Connections and Hamiltonian Mechanics

S. Benenti

Abstract. *The theory of connections on cotangent bundles is used in order to establish some basic results concerning the theory of separation of variables for the Hamilton-Jacobi equation.*

1. Introduction

In 1904, in a letter addressed to P. Stäckel and published in the Matematische Annalen [LC], Levi-Civita deals with the problem of the integration by separation of variables of the Hamilton-Jacobi equation

$$H(\underline{q}, \underline{p}) = h \qquad (\underline{p} = dW), \tag{1.1}$$

based on the existence of a complete solution of the form

$$W(\underline{q}, \underline{c}) = W_1(q^1, \underline{c}) + W_2(q^2, \underline{c}) + \ldots + W_n(q^n, \underline{c}). \tag{1.2}$$

In the introduction he writes: *"Ho notato che si possono facilmente assegnare (sotto forma esplicita di equazioni a derivate parziali ...) le condizioni necessarie e sufficienti cui deve soddisfare una H affinché l'equazione (1.1) ammetta un integrale completo della forma (1.2). Da queste condizioni scaturiscono alcune conseguenze di indole generale, che mi sembrano abbastanza interessanti, per quanto il dedurre da esse la completa risoluzione del problema apparisca ancora laborioso, e non vi sia nemmeno - oserei affermare - grande speranza di trovare tipi essenzialmente nuovi, oltre a quelli da Lei* [Stäckel] *scoperti."* Indeed, by a simple reasoning Levi-Civita shows that

Proposition 1. *The H-J equation* (1.1) *has a solution of the form* (1.2) *in a coordinate system* ($\underline{q}$) *if and only if the following* $\frac{n(n-1)}{2}$ *second order equations are satisfied,*

$$\begin{aligned} &\partial^i \partial^j H\, \partial_i H\, \partial_j H + \partial_i \partial_j H\, \partial^i H\, \partial^j H - \\ &\quad - \partial^i \partial_j H\, \partial_i H\, \partial^j H - \partial_i \partial^j H\, \partial^i H\, \partial_j H = 0 \quad (i \neq j). \end{aligned} \tag{1.3}$$

Here we use the notation $\partial_i = \partial/\partial q^i$ and $\partial^i = \partial/\partial p_i$. We call these equations the **separability conditions of Levi-Civita**. They refer to a generic Hamiltonian function. For a Hamiltonian of the kind

$$H = G + V = \tfrac{1}{2} g^{ij}\, p_i\, p_j + V, \tag{1.4}$$

corresponding to a holonomic mechanical system with kinetic energy G and potential V, the left hand sides of equations (1.3) are polynomials of 4th order in the momenta $\underline{p}$. The homogenous part of 4th order of these polynomials leads to a system of second order partial differential equations on the coefficients g^{ij} of G only, coinciding with the same equations derivable from (1.3) for $H = G$. This shows that

Proposition 2. *If the H-J equation of* $H = G + V$ *is integrable by separation of variables, then the same property holds for the geodesic Hamiltonian G (in the same coordinates).*

Hence, the separation of the Hamilton-Jacobi equation of the geodesics of a Riemannian manifold becomes a preliminary fundamental argument. However, the differential equations involving the metric tensor components g^{ij} arising from (1.3) appear to be extremely complicated to be handled. Nevertheless, Levi-Civita suggests a way for working on these equations, by dividing the separable coordinates $\underline{q} = (q^i)$ in two classes: in the first class he considers those coordinates q^i such that the corresponding partial derivative $\partial_i G$ (which is a second degree polynomial in the momenta $\underline{p}$) is divisible by $\partial^i G$ (which is a first degree polynomial in $\underline{p}$), in the second class he considers the remaining coordinates (the terminology of coordinates **of first class** and **of second class** is due to Dall'Acqua [DA], some years later). After this splitting, Levi-Civita shows how to extract from his separability conditions a system of first order equations. If we use Greek letters $\alpha, \beta, \dots$ for first class coordinates and the first Latin letters $a, b, \dots$ for those of second class, these first order equations are:

$$\partial_\alpha g_{ij} = 0 \qquad (i, j \neq \alpha), \tag{1.5}$$

and

$$\begin{cases} g^{aj}\, \partial_j g_{rs} = 0, \\ g^{al}\, \partial_j g_{rl} - g^{aj}\, \partial_j g_{rj} = 0, \qquad (j, r, s \neq a;\ r, s \neq j) \\ g^{il} \partial_j g_{jl} - \frac{1}{2}\, g^{aj}\, \partial_j g_{jj} = 0, \end{cases} \tag{1.6}$$

with no summation over the repeated indices.

However, Levi-Civita is able to deal with these equations, whose meaning is rather obscure, only in two particular cases: (1) when all the n coordinates are of first class, and (2) for $n = 2$. In this second very particular case he shows that all the cases found by Stäckel [ST] can be obtained. In the first case he shows by a direct calculation that the "Riemann symbols" are identically zero, i.e. that

Proposition 3. *If all the separable coordinates are of first class, then the Riemannian manifold is locally flat.*

After some preliminary remarks and just before entering in the discussion of these two cases, Levi-Civita makes a comment about the general case. He writes: *"Malgrado queste relative semplificazioni, se non si trova un* **artificio sintetico***, bisognerebbe passare in rassegna tutte le eventualità a priori possibili ...", cioè tutti i possibili valori dell'intero r ($0 \leq r \leq n$) numero delle coordinate di prima classe"*. A free translation is the following: in spite of the relative simplifications suggested before, if we do not find a "synthetic tool" (a "trick") we should examine separately all the possible cases corresponding to all the integers r (the number of first class coordinates).

The aim of the present lecture is to show that the "trick" wished by Levi-Civita is in fact the notion of connection defined by himself and Ricci some years before (the article on the "differential absolute calculus" is published in the Mat. Ann. in 1901 [RLC]). We shall consider the general notion of non-linear connection on a cotangent bundle (i.e. the parallel transport of covectors) and its relationships with symplectic geometry and Hamiltonian mechanics. In this approach, Proposition 3 of Levi-Civita is easily proved, as well as its extension to the general situation (when the separable coordinates are not all of first class) discovered by Agostinelli in 1937 [AG] after long calculations:

Proposition 4. *If $\underline{q} = (q^a, q^\alpha)$ are separable coordinates, with (q^α) of first class and (q^a) of second class, then the submanifolds $q^a = const.$ are locally Euclidean.*

2. Connections on cotangent bundles

A **connection** on a fibered manifold Z over a manifold Q is a smooth field of subspaces (i.e. a distribution) Γ transversal to the fibers (Ehresmann) [LM]. Any vector tangent to Z can be decomposed into the sum of its **vertical** part tangent to the fiber and its **horizontal** part belonging to Γ. There is an analogous decomposition for the vector fields on Z. Let us denote by D_Γ the space of the horizontal vector fields and by D_V the space of the vertical vector fields.

