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Orthogonal separable dynamical systems

S. Benenti

Abstract. *The general setting for a global geometrical approach to the orthogonal separation of variables is presented together with some applications to dynamical systems in the Euclidean spaces.*

1. Introduction

Let M_n be an n-dimensional manifold and T^*M_n its cotangent bundle. Let (q^i) be local coordinates of M_n and (q^i, p_i) the corresponding canonical coordinates on T^*M_n $(i = 1, \dots, n)$.

It is known that integrals in involution of a Hamiltonian dynamical system $\boldsymbol{X}_H$ on T^*M_n can be found by solving the corresponding Hamilton-Jacobi equation, that is by finding a complete solution $W(q^i, c_a)$ depending on suitable coordinates (q^i) of M_n and on n constants of integration (c_a), $a = 0, 1, \dots, n-1$. Such a complete solution is (locally) the generating function of a transversal Lagrangian foliation of T^*M_n parametrized by (c_a). These Lagrangian submanifolds are defined by equations

$$p_i = \frac{\partial W}{\partial q^i}, \tag{1.1}$$

which can be solved with respect to (c_a). Then the functions $c_a(q^j, p_h)$ give rise to n independent integrals in involution, $\{c_a, c_b\} = 0$ (Jacobi theorem), so that the system is integrable.

If it is possible to find a complete integral of the kind

$$W = W_1(q^1, c_a) + \ldots + W_n(u^q, c_a), \tag{1.2}$$

then we say that the Hamiltonian system is **integrable by separation of variable** (briefly, **separable**) and the coordinates (q^i) are said to be **separable** (with respect to the Hamiltonian H). The constants of integration (c_a) are called **separation constants**. Equations (1.1) show that this is the case when each p_i is a function of the corresponding coordinate q^i only (and in general of all the constants (c_a)).

Analytical Mechanics is mainly interested in dynamical systems defined by a triple $(M_n, \boldsymbol{g}, V)$ where $(M_n, \boldsymbol{g})$ is the Riemannian configuration manifold and $V: M_n \to \mathbb{R}$ the potential energy, and whose Hamiltonian is

$$H = \tfrac{1}{2}\, g^{ij}\, p_i p_j + V. \tag{1.3}$$

We say that such a system is **orthogonal separable** if it is separable with respect to orthogonal coordinates (q^i), $g^{ij} = 0$ for $i \neq j$. A celebrated theorem of Stäckel [22, 23] gives the general form of the integrals in involution in orthogonal separable coordinates. These integrals are quadratic polynomials in the momenta (p_i), so that their coefficients define Killing tensors of order 2, and it turns out to be convenient to deal with the algebraic properties of these tensors, interpreted as linear operators on vector fields and 1-forms, in order to establish an intrinsic characterization of the orthogonal separation. This characterization was firstly investigated in classical papers by Eisenhart [6, 7], and more recently by Kalnins and Miller [10], Woodhouse [27], Shapovalov [21]. The aim of this lecture, which is closely related to previous papers [1, 2], is to present an improved version of the geometrical characterization of the orthogonal separation, together with some applications to separable dynamical systems in the Euclidean spaces.

2. Stäckel webs and Stäckel systems

According to the general theory of the separation of variables (Levi-Civita [13]), a necessary condition for the separability of a dynamical system $(M_n, \boldsymbol{g}, V)$ is that the corresponding geodesic Hamiltonian system ($V = 0$) be separable. This means that the investigation of the separability of the geodesic flow of a Riemannian manifold is a preliminary problem, even if the geodesic flow is known (as in the case of an Euclidean space, for instance).

Definition 2.1. Local coordinates (q^i) on a Riemannian manifold $(M_n, \boldsymbol{g})$ are said to be **separable** if the corresponding geodesic Hamilton-Jacobi equation has a complete integral of the form (1.2).

Since the separation property is invariant under separated transformations of the coordinates (i.e. transformations whose Jacobian is diagonal), in order to have a geometrical picture of the separation it is better to think of the coordinate hypersurfaces rather than of the coordinates. This suggests the use of the notion of a *web*.

Definition 2.2. An **orthogonal web** on a Riemannian manifold $(M_n, \boldsymbol{g})$ is a family $(\mathcal{S}_i) = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ of n orthogonal foliations of hypersurfaces (submanifolds of dimension $n-1$), defined on all M_n with the exception of a closed **singular set** Ω. A **parametrization** of an orthogonal web is a set $(q^i) = (q^1, \ldots, q^n)$ of n real C^∞ functions on $M_n - \Omega$ such that $dq^i \neq 0$ everywhere and $dq^i|S_i = 0$ (i.e. $q^i|S_i =$ const.) for each leaf $S_i \in \mathcal{S}_i$. Locally, a parametrization gives rise to orthogonal coordinates which are **adapted** to the web.

Example 2.1. *Elliptic web on the Euclidean plane* $\mathbb{E}_2$. Let $\mathbb{E}_2$ be the Euclidean plane and let (x, y) be Cartesian (orthogonal) coordinates with origin in a point O. Let us consider two points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ in $\mathbb{E}_2$. Confocal ellipses and hyperbolae, with focuses (F_1, F_2) define an orthogonal web $(\mathcal{S}_1, \mathcal{S}_2)$ on $\mathbb{E}_2$ with singular set $\Omega = \{F_1, F_2\}$. In the foliation $\mathcal{S}_1$ we include the y-axis, all confocal hyperbolae, and the open intervals $I_1 = (-\infty, F_1)$ and $I_2 = (F_2, +\infty)$ of the x-axis. In the foliation $\mathcal{S}_2$ we include all confocal ellipses and the open interval $I_0 = (F_1, F_2)$ of the x-axis. We have two natural parametrizations of this web. A first parametrization (q^1, q^2) is defined by

$$q^1(P) = |PF_1| - |PF_2|, \qquad q^2(P) = |PF_1| + |PF_2|, \qquad P \in \mathbb{E}_2,$$

where $|PQ|$ denotes the distance between two points. A second parametrization (u^1, u^2) is defined by the roots of the equation

$$\frac{x^2}{u-a} + \frac{y^2}{u-b} = 1 \qquad \big(a < b, \quad b - a = \tfrac{1}{2}\,|F_1F_2|^2 = 2\,c^2\big).$$

These roots are (improperly) called **elliptic coordinates**. We observe that two hyperbolae which are symmetric with respect to the y-axis have the same value of u^1 but the opposite value of q^1.

Definition 2.3. A **Stäckel web** on a Riemannian manifold $(M_n, \boldsymbol{g})$ is an orthogonal web whose adapted coordinates are separable.

The question arises how to recognize geometrically if an orthogonal web is a Stäckel web. An answer to this problem can be found by considering particular sets of Killing tensors.

Definition 2.4. A **Stäckel system** on a Riemannian manifold $(M_n, \boldsymbol{g})$ is an n-dimensional subspace $\mathcal{S}$ of the space of Killing tensors of order 2 over the manifold, such that, with the exclusion of a closed **singular set** $\Omega \subset M_n$,

(i) it has a basis of n pointwise independent elements,
(ii) all the elements have common eigendirections,
(iii) these eigendirections are normal.

A vector field (or a distribution of dimension 1, i.e. a field of directions) is said to be **normal** if the orthogonal distribution is completely integrable. The integral manifolds of the normal eigendirections are called the **integral manifolds** of the Stäckel system.

The main fact is that there is a one to one correspondence between Stäckel webs and Stäckel systems. We can state this property as follows.

Proposition 2.1. *An orthogonal web is a Stäckel web if and only if its leaves are integral manifolds of a Stäckel system.*

According to this proposition we say that a Stäckel web is **generated** by a Stäckel system. For the proof we need two lemmas.

Notation. We denote by (∂_i) the partial derivatives with respect to coordinates (q^i) and by $(\boldsymbol{E}_i)$ the vector fields corresponding to these derivatives, that is the natural (local) frame corresponding to the coordinates (q^i).

