\mathbf{r}+\mathbf{r}\otimes\mathbf{A}\cdot\mathbf{r}, \tag{6.20}$$

where $\mathbf{A}$ is a constant linear operator with simple eigenvalues (a_α). Notice that for $n=2$ the spherical web reduces to the polar web, which is symmetric; thus in $\mathbb{R}^2$ we have only two asymmetric separable webs.

For $n=3$ the K-tensors (6.19) and (6.20) have an equivalent representation in terms of translations and rotations. A translation is a constant vector and a rotation is a vector field $\mathbf{R}=\boldsymbol{\omega}\times\mathbf{r}$, where $\boldsymbol{\omega}$ is a constant (unitary) vector and $\mathbf{r}$ is the generic position vector with respect to a fixed point O belonging to the axis $\mathscr{A}$ of rotation. If we consider in $\mathbb{R}^3$ an orthogonal unitary frame $(\mathbf{i},\mathbf{j},\mathbf{k})$ corresponding to Cartesian coordinates (x,y,z), then the (unitary) rotations around the axis are defined as follows:

$$\mathbf{R}_x=y\mathbf{k}-z\mathbf{j}, \quad \mathbf{R}_y=z\mathbf{i}-x\mathbf{k}, \quad \mathbf{R}_z=x\mathbf{j}-y\mathbf{i}. \tag{6.21}$$

If (a,b,c) are the distinct eigenvalues of the matrix $\mathbf{A}$ corresponding to the eigenvectors $(\mathbf{i},\mathbf{j},\mathbf{k})$, then it can be shown that (see Ref. 21 for any dimension n)

$$\begin{cases} \mathbf{K}_e=m(\mathbf{R}_x^2+\mathbf{R}_y^2+\mathbf{R}_z^2)+(b+c)\mathbf{i}^2+(c+a)\mathbf{j}^2+(a+b)\mathbf{k}^2, \\ \mathbf{K}_p=w(\mathbf{R}_z\cap\mathbf{j}-\mathbf{R}_y\cap\mathbf{k})+(b+c)\mathbf{i}^2+(c+a)\mathbf{j}^2+(a+b)\mathbf{k}^2, \\ \mathbf{K}_s=a\mathbf{R}_x^2+b\mathbf{R}_y^2+c\mathbf{R}_z^2, \end{cases} \tag{6.22}$$

where $\mathbf{R}_x^2=\mathbf{R}_x\otimes\mathbf{R}_x$ and so on. In the definition (6.19) of $\mathbf{K}_p$ we have considered $\mathbf{w}=w\mathbf{i}$.

Case $r=1$: D_1 is generated by a single K-vector. It is know that there are only two kinds of orthogonally integrable K-vectors: the translations and the rotations. Let us consider both cases separately.

Translational case: Let us take a plane Q' orthogonal to a translation (constant vector) $\mathbf{X}$. Let us consider on Q' the two asymmetric separable webs, the elliptic-hyperbolic web and the parabolic web, corresponding to K-tensors $\mathbf{K}_e'$ and $\mathbf{K}_p'$ defined as in (6.19) or in (6.22). In the present case the matrix $(g_{\alpha\beta})$ is of one element only, $g_{11}=\mathbf{X}\cdot\mathbf{X}=$constant and Eq. (5.3) is trivially satisfied. Hence we get two cylindrical separable webs: the *elliptic-hyperbolic cylindrical web* and the *parabolic cylindrical web*. The remaining two separable webs on the plane Q', the polar and the Cartesian ones, generates the *polar cylindrical web* and the *Cartesian web* on the space. But these webs are reducible to the cases $r=2$ and $r=3$, respectively (see Remark 4, Sec. III).

Rotational case: Let us take a plane Π orthogonal to a rotation vector $\mathbf{R}$. This plane contains the axis of rotation $\mathscr{A}$. One halfpane of Π having $\mathscr{A}$ as a boundary will play the role of the manifold Q' in Proposition 1 of Sec. V. Up to an inessential constant factor, $g_{11}=\mathbf{R}\cdot\mathbf{R}$ is now the square of the distance F from this line and Eq. (5.3) with $g^{11}=F^{-2}$ becomes

$$Fd(\mathbf{K}'_{*}\cdot dF)-3dF\wedge(\mathbf{K}'_{*}\cdot dF)=0. \tag{6.23}$$

Up to a constant factor $F(\mathbf{r})=\mathbf{u}\cdot\mathbf{r}+c$, where $\mathbf{u}$ is a constant vector orthogonal to $\mathscr{A}$ and $c\in\mathbb{R}$. Thus we have to find all pairs $(\mathbf{u},c)$ such that Eq. (6.23) is identically satisfied. In terms of vector operations this equation is equivalent to

$$F\ \mathrm{curl}(\mathbf{K}'_{*}\cdot\mathbf{u})-3\mathbf{u}\times(\mathbf{K}'_{*}\cdot\mathbf{u})=0. \tag{6.24}$$

We have to consider for $\mathbf{K}'_{*}$ the two possible cases (6.19). By developing the differential condition (6.24), we can prove the following propositions.