A curve γ on the base manifold Q connecting to points q_0 and q_1 can be lifted in a unique way to a curve $\overline{\gamma}$ on Z starting from a point z_0 over q_0 and tangent to Γ (this is the **horizontal lift** of γ). This curve, ending to a point z_1 over q_1, realizes the **parallel transport** of z_0. The parallel transport does not depend on the parametrization of the curve γ.

Any vector field X on the base manifold Q can be lifted in a unique way to a horizontal vector field $\overline{X}$ on the fibered manifold Z (the **horizonal lift** of X). In general the Lie bracket of two vector fields is not lifted to the Lie bracket of the corresponding horizontal lifts; however, it can be seen that the difference

$$R(X,Y) = [\overline{X}, \overline{Y}] - \overline{[X,Y]}$$

is a vertical vector field. The bilinear mapping $R(\cdot,\cdot)$ from vector fields on Q to vertical fields on Z is the **curvature** of the connection. This is zero if and only if the distribution Γ is completely integrable. In this case the connection is called **flat** or **locally flat**. When the integral manifolds are trivial coverings of the base manifold (i.e. they are diffeorphic to Q) the connection is **globally flat**. In this case the parallel transport does not depend on the path.

We can apply these considerations to two fundamental fibrations: $Z = TQ$, the tangent bundle of Q, and $Z = T^*Q$, the cotangent bundle. In these two cases a connection defines the parallel transport of vectors and covectors respectively. On the other hand, tangent and cotangent bundles provide the basic structures for the Lagrangian and the Hamiltonian description of dynamical systems, so that some relationships between mechanics and the theory of connections on these fibered manifolds is expected. The relationships between Lagrangian mechanics and connections on tangent bundles has been explored by various authors (see for instance [CM] [GR] [KL1] [KL2]). However, the presence on the cotangent bundles of a canonical symplectic structure, which is an essential tool for Hamiltonian dynamics, stimulates to a deeper analysis of the connections on these fiber bundles.

Although we shall deal with intrinsic objects, we shall use their local coordinate representations. Local coordinates $\underline{q} = (q^i)$ on Q generate **natural canonical coordinates** $(\underline{q}, \underline{p}) = (q^i, p_i)$ on T^*Q and fibered coordinates $(q^i, p_i, \dot{q}^i, \dot{p}_i)$ on TT^*Q. We shall use the short notation

$$\partial_i = \frac{\partial}{\partial q^i}, \qquad \partial^i = \frac{\partial}{\partial p_i}.$$

As for any connection, a connection Γ on T^*Q can be represented in three ways: by **equations**,

$$\dot{p}_i = \Gamma_{ji}\, \dot{q}^j, \tag{2.1}$$

by **generators**,

$$D_i = \partial_i + \Gamma_{ij}\, \partial^j, \tag{2.2}$$

and by **characteristic forms**,

$$\theta_i = dp_i - \Gamma_{ji}\, dq^j. \tag{2.3}$$

The symbols Γ_{ij} are the **coefficients** of the connection. They are functions on T^*Q, thus locally represented by functions of $(\underline{q}, \underline{p})$. The representation (2.1) follows from the assumption that Γ is a subbundle of TT^*Q transversal to the fibers, so that the $\underline{\dot{p}}$ are linear functions of the $\underline{\dot{q}}$. The generators D_i are derivations. Interpreted as vector fields they are horizontal and span at each point $p \in T^*Q$ the subspace $\Gamma_p = \Gamma \cap T_p(T^*Q)$. The characteristic forms annihilates the horizontal vectors: $\langle v, \theta_i \rangle = 0,\ \forall i \iff v \in \Gamma$.

The representation (2.1) leads to the differential system

$$\frac{dp_i}{dt} = \Gamma_{ji}(\underline{q}, \underline{p})\, \frac{dq^j}{dt}. \tag{2.4}$$

These are the **transport equations**. For any given curve $q(t)$ on Q and any given covector p_0 at $q_0 = q(0)$ they provide a unique solution $p(t)$ such that $p(0) = p_0$; this represents the parallel transport of p_0. We say that $p(t)$ is a **parallel covector** on $q(t)$.

The representation by generators (2.2) shows that the horizontal lift of vector fields is defined by

$$X = X^i\, \partial_i \quad \Longrightarrow \quad \overline{X} = X^i\, D_i. \tag{2.5}$$

In particular,

(2.6)
$$D_i = \overline{\partial}_i.$$

Since $[\partial_i, \partial_j] = 0$, it follows that

$$[D_i, D_j] = R(\partial_i, \partial_j) = R_{ijh}\,\partial^h,$$

with

(2.7)
$$R_{ijh} = \Gamma_{ik}\,\partial^k\Gamma_{jh} - \Gamma_{jk}\,\partial^k\Gamma_{ih} + \partial_i\Gamma_{jh} - \partial_j\Gamma_{ih}.$$

These are the **curvature coefficients** of the connection; they are skew-symmetric in the first two indices. A connection is called **linear** if the parallel transport is linear. In this case the connection coefficients are linear in $\underline{p}$:

(2.8)
$$\Gamma_{ij}(\underline{q}, \underline{p}) = \Gamma^k_{ij}(\underline{q})\,p_k,$$

as well as the curvature coefficients,

(2.9)
$$R_{ijh} = R^k_{ijh}\,p_k, \qquad R^k_{ijh} = \partial_i\Gamma^k_{jh} - \partial_j\Gamma^h_{ih} + \Gamma^k_{il}\,\Gamma^l_{jh} - \Gamma^k_{jl}\,\Gamma^l_{ih}.$$

A 1-form φ on Q can be interpreted as a section $\varphi\colon Q \to T^*Q$ of the cotangent bundle. Its image $L = \varphi(Q)$ is a submanifold of T^*Q transversal to the fibers. If locally $\varphi = \varphi_i\,dq^i$ then L is described by equations

(2.10)
$$p_i = \varphi_i(\underline{q}).$$

We say that φ is an **invariant form** of Γ (or Γ-invariant, or a **parallel form**) if its image L is an integral manifold of Γ. The components of the invariant forms satisfy the differential system

(2.11)
$$\partial_i p_j = \Gamma_{ij}(\underline{q}, \underline{p}).$$

This system is completely integrable iff $R = 0$.

3. Torsion and symmetric connections

The presence on T^*Q of a canonical symplectic structure leads to the notions of **dual connection** and **torsion**.

