Inertia tensors and Stäckel systems in Euclidean spaces

S. Benenti

Abstract. *The Cartesian components of the Killing tensors belonging to the fundamental Stäckel systems in the Euclidean affine space* $\mathbf{E}_n$ *(elliptic, parabolic and spherical-elliptic) are computed within the Stäckel-Eisenhart theory of the orthogonal separation on a Riemannian manifold, by using the fundamental properties of the elementary symmetric functions and of the inertia tensors of a massive body.*

1. Introduction

A system of orthogonal separable coordinates on a Riemannian manifold $(M_n, \boldsymbol{g})$ is geometrically characterized by an n-dimensional space $\mathcal{S}$ of Killing 2-tensors over M, whose elements are in involution and have n eigenvectors in common (the intrinsic characterization of the orthogonal separation, after the crucial results of Eisenhart [EI1] [EI2] [EI3], have been treated in [WO] [KM1] [SH] [BKW] [BKM] [BE1]). Let us call such a space $\mathcal{S}$ a **Stäckel system**.

The aim of the present lecture is to illustrate a method for calculating the Stäckel systems in the affine Euclidean spaces $\mathbf{E}_n$.

Separable coordinates in the Euclidean spaces $\mathbf{E}_n$ have been fully discussed and classified by Kalnins and Miller [KM2] [KA] after proving that in these spaces every separable coordinate system has an orthogonal equivalent (this analysis is extended to other spaces of constant curvature, the spheres $\mathbf{S}_n$ [KM2] and the hyperboloids $\mathbf{H}_n$ [KM3], to the conformally Euclidean spaces and other spaces [KMR] [BKM] [BKW]).

In the Euclidean spaces $\mathbf{E}_n$ the **elliptic coordinates** play a fundamental role. In elliptic coordinates a basis of the corresponding Stäckel system can be computed by a method suggested by Jacobi himself (see [MO1] [MO2]), or, in a more general way, by the theorem of Stäckel [ST1] [ST2] (see [KM2]). However, for a better understanding of the separation in $\mathbf{E}_n$, it is of some interest to know the Cartesian components of the elements of the elliptic Stäckel systems. These components can be computed by using the transformation rule of the components of a tensor. However, looking at the transformations from elliptic coordinates to Cartesian coordinates, this seems to be an unpracticable way. So that some special procedure is needed.

A solution to this problem can be found in [MW]. We present here a different approach, which is based on the general theory of orthogonal separation and it is suggested by a known fact concerning the inertia tensor of a massive body: the inertia tensor is a symmetric 2-tensor field whose eigendirections span orthogonal distributions which are completely integrable and whose integral manifolds are confocal quadrics. Indeed, as it is classically known [EI1] [BL] [WE], the coordinate surfaces corresponding to a separable system in $\mathbf{E}_n$ are confocal quadrics (possibly with degeneracy). Thus there is a rather surprising link between these two apparently distant subjects, the separablity of the Hamilton-Jacobi equation and the geometry of massive bodies. This is simply due to the fact that *the inertia tensor is a Killing tensor.*

We shall use this fact (Section 3), together with the main properties of the elementary symmetric functions (which are listed in Section 2), as a tool for dealing with the separation in the Euclidean space $\mathbf{E}_n$. In the present lecture we will consider only three kinds of Stäckel systems, **elliptic** (Section 4) **parabolic**

(Section 5) and **spherical-elliptic** (Section 6), which are the *building blocks* for all Stäckel systems and all separable dynamical systems in $\mathbf{E}_n$. Elliptic and parabolic coordinates are naturally defined by the inertia tensor of a body, more precisely by the planar moment of inertia, denoted by $\boldsymbol{L}$, which is a conformal Killing tensor and plays the role of generator of the whole Stäckel system according to a rule concerning a particular kind of orthogonal separable coordinates (Section 2). We allow the masses of the single points of the body to be either positive or negative numbers. When the total mass m is not zero, the center of mass is well defined, and when the principal moments of inertia at the center of mass are distinct numbers, we get the elliptic coordinates. When $m = 0$, the center of mass is not defined and it is substituted by a point O and a vector $\boldsymbol{w}$, such that $\boldsymbol{L}_O \cdot \boldsymbol{w} = 0$. If the tensor $\boldsymbol{L}$ has all distinct eigenvalues at O, then we get the parabolic coordinates (the parabolic axis is determined by the pair $(O, \boldsymbol{w})$). Finally, the Stäckel system corresponding to spherical-elliptic coordinates (also called conical coordinates) can be obtained as the part proportional to the total mass m of the elliptic Stäckel system, or as the limit when the total mass increases to infinity.

2. Elementary symmetric functions and orthogonal separation

Before dealing with the orthogonal separation in the Euclidean spaces we ought to consider a very special kind of separable coordinates on a generic Riemannian manifold $(M_n, \boldsymbol{g})$, whose properties are strictly related to the properties of the elementary symmetric functions.