Lemma 2.1. [8, 11] *Let $\boldsymbol{K}$ be a symmetric tensor which is diagonalized with respect to orthogonal coordinates (u^i). Let (ϱ_i) be the corresponding eigenvalues,*

$$\boldsymbol{K} = \sum_{i=1}^{n} \varrho_i \, g^{ii} \, \boldsymbol{E}_i \otimes \boldsymbol{E}_i. \tag{2.1}$$

Then $\boldsymbol{K}$ is a Killing tensor if and only if the following equations are satisfied

$$\partial_i \varrho_j = (\varrho_i - \varrho_j)\, \partial_i \ln g^{jj} \qquad (i \neq j), \qquad \partial_i \varrho_i = 0. \tag{2.2}$$

Proof. There is a bijective correspondence between symmetric contravariant tensor fields on a differentiable manifold M_n and the polynomial functions on the cotangent bundle T^*M_n:

$$\boldsymbol{K} = (K^{i_1 \dots i_k}) \quad \leftrightarrow \quad E_{\boldsymbol{K}} = \frac{1}{k!}\, K^{i_1 \dots i_k}\, p_{i_1} \, \dots p_{i_k}.$$

A symmetric tensor $\boldsymbol{K}$ on a Riemannian manifold $(M_n, \boldsymbol{g})$ is by definition a Killing tensor if

$$\{E_{\boldsymbol{K}}, E_{\boldsymbol{g}}\} = 0, \tag{2.3}$$

where $\{\ ,\ \}$ are the canonical Poisson brackets of the real functions on T^*M_n. This means that the function $E_{\boldsymbol{K}}$ is an integral of the geodesic flow. Under the assumption (2.1) equation (2.3) implies

$$\sum_i \sum_j g^{ii} \left(g^{jj}\, \partial_i \varrho_j - (\varrho_i - \varrho_h)\, \partial_i g^{jj} \right) \, p_i \, p_j^2 = 0,$$

and this condition is equivalent to system (2.2). ∎

Equations (2.2) can be derived from more general equations written by Eisenhart [8], concerning the characterization of a Killing 2-tensor in the frame made by unitary eigenvectors. For a more general version of Eisenhart's equations see [1].

Lemma 2.2. [6] *Local orthogonal coordinates (q^i) are separable if and only if the following $\frac{1}{2}n^2(n-1)$ equations are satisfied,*

$$\partial_i \partial_j g^{hh} = \partial_i \ln g^{jj}\, \partial_j g^{hh} + \partial_j \ln g^{ii}\, \partial_i g^{hh}, \qquad i \neq j. \tag{2.4}$$

Proof. The additive separation of a complete solution $W(u^i, c_a)$ is equivalent to the complete integrability of the following differential system (see Levi-Civita [13]):

$$\partial_i p_j = 0 \quad (i \neq j), \qquad \partial_i p_i = -\frac{\partial_i H}{\partial^i H},$$

where

$$\partial_i = \frac{\partial}{\partial q^i}, \qquad \partial^i = \frac{\partial}{\partial p_i}.$$

For the geodesic Hamiltonian $H = \frac{1}{2}\, g^{ii}\, p_i^2$ these integrability conditions coincide with equations (2.4). ∎

Proof of Proposition 2.1. Let us consider the differential system (2.2). The complete integrability conditions of this system,

$$\partial_i(\partial_j \varrho_h) - \partial_j(\partial_i \varrho_h) = 0, \tag{2.5}$$

are equivalent to equations

$$(\varrho_i - \varrho_j)\big(\partial_i\partial_j g^{hh} - \partial_i \ln g^{jj}\, \partial_j g^{hh} - \partial_j \ln g^{ii}\, \partial_i g^{hh}\big) = 0 \qquad (i \neq j). \tag{2.6}$$

Assume that the orthogonal coordinates (q^i) are separable. Then equations (2.4) are identically satisfied as well as equations (2.5). As a consequence, the linear system (2.2) has a set of independent solutions (ϱ_{ai}), $a = 0, \dots, n-1$, so that $\det(\varrho_{ai}) \neq 0$. Due to Lemma 1 the tensors

$$\boldsymbol{K}_a = \varrho_{ai}\, g^{ii}\, \boldsymbol{E}_i \otimes \boldsymbol{E}_i \tag{2.7}$$

form a basis of an n-dimensional space $\mathcal{S}$ of Killing 2-tensors which is a Stäckel system in the domain of definition of the coordinates. Conversely, if such a subspace exists, we can represent a basis as in (2.7) and the functions (ϱ_{ai}) form a complete solution of system (2.2). Hence, the complete integrability conditions (2.4) hold. For each pair of distinct indices (i, j) there is at least one index a such that $\varrho_{ai} - \varrho_{aj} \neq 0$, otherwise $\det(\varrho_{ai}) = 0$. Thus equations (6) imply the separability conditions (2.4). ∎

Some remarks concerning Proposition 2.1 and its proof are in order.

Remark 2.1. The complete integrability of the differential system (2.2) is equivalent to the separability of the coordinates (q^i). Hence, equations (2.2) are the characteristic differential equations of the Stäckel systems.

Remark 2.2. Every Stäckel system contains the metric tensor $\boldsymbol{g}$. Indeed, $\varrho_i = 1$ is always a solution of equations (2.2).

Remark 2.3. All elements of a Stäckel system $\mathcal{S}$ commute as linear operators and are in involution, i.e.

$$\boldsymbol{K} \cdot \boldsymbol{K}' - \boldsymbol{K}' \cdot \boldsymbol{K} = 0, \qquad [\boldsymbol{K}, \boldsymbol{K}'] = 0, \qquad \forall \boldsymbol{K}, \boldsymbol{K}' \in \mathcal{S}. \tag{2.8}$$

We denote by $\boldsymbol{K} \cdot \boldsymbol{K}'$ the composition of two linear operators and by $[\boldsymbol{K}, \boldsymbol{K}']$ the Lie brackets defined by equation

$$E_{[\boldsymbol{K},\boldsymbol{K}']} = \{E_{\boldsymbol{K}}, E_{\boldsymbol{K}'}\}. \tag{2.9}$$

The first property is obvious since all elements of $\mathcal{S}$ have the same eigendirections. The second is a general property of solutions of linear differential systems like (2.2). In the present case, for two elements of $\mathcal{S}$ we have:

$$\begin{aligned}
\{E_{\boldsymbol{K}}, E_{\boldsymbol{K}'}\} &= \sum_i \big\{\tfrac{1}{2}\, \varrho_i\, g^{ii}\, p_i^2, \tfrac{1}{2}\, \varrho_i'\, g^{ii}\, p_i^2\big\} \\
&= \sum_i \sum_j \big[\big(\partial_j \varrho_i\, g^{ii} + \partial_j g^{ii}\, \varrho_i\big)\, \varrho_j' - \big(\partial_j \varrho_i'\, g^{ii} + \partial_j g^{ii}\, \varrho_i'\big)\, \varrho_j\big]\; p_i^2\, g^{jj}\, p_j \\
&= \sum_i \sum_j g^{ii} g^{jj} p_i^2 p_j\, \big(\varrho_j'\, \partial_j \varrho_i - \varrho_j\, \partial_j \varrho_i' + (\varrho_j' \varrho_i - \varrho_j \varrho_i')\partial_j \ln g^{ii}\big) \\
&= 0
\end{aligned}$$

due to equations (2.2).

Remark 2.4. A Stäckel system can be represented by a single Killing tensor with pointwise simple eigenvalues. Indeed, if a Killing tensor $\boldsymbol{K}$ having normal eigendirections exists, since the eigenvalues are all different, equation (2.5) written for the eigenvalues of $\boldsymbol{K}$ produces equations (2.6) and consequently the separability conditions (2.4), since $\varrho_i \neq \varrho_j$. But these separability conditions imply the existence of a Stäckel system. Conversely, in a Stäckel system we can always find an element with pointwise distinct eigenvalues. Indeed, if we consider a complete solution (ϱ_{ai}) at a point as a set of components of independent vectors $\boldsymbol{\varrho}_a$ with respect to a basis $\boldsymbol{\varepsilon}^i$ of an n-dimensional vector space, up to a linear transformation with constant coefficients, we can obviously find a vector $\boldsymbol{\varrho}$ whose components with respect to this basis are all different. This means that at that point $\varrho_i \neq \varrho_j$. However this condition remains valid in an open neighborhood of that point. We do not discuss here the possibility of globalizing this local process. We conclude that

Proposition 2.2. *An orthogonal web is a Stäckel web if and only if its leaves are (locally) integral manifolds of a Killing tensor with pointwise simple eigenvalues.*

Then we say that a Stäckel web is **generated** by a Killing tensor with pointwise simple eigenvalues.

Remark 2.5. It can be shown that the commutability condition $(2.8)_1$ can replace condition (ii) in the Definition 2.4 of Stäckel system (for positive metrics), and that the involutive condition $(2.8)_2$ can replace condition (iii) [1]. It can also be shown that, in terms of eigenforms, conditions (ii) and (iii) can be replaced with the following one: the Killing tensors have simultaneous closed eigenforms (this is in fact the characteristic property of the orthogonal separation proposed in [4]).

The properties of a Stäckel system considered in the Remarks (2.2), (2.3), (2.4) have been included in the set of necessary and sufficient conditions for the existence of orthogonal separable coordinates (see [6] and [27]). In fact, after Proposition 2.1 and Definition 2.4, they are redundant.

Example 2.2. For $n = 2$ a Stäckel system always has a basis of the kind $(\boldsymbol{g}, \boldsymbol{K})$ where $\boldsymbol{K}$ is a Killing tensor which is not proportional to the metric tensor. Thus a Stäckel system is always represented by such a Killing tensor $\boldsymbol{K}$. The critical set Ω is the set of points where the Killing tensor $\boldsymbol{K}$ has double eigenvalues (i.e. where it is proportional to the metric tensor). The condition that the eigenvectors are normal is obviously satisfied. If the eigenvalues (ϱ_1, ϱ_2) of $\boldsymbol{K}$ are independent functions, then they give a parametrization of the Stäckel web. For more details see [3].