On T^*Q the **Liouville 1-form** θ and the **canonical symplectic form** ω are locally defined by

(3.1)
$$\theta = p_i\,dq^i, \qquad \omega = d\theta = dp_i \wedge dq^i.$$

A connection Γ is a subbundle of TT^*Q. Let us consider the symplectic dual (or orthogonal) subbundle Γ° defided by

(3.2)
$$\omega(\Gamma, \Gamma^\circ) = 0$$

i.e. by

$$\omega(u, v) = 0, \quad \forall u \in \Gamma, \ \ \forall v \in \Gamma^\circ.$$

This subbundle is again a connection, called the **dual connection**. To see this, let us consider the vertical subbundle V of TT^*Q. This subbundle is Lagrangian (i.e. isotropic of maximal dimension). This means that $V^\circ = V$. By definition of connection we have

$$V \cap \Gamma = 0, \qquad V + \Gamma = TT^*M.$$

By applying the symplectic dual operator $^\circ$ to these equalities we get

$$V^\circ + \Gamma^\circ = TT^*M, \qquad V^\circ \cap \Gamma^\circ = 0,$$

and consequently

$$V + \Gamma^\circ = TT^*M, \qquad V \cap \Gamma^\circ = 0.$$

This shows that also Γ° is a connection.
Let Γ°_{ij} be the coefficients of the dual connection Γ°. The generators of Γ° are

$$D^\circ_i = \partial_i + \Gamma^\circ_{ij}\,\partial^j.$$

Since

$$\begin{aligned}\omega(D_i, D^\circ_j) &= (dp_h \wedge dq^h)(D_i, D^\circ_j)\\ &= \langle D_i, dp_h\rangle\langle D^\circ_j, dq^h\rangle - \langle D^\circ_j, dp_h\rangle\langle D_i, dq^h\rangle\\ &= \Gamma_{ih}\delta^h_j - \Gamma^\circ_{jh}\delta^h_i\\ &= \Gamma_{ij} - \Gamma^\circ_{ji},\end{aligned}$$

from the condition $\omega(D_i, D^\circ_j) = 0$ we see that the coefficients of the dual connection are

$$\Gamma^\circ_{ij} = \Gamma_{ji}. \tag{3.3}$$

We say that the connection is **symmetric** (or **torsion free**) if $\Gamma^\circ = \Gamma$ i.e. if Γ is Lagrangian. Thus the connection is symmetric iff its coefficients are symmetric

$$\Gamma_{ij} = \Gamma_{ji}.$$

We can measure the lack of symmetry of a connection by means of the **torsion form**

$$T(X, Y) = \omega(\overline{X}, \overline{Y}) \tag{3.3}$$

which is a skew-symmetric bilinear form on vector fields over Q with values on the scalar fields on T^*Q. Indeed, since the horizontal lifts $\overline{X}$ of the vector fields X on Q span the distribution Γ, we see from (3.3) that Γ is isotropic (thus Lagrangian) if and only if $T = 0$. The components of the torsion form are

$$T_{ij} = T(\partial_i, \partial_j) = \omega(D_i, D_j) = \Gamma_{ij} - \Gamma_{ji}. \tag{3.4}$$

In a linear connection the components of the torsion form are linear in $\underline{p}$:

$$T_{ij}(\underline{q}, \underline{p}) = T^k_{ij}(\underline{q})\,p_k. \tag{3.5}$$

We can give a dynamical interpretation of the torsion. A smooth function $H\colon T^*Q \to \mathbb{R}$ constant along all the horizontal curves (i.e. constant on the parallel covectors) is said to be an **invariant function** of Γ (or Γ-invariant). This is equivalent to $\langle \Gamma, dH\rangle = 0$ or $D_i H = 0$, that is

$$\partial_i H + \Gamma_{ij}\,\partial^j H = 0. \tag{3.6}$$

We recall that on a cotangent bundle a smooth function $H\colon T^*Q \to \mathbb{R}$ (i.e. a **Hamiltonian function**) generates a **Hamiltonian vector field** X_H by the equation

$$i_{X_H}\omega = -\,dH, \tag{3.7}$$

i.e.

$$\langle v, dH\rangle = \omega(v, X_H), \ \ \forall v \in TT^*Q.$$

In canonical coordinates the first order equations of X_H are the **Hamilton equations**

$$\dot{q}^i = \partial^i H, \qquad \dot{p}_i = -\,\partial_i H. \tag{3.8}$$

The Hamiltonian H is a first integral of X_H, i.e. it is constant along the integral curves of X_H.

Proposition 1. *A function H is invariant in a connection Γ iff the Hamiltonian vector field X_H is horizontal in the dual connection Γ°. For a symmetric connection a function H is invariant iff X_H is horizontal.*

Proof. A function H is invariant if $\langle v, dH\rangle = 0$, $\forall\ v \in \Gamma$. By (3.7) this condition is equivalent to $\omega(X_H, v) = 0$, $\forall\ v \in \Gamma$ and this is equivalent to $X_H(p) \in \Gamma_p^\circ$ for each $p \in T^*Q$. ∎

Now, let us consider a Γ-invariant function H. Let $p(t)$ be an integral curve of X_H and let $q(t)$ be its projection in Q. Assume that this projected curve does not reduce to a point. Since X_H is horizontal in the dual connection Γ°, the covector $p(t)$ is Γ°-parallel along $q(t)$. This is the **dynamical transport** of the initial covector $p_0 = p(0)$. Let $\overline{q}(t)$ be the Γ-horizontal lift of $q(t)$ starting at p_0. If H is not Γ°-invariant, then $p(t) \neq \overline{q}(t)$. Hence, the torsion is responsible of the difference between the dynamical transport and the parallel transport when the Hamiltonian is Γ-invariant but not Γ°-invariant. We notice that the Hamiltonian H is constant on both the horizontal lifts: on $p(t)$ since it is a first integral, on $\overline{q}(t)$ since H is Γ-invariant.

Let us consider further properties of the symmetric connections.