Let σ_a be the elementary symmetric function of order $a = 1, \ldots, n$ of the n variables (u^i), that is the sum of all possible products of a variables with distinct indices. Let $\sigma_a^{i\ldots j}$ be the symmetric function of order a over the same variables, with the exclusion of those of index $(i, \ldots, j)$, that is

$$\sigma_a^{i\ldots j} = \sigma_a\big|_{u^i=\cdots=u^j=0}.$$

Let us set

$$\sigma_0 = \sigma_0^i = \sigma_0^{ij} = 1, \quad \sigma_{-1} = \sigma_{-1}^i = \sigma_{-1}^{ij} = 0, \qquad (2.1)$$
$$\sigma_n^i = \sigma_n^{ij} = 0, \quad \sigma_{n-1}^{ij} = 0, \quad \sigma_a^{ii} = 0.$$

For $a = 0, 1, \ldots, n$ these functions satisfy the algebraic identity

$$\boxed{\sigma_a = \sigma_a^i + u^i\,\sigma_{a-1}^i} \qquad (2.2)$$

and the differential identity

$$\boxed{\partial_i \sigma_a = \sigma_{a-1}^i} \qquad (2.3)$$

where

$$\partial_i = \frac{\partial}{\partial u^i},$$

from which we derive the identities

$$\boxed{\begin{array}{l} \partial_i \sigma_a^j = \sigma_{a-1}^{ij} \\ \sigma_a^j = \sigma_a^{ij} + u^i\,\sigma_{a-1}^{ij} \\ \sigma_a^i - \sigma_a^j = (u^j - u^i)\,\sigma_{a-1}^{ij} \\ \displaystyle\sum_i u^i \sigma_{a-1}^i = a\,\sigma_a \end{array}} \qquad (2.4)$$

The last one is a consequence of (2.3) and the Euler theorem for homogeneous functions,

$$\sum_i u^i\,\partial_i \sigma_a = a\,\sigma_a.$$

By combining $(2.4)_1$ with $(2.4)_3$ we get

$$\boxed{(u^j - u^i)\,\partial_i \sigma_a^j = \sigma_a^i - \sigma_a^j} \tag{2.5}$$

The elementary symmetric functions are defined by the polynomial identity

$$\sum_{k=0}^{n} (-1)^k \, \lambda^{n-k} \, \sigma_k = U(\lambda), \tag{2.6}$$

where

$$U(\lambda) = \prod_{i=1}^{n} (\lambda - u^i), \tag{2.7}$$

from which it follows that

$$\sum_{k=0}^{n} (-1)^k \, (n-k) \, \lambda^{n-k-1} \, \sigma_k = U'(\lambda), \tag{2.8}$$

and

$$\sum_{k=0}^{n} (-1)^k \, (u^i)^{n-k} \, \sigma_k = 0. \tag{2.9}$$

By applying to this last identity the partial derivative ∂_i, due to (2.3) and (2.8), we get

$$\sum_{k=0}^{n} (-1)^k \Big((n-k)\,(u^i)^{n-k-1}\,\sigma_k + (u^i)^{n-k}\,\sigma_{k-1}^i \Big) = 0,$$

i.e.

$$\sum_{k=1}^{n} (-1)^k \, (u^i)^{n-k} \, \sigma_{k-1}^i = -\,U'(u^i). \tag{2.10}$$

By applying ∂_j to (2.9), with $j \neq i$, we get

$$\sum_{k=1}^{n} (-1)^k (u^i)^{n-k} \, \sigma_{k-1}^j = 0. \tag{2.11}$$

From (2.10) and (2.11) it follows that if we set

$$\boxed{\bar{\sigma}_i^a = (-1)^a \, \frac{(u^i)^{n-a-1}}{U'(u^i)}} \qquad a = 0, \dots, n-1, \tag{2.12}$$

then the matrix $(\bar{\sigma}_i^a)$ is the inverse matrix of (σ_a^i), with $a = 0, \dots, n-1$:

$$\sum_{a=0}^{n-1} \sigma_a^j \, \bar{\sigma}_i^a = \delta_i^j.$$

The determinant of the matrix $(\bar{\sigma}_i^a)$, up to the factors $(-1)^a (U'(u^i))^{-1}$ is the Vandermonde determinant, so that

$$\det(\bar{\sigma}_i^a) = \frac{\prod_{a=0}^{n-1}(-1)^a}{\prod_{i=1}^{n} U'(u^i)} \prod_{j>i}(u^i - u^j).$$

where

$$\prod_{a=0}^{n-1}(-1)^a = (-1)^{\frac{1}{2}n(n-1)}, \qquad \prod_{i=1}^{n} U'(u^i) = (-1)^{\frac{1}{2}n(n-1)}(u^i - u^j)^2.$$

Thus

$$\det(\bar{\sigma}_i^a) = \frac{1}{\prod_{j>i}(u^i - u^j)}, \tag{2.13}$$

and

$$\det(\sigma_a^i) = \prod_{j>i}(u^i - u^j). \tag{2.14}$$

Finally, from $(2.4)_3$ we get the last identity we will use in the following discussion,

$$\boxed{\sum_{a=0}^{n-1} \bar{\sigma}_h^a \, \sigma_{a-1}^{ij} \, (u^j - u^i) = \delta_h^i - \delta_h^j} \tag{2.15}$$