Example 2.3. *The Euclidean plane* $\mathbb{E}_2$. Let us denote by $\boldsymbol{R}_O$ the *unitary rotation around a point* O of the Euclidean plain. This is a vector field defined by

$$\boldsymbol{R}_O(\boldsymbol{x}) = \boldsymbol{\omega} \times \boldsymbol{x},$$

where $\boldsymbol{x}$ is the position vector of a generic point with respect to O, and $\boldsymbol{\omega}$ is a unitary vector orthogonal to the plain. We can extend the meaning of the symbol $\boldsymbol{R}_O$ to the case in which the point O goes to infinity. Then $\boldsymbol{R}_O$ is a *translation*, a constant unitary vector field. With this convention, it can be seen that every Killing tensor in the plain is of the kind

$$\boldsymbol{K} = a\, \boldsymbol{R}_P \odot \boldsymbol{R}_Q + b\, \boldsymbol{g}, \qquad a, b \in \mathbb{R},$$

where $\odot$ denotes the symmetric tensor product. It follows that all possible Stäckel systems are characterized by a Killing tensor of the kind

$$\boldsymbol{K} = \boldsymbol{R}_P \odot \boldsymbol{R}_Q,$$

thus there are four kinds of Stäckel webs in the plain, corresponding to the following four cases (Ω is the critical set):

$P \neq Q$	$\Omega = \{P, Q\}$	elliptic web
$P = Q$	$\Omega = \{P\}$	polar web
$P \to \infty$	$\Omega = \{Q\}$	parabolic web
$P, Q \to \infty$	$\Omega = \emptyset$	cartesian web

All these webs are made of confocal conics (with degeneration in the second and fourth case; a detailed discussion is in [3]). A similar approach can be used for the hyperbolic plane, for the 2-sphere and the 2-pseudosphere.

3. Separable potentials

Definition 3.1. We say that a real differentiable function $V: M_n \to \mathbb{R}$ is **separable** with respect to (or **compatible** with) a Stäckel web if the dynamical system $(M_n, \boldsymbol{g}, V)$ is separable in any coordinate system adapted to the web.

Proposition 3.1. *A potential function V is compatible with respect to a Stäckel web generated by a Stäckel system $\mathcal{S}$ if and only if*

$$d(\boldsymbol{K} \cdot dV) = 0 \tag{3.1}$$

for all elements $\boldsymbol{K}$ of $\mathcal{S}$ or for at least an element of $\mathcal{S}$ with simple eigenvalues.

To prove this statement we need the following lemma (see also [4]).

Lemma 3.1. *A potential V is separable with respect to orthogonal separable coordinates (q^i) if and only if the following $\frac{1}{2}n(n-1)$ equations are satisfied,*

$$\partial_i \partial_j V = \partial_i \ln g^{jj}\, \partial_j V + \partial_j \ln g^{ii}\, \partial_i V, \qquad i \neq j, \tag{3.2}$$

Proof. These equations follow from the separability conditions of Levi-Civita [13] applied to a Hamiltonian of the kind $H = \frac{1}{2} g^{ii} p_i^2 + V$ (see the proof of Lemma 2.2). ∎

Proof of Proposition 3.1. Let us consider an element $\boldsymbol{K} \in \mathcal{S}$ expressed as in (2.1). The covariant components of $\boldsymbol{K} \cdot dV$ are $(\varrho_i \partial_i V)$, thus equation $d(\boldsymbol{K} \cdot dV) = 0$ is equivalent to

$$\partial_i(\varrho_j\, \partial_j V) - \partial_j(\varrho_i\, \partial_i V) = 0.$$

Due to equations (3.2), it follows that this is equivalent to

$$(\varrho_i - \varrho_j)\big(\partial_i \partial_j V - \partial_i \ln g^{jj}\, \partial_j V - \partial_j \ln g^{ii}\, \partial_i V\big) = 0, \qquad i \neq j.$$

The rest of the proof follows the same pattern as that of Proposition 2.1. See also Remark 2.4. ∎

Proposition 3.2. *Let $(\boldsymbol{K}_a) = (\boldsymbol{K}_1, \boldsymbol{K}_2, \dots, \boldsymbol{K}_n)$ be a basis of a Stäckel system $\mathcal{S}$, and V a separable potential. Let $V_a: M \to \mathbb{R}$ be functions (locally) defined by*

$$dV_a = \boldsymbol{K}_a \cdot dV, \tag{3.3}$$

then the functions $c_a: T^*M \to \mathbb{R}$ defined by

$$c_a = E_{\boldsymbol{K}_a} + \pi^* V_a, \tag{3.4}$$

where $\pi: T^*M \to M$ is the cotangent fibration, are independent integrals in involution.

Proof. This is a consequence of the theorems of Jacobi and Stäckel. If we choose $\boldsymbol{K}_0 = \boldsymbol{g}$, then $c_0 = H$. However, due to the preceding theorems and Remark 2.3, we can prove this fact directly:

$$\begin{aligned}
\{c_a, c_b\} &= \{E_{\boldsymbol{K}_a} + \pi^* V_a, E_{\boldsymbol{K}_b} + \pi^* V_b\} \\
&= \{E_{\boldsymbol{K}_a}, E_{\boldsymbol{K}_b}\} + \{E_{\boldsymbol{K}_a}, \pi^* V_b\} + \{\pi^* V_a, E_{\boldsymbol{K}_b}\} \\
&= E_{[\boldsymbol{K}_a, \boldsymbol{K}_b]} + E_{\boldsymbol{K}_a \cdot dV_b} - E_{\boldsymbol{K}_b \cdot dV_a} \\
&= E_{(\boldsymbol{K}_a \cdot \boldsymbol{K}_b - \boldsymbol{K}_b \cdot \boldsymbol{K}_a) \cdot dV} \\
&= 0. \qquad ∎
\end{aligned}$$

Remark 3.1. Equations (3.2) characterize the separability of a potential V by means of the separable coordinates (q^i) and the contravariant components of the metric tensor in these coordinates. It is classically known that the most general solution of equations (3.2) is

$$V = V_i \, g^{ii} \tag{3.5}$$

where each V_i is a function of the coordinate q^i (a function like V is called a **Stäckel multiplier** [11, 4]). Instead, equation (3.1) characterizes the separability of V in an intrinsic way, by means of the elements (or one element with simple eigenvalues) of the Stäckel system. The fact is remarkable that in applying equation (3.1) the choice of coordinates is completely free. Another characterization of the orthogonal separation of a dynamical system, which involves simultaneously the Killing tensors $(\boldsymbol{K}_a)$ and the functions (V_a), is considered in [21] and [4].

Remark 3.2. The separability conditions (2.4) hold for any coordinate hypersurface q^i = const., so that orthogonal separable coordinates induce separable coordinates on all the coordinate hypersurfaces (this fact has been pointed out in [4], after different considerations). Moreover, equations (3.1) are also reducible on any coordinate hypersurface. We conclude that

Proposition 3.3. *If a dynamical system $(M_n, \boldsymbol{g}, V)$ is orthogonal separable then every restriction $(S, \boldsymbol{g}|S, V|S)$ to a submanifold $S \subset M_n$ which is the intersection of leaves of the corresponding Stäckel web is separable.*

However, there are potentials which are separable on the leaves of the web which are not the restriction of a separable potential defined in the whole space, as shown by the following proposition.

Proposition 3.4. *Let $\mathbb{S}_i$ be a foliation belonging to a Stäckel web corresponding to a Stäckel system $\mathcal{S}$. Let $\boldsymbol{X}_i$ be a vector field orthogonal to this foliation (without singular points). Then the restriction of a potential $V \colon M_n \to \mathbb{R}$ to every submanifold of $\mathbb{S}_i$ is separable if and only*

$$d(\boldsymbol{K} \cdot dV) \wedge \boldsymbol{X}_i = 0, \tag{3.6}$$

for all $\boldsymbol{K} \in \mathcal{S}$ or for at least one element $\boldsymbol{K} \in \mathcal{S}$ with pointwise simple eigenvalues.

Proof. According to Proposition 3.1, the induced potential $V|S$ is separable on all submanifolds $S \in \mathbb{S}_i$ if and only if

$$d(\boldsymbol{K}|S \cdot dV|S) = 0,$$

for all $\boldsymbol{K} \in \mathcal{S}$, or for a $\boldsymbol{K}$ with simple eigenvalues. By considering separable coordinates (q^i) adapted to the web, the right hand side of this equation can be written $d\sum_{k \neq i}(\varrho_k \partial_k V \, dq^k)$, so that it becomes equivalent to

$$d\sum_k(\varrho_k \partial_k V \, dq^k) \wedge dq^i = 0,$$

and this is equivalent to (3.6), since by using the natural identification between 1-forms and vector fields the differential dq^i can be substituted with any vector field orthogonal to the submanifolds of $\mathbb{S}_i$. ∎

Remark 3.3. All the above considerations hold in a pseudo-Riemannian manifold, where the metric is not positive-definite, provided that in the formulae we replace $\ln g^{ii}$ with $\ln |g^{ii}|$ and we add the requirement that the Killing tensors involved have real eigenvalues with non-null eigenvectors.