Proposition 2. *An invariant form of a symmetric connection is closed.*

Proof. A submanifold $L \subset T^*Q$ is called **isotropic** if $\omega|L = 0$ (the restriction of the symplectic form to vectors tangent to L is zero). In this case $\dim(L) \leq n$ where $n = \dim(Q)$. A **Lagrangian submanifold** is an isotropic submanifold of maximal dimension n. Let φ be a 1-form on Q interpreted as a section of the cotangent bundle (see §1). It is a well known fact that its image $L = \varphi(Q)$ is a Lagrangian submanifold of T^*Q iff $d\varphi = 0$. If φ is Γ-invariant and Γ is symmetric, then L is Lagrangian and $d\varphi = 0$. ∎

This property can be easily derived from the differential equations of the invariant forms (2.11). If the coefficients Γ_{ij} are symmetric then $\partial_i p_j - \partial_j p_i = 0$. From this property and the fact that a closed form is locally exact one can derive the following result

Proposition 3. *A symmetric connection is flat iff there are local coordinates such that the coefficients are zero.*

Proof. Assume that the connection Γ is flat. Take a point $q \in Q$ and n independent covectors $p^{(i)}$ on the fiber T_q^*Q. Due to the complete integrability of Γ there are n invariant forms $\varphi^{(i)}$ defined on a neighborhood of q such that $\varphi^{(i)}(q) = p^{(i)}$. Since Γ is symmetric, these forms are locally exact; so that there are functions (x^i) such that $dx^i = \varphi^{(i)}$. These functions are independent in a neighborhood of q, thus they define a coordinate system. In these coordinates the invariant forms have constant coefficients, so that from the differential equations of the invariant forms we see that the corresponding coefficients are zero. The converse is trivial. ∎

4. Geodesics and metric connections

A connection Γ on T^*Q defines special curves on Q called **geodesics**. A parametrized curve $q(t)$ on Q is a geodesic if

$$\langle p(t), \dot{q}(t)\rangle = \text{constant} \tag{4.1}$$

for any parallel covector $p(t)$ along it. Here, $\dot{q}(t)$ is the tangent of $q(t)$. This definition is invariant under an affine transformation of the parameter t. A **geodesic path** is the image of a curve $q(t)$ such that

$$\langle p(t), \dot{q}(t)\rangle = 0 \tag{4.2}$$

for any parallel covector $p(t)$ along it such that $\langle p(0), \dot{q}(0)\rangle = 0$. This definition does not depend on the parametrization of $q(t)$. Definition (4.1) implies (4.2). Conversely, it can be shown that if $q(t)$ satisfies (4.2) then it can be reparametrized in such a way that (4.1) is satisfied.

Il local coordinates definition (4.1) is equivalent to the differential condition

$$\frac{d}{dt}\left(p_i\,\frac{dq^i}{dt}\right) = 0. \tag{4.3}$$

By combining this equation with the transport equations we find

$$p_i \frac{d^2 q^i}{dt^2} + \Gamma_{ji} \frac{dq^j}{dt} \frac{dq^i}{dt} = 0. \tag{4.4}$$

For a linear connection,

$$p_k \left(\frac{d^2 q^k}{dt^2} + \Gamma^k_{ji} \frac{dq^j}{dt} \frac{dq^i}{dt} \right) = 0. \tag{4.5}$$

Since p_k take arbitrary values at any fixed point of the curve, we find the well known **geodesic differential equations**

$$\frac{d^2 q^k}{dt^2} + \Gamma^k_{ji} \frac{dq^j}{dt} \frac{dq^i}{dt} = 0 \tag{4.6}$$

from which it follows that for each vector $v_0 \in TQ$ there is a unique geodesic curve $q(t)$ such that $\dot{q}(0) = v_0$. If $v_0' = \lambda v_0$ then the corresponding geodesic $q'(t)$ is such that $q'(t) = q(\lambda t)$.
For a non-linear connection the discussion of the existence of geodesics is more complicated. However, for constructing geodesics one can use homogeneous invariant functions.

Proposition 1. *Let $H\colon T^*Q \to \mathbb{R}$ be homogeneous on the fibers. Then the projections on Q of the integral curves of the Hamiltonian vector field X_H are geodesic curves of a connection Γ iff H is Γ°-invariant.*

Proof. Homogenous means that

$$\langle X_H, \theta \rangle = p_i\, \partial^i H = \alpha\, H$$

with $\alpha \in \mathbb{R}$. Since H is a first integral of X_H, the same is for this function, so that along the integral curves $p(t)$ of X_H

$$p_i\, \dot{q}^i = \text{constant}.$$

Every integral curve $p(t)$ is Γ-parallel iff X_H is Γ-horizontal which means that H is Γ°-invariant. ∎
Let us consider a Riemannian manifold (Q, g^{ij}) and the function

$$G = \tfrac{1}{2}\, g^{ij}(\underline{q})\, p_i\, p_j.$$

A connection Γ on T^*Q is said to be **metric** if G is Γ-invariant. This means that the norm $G(p)$ is constant along the parallel transport. For metric connections the classical theorem of Levi-Civita holds,

Proposition 2. *There is a unique symmetric, linear and metric connection (called the* **Levi-Civita connection***).*

Proof. The invariance condition $\partial_i G + \Gamma^k_{ij}\, p_k\, \partial^j G = 0$ reads

$$\tfrac{1}{2}\, \partial_i g^{rs}\, p_r\, p_s + \Gamma^k_{ij}\, p_k\, g^{js}\, p_s = 0.$$

This is equivalent to

$$\partial_i g^{rs} + \Gamma^r_{ij}\, g^{js} + \Gamma^s_{ij}\, g^{jr} = 0.$$

From this equation, by lowering the indices and by the well known procedure we find that

$$\Gamma^k_{ij} = \tfrac{1}{2}\, g^{kh} \left(\partial_i g_{jh} + \partial_j g_{hi} - \partial_h g_{ij} \right). \qquad ∎$$

By this theorem and the existence and uniqueness theorem for the geodesics of a linear connection it is possible to prove that

Proposition 3. *The integral curves of the Hamiltonian vector field X_G project onto all the geodesic curves of the Levi-Civita connection Γ.*

Remark 1. A metric tensor defines an canonical bundle isomorphism from T^*Q to TQ thus a bijective mapping between connections on T^*Q and connections on TQ.