Proposition 2.1. *Let (u^i) be orthogonal coordinates on a Riemannian manifold $(M_n, \boldsymbol{g})$. Let $(\boldsymbol{E}_i)$ be the corresponding frame of vector fields. If*

$$\boxed{u^i \neq u^j, \quad \partial_i \ln g^{jj} = \frac{1}{u^j - u^i} \qquad (i \neq j)} \tag{2.16}$$

or equivalently, if the 2-tensor field $\boldsymbol{L}$ with eigenvalues (u^i) and eigenvectors $(\boldsymbol{E}_i)$,

$$\boxed{\boldsymbol{L} = \sum_{i=1}^{n} u^i \, g^{ii} \, \boldsymbol{E}_i \otimes \boldsymbol{E}_i} \tag{2.17}$$

is a conformal Killing tensor, then the coordinates (u^i) are separable and the corresponding Stäckel system $\mathcal{S}$ is generated by the independent Killing tensors $(\boldsymbol{K}_a)$ $(a = 0, 1, \ldots, n-1)$ defined by one of the following equivalent formulae

$$\boxed{\boldsymbol{K}_a = \sum_{i=1}^{n} \sigma_a^i \, g^{ii} \, \boldsymbol{E}_i \otimes \boldsymbol{E}_i} \tag{2.18}$$

$$\boxed{\boldsymbol{K}_a = \sigma_a \boldsymbol{g} - \boldsymbol{K}_{a-1} \cdot \boldsymbol{L}, \quad \boldsymbol{K}_{-1} = 0} \tag{2.19}$$

$$\boxed{\boldsymbol{K}_a = \sum_{k=0}^{a} (-1)^k \, \sigma_{a-k} \, \boldsymbol{L}^k} \tag{2.20}$$

$$\boxed{\boldsymbol{K}_0 = \boldsymbol{g}, \qquad \boldsymbol{K}_a = \frac{1}{a}\, tr(\boldsymbol{K}_{a-1} \cdot \boldsymbol{L})\, \boldsymbol{g} - \boldsymbol{K}_{a-1} \cdot \boldsymbol{L} \qquad (a = 1, \dots, n-1)} \tag{2.21}$$

To prove this proposition we need two lemmas.

Lemma 2.1. *Let $\boldsymbol{L}$ be a symmetric tensor which is diagonalized with respect to orthogonal coordinates (u^i). Let (ϱ_i) be the corresponding eigenvalues,*

$$\boldsymbol{L} = \sum_{i=1}^{n} \varrho_i\, g^{ii}\, \boldsymbol{E}_i \otimes \boldsymbol{E}_i. \tag{2.22}$$

Then $\boldsymbol{L}$ is a conformal Killing tensor if and only if there exists a vector field $\boldsymbol{C} = C^i \boldsymbol{E}_i$ such that the following equations are satisfied

$$\partial_i \varrho_j = (\varrho_i - \varrho_j)\, \partial_i \ln g^{jj} + C_i \qquad (i \neq j), \qquad \partial_i \varrho_i = C_i. \tag{2.23}$$

$\boldsymbol{L}$ is a Killing tensor when $\boldsymbol{C} = 0$.

Proof. A tensor field $\boldsymbol{L}$ is a conformal Killing tensor if and only if

$$\{E_{\boldsymbol{L}}, E_{\boldsymbol{g}}\} = E_{\boldsymbol{C}}\, E_{\boldsymbol{g}}.$$

where

$$E_{\boldsymbol{L}} = \tfrac{1}{2}\, L^{ij}\, p_i p_j, \qquad E_{\boldsymbol{g}} = \tfrac{1}{2}\, g^{ij}\, p_i p_j, \qquad E_{\boldsymbol{C}} = C^i\, p_i.$$

Under the assumption (2.22) we get

$$\sum_i \sum_j g^{ii} \left[g^{jj}\, \partial_i \varrho_j - (\varrho_i - \varrho_h)\, \partial_i g^{jj} - g^{jj}\, C_i \right]\, p_i\, p_j^2 = 0,$$

and this condition is equivalent to equations (2.23). ∎

Lemma 2.2. *An orthogonal coordinate system (u^i) is separable if and only if the linear differential system*

$$\partial_i \varrho_j = (\varrho_i - \varrho_j) \partial_i \ln g^{jj} \quad (i \neq j), \qquad \partial_i \varrho_i = 0. \tag{2.24}$$

is completely integrable. If (ϱ_{ai}) $(a = 0, \dots, n-1)$ is a complete solution, then the tensors

$$\boldsymbol{K}_a = \sum_{i=1}^{n} \varrho_{ai}\, g^{ii}\, \boldsymbol{E}_i \otimes \boldsymbol{E}_i, \tag{2.25}$$

form a basis of the Stäckel system $\mathcal{S}$ corresponding to the separable coordinates (u^i).