4. The elliptic Stäckel web in the Euclidean plane $\mathbb{E}_2$

Let us consider the elliptic web in the Euclidean plane $\mathbb{E}_2$ (see Example 2.3), with focuses $F_1 \neq F_2$. Let $\boldsymbol{R}_i$ $(i = 1, 2)$ be the rotation around F_i, $\boldsymbol{R}_i(P) = \boldsymbol{\omega} \times \boldsymbol{r}_i$, $\boldsymbol{r}_i = F_iP$, $P \in \mathbb{E}_2$, and $\boldsymbol{u}_i = F_iP/r_i$, $r_i = |F_iP|$. The elliptic Stäckel system is completely determined by the Killing tensor

$$\boldsymbol{K} = \boldsymbol{R}_1 \odot \boldsymbol{R}_2 = (\boldsymbol{\omega} \times \boldsymbol{r}_1) \odot (\boldsymbol{\omega} \times \boldsymbol{r}_2) = r_1 \, r_2 \, (\boldsymbol{\omega} \times \boldsymbol{u}_1) \odot (\boldsymbol{\omega} \times \boldsymbol{u}_2). \tag{4.1}$$

Let us consider a *bicentered symmetric potential*, i.e. a potential V of the form

$$V = V_1(r_1) + V_2(r_2). \tag{4.2}$$

Let us impose condition (3.1) on this potential. We can use (r_1, r_2) as coordinates. The local frame $\boldsymbol{E}_i = \frac{\partial}{\partial r_i}$ corresponding to these coordinates is given by

$$\boldsymbol{E}_1 = \frac{1}{\sin^2 \vartheta}(\boldsymbol{u}_1 - \cos\vartheta\, \boldsymbol{u}_2), \qquad \boldsymbol{E}_2 = \frac{1}{\sin^2 \vartheta}(\boldsymbol{u}_2 - \cos\vartheta\, \boldsymbol{u}_1),$$

where ϑ is the angle between the unitary vectors $(\boldsymbol{u}_i)$, $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \cos\vartheta$. The gradient of V is the vector field

$$\mathrm{grad}(V) = V_1'\, \boldsymbol{u}_1 + V_2'\, \boldsymbol{u}_2,$$

so that

$$\begin{aligned} \boldsymbol{X} = \boldsymbol{K} \cdot \mathrm{grad}(V) &= r_1\, r_2\, \boldsymbol{\omega} \cdot \boldsymbol{u}_1 \times \boldsymbol{u}_2\, (V_2'\, \boldsymbol{\omega} \times \boldsymbol{u}_2 - V_1'\, \boldsymbol{\omega} \times \boldsymbol{u}_1) \\ &= r_1\, r_2\, \sin\vartheta\, (V_2'\, \boldsymbol{\omega} \times \boldsymbol{u}_2 - V_1'\, \boldsymbol{\omega} \times \boldsymbol{u}_1). \end{aligned}$$

The covariant components of the vector field $\boldsymbol{X}$ are

$$\begin{aligned} X_1 &= \boldsymbol{X} \cdot \boldsymbol{E}_1 = r_1\, r_2\, (V_1' \cos\vartheta - V_2') = \tfrac{1}{2}\, V_1'\, (r_1^2 + r_2^2 - a^2) - r_1\, r_2\, V_2', \\ X_2 &= \boldsymbol{X} \cdot \boldsymbol{E}_2 = r_1\, r_2\, (V_2' \cos\vartheta - V_1') = \tfrac{1}{2}\, V_2'\, (r_1^2 + r_2^2 - a^2) - r_1\, r_2\, V_1', \end{aligned} \tag{4.3}$$

where $a = |F_1 F_2|$ is the distance between the two focuses, and

$$\cos\vartheta = \frac{r_1^2 + r_2^2 - a^2}{2\, r_1\, r_2}.$$

Equation (3.1) is in the present case equivalent to equation

$$\frac{\partial X_1}{\partial r_2} = \frac{\partial X_2}{\partial r_1},$$

that is, after (4.3), to the separated differential equation

$$\frac{2}{r_1}\, V_1' + V_1'' = \frac{2}{r_2}\, V_2' + V_2'',$$

so that

$$V = \alpha_0 (r_1^2 + r_2^2) + \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2} \qquad (\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}). \tag{4.4}$$

Hence, we have proved that (a different proof is given in [3])

Proposition 4.1. *The most general bicentered symmetric potential compatible with the Stäckel web in the plane is of the kind* (4.4).

Now, in order to find the integrals corresponding to the dynamical system with potential (4.4), let us apply Proposition 3.2. Let U be the function such that $dU = \boldsymbol{K} \cdot dV$ where V is the potential (4.4). According to (3.4) and (4.1) the corresponding integral is

$$c = \boldsymbol{v} \cdot \boldsymbol{K} \cdot \boldsymbol{v} + U = (\boldsymbol{r}_1 \times \boldsymbol{v}) \cdot (\boldsymbol{r}_2 \times \boldsymbol{v}) + U. \tag{4.5}$$

We consider three cases separately.

Case I: $\alpha_0 = \alpha_2 = 0$. This is the case of a particle moving in a Newtonian or Coulombian field centered at a point $O = F_1$. Hence, this dynamical system is integrable by separation of variables not only in polar coordinates centered at O but also in elliptic coordinates with one of the focuses coinciding with the center of the field and the other arbitrarily chosen in the plane. In this case, see (4.3),

$$X_1 = -\,\alpha_1\,\frac{r_2}{r_1}\,\cos\vartheta = \frac{\partial U}{\partial r_1}, \qquad X_2 = \alpha_1\,\frac{r_2}{r_1} = \frac{\partial U}{\partial r_2},$$

where

$$U = \alpha_1\,(r_2 \cos\vartheta - r_1).$$

Indeed we have

$$\frac{\partial}{\partial r_i}\cos\vartheta = \frac{1}{r_{i+1}} - \frac{1}{r_i}\cos\vartheta.$$

If we introduce the vector

$$\boldsymbol{a} = F_1F_2 = \boldsymbol{r}_1 - \boldsymbol{r}_2, \tag{4.6}$$

then this potential can be written

$$U = -\,\alpha_1\,\boldsymbol{a}\cdot\boldsymbol{u}_1. \tag{4.7}$$

According to formula (4.5) the corresponding separation constant is

$$c = (\boldsymbol{r}_1 \times \boldsymbol{v})^2 - \boldsymbol{a}\cdot\boldsymbol{A} \tag{4.8}$$

where

$$\boldsymbol{A} = \boldsymbol{v} \times (\boldsymbol{r}_1 \times \boldsymbol{v}) + \frac{\alpha_1}{r_1}\boldsymbol{r}_1. \tag{4.9}$$

This is in fact the celebrated Laplace-Runge-Lenz vector. Hence, we recognize that this vector integral is intimately connected with the separation in elliptic coordinates. Note that the first part of the separation constant (4.8) is the square of the angular momentum.

Case II: $\alpha_0 = 0$. This is a classical dynamical system: a point moving in a field generated by two fixed charges or masses. Its integrability has been investigated by Euler, Lagrange and Legendre (for the treatment of this problem and its bibliography see [9, 20, 24]). It follows from (4.7) that the function U is

$$U = \boldsymbol{a}\cdot(\alpha_2\boldsymbol{u}_2 - \alpha_1\boldsymbol{u}_1), \tag{4.10}$$

and the separation constant is again given by (4.8) where in this case

$$\boldsymbol{A} = \boldsymbol{v} \times (\boldsymbol{r}_1 \times \boldsymbol{v}) + \left(\frac{\alpha_1}{r_1} - \frac{\alpha_2}{r_2}\right)\boldsymbol{r}_1 - \frac{\alpha_2}{r_2}\,\boldsymbol{a}. \tag{4.11}$$

Case III: $\alpha_1 = \alpha_2 = 0$. This is the case of a point moving in a symmetric linear field (for instance, the force of an ideal spring) centered at a point O. This dynamical system is integrable by separation of variables in Cartesian coordinates with origin O, in polar coordinates centered at O and in elliptic coordinates with symmetric focuses with respect to O. In this case, see (4.3),

$$X_1 = \alpha_0\,r_1\,(r_1^2 - r_2^2 - a^2), \qquad X_2 = \alpha_0\,r_2\,(r_2^2 - r_1^2 - a^2),$$

so that

$$U = \frac{\alpha_0}{4}\left(\left(r_1^2 - r_2^2\right)^2 - 2a^2\left(r_1^2 + r_2^2\right)\right).$$