5. Canonical lifts and invariant connections

There is a **canonical lift** of objects on Q to "symplectic" objects on T^*Q. The canonical lift of a vector field X on Q is the Hamiltonian vector field $\hat{X}$ generated by the function $E_X : T^*Q \to \mathbb{R}$ defined by

$$E_X(p) = \langle p, X \rangle. \tag{5.1}$$

In canonical coordinates $E_X = X^i\, p_i$, and the Hamilton equations of $\hat{X}$ are

$$\dot{q}^i = X^i, \qquad \dot{p}_i = -\,\partial_i X^j\, p_j. \tag{5.2}$$

Thus,

$$\hat{X} = X^i\, \partial_i - \partial_i X^j\, p_j\, \partial^i. \tag{5.3}$$

The canonical lift $\hat{X}$ is projectable onto X.
The canonical lift of an automorphism $\psi : Q \to Q$ is the bundle automorphism $\hat{\psi} : T^*Q \to T^*Q$ defined by the equation

$$\langle v, \hat{\psi}(p) \rangle = \langle T\psi^{-1}(v), p \rangle, \tag{5.4}$$

where T is the tangent functor. It can be proved that if ψ_t is the one-parameter group of transformations of a vector field X then $\hat{\psi}_t$ is the one-parameter group of $\hat{X}$.
We say that a connection Γ on T^*Q is invariant under an automorphism $\psi : Q \to Q$ (or ψ-invariant) if the canonical lift $\hat{\psi}$ transforms parallel covector into parallel covectors. This means that if $p(t)$ is a parallel covector on a curve $q(t)$, the $\hat{\psi} \circ p(t)$ is a parallel covector on the curve $\psi \circ q(t)$. It can be shown that this is equivalent to the condition

$$T\hat{\psi}(\Gamma) = \Gamma. \tag{5.5}$$

We say that a connection Γ on T^*Q is invariant under a vector field X on Q (or X-invariant) if it is invariant under the corresponding (local) group ψ_t. It can be shown that this is equivalent to the condition

$$[\hat{X}, \Gamma] \subseteq \Gamma. \tag{5.6}$$

By using generators and characteristic forms we can translate this last condition in a set of partial differential equations involving the coefficients of the connection and the components of X. However, in a neighborhood of a non singular point of X we can consider adapted coordinates (q^i) such that $X = \partial_1$ and conclude that (5.6) is equivalent to

$$\partial_1 \Gamma_{ij} = 0. \tag{5.7}$$

Then we say that the coordinate q^1 is **ignorable**.

Remark 1. A vector field X on a Riemannian manifold Q is a **Killing vector** if the geodesic Hamiltonian G is invariant under $\hat{X}$, that is $\langle \hat{X}, dG \rangle = 0$. This means that E_X is a first integral of X_G. In adapted coordinates this is equivalent to $\partial_1 g^{ij} = 0$, and the coordinates q^1 is called **ignorable**. I follows that the Levi-Civita connection is X-invariant.

6. Reductions of connections

Let S be a submanifold of Q. There is a natural **symplectic reduction** [LM] from T^*Q to T^*S represented by a surjective submersion $\rho : C \to T^*S$ where $C = T^*_S Q$ (the covectors at the points of S). By definition $y = \rho(p)$ if y and p are covectors at the same point $q \in S$ and $\langle v, y \rangle = \langle v, p \rangle$ for each $v \in T_q S$. The inverse images by ρ of the covectors on S are called **fibers** of the reduction. The difference of two elements p and p' of the same fiber annihilates the vectors tangent to S: $p - p' \in T^\circ S$.

We say that a connection Γ on T^*Q is **reducible** to the submanifold S if the parallel transport on Q preserves the fibers of the reduction ρ. This means that for two different parallel covectors $p(t)$ and $p'(t)$ on a curve $q(t)$ on S such that $p(0)-p'(0) \in T^\circ S$ we have $p(t)-p'(t) \in T^\circ S$ for all values of t.

Let $\underline{q} = (q^i) = (x^a, x^\alpha)$ $(a = 1, \ldots, m,\ \alpha = m+1, \ldots, n)$ be a coordinate system on Q **adapted** to S. This means that S is locally represented by equations $q^a = 0$. Since $\langle v, p \rangle = v^a p_a + v^\alpha p_\alpha$, and $v \in TS \Leftrightarrow v^a = 0$, it follows that for each vector $v \in TS$, $\langle v, p \rangle = v^\alpha p_\alpha$. This shows that the coordinates (q^α) can be interpreted as coordinates on S and that the components (p_α) of a covector $p \in T^*_S Q$ can be interpreted as the components of the corresponding reduced element of T^*S.

Let us split the transport equations as follows:

$$\text{(6.1)} \qquad \begin{cases} \dfrac{dp_a}{dt} = \Gamma_{\alpha a} \dfrac{dq^\alpha}{dt} + \Gamma_{ba} \dfrac{dq^b}{dt} \\ \dfrac{dp_\alpha}{dt} = \Gamma_{\beta\alpha} \dfrac{dq^\beta}{dt} + \Gamma_{b\alpha} \dfrac{dq^b}{dt}. \end{cases}$$

For a curve on S (where $q^a = 0$) they become

$$\text{(6.2)} \qquad \begin{cases} \dfrac{dp_a}{dt} = \Gamma_{\alpha a} \dfrac{dq^\alpha}{dt} \\ \dfrac{dp_\beta}{dt} = \Gamma_{\alpha\beta} \dfrac{dq^\alpha}{dt}, \end{cases}$$

where the coefficients of the connection are computed for $q^a = 0$.

The reducibility of the connection means that two solutions along a curve $q^\alpha(t)$ differ by an element of $T^\circ N$, that is

$$p_\alpha(0) - p'_\alpha(0) = 0 \quad \Longrightarrow \quad p_\alpha(t) - p'_\alpha(t) = 0.$$

This uniqueness condition means that the second subsystem (6.2) is not influenced by the solutions $p_a(t)$, and this happens if and only if the coefficients $\Gamma_{\alpha\beta}$ do not depend on the (p_a),

$$\text{(6.3)} \qquad \partial^a \Gamma_{\alpha\beta}|_{(q^a=0)} = 0.$$

This proves

Proposition 1. *A connection* Γ *is reducible to a submanifold* $S \subset Q$ *iff in adapted coordinates* (6.3) *holds.*

Remark 1. This conditions is equivalent to the fact that the subsystem $(6.2)_2$ is separated, i.e. it involves the variables (q^α, p_α) only (which are in fact canonical coordinates of T^*S). Moreover, if $\Gamma_{\alpha a} = 0$ then the first subsystem reduces to

$$\text{(6.4)} \qquad \frac{dp_a}{dt} = 0$$

which means that along any curve $p_a =$ constant.

Let $\pi \colon Q \to N$ be a surjective submersion. There is a natural symplectic reduction from T^*Q to T^*N represented by the surjective submersion $\rho \colon C \to T^*N$, where $C = V^\circ Q$ is the set of covectors annihilating the vertical vectors of Q (i.e. the vectors tangent to the fibers of π). It is defined as follows: $y = \rho(p)$ if $y \in T^*_x N$ and $p \in T^*_q Q$, $x = \pi(q)$, and $\langle v, p \rangle = T\pi(v), y\rangle$ for each $v \in T_q Q$.