Proof. The separability conditions for an orthogonal coordinate system, that is equations

$$\partial_i \partial_j g^{hh} - \partial_i \ln g^{jj}\, \partial_j g^{hh} - \partial_j \ln g^{ii}\, \partial_i g^{hh} = 0 \qquad (i \neq j)$$

coincides with the integrability conditions of (2.24) (for a discussion on this topic see [BE1]). ∎

Proof of Proposition 2.1. Due to Lemma 1 (equations (2.23)) the tensor $\boldsymbol{L}$ defined by (2.17) is a conformal Killing tensor if and only if conditions (2.16) are satisfied (when $\varrho_i = u^i$ equations (2.23) become $0 = (u^i - u^j) \partial_i \ln g^{jj} + C_i$ and $C_i = 1$). Due to equations (2.16) the differential system (2.24) becomes

$$\partial_i \varrho_j = \frac{\varrho_i - \varrho_j}{u^j - u^i} \qquad (i \neq j), \qquad \partial_i \varrho_i = 0, \tag{2.26}$$

Formula (2.5) shows that this system is identically satisfied by the functions

$$\varrho_{ai} = \sigma_a^i \qquad (a = 0, \dots, n-1) \tag{2.27}$$

on the domain $D = \{u^i \neq u^j, i \neq j\}$. Formula (2.14) shows that in this domain $\det(\varrho_{ai}) \neq 0$. So we have proved that the linear differential system (2.26) is completely integrable and we know that the functions (2.27) form a basis of solutions. Then, due to Lemma 2.2, the coordinates (u^i) are separable and the tensors (2.18) provide a basis for the corresponding Stäckel system. Let us consider these tensors as linear operators (the metric tensor $\boldsymbol{g} = \boldsymbol{K}_0$ is the identity) and use the formula (2.2) which shows that (σ_a^i) are the eigenvalues of the tensor defined by (2.19) since, due to the definition (2.17), the coordinates (u^i) are the eigenvalues of the tensor $\boldsymbol{L}$. Formula (2.20) follows by iteration from (2.19). Finally, identity $(2.4)_4$ implies

$$\sigma_a = \frac{1}{a}\,\mathrm{tr}(\boldsymbol{K}_{a-1} \cdot \boldsymbol{L}),$$

and (2.21) follows from (2.19). ∎

Remark 2.1. A first consequence of Proposition 2.1 is that the functions (σ_a^i) are the principal invariants of the tensor $\boldsymbol{L}$ interpreted as a linear operator. So that by using one of the formulae (2.19)-(2.21) we can build the basis $(\boldsymbol{K}_a)$ of the Stäckel system by knowing the expression of $\boldsymbol{L}$ with respect to any coordinate system. Hence, the tensor $\boldsymbol{L}$ plays the role of **generator** of the Stäckel system. Due to Lemma 2.1, we note that a tensor $\boldsymbol{L}$ is a generator of a Stäckel system by means of formulae (2.19)-(2.21) if and only if: (i) $\boldsymbol{L}$ is a conformal Killing tensor; (ii) the eigenvalues (u^i) of $\boldsymbol{L}$ are real independent functions (so that they can be interpreted as coordinates) with distinct values; (iii) all the eigendirections of $\boldsymbol{L}$ are normal (the orthogonal distributions are completely integrable) and the corresponding eigenvalues (u^i) are constant on the integral manifolds. The requirement "real" is automatically satisfied in a proper Riemannian manifold (that is for positive definite metrics). Tensors having properties (ii) and (iii) are characterized by the vanishing of the Nijenhuis tensor [NI].

Remark 2.2. It can be seen that the assumption (2.16) for the metric tensor implies $R_{ij} = 0$ for $i \neq j$. This means that the coordinate curves are tangent to the Ricci principal directions. This is a necessary and sufficient condition for an additive separable orthogonal system (for the Hamilton-Jacobi equation) to be multiplicative separable for the Helmholtz and related equations. It is called the **Robertson condition** [RO] [KM1] [KA].

3. Inertia tensors and the orthogonal separation

Let $\mathcal{M} = \{(P_\iota, m_\iota) \in (\mathbf{E}_n, \mathbf{R}); \iota = 1, \dots, N\}$ be a finite system of mass points in the Euclidean n-dimensional space. The masses m_ι can be either positive or negative numbers. We denote by m the **total mass**, the sum of all masses. If $m \neq 0$ we can define the **center of mass** G through one of the equivalent equations

$$\sum_\iota m_\iota GP_\iota = 0 \quad \Longleftrightarrow \quad PG = \frac{1}{m}\sum_\iota m_\iota\, PP_\iota \quad (P \in \mathbf{E}_n). \tag{3.1}$$

If $m = 0$ the vector $\boldsymbol{w}$ defined by equation

$$\sum_\iota m_\iota\, PP_\iota = \boldsymbol{w} \tag{3.2}$$

does not depend on the point P. With a system of masses $\mathcal{M}$ we can associate three fundamental fields. A scalar field $M \colon \mathbf{E}_n \to \mathbf{R}$, called the **polar moment** of inertia, defined by

$$M_P = \sum_\iota m_\iota (PP_\iota)^2, \tag{3.3}$$

a symmetric tensor field $\boldsymbol{L}$ on $\mathbf{E}_n$, called the **planar inertia tensor**, defined by

$$\boldsymbol{L}_P(\boldsymbol{v}) = \sum_{\iota} m_\iota \, (PP_\iota \cdot \boldsymbol{v})^2, \tag{3.4}$$

and a symmetric tensor field $\boldsymbol{I}$ on $\mathbf{E}_n$, called the **inertia tensor**, defined by

$$\boldsymbol{I}_P(\boldsymbol{v}) = \sum_{\iota} m_\iota \big((PP_\iota)^2 \boldsymbol{v}^2 - (PP_\iota \cdot \boldsymbol{v})^2\big). \tag{3.5}$$