If we introduce the radius vector

$$\boldsymbol{r} = OP = \tfrac{1}{2}\,(\boldsymbol{r}_1 - \boldsymbol{r}_2), \tag{4.12}$$

since

$$r^2 = \tfrac{1}{2}\left(r_1^2 + r_2^2\right) - \frac{1}{4}a^2, \qquad \boldsymbol{a} \cdot \boldsymbol{r} = \tfrac{1}{2}\left(r_1^2 - r_2^2\right),$$

we find

$$U = \alpha_0 \left((\boldsymbol{a} \cdot \boldsymbol{r})^2 - a^2 r^2 - \frac{a^4}{4} \right).$$

Thus, by neglecting an inessential constant,

$$U = -\,\alpha_0\,(\boldsymbol{a} \times \boldsymbol{r})^2. \tag{4.13}$$

Since $\boldsymbol{r}_1 = \boldsymbol{r} + \frac{1}{2}\,\boldsymbol{a}$ and $\boldsymbol{r}_2 = \boldsymbol{r} - \frac{1}{2}\,\boldsymbol{a}$, the separation constant (4.5) becomes

$$c = (\boldsymbol{r} \times \boldsymbol{v})^2 - \frac{1}{4}(\boldsymbol{a} \times \boldsymbol{v})^2 - \alpha_0\,(\boldsymbol{a} \times \boldsymbol{r})^2. \tag{4.14}$$

Again the first term is the square of the angular momentum. The remaining part is equivalent to the first integral $\dot{y}^2 + 2\alpha_0\,y^2$, in Cartesian coordinates (x, y) such that $F_1 = (-\frac{a}{2}, 0)$ and $F_2 = (\frac{a}{2}, 0)$.

5. The elliptic Stäckel web in the Euclidean space $\mathbb{E}_n$

Let (x_α) $(\alpha = 1, \dots, n)$ be Cartesian coordinates in the Euclidean affine space $\mathbb{E}_n$. Let us consider the symmetric 2-tensor $\boldsymbol{K}$ with Cartesian components

$$K^{\alpha\alpha} = \varsigma^\alpha + m \sum_{\gamma \neq \alpha} x_\gamma^2, \qquad K^{\alpha\beta} = -\,m\,x_\alpha\,x_\beta \quad (\alpha \neq \beta), \tag{5.1}$$

where ς^α, $m \in \mathbb{R}$ $(m \neq 0)$. *(i) The tensor $\boldsymbol{K}$ is a Killing tensor.* Indeed,

$$E_{\boldsymbol{K}} = \tfrac{1}{2} \sum_\alpha \varsigma^\alpha\, p_\alpha^2 + \frac{m}{4} \sum_{\alpha \neq \beta} (x_\alpha p_\beta - x_\beta p_\alpha)^2, \qquad E_{\boldsymbol{g}} = \tfrac{1}{2} \sum_\alpha p_\alpha^2.$$

Thus,

$$\begin{aligned} \{E_{\boldsymbol{K}}, E_{\boldsymbol{g}}\} &= \sum_\gamma \partial_\gamma E_{\boldsymbol{K}}\, p_\gamma = \frac{m}{2} \sum_{\alpha,\beta,\gamma} \left(\delta_{\alpha\gamma} p_\beta - p_\alpha \delta_{\beta\gamma}\right) p_\gamma = \\ &= \frac{m}{2} \sum_{\alpha\beta} (p_\alpha p_\beta - p_\beta p_\alpha) = 0. \end{aligned}$$

Let us consider the tensor

$$\boldsymbol{L} = \frac{1}{n-1}\,\mathrm{tr}(\boldsymbol{K})\,\boldsymbol{g} - \boldsymbol{K}, \tag{5.2}$$

whose Cartesian components are

$$L_{\alpha\beta} = \delta_{\alpha\beta}\,a_\alpha + m\,x_\alpha\,x_\beta, \tag{5.3}$$

where

$$a_\alpha = \frac{1}{n-1} \sum_\gamma \varsigma^\gamma - \varsigma^\alpha, \tag{5.4}$$

so that, conversely,

$$\varsigma^\alpha = \sum_\gamma a_\gamma - a_\alpha. \tag{5.5}$$

Obviously, *(ii) the tensors* $\boldsymbol{L}$ *and* $\boldsymbol{K}$ *have the same eigenvectors.* Moreover, *(iii) the Nijenhuis torsion of* $\boldsymbol{L}$ *vanishes.* The Nijenhuis torsion [19] of a (1,1)-tensor field $\boldsymbol{L} = (L_i^j)$ is the (1,2)-tensor $\boldsymbol{N}(\boldsymbol{L})$ with components

$$N_{ij}^h = L_{[i}^k \partial_{|k|} L_{j]}^h - L_k^h \partial_{[i} L_{j]}^k. \tag{5.6}$$

In the present case, with respect to Cartesian coordinates,

$$\begin{aligned}
\sum_\varrho L_{\alpha\varrho}\partial_\varrho L_{\beta\gamma} &= \sum_\varrho L_{\alpha\varrho}\big(\delta_{\varrho\beta}x_\gamma + \delta_{\varrho\gamma}x_\beta\big) = L_{\alpha\beta}x_\gamma + L_{\alpha\gamma}x_\beta, \\
\sum_\varrho L_{\varrho\gamma}\partial_\alpha L_{\beta\varrho} &= \sum_\varrho L_{\varrho\gamma}\big(\delta_{\alpha\beta}x_\varrho + \delta_{\alpha\varrho}x_\beta\big) = \sum_\varrho L_{\varrho\gamma}x_\varrho\delta_{\alpha\beta} + L_{\alpha\gamma}x_\beta,
\end{aligned}$$

so that $N_{\alpha\beta\gamma} = 0$. Due to (5.3), the following identity holds,

$$\sum_\beta L_{\alpha\beta}\frac{x_\beta}{u - a_\beta} = a_\alpha \frac{x_\alpha}{u - a_\alpha} + m\, x_\alpha \sum_\beta \frac{x_\beta^2}{u - a_\beta}.$$

Hence, if u is a root of equation

$$\sum_\alpha \frac{x_\alpha^2}{u - a_\alpha} = \frac{1}{m}, \tag{5.7}$$

then

$$\sum_\beta L_{\alpha\beta}\frac{x_\beta}{u - a_\beta} = u\, \frac{x_\alpha}{u - a_\alpha}.$$

This shows that *(iv) the roots* (u_i) *of equation* (5.7) *are eigenvalues of* $\boldsymbol{L}$ and *(v) the vectors* $(\boldsymbol{E}_i)$ *with Cartesian components*

$$E_i^\alpha = \frac{x_\alpha}{u^i - a_\alpha} \tag{5.8}$$

are the corresponding eigenvectors. Moreover, equation (5.7) shows that *(vi) if the constants* (a_α) *are distinct, then they separate the roots* (u^i),

$$a_1 < u^1 < a_2 < u^2 < \ldots < a_n < u^n.$$

By the theorem of Nijenhuis [19], from properties *(ii) (iii) (iv) (vi)* we conclude that *(vii) the Killing tensor* $\boldsymbol{K}$ *has normal eigenvectors whose integral manifolds are defined by equations* $u^i = const.$, *where* (u^i) *are the roots of equation* (5.7). Furthermore, *(viii) the eigenvalues of* $\boldsymbol{K}$ *are pointwise distinct.* Indeed, (5.2) implies

$$\boldsymbol{K} = \mathrm{tr}(\boldsymbol{L})\boldsymbol{g} - \boldsymbol{L},$$

so that the eigenvalues of $\boldsymbol{K}$ are $\varrho_i = \sum_{j\neq i} u^j$. Due to Proposition 2.2, from properties *(i)* and *(viii)* we conclude that

Proposition 5.1. *If the numbers* (ς^α) *(or* (a_α)*) are all distinct then the tensor* $\boldsymbol{K}$ *defined in* (5.1) *is a Killing tensor generating a Stäckel web on* $\mathbb{E}_n$.

We call this web the **elliptic web** of $\mathbb{E}_n$ corresponding to the constants (a_α). This web is invariant under the translation $a_\alpha \to a_\alpha + b$ (that is, it depends on the differences $a_\alpha - a_\beta$ only). The roots of equation (5.7) (with m=1) are classically known as **elliptic coordinates** of $\mathbb{E}_n$. The elliptic web is made of **confocal quadrics**, and its singular set is the union of the sets Ω_α defined by equations

$$x_\alpha = 0, \qquad \sum_{\gamma\neq\alpha} \frac{x_\gamma^2}{a_\alpha - a_\gamma} = 1.$$

These are called the **focal quadrics**. Now, let us consider the potentials which are separable in this web.

Proposition 5.2. *A potential V is compatible with the elliptic Stäckel web if and only if the $\frac{1}{2}n(n-1)$ linear differential equations*

$$(a_\beta - a_\alpha)\,\partial_\alpha\partial_\beta V + \big(x_\beta\partial_\alpha - x_\alpha\partial_\beta\big)\left(2V + \sum_{\gamma=1}^{n} x_\gamma\partial_\gamma V\right) = 0 \tag{5.9}$$

are satisfied.