We say that a connection Γ on T^*Q is **reducible** with respect to the submersion π if along two curves on Q which project by π on the curve on N the parallel transport of elements of C preserves the fibers of ρ. This means that if $q(t)$ and $q'(t)$ are two curves on Q projecting on the curve $x(t)$ of N, and p_0 and p'_0 are two covectors belonging to C at the points $q(0)$ and $q'(0)$, then the corresponding parallel covectors $p(t)$ and $p'(t)$ belong to the same fiber of ρ for all values of t.

Let (q^a, q^α) be coordinates adapted to π, such that (q^α) are constant on the fibers of π. The coordinates (q^a) can be interpreted as coordinates on N. Let $(q^a, q^\alpha, p_a, p_\alpha)$ and (q^a, y_a) be the corresponding canonical coordinates on T^*Q and T^*N respectively. Then the submanifold C is represented by equations $p_\alpha = 0$ and the submersion ρ by $y_a = p_a$.

Let us consider the transport equations splitted as in (6.2). The parallel transport must preserve C, that is condition $p_\alpha = 0$. It follows that $\Gamma_{i\alpha} = 0$ on C. Moreover, the solutions $p_a(t)$ must depend only on $q^a(t)$ and not on $q^\alpha(t)$. So that $\Gamma_{\alpha a} = 0$ on C. This shows that

Proposition 2. *A connection Γ is reducible with respect to a submersion $\pi\colon Q \to N$ iff in adapted coordinates*

$$\Gamma_{i\alpha}|_{(p_\alpha=0)} = 0, \qquad \Gamma_{\alpha a}|_{(p_\alpha=0)} = 0. \tag{6.5}$$

Remark 2. There is a second way for defining the fibers of this reduction: they are the integral manifolds of the canonical lifts $\hat{X}$ of all the vector fields X tangent to the fibers of π. It can be shown that the reducibility can be expressed by the intrinsic conditions

$$\begin{cases} \Gamma_C \subset TC, \\ \hat{X} \in D_\Gamma, \qquad \forall\, X \text{ tangent to the fibres of } \pi. \end{cases} \tag{6.6}$$

Remark 3. Since flatness means the independence of the transport by the path, if a connection is flat then any reduced connection is flat.

7. Flat connections and complete integrals

A submanifold L of T^*Q transversal to the fibers can be locally interpreted as the image of a 1-form $\varphi = \varphi_i\, dq^i$ on Q and represented by equations

$$p_i = \varphi_i(\underline{q}). \tag{7.1}$$

We observed that L is Lagrangian iff φ is closed. In this case φ is locally exact: there are local functions W on Q such that $\varphi = dW$. They are called **generating functions** of L. This means that any Lagrangian submanifold L transversal to the fibers can be locally described by equations

$$p_i = \frac{\partial W}{\partial q^i}. \tag{7.2}$$

Any submanifold L of dimension n can also be locally described by equations

$$F_i(\underline{q}, \underline{p}) = 0, \tag{7.3}$$

where (F_i) are n independent functions on T^*Q. The submanifold L is transversal to the fibers iff the functions (F_i) are **vertically independent**, i.e. iff they are independent as functions on the fibers. Then locally we can reverse the equations (7.3) and get equations of the kind (7.1). It is known that L is Lagrangian iff these functions are in involution on L, that is

$$\{F_i, F_j\}|L = 0. \tag{7.4}$$

Here $\{\cdot,\cdot\}$ are the Poisson brackets associated with the canonical symplectic form ω. As a consequence (7.4) is equivalent to $d\varphi = 0$.

Let X_H be the Hamiltonian vector field corresponding to a Hamiltonian function $H\colon T^*Q \to \mathbb{R}$. It can be proved that X_H is tangent to a Lagrangian submanifold iff H is constant on L or, equivalently, iff H is in involution on L with all functions constant on L. In the case of a Lagrangian submanifold transversal to the fibers and described by a generating function this condition is expressed by equation

$$H(\underline{q}, dW) = h \qquad (h = \text{constant}). \tag{7.5}$$

When L is represented by functions in involution, this condition is expressed by equations

$$\{H, F_i\}|L = 0. \tag{7.6}$$

About these last equations (which in fact hold also for a non-transversal submanifold) we recall that a function F is in involution with H, that is $\{H, F\} = 0$ iff F is a first integral of X_H (i.e. it is constant along the integral curves of X_H). We recognize in (7.5) the **Hamilton-Jacobi equation** generated by the function $H(\underline{q}, \underline{p})$.

A **complete integral** of X_H is a Lagrangian transversal foliation of T^*Q tangent to X_H. Let $\{L_a \mid a \in A\}$ be the submanifolds (the leaves) of T^*Q forming this foliation. This means that

(i) Each leaf L_a is Lagrangian,

(ii) The leaves are transversal to the fibers of T^*Q.

(iii) The vector field X_H is tangent to the leaves (i.e. $H|L_a$ =constant).

We assume that the quotient set A is a differential manifold such that the canonical projection $T^*Q \to A$ is a submersion. The dimension of A is n. We denote by $\underline{a} = (a_i)$ a coordinate system on A.

Such a foliation can be represented (at least locally) in two ways: (1) by means of a generating function $W(\underline{q}, \underline{a})$ or (2) by means of a complete set of first integrals in involution (F_i).

In the first case W is a **complete solution** of the H-J equation (7.5). This means that W depends on the parameters $\underline{a}$ in such a way that

$$\det\left(\frac{\partial^2 W}{\partial q^i \partial a_j}\right) \neq 0. \tag{7.7}$$

In fact this completeness condition is equivalent to the fact the Lagrangian submanifolds generated by W according to equations (7.2) form a foliation: for each $p \in T^*Q$ there is a unique $a \in A$ such that $p \in L_a$.

In the second case the functions (F_i) are first integrals in involution

$$\{H, F_i\} = 0, \qquad \{F_i, F_j\} = 0. \tag{7.8}$$

It is known in Hamiltonian mechanics that when a complete integral is known then the integral curves of X_H can be determined by simple integrals and by "algebraic" operations. Then the Hamiltonian system is said to be **integrable**. This property is known as the **Jacobi theorem** when the complete integral is represented by a generating function, as the **Liouville theorem** when it is represented by first integrals in involution.

A complete integral can be reinterpreted in terms of connections: a complete integral is a symmetric flat connection Γ on T^*Q such that H is invariant. Indeed:

(i) Γ is flat; this means that Γ is completely integrable and the integral manifolds are transversal to the fibers.

(ii) Γ is symmetric; this means that Γ is a field of Lagrangian subspaces, so that the integral manifolds are Lagrangian.

(iii) H is Γ-invariant; it means that H is constant on the integral manifolds.