Our notation is such that $\boldsymbol{L}_P(\boldsymbol{v})$ is the value on the vector $\boldsymbol{v}$ at the point $P \in \mathbf{E}_n$ of the quadratic form associated with the symmetric tensor $\boldsymbol{L}$. It can be seen that

$$\boldsymbol{I} = M\,\boldsymbol{g} - \boldsymbol{L}, \qquad M = \mathrm{tr}(\boldsymbol{L}) = I_1(\boldsymbol{L}). \tag{3.6}$$

With each straight line r in the space $\mathbf{E}_n$ we associate a number I^r, the **moment of inertia** of r, which is the sum of the products of the masses times the square of the distance from the line r. It follows that

$$I^r = \boldsymbol{I}_P(\boldsymbol{v}), \tag{3.7}$$

where $\boldsymbol{v}$ is a unitary vector parallel to r and P is an arbitrary point belonging to r. The following **transposition formulae** hold for the tensor $\boldsymbol{L}$,

$$\begin{cases} \boldsymbol{L}_P(\boldsymbol{v}) = \boldsymbol{L}_G(\boldsymbol{v}) + m\,(GP \cdot \boldsymbol{v})^2, & m \neq 0, \\ \boldsymbol{L}_P(\boldsymbol{v}) = \boldsymbol{L}_Q(\boldsymbol{v}) + 2\; PQ \cdot \boldsymbol{v}\; \boldsymbol{w} \cdot \boldsymbol{v}, & m = 0. \end{cases} \tag{3.8}$$

Proposition 3.1. *The planar inertia tensor* $\boldsymbol{L}$ *is a conformal Killing tensor.*

Proof. By definition, a symmetric 2-tensor field $\boldsymbol{L}$ is conformal Killing tensor if along all geodesics

$$\frac{dF_{\boldsymbol{K}}}{dt} = F_{\boldsymbol{C}}\, \boldsymbol{v}^2, \tag{3.9}$$

where $\boldsymbol{C}$ is a vector field and

$$F_{\boldsymbol{L}} = \tfrac{1}{2}\, L_{ij}\, v^i\, v^j, \qquad F_{\boldsymbol{C}} = C_i\, v^i,$$

being (v^i) the components of the vector tangent to the geodesics. Along any curve $P(t)$ in $\mathbf{E}_n$ we have

$$\frac{d}{dt}\boldsymbol{L}(\boldsymbol{v}) = 2 \sum_{\iota} m_\iota\, PP_\iota \cdot \boldsymbol{v} \left(PP_\iota \cdot \frac{d\boldsymbol{v}}{dt} - \boldsymbol{v} \cdot \boldsymbol{v} \right).$$

If the curve is a geodesic curve (that is, a straight line), then $\frac{d\boldsymbol{v}}{dt} = 0$, so that

$$\frac{d\boldsymbol{L}}{dt} = -\,\boldsymbol{C} \cdot \boldsymbol{v}\; \boldsymbol{v}^2,$$

where $\boldsymbol{C}$ is the vector field defined by

$$\boldsymbol{C}_P = -\,2 \sum_{\iota} m_\iota\, PP_\iota = \begin{cases} \boldsymbol{C}_P = 2\, m\, GP & (m \neq 0) \\ \boldsymbol{C}_P = -\,2\, \boldsymbol{w} = \mathrm{const.} & (m = 0) \end{cases} \tag{3.10}$$

This proves the statement. ∎

The vector field $\boldsymbol{C}$ is a gradient of a function f (we say that $\boldsymbol{L}$ is an **exact conformal Killing tensor**):

$$\begin{cases} f(P) = \tfrac{1}{2}\, m\, \boldsymbol{r}^2, & \boldsymbol{r} = GP, \\ f(P) = -2\, \boldsymbol{w} \cdot \boldsymbol{r}, & \boldsymbol{r} = OP \quad (O = \text{arbitrary point}). \end{cases}$$

Proposition 3.2. *The inertia tensor $\boldsymbol{I}$ is a Killing tensor.*

Proof. We have two simple proofs. (i) A Killing tensor, on a Riemannian manifold, is a symmetric 2-tensor $\boldsymbol{K}$ such that the quantity

$$F_{\boldsymbol{K}}(P, \boldsymbol{v}) = \tfrac{1}{2} K_P(\boldsymbol{v})$$

is constant along every geodesic curve $P(t)$, where $\boldsymbol{v}$ is the unitary vector tangent to the geodesic. In the Euclidean space the geodesics are the straight lines and moreover, due to the definition (3.7) of the moment of inertia, the quantity $F_{\boldsymbol{I}}(P)$ does not depend on the choice of the point P over a line. (ii) We have

$$\frac{d\boldsymbol{I}}{dt} = -2 \sum_{\iota} m_\iota \, PP_\iota \cdot \boldsymbol{v} \, \boldsymbol{v}^2 - \frac{d\boldsymbol{L}}{dt} = 0. \qquad \blacksquare$$

These propositions and the fact that equations (3.6) fit with the step $a = 1$ in the iterative formula (2.18), with $\boldsymbol{K}_1 = \boldsymbol{I}$, suggests that the planar inertia tensor $\boldsymbol{L}$ is a good candidate for playing the role of generator of a Stäckel system of the kind considered in the preceding section.