Proposition 3: If $\mathbf{K}'_{e}$ is the inertia K-tensor of a planar distribution of masses with total mass $m\neq 0$, then the rotation around a straight line $\mathscr{A}$ belonging to this plane generates a separable web in $\mathbb{R}^3$ if and only if $\mathscr{A}$ is a central axis of inertia (i.e., a principal axis of inertia relative to the center of mass).

Since the central moments of inertia (i.e., the eigenvalues of $\mathbf{A}$) are different, according to our assumptions we generate in this way two different separable webs: the *oblate spheroidal web* corresponding to the maximal moment of inertia (it generates, in particular, the so-called oblate spheroidal coordinates[28,35]) and the *prolate spheroidal web* corresponding to the minimal one.

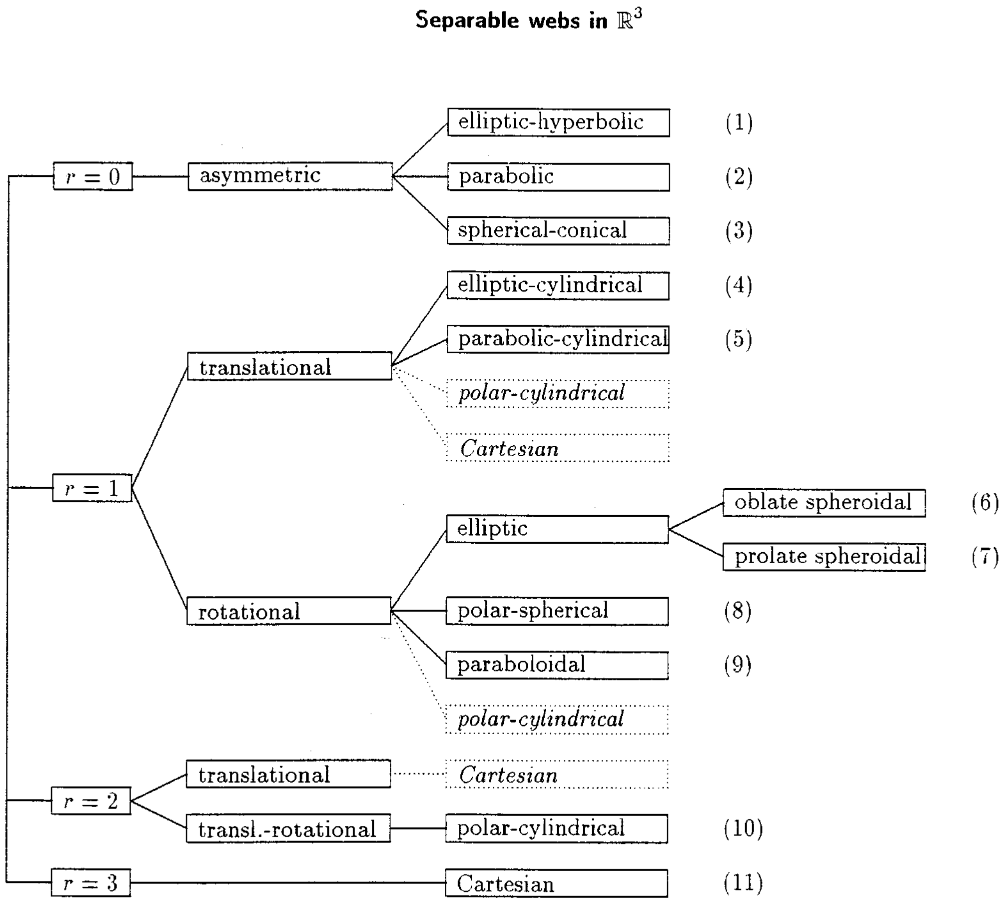
Proposition 4: If $\mathbf{K}'_{p}$ is the inertia K-tensor of a planar distribution of masses with total mass $m=0$, then the rotation around a straight line $\mathscr{A}$ belonging to this plain generates a separable web in $\mathbb{R}^3$ if and only if $\mathscr{A}$ is the central axis of inertia (i.e., the line parallel to $\mathbf{w}$ and containing the focus O).

The separable web obtained in this way is the *parabolic spheroidal web* (generating, in particular, the so-called parabolic spheroidal coordinates). For the rotational case it remains to consider on Π the polar and the Cartesian webs. For the polar web it is easy to check that it generates in $\mathbb{R}^3$ a separable web if and only if the center belongs to the axis of rotation $\mathscr{A}$. Thus we get the *spherical polar web* only (corresponding to the spherical coordinates). The Cartesian web on Π generates the cylindrical polar web in $\mathbb{R}^3$ when one of the two orthogonal K-vectors on Π is parallel to $\mathscr{A}$. This belongs, as we have already remarked, to the case $r=2$. Thus in the rotational case we have four separable webs with $r=1$.