Since a connection Γ is locally represented by its coefficients Γ_{ij}, we consider the relationships of these coefficients with the two representations of a complete integral. If the complete integral is represented by a generating functions W then

$$\Gamma_{ij} = \partial_i \partial_j W, \tag{7.9}$$

where in the right hand side the parameters $\underline{a}$ appearing in W must be replaced by their functions in $(\underline{q}, \underline{p})$ obtained by reversing equations (7.2). Indeed, when we fix a leaf L_a, that is a value of the parameters $\underline{a}$, from equations (7.2) $p_i = \partial_i W$ it follows $\dot{p}_i = \partial_i \partial_j W \, \dot{q}^j$. On the other hand, $\dot{p}_i = \Gamma_{ji} \, \dot{q}^j$. This implies (7.9).

If the complete integral is represented by first integrals in involution (F_i), then these functions are constant on the integral manifolds thus Γ-invariant, so that $D_i F_k = 0$ i.e.

$$\partial_i F_k + \Gamma_{ij} \, \partial^j F_k = 0. \tag{7.10}$$

These equations define the coefficients Γ_{ij} since $\det(\partial^j F_k) \neq 0$ due to the vertical independence.

8. Separable connections

A connection Γ on T^*Q is called **separable** if there are local coordinates on Q such that the corresponding coefficients Γ_{ij} are diagonalized,

$$\Gamma_{ij} = 0, \quad i \neq j. \tag{8.1}$$

These coordinates are called **separable** with respect to the connection Γ. This definition is strictly related to the separation of the H-J equation, as shown by the following remarks.

Remark 1. A separable connection is symmetric.

Remark 2. A separable connection is uniquely determined by a non-trivial invariant function. Indeed, if H is an invariant function then $\partial_i H + \Gamma_{ij}\, \partial^j H = 0$. Since the connection is separable, $\partial_i H + \Gamma_{ii}\, \partial^i H = 0$ so that

$$\Gamma_{ii} = -\frac{\partial_i H}{\partial^i H}. \tag{8.2}$$

We exclude those points of T^*Q where $\partial^i H = 0$.

Remark 3. If a separable connection is separable and it is defined by an invariant function H then it is flat iff the Levi-Civita separability conditions are satisfied. Indeed, because of (8.2), the generators are

$$D_i = \partial_i - \frac{\partial_i H}{\partial^i H}\, \partial^i,$$

and a straightforward calculation shows that the flatness conditions $[D_i, D_j] = 0$ are equivalent to the Levi-Civita equations (1.3).

Remark 4. From the discussion in the preceding section, in particular from (7.9), it follows that a separable flat connection with an invariant Hamiltonian function H corresponds to a separable complete solution W of the H-J equation.

Remark 5. Let us consider a metric Hamiltonian $G = \frac{1}{2}\, g^{ij}\, p_i\, p_j$. Assume that for a coordinate system $\underline{q}$ there is a separable complete solution W of the H-J equation. This means that there is a connection Γ which is

(i) flat,

(ii) symmetric (since it is separable),

(iii) metric (since G is invariant).

Furthermore, let us assume that all the coordinates are of first class according to Levi-Civita, i.e. that the functions

$$\frac{\partial_i G}{\partial^i G}$$

are linear in $\underline{p}$. This means that

$$\Gamma_{ii} = B_i^k\, p_k$$

i.e. that

(iv) Γ is linear.

Conditions (ii)+(iii)+(iv) show that Γ is the Levi-Civita connection. Hence, condition (i) implies that the Riemannian manifold Q is flat. This is a simple proof of Proposition 3 of the Introduction.

Let us return to the general case. Let Γ be a separable connection in coordinates $\underline{q}$. We say that the coordinates (q^α) $(\alpha = m+1, \ldots, n)$ are of **first class** if the corresponding coefficients are linear in $\underline{p}$,

$$\Gamma_{\alpha\alpha} = B_\alpha^i(\underline{q})\, p_i. \tag{8.3}$$

The remaining coordinates (q^a) $(a = 1, \ldots, m)$ are of **second class**.

Let us consider a coordinate transformation from (q^i) to another system $(q^{i'})$ of separable coordinates and set

$$A_i^{i'} = \frac{\partial q^{i'}}{\partial q^i}, \qquad A_{i'}^i = \frac{\partial q^i}{\partial q^{i'}}.$$

The corresponding momenta are related by equations

$$p_i = A_i^{i'}\, p_{i'}, \qquad p_{i'} = A_{i'}^i\, p_i. \tag{8.4}$$

Assume that coordinates (q^α) and $(q^{\alpha'})$ are of first class and (q^a) and $(q^{a'})$ of second class. Assume that there is an invariant form φ. The components $p_i = \varphi_i(\underline{q})$ are solutions of the system

$$\partial_i p_i = B_i, \qquad \partial_i p_j = 0 \qquad (\neq j), \tag{8.5}$$

where

$$B_i = \Gamma_{ii}.$$

Let us substitute these components in equations $(8.4)_1$ and apply the partial derivative

$$\partial_{j'} = A_{j'}^j\, \partial_j$$

to both sides. From

$$A_{j'}^j\, \partial_j p_i = A_{j'}^j\, \partial_j A_i^{i'}\, p_{i'} + A_i^{i'}\, \partial_{j'} p_{i'},$$

due to (8.5) and the analogous system written for the coordinates $(q^{i'})$ it follows that

$$A_{j'}^i\, \partial_i p_i = A_{j'}^j\, \partial_j A_i^{i'}\, p_{i'} + A_i^{j'}\, \partial_{j'} p_{j'},$$

with no summation over the indices i and j'. If the connection is flat, i.e. completely integrable, then from this equation we can derive an analogous equation on the coefficients $B_i = \Gamma_{ii}$ of the connection:

$$A_{j'}^i\, B_i = A_{j'}^j\, \partial_j A_i^{i'}\, p_{i'} + A_i^{j'}\, B_{j'}. \tag{8.6}$$

Let us consider the particular case $j' = a'$ (index of second class) and $i = \alpha$ (index of first class):

$$A_{a'}^\alpha\, B_\alpha = A_{a'}^j\, \partial_j A_\alpha^{i'}\, p_{i'} + A_\alpha^{a'}\, B_{a'}. \tag{8.7}$$

Since B_α is linear in $\underline{p}$, from $A_\alpha^{a'} \neq 0$ it follows that also $B_{a'}$ is linear, which is in contradiction with the hypothesis that $q^{a'}$ is of second class. This proves that $A_\alpha^{a'} = 0$ and by symmetry that $A_{\alpha'}^a = 0$. This means that the coordinates $(q^{a'})$ depend on the coordinates (q^a) only. As a consequence, if a separable connection is flat then two separable coordinate systems have the same number of second class coordinates (thus the same number of first class coordinates) (see also [B1]), so that

Proposition 1. *A separable and flat connection Γ gives rise to a foliation of submanifolds of codimension m defined by equations $q^a = constant$ $(a = 1, \ldots, m)$, where (q^a) are second class coordinates.*

We call these submanifolds the **characteristic manifolds** of Γ. They form the **characteristic foliation** of Γ. We remark that the vector fields ∂_α are tangent to the characteristic foliation.