4. Elliptic Stäckel systems

Let us consider the conformal Killing tensor $\boldsymbol{L}$ defined by (3.8) in the case $m \neq 0$,

$$\boldsymbol{L} = \boldsymbol{L}_G + m \, \boldsymbol{r} \otimes \boldsymbol{r} \qquad (\boldsymbol{r} = GP). \tag{4.1}$$

Let (x_α) be Cartesian coordinates with origin $O = G$ defined by the eigendirections of $\boldsymbol{L}_G$. Let (a_α) be the eigenvalues of $\boldsymbol{L}_G$ and (u^i) the eigenvalues of $\boldsymbol{L}$ at a generic point. Equation (4.1) is equivalent to

$$L^{\alpha\beta} = a_\alpha \, \delta_{\alpha\beta} + m \, x_\alpha \, x_\beta. \tag{4.2}$$

Let $(\boldsymbol{X}_\alpha)$ be the unitary constant vector fields corresponding to the Cartesian coordinates (x_α).

Proposition 4.1. *If $a_1 < a_2 < \ldots < a_n$, then the eigenvalues (u^i) of $\boldsymbol{L}$ coincide with the elliptic coordinates and the tensor $\boldsymbol{L}$ generates a basis of the elliptic Stäckel system according to Proposition 2.1.*

Proof. Let us consider the vector fields

$$\boldsymbol{E}_i = \tfrac{1}{2} \sum_{\alpha=1}^{n} \frac{x_\alpha}{u^i - a_\alpha} \, \boldsymbol{X}_\alpha, \tag{4.3}$$

where (u^i) are the roots of equation

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\lambda - a_\alpha} = \frac{1}{m}. \tag{4.4}$$

It follows that

$$\begin{aligned}
(\boldsymbol{L} \cdot \boldsymbol{E}_i)^\alpha &= \tfrac{1}{2} \, x_\alpha \left(\frac{a_\alpha}{u^i - a_\alpha} + m \sum_{\beta=1}^{n} \frac{x_\beta^2}{u^i - a_\beta} \right) \\
&= \tfrac{1}{2} \, x_\alpha \left(\frac{a_\alpha}{u^i - a_\alpha} + 1 \right) \\
&= \tfrac{1}{2} \, \frac{x_\alpha}{u^i - a_\alpha} \, u^i,
\end{aligned}$$

that is $\boldsymbol{L} \cdot \boldsymbol{E}_i = u^i \, \boldsymbol{E}_i$. This shows that the elliptic coordinates (u^i), which are defined by equation (4.4) (usually with $m = 1$) [MO1] and are separated by the constants (a_α),

$$a_1 < u^1 < a_2 < u^2 < \ldots < u^{n-1} < a_n < u^n,$$

are eigenvalues of the conformal Killing tensor $\boldsymbol{L}$. Furthermore, the eigenvectors $(\boldsymbol{E}_i)$ form the natural frame corresponding to these coordinates. Indeed, equation (4.4) is equivalent to equation

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\lambda - a_\alpha} - \frac{1}{m} = -\frac{U(\lambda)}{A(\lambda)} \tag{4.5}$$

where

$$U(\lambda) = \prod_{i=1}^{n} (\lambda - u^i), \qquad A(\lambda) = \prod_{\alpha=1}^{n} (\lambda - a_\alpha). \tag{4.6}$$

If we multiply by $\lambda - a_\alpha$ and evaluate the result for $\lambda = a_\alpha$, then we get equations

$$x_\alpha^2 = -\frac{U(a_\alpha)}{A'(a_\alpha)}, \tag{4.7}$$

which relate the Cartesian coordinates (x_α) with the elliptic coordinates (u^i). From (4.7) it follows that

$$\frac{\partial x_\alpha}{\partial u^i} = \tfrac{1}{2} \frac{x_\alpha}{u^i - a_\alpha}, \tag{4.8}$$

and we find the components of the vector fields $(\boldsymbol{E}_i)$. Hence, the tensor field $\boldsymbol{L}$ satisfies the requirements of Proposition 2.1. ∎

Remark 4.1. The relevance of a tensor like (4.1), in connection with elliptic coordinates, is pointed out by Moser [MO1] [MO2] within a different framework.

Remark 4.2. According to Proposition 4.1, the roots of equation (4.4) are the eigenvalues of $\boldsymbol{L}$, thus equation (4.4) is equivalent to the characteristic equation of $\boldsymbol{L}$, so that the elementary symmetric functions (σ_a) of the roots (u^i) are the coefficients of this equation and coincide with the principal invariants of $\boldsymbol{L}$. The comparison between the characteristic equation of $\boldsymbol{L}$ and equation (4.4) shows that

$$\boxed{\sigma_a = \varsigma_a + m \sum_{\gamma=1}^{n} \varsigma_{a-1}^{\gamma}\, x_\gamma^2} \qquad a = 0, 1, \ldots, n, \tag{4.9}$$

where (ς_a) are the symmetric functions of the variables (a_α) and

$$\sigma_0 = \varsigma_0 = 1, \qquad \varsigma_{-1} = 0. \tag{4.10}$$

Proposition 4.2. *The elliptic Stäckel system corresponding to the distinct constants (a_α) is generated by the Killing tensors $(\boldsymbol{K}_a)$ $(a = 0, 1, \ldots, n-1)$ whose Cartesian components are*

$$\boxed{\begin{aligned} &K_a^{\alpha\alpha} = \varsigma_a^{\alpha} + m \sum_{\gamma=1}^{n} \varsigma_{a-1}^{\alpha\gamma}\, x_\gamma^2, \\ &K_a^{\alpha\beta} = -\, m\, \varsigma_{a-1}^{\alpha\beta}\, x_\alpha\, x_\beta \qquad (\alpha \neq \beta) \end{aligned}} \tag{4.11}$$

where (ς_a) are the elementary symmetric functions of the constants (a_α).