Proof. Since the Killing tensor $\boldsymbol{K}$ has distinct eigenvalues, we can use Proposition 3.1 and write equation (3.1) for $\boldsymbol{K}$ in Cartesian coordinates, $\partial_\alpha X_\beta - \partial_\beta X_\alpha = 0$, where

$$\begin{aligned} X_\alpha = (\boldsymbol{K}\cdot dV)_\alpha &= \sum_\gamma K^{\alpha\gamma}\partial_\gamma V = K^{\alpha\alpha}\partial_\alpha V + \sum_{\gamma\neq\alpha} K^{\alpha\gamma}\partial_\gamma V \\ &= \left(\varsigma^\alpha + \sum_{\gamma\neq\alpha} x_\gamma^2\right)\partial_\alpha V - \sum_{\gamma\neq\alpha} x_\alpha\, x_\gamma\, \partial_\gamma V, \end{aligned}$$

By a straightforward calculation we derive equations (5.9). ■

The linear differential equations (5.9) in Cartesian coordinates (x_α) are equivalent to equations (3.2) in any coordinate system adapted to the elliptic web (for instance, in the elliptic coordinates (u^i)). If we consider the potential V as a function of the squares (x_α^2) of the Cartesian coordinates, and if we set

$$\partial_{\bar\alpha} = \frac{\partial}{\partial x_\alpha^2},$$

so that

$$\partial_\alpha V = 2\,x_\alpha\,\partial_{\bar\alpha}, \qquad \partial_\alpha\partial_\beta V = 4\,x_\alpha\,x_\beta\,\partial_{\bar\alpha}\partial_{\bar\beta}V \quad (\alpha\neq\beta),$$

then equations (5.9) become equivalent to

$$(a_\beta - a_\alpha)\partial_{\bar\alpha}\partial_{\bar\beta}V + \big(\partial_{\bar\alpha} - \partial_{\bar\beta}\big)\left(V + \sum_{\gamma=1}^{n} x_\gamma^2\partial_{\bar\gamma}V\right) = 0. \tag{5.10}$$

Remark 5.1. The most general separable potential of the kind

$$V = \sum_{\alpha=1}^{n} V_\alpha(x_\alpha) = V_1(x_1) + \ldots + V_n(x_n) \tag{5.11}$$

is

$$V = \sum_{\alpha=1}^{n}\left(c\,x_\alpha^2 + \frac{b_\alpha}{x_\alpha^2}\right) \tag{5.12}$$

where c and (b_α) are arbitrary constants. Indeed, when $\partial_\alpha\partial_\beta V = 0$ for $\alpha \neq \beta$, equations (5.9) become

$$3\big(x_\beta\,\partial_\alpha V - x_\alpha\,\partial_\beta V\big) + x_\alpha\,x_\beta\,\big(\partial_\alpha^2 V - \partial_\beta^2 V\big) = 0.$$

i.e.

$$\partial_\alpha^2 V + \frac{3}{x_\alpha}\partial_\alpha V = \partial_\beta^2 V + \frac{3}{x_\beta}\partial_\beta V.$$

These equations can be satisfied if and only if

$$\partial_\alpha^2 V + \frac{3}{x_\alpha}\partial_\alpha V = c \qquad (c \in \mathbb{R}),$$

and this implies (5.12).

Remark 5.2. If the potential V is assumed to be homogeneous of degree k in the Cartesian coordinates, equations (5.9) imply

$$\partial_\alpha\partial_\beta V = (2+k)\frac{x_\beta\partial_\alpha V - x_\alpha\partial_\beta V}{a_\alpha - a_\beta}. \tag{5.13}$$

These equations are identically satisfied when, for instance, $k=-2$ and $\partial_\alpha\partial_\beta V = 0$, or $x_\alpha\partial_\beta V = x_\beta\partial_\alpha V$. We find again the potential (5.12).

Remark 5.3. Due to Proposition 3.3 all the potentials satisfying equations (5.9) give rise to separable dynamical systems when restricted to the quadrics of the elliptic Stäckel web and to their intersections. For instance, it follows from (5.12) that a linear spherically symmetric force (like an elastic force) centered at a point O and acting on a mass point constrained on a smooth quadric with center O, or on the intersection of two or more confocal quadrics centered at O, is separable. The separability of the elastic potential $V = r^2 = \sum_{\alpha=1}^n x_\alpha^2$ over an ellipsoid was discovered by Jacobi.

We call the Stäckel system corresponding to the elliptic web the **elliptic Stäckel system**.

Let ς_a be the elementary symmetric functions of order $a = 1,\dots,n$ of the distinct real numbers (a_α). Let ς_a^α and $\varsigma_a^{\alpha\beta}$ be the elementary symmetric functions with the exclusion of a_α and (a_α, a_β), $\alpha \neq \beta$, respectively. Let us set $\varsigma_0^\alpha = \varsigma_0^{\alpha\beta} = 1$ and $\varsigma_{-1}^{\alpha\beta} = 0$.

Proposition 5.3. *The 2-tensors* $(\boldsymbol{K}_a)$ $(a = 0,1,\dots,n-1)$ *with Cartesian components*

$$K_a^{\alpha\alpha} = \varsigma_a^\alpha + m\sum_{\gamma\neq\alpha}\varsigma_{a-1}^{\alpha\gamma}x_\gamma^2, \qquad K_a^{\alpha\beta} = -\,m\,\varsigma_{a-1}^{\alpha\beta}\,x_\alpha\,x_\beta \quad (\alpha\neq\beta), \tag{5.14}$$

form a basis of the elliptic Stäckel system.

Note that $\boldsymbol{K}_0 = \boldsymbol{g}$ and $\boldsymbol{K}_1 = \boldsymbol{K}$.

Proof. These tensors have common eigenvectors $(\boldsymbol{E}_i)$ defined by (5.8):

$$\begin{aligned}
\big(\boldsymbol{K}_a\cdot\boldsymbol{E}_i\big)^\alpha &= K_a^{\alpha\alpha}E_i^\alpha + \sum_{\gamma\neq\alpha}K_a^{\alpha\gamma}E_i^\gamma \\
&= \tfrac12\left(\varsigma_a^\alpha + m\sum_{\gamma\neq\alpha}\varsigma_{a-1}^{\alpha\gamma}x_\gamma^2\right)\frac{x_\alpha}{u^i - a_\alpha} - \tfrac12\sum_{\gamma\neq\alpha}\varsigma_{a-1}^{\alpha\gamma}\,x_\alpha x_\gamma\,\frac{x_\gamma}{u^i - a_\gamma} \\
&= \tfrac12\,\frac{x_\alpha}{u^i - a_\alpha}\left(\varsigma_a^\alpha + m\sum_{\gamma\neq\alpha}\varsigma_{a-1}^{\alpha\gamma}(a_\alpha - a_\gamma)\frac{x_\gamma^2}{u^i - a_\gamma}\right) \\
&= \tfrac12\,\frac{x_\alpha}{u^i - a_\alpha}\left(\varsigma_a^\alpha - m\sum_{\gamma}\big(\varsigma_a^\alpha - \varsigma_a^\gamma\big)\frac{x_\gamma^2}{u^i - a_\gamma}\right) \\
&= \tfrac12\,\frac{x_\alpha}{u^i - a_\alpha}\sum_\gamma\frac{\varsigma_a^\gamma x_\gamma^2}{u^i - a_\gamma},
\end{aligned}$$

i.e.,

$$\boldsymbol{K}_a \cdot \boldsymbol{E}_i = \sum_\gamma \frac{\varsigma_a^\gamma x_\gamma^2}{u^i - a_\gamma}\, \boldsymbol{E}_i.$$

Here we used the identity [2]

$$\varsigma_a^\alpha - \varsigma_a^\beta = (a_\beta - a_\alpha)\varsigma_{a-1}^{\alpha\beta}.$$

The tensors $(\boldsymbol{K}_a)$ are pointwise independent (out of the singular set), since at the point O the non-vanishing components are $K_a^{\alpha\alpha} = \varsigma_a^{\alpha\alpha}$ and $\det(\varsigma_a^\alpha) = \prod_{\alpha>\beta}(a_\beta - a_\alpha) \neq 0$. The tensors $(\boldsymbol{K}_a)$ are Killing tensors. Indeed,

$$E_{\boldsymbol{K}_a} = \tfrac{1}{2} \sum_\alpha \varsigma_a^\alpha\, p_\alpha^2 + \frac{m}{4} \sum_{\alpha\neq\beta} \varsigma_{a-1}^{\alpha\beta}(x_\alpha p_\beta - x_\beta p_\alpha)^2. \tag{5.15}$$

Thus,

$$\begin{aligned}\{E_{\boldsymbol{K}_a}, E_{\boldsymbol{g}}\} &= \sum_\gamma \partial_\gamma E_{\boldsymbol{K}_a}\, p_\gamma = \frac{m}{2} \sum_{\alpha,\beta,\gamma} \varsigma_{a-1}^{\alpha\beta}\big(\delta_{\alpha\gamma} p_\beta - p_\alpha \delta_{\beta\gamma}\big) p_\gamma = \\ &= \frac{m}{2} \sum_{\alpha\beta} \varsigma_{a-1}^{\alpha\beta}(p_\alpha p_\beta - p_\beta p_\alpha) = 0. \qquad \blacksquare\end{aligned}$$

Remark 5.4. The functions (V_a) such that $dV_a = \boldsymbol{K} \cdot dV$, where V is of the kind (5.12), are

$$V_a = \sum_{\alpha=1}^n \varsigma_a^\alpha \left(c\, x_\alpha^2 + \frac{b_\alpha}{x_\alpha^2} \right) + \sum_{\alpha\neq\beta} \varsigma_{a-1}^{\alpha\beta}\, b_\alpha\, \frac{x_\beta^2}{x_\alpha^2}. \tag{5.16}$$

Thus, by adding these functions to the corresponding functions (5.15), according to Proposition 3.2, we get a complete system of integrals in involution.