Now we consider the reducibility of a separable flat connection.

Proposition 2. *A separable flat connection is reducible to the characteristic manifolds and the reduced connections are linear and flat.*

Proof. For a separable connection the generators are

$$D_a = \partial_a + B_a\, \partial^a, \qquad D_\alpha = \partial_\alpha + B_\alpha^i\, p_i\, \partial^\alpha \tag{8.8}$$

where $B_a = \Gamma_{aa}$. If the connections is flat then $[D_a, D_\alpha] = 0$, so that

$$\left(\partial_a B_\alpha^i p_i + B_a\, B_\alpha^a\right) \partial^\alpha - \left(\partial_\alpha B_a + \partial^\alpha B_a\, B_\alpha^i p_i\right) \partial^a = 0. \tag{8.9}$$

From

$$\partial_a B^i_\alpha p_i + B_a\, B^a_\alpha = 0 \tag{8.10}$$

it follows that if $B^a_\alpha \neq 0$ then B_a is linear (or null): absurd. Thus $B^a_\alpha = \partial^a \Gamma_{\alpha\alpha} = 0$ and Γ is reducible. On a characteristic manifold the transport equations are (see Prop. 1, §6)

$$\frac{dp_\beta}{dt} = B^\alpha_\beta\, p_\alpha \frac{dx^\beta}{dt}, \tag{8.11}$$

and the connection is linear. ∎

From (8.9) it follows that $\partial_a B^\beta_\alpha = 0$, thus system (8.11) does not involve coordinates (x^a, p_a). Since the reduced connections are flat and linear, there is a coordinate transformation, not involving the second class coordinates (x^a), such that the coefficients $\Gamma_{\alpha\alpha}$ are zero. Indeed, in our case system (8.5) has $r = n - m$ independent solutions

$$\varphi^{(\beta)} = dx^{(\beta)} = p^{(\beta)}_\alpha dx^\alpha$$

which define new coordinates $x^{(\beta)}$. In these coordinates the coefficients of the connection are zero, so that the corresponding momenta $p_{(\alpha)}$ are constant along the transport. These new coordinates are still separable, since $\partial_\beta p_\alpha = 0$ for $\alpha \neq \beta$. Since $B^i_\alpha = 0$, the flatness condition (8.9) reduces to $\partial_\alpha B_a = 0$. This means that the new first class coordinates do not appear at all in the coefficients of the connection. This proves

Proposition 3. *If a separable connection is flat then it is separable in a coordinate system such that all the first class coordinates are ignorable and $\Gamma_{\alpha\alpha} = 0$.*

Let N be the quotient manifold of the characteristic foliation of Γ (assume that it is globally defined). When all first class coordinates are ignorable equations (6.5) are fulfilled. Hence,

Proposition 4. *A separable flat connection is reducible to the quotient of the characteristic foliation and the reduced connection is flat.*

We remark that all the preceding results does involve a separable connection only, without any reference to a Hamiltonian H or, in particular, to a metric. Assuming that there is an invariant function H of the separable flat connection Γ, from (8.2) it follows that $\partial_\alpha H = 0$. This shows that the ignorable first class coordinates are also ignorable for H. By extending the definition usually given for a metric Hamiltonian (see §5), we say that a vector field X on Q is a **Killing vector** of H if $\langle \hat{X}, dH \rangle = 0$. Hence, the vector fields ∂_α are commuting Killing vectors of H. Since H is constant on the characteristic manifold, it is reducible to a function on the quotient set. Since H is constant on the parallel covectors of Γ, from the definition of reduction it follows that it is also constant on the reduced parallel covectors. Hence,

Proposition 5. *If H is invariant in a separable flat connection then the characteristic foliation is generated by pointwise independent commuting Killing vectors. The Hamiltonian is reducible to the quotient set and it is invariant with respect to the reduced connection.*

In particular we conclude that if the connection is metric and the characteristic manifolds are Riemannian manifolds (i.e. the induced metric is not singular), then the induced metric is flat and Proposition 3 of the Introduction is proved.

These results help to understand the geometrical and intrinsic meaning of the separation of the Hamilton-Jacobi [B1] [B2] [KA]. Here they have been obtained in the framework of the theory of connections, which should be improved in order to avoid as much as possible the use of coordinate representations.

References

[AG] C. Agostinelli, *Sopra l'integrazione per separazione di variabili dell'equazione dinamica di Hamilton-Jacobi*, Memorie Acc. Scienze Torino **15**, 3-54 (1937).

[B1] S. Benenti, *Separability structures on Riemannian manifolds*, in L. N. Math. **863**, 512-538 (1980).

[B2] S. Benenti, *Separation of variables in the geodesic Hamilton-Jacobi equation*, Progress in Mathematics **9**, 1-36 (1991).

[CM] J.F. Cariñena, E. Martínez, *A new geometrical setting for Lax equations*, Int. J. Modern Physics **9**, 4973-4986 (1994).

[DA] F. A. Dall'Acqua, *Le equazioni di Hamilton-Jacobi che si integrano per separazione delle variabili*, Rend. Circ. Mat. Palermo **33**, 341-351 (1912).

[GR] J. Grifone, *Structures presque tangentes et connexions*, Ann. Ist. Fourier **22-1**, 287-334 (1972).

[KA] E.G. Kalnins, *Separation of Variables for Riemannian Spaces of Constant Curvature*, Pitman Monographs **28** (1986).

[KL1] J. Klein, *Geometry of sprays. Lagrangian case. Principle of least curvature.* in Modern Developments in Analytical Mechanics, Atti Accad. Sci. Torino **117**, 177-196 (1983).

[KL2] J. Klein, *Connections in Lagrangian dynamics*, in La "Mècaique Analytique" de Lagrange et son héritage, Atti Accad. Sci. Torino **126**, 33-91 (1992).

[LC] T. Levi-Civita, *Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili*, Math. Ann. **59**, 383-397 (1904).

[LM] P. Libermann, C.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel (1987).

[RLC] G. Ricci, T. Levi-Civita, *Méthodes de calcul différentiel absolu et leurs applications*, Math. Ann. **54**, 125-201 (1901).