Proof. We can apply one of the formulae (2.19)-(2.21). Let us choose for instance the iterative formula (2.19). For $a = 0$, from (11) we get $K_0^{\alpha\alpha} = 1$ and $K_0^{\alpha\beta} = 0$, that is $\boldsymbol{K}_0 = \boldsymbol{g}$, in agreement with (2.19).

Assume that the expressions (4.11) hold for the index $a-1$ and let us check if they hold for the following index a. According to (2.19) we have,

$$\begin{aligned}
K_a^{\alpha\alpha} &= \sigma_a - \sum_\gamma K_{a-1}^{\alpha\gamma}\, L^{\gamma\alpha} \\
&= \sigma_a - \sum_{\gamma\neq\alpha} K_{a-1}^{\alpha\gamma}\, L^{\gamma\alpha} - K_{a-1}^{\alpha\alpha}\, L^{\alpha\alpha} \\
&= \varsigma_a + m\sum_\gamma \varsigma_{a-1}^{\gamma} x_\gamma^2 + m^2 \sum_{\gamma\neq\alpha} \varsigma_{a-2}^{\alpha\gamma} x_\gamma^2\, x_\alpha^2 \\
&\qquad - \Big(\varsigma_{a-1}^{\alpha} + m\sum_\gamma \varsigma_{a-2}^{\alpha\gamma} x_\gamma^2\Big)\big(a_\alpha + m\, x_\alpha^2\big) \\
&= \varsigma_a - a_\alpha \varsigma_{a-1}^{\alpha} + m \sum_{\gamma\neq\alpha} \big(\varsigma_{a-1}^{\gamma} - a_\alpha \varsigma_{a-1}^{\alpha\gamma}\big)\, x_\gamma^2 \\
&= \varsigma_a^{\alpha} + m\sum_\gamma \varsigma_{a-1}^{\alpha\gamma} x_\gamma^2 .
\end{aligned}$$

So that $(4.11)_1$ is proved. Here we used formulae (4.2) and (4.9) and the identities (2.2) and $(2.4)_2$ for the symmetric functions (ς_a) of the variables (a_α). In an analogous way we can prove $(4.11)_2$. ∎

Remark 4.3. In the Cartesian frame $(\boldsymbol{X}_\alpha)$ the Killing tensors (4.11) can be written as follows,

$$\boxed{\boldsymbol{K}_a = \sum_{\alpha=1}^{n} \varsigma_a^{\alpha}\ \boldsymbol{X}_\alpha \otimes \boldsymbol{X}_\alpha + m \sum_{\alpha,\beta=1}^{n} \varsigma_{a-1}^{\alpha\beta}\ \boldsymbol{R}_{\alpha\beta}} \tag{4.12}$$

where, for $\alpha \neq \beta$,

$$\boxed{\begin{aligned}
\boldsymbol{R}_{\alpha\beta} &= \big(x_\alpha\, \boldsymbol{X}_\beta - x_\beta\, \boldsymbol{X}_\alpha\big) \otimes \big(x_\alpha\, \boldsymbol{X}_\beta - x_\beta\, \boldsymbol{X}_\alpha\big) \\
&= x_\alpha^2\, \boldsymbol{X}_\beta \otimes \boldsymbol{X}_\beta + x_\beta^2\, \boldsymbol{X}_\alpha \otimes \boldsymbol{X}_\alpha - x_\alpha\, x_\beta \big(\boldsymbol{X}_\alpha \otimes \boldsymbol{X}_\beta + \boldsymbol{X}_\beta \otimes \boldsymbol{X}_\alpha\big)
\end{aligned}} \tag{4.13}$$

We note that $\boldsymbol{R}_{\alpha\beta} = \boldsymbol{R}_{\beta\alpha}$ and that the vector field $x_\alpha\, \boldsymbol{X}_\beta - x_\beta\, \boldsymbol{X}_\alpha$ is a unitary rotation on the (x_α, x_β) plane. The tensor product $\boldsymbol{R}_{\alpha\beta}$ does not depend on the choice of the orientation of this rotation.