A method for constructing the Killing tensors forming the elliptic Stäckel system (5.14) is illustrated in [2], where the fundamental Killing tensor $\boldsymbol{K}_1$ is interpreted as the **inertia tensor** of a system of masses, with total mass m. Another approach to the construction of the elliptic separable potentials and the separation constants is discussed in [14, 25, 26] (however, following the approach in [14] the number of differential equations considered as characterizing the separability turns out to be larger than $\frac{1}{2}n(n-1)$). The construction of the components in elliptic coordinates of the separation constants dates back to Jacobi [15], and it is also developed in [12].

6. The spherical-conical Stäckel web in the Euclidean space $\mathbb{E}_n$

With a set of n real numbers (a_α) such that $a_1 < a_2 < \ldots < a_n$ we associate a family of cones

$$\sum_{\alpha=1}^n \frac{x_\alpha^2}{u - a_\alpha} = 0 \tag{6.1}$$

parametrized by $u \in \mathbb{R}$, with vertex at the origin O of the Cartesian coordinates. If we add to these cones the foliation made of the spheres centered at O we get an orthogonal web, which we call the **spherical-conical web** (corresponding to the numbers (a_α)). When u takes the values (a_α) the cones degenerate into cones of dimension $n-2$ which are called **focal cones**. The union of these focal cones is the critical set of the web.

For each point of $\mathbb{E}_n$ not belonging to the focal cones equation (6.1) has $n-1$ real roots (u^s) ($s = 1, \ldots, n-1$) such that $a_1 < u^1 < a_2 < u^2 < \ldots < u^{n-1} < a_n$. To these roots we add the distance from the origin $u^n = r$. The functions $(u^i) = (u^s, r)$, which give a parametrization of the web, are called **spherical-elliptic coordinates** or **conical coordinates**. It is known that these coordinates are separable, i.e. that the spherical-elliptic web is a Stäckel web. In our approach this fact can be recognized by the following proposition.

Proposition 6.1. *The 2-tensors* $(\boldsymbol{H}_a)$ $(a = 0, 1, \dots, n-1)$, *where* $\boldsymbol{H}_0 = \boldsymbol{g}$ *and*

$$H_a^{\alpha\alpha} = \sum_{\gamma\neq\alpha} \varsigma_{a-1}^{\alpha\gamma}\, x_\gamma^2, \qquad H_a^{\alpha\beta} = -\,\varsigma_{a-1}^{\alpha\beta} \quad (\alpha \neq \beta), \qquad a = 1, \dots, n-1, \tag{6.2}$$

generate a Stäckel system whose integral manifolds form the spherical-conical web.

Proof. Note that the tensors $(\boldsymbol{H}_a)$, for $a \neq 0$, come from the coefficients of m of the tensors $(\boldsymbol{K}_a)$ defined in (5.14), or, equivalently, that

$$\boldsymbol{H}_a = \frac{1}{m}\big(\boldsymbol{K}_a - \boldsymbol{K}_a(O)\big) \qquad (a = 1, \dots, n-1). \tag{6.3}$$

It follows that these tensors are independent Killing tensors. The radius vector $\boldsymbol{r}$ is a common eigenvector, corresponding to the eigenvalue 0 for $a \neq 0$ and the eigenvalue 1 for $a = 0$. Indeed $\boldsymbol{H}_a \cdot \boldsymbol{r} = 0$ for $a \neq 0$. The radius vector is normal and generates the spherical foliation. Following the same pattern of the proof of Proposition 5.2 it can be seen that the vectors $\boldsymbol{E}_i$ defined in (5.8), where (u^i) are now the roots of equation (6.1) (so that $i = 1, \dots, n-1$), are eigenvectors of $(\boldsymbol{H}_a)$. Moreover, the identity

$$\begin{aligned}\sum_\alpha E_i^\alpha \partial_\alpha \left(\sum_\gamma \frac{x_\gamma^2}{u^j - a_\gamma} \right) &= 2 \sum_\alpha \frac{x_\alpha^2}{(u^i - a_\alpha)(u^j - a_\alpha)} \\ &= \frac{2}{u^j - u^i} \sum_\alpha \left(\frac{x_\alpha^2}{u^i - a_\alpha} - \frac{x_\alpha^2}{u^j - a_\alpha} \right) = 0, \qquad i \neq j,\end{aligned}$$

shows that the vector fields $(\boldsymbol{E}_i)$ are normal and generate the surfaces defined by equation (6.1). This proves that the system $(\boldsymbol{H}_a)$ forms a Stäckel system whose web is a spherical-conical web. ∎

Proposition 6.2. *A potential* V *is compatible with the spherical-conical web if and only if the* $\frac{1}{2}n(n-1)$ *linear differential equations*

$$\begin{aligned}&3\varsigma_1^{\alpha\beta}(x_\beta \partial_\alpha V - x_\alpha \partial_\beta V) + (a_\beta - a_\alpha)\, r^2\, \partial_\alpha \partial_\beta V \\ &\qquad + x_\beta \sum_\gamma \varsigma_1^{\beta\gamma}\, x_\gamma\, \partial_\alpha \partial_\gamma V - x_\alpha \sum_\gamma \varsigma_1^{\alpha\gamma}\, x_\gamma\, \partial_\beta \partial_\gamma V = 0\end{aligned} \tag{6.4}$$

are satisfied.

Proof. We write equation (3.1) in Cartesian coordinates, by choosing $\boldsymbol{K} = \boldsymbol{H}_2$. Indeed this tensor is the first tensor in the sequence (6.3) with simple eigenvalues [2]. ∎

Remark 6.1. The restriction of $\boldsymbol{H}_1$ to a sphere centered at O coincides with the induced metric tensor. Indeed, the Cartesian components of $\boldsymbol{H}_1$ are

$$H_1^{\alpha\alpha} = \sum_{\gamma\neq\alpha} x_\gamma^2 = 1 - x_\alpha^2, \qquad H_1^{\alpha\beta} = -\,x_\alpha x_\beta,$$

and a vector $\boldsymbol{v} = (v^\alpha)$ is tangent to the sphere if and only if $\boldsymbol{v} \cdot \boldsymbol{r} = v^\alpha x_\alpha = 0$. Thus

$$\begin{aligned}(\boldsymbol{H}_1 \cdot \boldsymbol{v})^\alpha &= (1 - x_\alpha^2) v^\alpha - \sum_{\gamma\neq\alpha} x_\alpha x_\gamma v^\gamma \\ &= (1 - x_\alpha^2) v^\alpha + x_\alpha^2 v^\alpha = v^\alpha\end{aligned}$$

that is, $\boldsymbol{H}_1 \cdot \boldsymbol{v} = \boldsymbol{v}$. Thus the spectrum of the Killing tensor $\boldsymbol{H}_1$ is $(1, \dots, 1, 0)$.

Remark 6.2. If we consider the potential V as a function of (x_α^2), then equations (6.4) are equivalent to equations

$$\begin{aligned} 2\varsigma_1^{\alpha\beta}\big(\partial_{\bar\alpha}V - \partial_{\bar\beta}V\big) + r^2(a_\beta - a_\alpha)\partial_{\bar\alpha}\partial_{\bar\beta}V \\ + \sum_\gamma \Big(\varsigma_1^{\beta\gamma}x_\gamma^2\partial_{\bar\alpha}\partial_{\bar\gamma}V - \varsigma_1^{\alpha\gamma}x_\gamma^2\partial_{\bar\beta}\partial_{\bar\gamma}V\Big) = 0. \end{aligned} \tag{6.5}$$

Remark 6.3. A straightforward calculation shows that the most general potential of the form $V = \sum_\alpha V_\alpha(x_\alpha)$ which is compatible with the spherical-conical Stäckel web is again the potential (5.12).