Case $r=2$: There are two kinds of orthogonally integrable D_2. One is generated by two orthogonal translations $(\mathbf{X}_1,\mathbf{X}_2)$, the second by a translation $\mathbf{X}$ and a rotation $\mathbf{R}$ with axis parallel to $\mathbf{X}$. In both cases the foliations orthogonal to D_2 are made of straight lines. In the first case these lines are orthogonal to the translations, and the web generated in $\mathbb{R}^3$ is a Cartesian web (case $r=3$). In the second case they are the half-lines orthogonal to the axis of rotation, and we get a *polar cylindrical web*.

Case $r=3$: This case corresponds to the Cartesian rectangular web, i.e., to the Cartesian rectangular coordinates, generated by three constant and orthogonal K-vectors.

In conclusion, within this approach the 11 orthogonal separable webs in the Euclidean 3-space can be classified as in the following table (to be compared with those of Refs. 27, 28, 35).



Separable webs in $\mathbb{R}^3$

D. Separation in the spherically symmetric space–times

Many examples of separable geodesic Hamiltonian come from exact solutions of Einstein field equations (see for instance Refs. 10, 15, 16, 36) which could be re-examined within the framework presented here; for instance the Kerr metric, where in the equatorial plane we find a web of confocal conics ($r=$constant, $\theta=$constant).[37] Another example is the connection between separation and the existence of an Abelian orthogonally transitive isometry group (i.e., of an Abelian and orthogonally integrable algebra of K-vectors, according to our terminology) which is considered in Refs. 36, 38, 39.

Here we briefly discuss the simple case of a spherically symmetric space-time. In $Q=\mathbb{R}^4$ with coordinates (t,x,y,z) let us consider a metric of the kind

$$ds^2=A(r)(dx^2+dy^2+dz^2)+B(r)dt^2,$$

where $r^2=x^2+y^2+z^2$ and $A(r)$ and $B(r)$ are smooth functions of $r>0$. This kind of metric has four fundamental K-vectors; the time translation $\mathbf{T}=\partial_t$ and the rotations $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$, defined as in (6.28). Thus we recognize the existence of two orthogonal separable webs with degree of symmetry $r=1$ and $r=2$.

Case $r=1$: The vector field $\mathbf{T}$ is orthogonally integrable. The orthogonal manifolds are $t=$constant with coordinates (x,y,z) and metric $d\sigma^2=A(r)(dx^2+dy^2+dz^2)$. On the manifold Q'

defined by $t=0$ we have a separable web, the spherical-conical web characterized by the K-tensor $\mathbf{K}_s$ defined as in $(6.22)_3$. Since $g^{tt}=B^{-1}$ is a function of r alone, it follows that $\mathbf{K}_s \cdot dg^{tt}=0$ so that Eq. (5.3) is satisfied. Thus a separable web is defined on Q.

Case $r=2$: The space D_2 generated by the K-vectors $(\mathbf{X}_0,\mathbf{X}_1)=(\mathbf{T},\mathbf{R}_z)$ is orthogonally integrable. The orthogonal 2-manifolds are defined by equations $t=$constant and $ax+by=0 (a,b \in \mathbb{R})$. We can consider the manifold Q' defined by $t=0$ and $y=0$ with coordinates (x,z) and metric $d\sigma^2=A(r)(dx^2+dz^2)$, $r^2=x^2+z^2$. The manifold Q' admits the polar web as a separable web, whose characteristic K-tensor is $\mathbf{K}'_*=\mathbf{R}_y^2$, where $\mathbf{R}_y=z\mathbf{i}-x\mathbf{k}$. Since

$$g^{00}=\frac{1}{\mathbf{X}_0\cdot\mathbf{X}_0}=B^{-1}, \quad g^{11}=\frac{1}{\mathbf{X}_1\cdot\mathbf{X}_1}=A^{-1}x^{-2},$$

a straightforward calculation shows that $\mathbf{K}'_*\cdot dg^{00}=0$ and that $\varphi=\mathbf{K}'_*\cdot dg^{11}$ is the 1-form

$$\varphi=2\left(\frac{z}{x^2}\,dz-\frac{z^2}{x^3}\,dx\right)=d\,\frac{z^2}{x^2} \tag{6.25}$$

so that Eqs. (5.3) are fulfilled. The separable web on Q generated by D_2 and $\mathbf{K}'_*$ is that commonly used in a space–time of this kind. Adapted coordinates are the spherical coordinates (t,ϕ,r,θ) for which the metric has the form $ds^2=A(r)(dr^2+r^2(d\theta^2+\cos^2\theta d\phi^2))+B(r)dt^2$. It follows from (6.25) that the characteristic K-tensor $\mathbf{K}$ determined by Eq. (5.4) is

$$\mathbf{K}=\mathbf{R}_y^2+\frac{z^2}{x^2}\,\mathbf{R}_z^2.$$

In spherical coordinates this corresponds to the well known first integral $E_{\mathbf{K}}=p_\theta^2+\tan^2\theta p_\phi^2$.

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