7. The elliptic Stäckel web on the sphere $\mathbb{S}_{n-1}$

The spherical-conical Stäckel web in $\mathbb{E}_n$ induces Stäckel webs on each one of the spheres of the web (Remark 3.2). We call such a web the **spherical-elliptic Stäckel web**. Let us consider one of these spheres, namely the sphere $\mathbb{S}_{n-1}$ centered at the origin O with radius $r = 1$. The restrictions of the Killing tensors $(\boldsymbol{H}_1, \ldots, \boldsymbol{H}_{n-1})$ to $\mathbb{S}_{n-1}$ (the restriction of $\boldsymbol{H}_1$ is the metric tensor, Remark 6.1) form a basis of the Stäckel system generating the spherical-elliptic Stäckel web.

The potentials compatible with the spherical-elliptic web on $\mathbb{S}_{n-1}$ can be divided into three classes. (I) Potentials which are the restriction to the sphere of potentials in the whole space $\mathbb{E}_n$ compatible with the spherical-conical web. In Cartesian coordinates they are characterized by equations (6.4) or (6.5). (II) Potentials which are not compatible with the spherical-conical web in the whole space but are compatible with the spherical-elliptic web when restricted to every sphere. (III) Potentials defined on the whole space, which are compatible with the spherical-elliptic web on a sphere of fixed radius only.

The potentials of class (I) and (II) are characterized by the following proposition, which is a corollary of Proposition 3.4.

Proposition 7.1. *The restriction of a potential V to every sphere centered at a point $O \in \mathbb{E}_n$ is compatible with the spherical-elliptic webs if and only if*

$$d(\boldsymbol{H}_2 \cdot dV) \wedge \boldsymbol{r} = 0, \tag{7.1}$$

i.e. if and only if

$$\xi_{\alpha\beta}\, x_\gamma + \xi_{\beta\gamma}\, x_\alpha + \xi_{\gamma\alpha}\, x_\beta = 0 \qquad (\alpha, \beta, \gamma \neq), \tag{7.2}$$

where $\xi_{\alpha\beta} = -\,\xi_{\beta\alpha}$ is the right hand side of equation (6.4).

A method of constructing a large class of potentials compatible with the spherical-elliptic web on the sphere can be found in [25, 26]. We mention here three examples. The first two belong to class (I), the third to class (III).

Example 7.1. The potential (5.12) is a solution of (6.5), thus it is compatible with the spherical-conical web (Remark 6.3). The potential $c\sum_\alpha x_\alpha^2$ is trivial on each sphere. The potential $\sum_\alpha b_\alpha x_\alpha^{-2}$ is due to Rosochatius [17].

Example 7.2. The potential

$$V = \left(\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{a_\alpha}\right)^{-1} \tag{7.3}$$

is a solution of equations (6.5) (it is found in [5]).

Example 7.3. The Neumann potential [18]

$$V = \sum_{\alpha=1}^{n} a_\alpha\, x_\alpha^2 \tag{7.4}$$

is a solution of equations (7.2). Indeed, the left hand side of equation (6.5) becomes, up to a factor 2, $\varsigma_1^{\alpha\beta}(a_\alpha - a_\beta)$, that is $\varsigma_2^\beta - \varsigma_2^\alpha$. This means that $\xi_{\alpha\beta} x_\gamma = x_\alpha x_\beta x_\gamma (\varsigma_2^\beta - \varsigma_2^\alpha)$, and equations (7.2) are identically satisfied.

By using equations (7.2) it can be shown that

Proposition 7.1. *The most general potential of the kind* (5.11) *which is compatible with the elliptic web on* $\mathbb{S}_{n-1}$ *determined by the distinct numbers* (a_α) *is the sum of the Rosochatius and Neumann potentials,* $V = \sum_\alpha b_\alpha\, x_\alpha^{-2} + a_\alpha\, x_\alpha^2$.

8. The parabolic Stäckel web in the Euclidean space $\mathbb{E}_n$

Let us consider the 2-tensor $\boldsymbol{K}$ with Cartesian components

(8.1)
$$\begin{cases} K^{11} = \varsigma^1, \\ K^{1\alpha} = -\, w\, x_\alpha, \\ K^{\alpha\alpha} = \varsigma^\alpha + 2\, w\, x_1, \\ K^{\alpha\beta} = 0 \quad (\alpha \neq \beta, \quad \alpha, \beta = 2, \ldots, n), \end{cases}$$

where $\varsigma^\alpha, w \in \mathbb{R} (w \neq 0)$. Following the same procedure of §5, it can be shown that this is a Killing tensor with normal eigenvectors. The tensor $\boldsymbol{L}$ defined in (5.2) plays in the present case the same role as in the case of §5. If the constants (a_α) defined in (5.4) are all distinct, then the eigenvalues (u^i) give a parametrization of a Stäckel web of $\mathbb{E}_n$, called the **parabolic web**. By assuming for convenience (without loss of generality) that $a_1 = 0$, the eigenvalues (u^i) are the roots of equation

(8.2)
$$\sum_{\beta \neq 1} \frac{x_\beta^2}{u - a_\beta} = \frac{1}{w^2}(u - 2\, w\, x_1)$$

and the vectors $(\boldsymbol{E}_i)$ of components

(8.3)
$$E_i^1 = \frac{1}{w}, \qquad E_i^\alpha = \frac{x_\alpha}{u_i - a_\alpha},$$

are eigenvectors. The roots (u^i) (for $w = 1$) are called **parabolic coordinates**.

The compatible potentials are characterized by equation (3.1) applied to the tensor (8.1). By a straightforward calculation it can be shown that

Proposition 8.1. *A potential* V *is compatible with the parabolic Stäckel web if and only if the* $\frac{1}{2}n(n-1)$ *linear differential equations*

(8.4)
$$\begin{cases} (a_\alpha - 2x_1)\partial_1\partial_\alpha V + x_\alpha \partial_1^2 V - \sum_{\gamma=1}^n x_\gamma \partial_\alpha \partial_\gamma V - 3\partial_\alpha V = 0, \\ \\ x_\alpha \partial_1 \partial_\beta V - x_\beta \partial_1 \partial_\alpha V + (a_\alpha - a_\beta)\partial_\alpha \partial_\beta V = 0. \end{cases} \qquad (\alpha, \beta \neq 1)$$

are satisfied.

Remark 8.1. The most general separable potential of the kind $V = V_1(x_1) + \ldots + V_n(x_n)$ is

(8.5)
$$V = c\, x^2 + \sum_{\alpha=1}^{n} \left(\frac{1}{4}\, c\, x_\alpha^2 + \frac{b_\alpha}{x_\alpha^2} \right)$$

where c and (b_α) are arbitrary constants.

Following a method similar to that of §5 it can be proved that

Proposition 8.2. *The 2-tensors* $(\boldsymbol{K}_a)$ $(a = 0, 1, \dots, n-1)$ *with Cartesian components*

$$\begin{cases} K^{11} = \varsigma_a^1, \\ K^{1\alpha} = -\, w\, \varsigma_{a-1}^{\alpha}\, x_\alpha \\ K^{\alpha\alpha} = \varsigma_a^{\alpha} + 2\, w\, \varsigma_{a-1}^{\alpha}\, x_1 - w^2 \sum_{\gamma\neq 1} \varsigma_{a-2}^{\alpha\gamma}\, x_\gamma^2, \\ K^{\alpha\beta} = w^2\, \varsigma_{a-2}^{\alpha\beta}\, x_\alpha\, x_\beta \qquad (\alpha \neq \beta, \quad \alpha, \beta = 2, \dots, n), \end{cases} \tag{8.6}$$

form a basis of the parabolic Stäckel system.

A constructive proof of this Proposition is given in [2].

Remark 8.2. The functions (V_a) such that $dV_a = \boldsymbol{K}_a \cdot dV$ (Proposition 3.2), where V is of the kind (8.5), are

$$\begin{aligned} V_a = c\, \varsigma_a\, x^2 + x \sum_{\alpha=2}^{n} \varsigma_a^{\alpha} \left(\frac{b_\alpha}{x_\alpha^2} + \frac{c}{4}\, x_\alpha^2 \right) + \\ \sum_{\alpha=2}^{n} \varsigma_{a-1}^{\alpha} \left(2 \frac{b_\alpha}{x_\alpha^2} - \frac{c}{2}\, x_\alpha^2 \right) - \sum_{\alpha,\beta=2}^{n} \varsigma_{a-2}^{\alpha\beta} b_\alpha \frac{x_\beta^2}{x_\alpha^2}. \end{aligned} \tag{8.7}$$

Remark 8.3. Due to Proposition 3.2, all the potentials satisfying equations (8.4) give rise to orthogonal separable dynamical systems when restricted to the quadrics (paraboloids) of the parabolic Stäckel web and to their intersections.

Remark 8.4. The Stäckel webs considered in this lecture are in a sense the *building blocks* for constructing all the Stäckel webs in Euclidean spaces and on spheres. Indeed, by also considering Stäckel webs with symmetries it is possible to perform an exhaustive classification similar to that proposed in [12].

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