Remark 4.4. Let us denote by (p_α) the momenta corresponding to the Cartesian coordinates (x_α) and set

$$\boxed{l_{\alpha\beta} = x_\alpha\, p_\beta - x_\beta\, p_\alpha} \tag{4.14}$$

Then the integrals

$$E_{\boldsymbol{K}_a} = \tfrac{1}{2}\, K_a^{\alpha\beta}\, p_\alpha\, p_\beta$$

corresponding to the Killing tensors $(\boldsymbol{K}_a)$ are

$$\boxed{E_{\boldsymbol{K}_a} = \tfrac{1}{2}\left(\sum_{\alpha=1}^{n} \varsigma_a^{\alpha}\, p_\alpha^2 + \tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} \varsigma_{a-1}^{\alpha\beta}\, l_{\alpha\beta}^2\right)} \tag{4.15}$$

Remark 4.5. If we define

$$\boxed{\widetilde{\boldsymbol{K}}_\alpha = \sum_{a=0}^{n-1} \varsigma_\alpha^{a}\, \boldsymbol{K}_a} \quad (\alpha = 1, \ldots, n), \tag{4.16}$$

where $(\bar{\varsigma}^a_\alpha)$ is the inverse matrix of (ς^α_a), that is (compare with (2.12))

$$\bar{\varsigma}^a_\alpha = \frac{(-1)^a}{A'(a_\alpha)}(a_\alpha)^{n-a-1}, \qquad A(z) = \prod_{\alpha=1}^{n}(z - a_\alpha), \tag{4.17}$$

then we get (we need identity (2.15) for the symmetric functions (ς_a)) another basis $(\widetilde{\boldsymbol{K}}_\alpha)$ of the elliptic Stäckel system,

$$\boxed{\widetilde{\boldsymbol{K}}_\alpha = \boldsymbol{X}_\alpha \otimes \boldsymbol{X}_\alpha - m \sum_{\gamma \neq \alpha} \frac{1}{a_\alpha - a_\gamma} \boldsymbol{R}_{\alpha\gamma}} \tag{4.18}$$

The Cartesian components of the tensors $(\widetilde{\boldsymbol{K}}_\alpha)$ are

$$\boxed{\begin{aligned} \widetilde{K}^{\alpha\alpha}_\alpha &= 1 - m \sum_{\gamma \neq \alpha} \frac{x_\gamma^2}{a_\alpha - a_\gamma} \\ \widetilde{K}^{\beta\beta}_\alpha &= -\, m \frac{x_\alpha^2}{a_\alpha - a_\beta} \\ \widetilde{K}^{\alpha\beta}_\alpha &= m \frac{x_\alpha x_\beta}{a_\alpha - a_\beta} \\ \widetilde{K}^{\beta\gamma}_\alpha &= 0 \end{aligned}} \qquad (\alpha, \beta, \gamma \neq) \tag{4.19}$$

and the corresponding integrals

$$E_{\widetilde{\boldsymbol{K}}_\alpha} = \tfrac{1}{2}\left(p_\alpha^2 - m \sum_{\gamma \neq \alpha} \frac{l_{\alpha\gamma}^2}{a_\alpha - a_\gamma}\right). \tag{4.20}$$

For $m = -1$ they correspond to the involutive integrals found by Marshall and Wojciechowski [MW] (with constant potentials).

Remark 4.6. If the coefficients (a_α) are not all distinct, then the Killing tensors $(\boldsymbol{K}_a)$ are well defined but not all independent, while some of the Killing tensors $(\widetilde{\boldsymbol{K}}_\alpha)$ are not defined (however, those which can be defined by (4.14) are independent). Under the assumption that all (a_α) are distinct, all the Killing tensors $(\boldsymbol{K}_a)$, with the exception of $\boldsymbol{K}_0 = \boldsymbol{g}$, have distinct eigenvalues

$$\varrho_{ai} = \sigma^i_a,$$

which are functions of the coordinates (u^i). The eigenvalues of the tensors $(\widetilde{\boldsymbol{K}}_\alpha)$ are the functions

$$\widetilde{\varrho}_{\alpha i} = \sum_{a=0}^{n-1} \bar{\varsigma}^a_\alpha \, \sigma^i_a \tag{4.21}$$

of the $2n$ variables (u^i, a_α).

5. Parabolic Stäckel systems

Equation (3.8) for $m = 0$ shows that

$$\boldsymbol{L} = \boldsymbol{L}_Q + \boldsymbol{r} \otimes \boldsymbol{w} + \boldsymbol{w} \otimes \boldsymbol{r} \qquad (\boldsymbol{r} = QP), \tag{5.1}$$

where Q is any fixed point of the space. We assume $\boldsymbol{w} \neq 0$.

Proposition 5.1. *There exists a unique point O such that $\boldsymbol{L}_O \cdot \boldsymbol{w} = 0$. The vector $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{L}_P$ for every point P of the straight line parallel to the vector $\boldsymbol{w}$ and containing O.*

Proof. Due to (5.1), condition $\boldsymbol{L}_O \cdot \boldsymbol{w} = 0$ is equivalent to

$$\boldsymbol{L}_Q \cdot \boldsymbol{w} + w^2\, \boldsymbol{r} + \boldsymbol{r} \cdot \boldsymbol{w}\, \boldsymbol{w} = 0, \qquad \boldsymbol{r} = QO.$$

This is a linear equation in $\boldsymbol{r}$ which implies

$$\boldsymbol{r} \cdot \boldsymbol{w} = -\frac{1}{2w^2}\, \boldsymbol{w} \cdot \boldsymbol{L}_Q \cdot \boldsymbol{w},$$

thus

$$\boldsymbol{r} = \frac{1}{w^2}\left(\frac{\boldsymbol{w} \cdot \boldsymbol{L}_Q \cdot \boldsymbol{w}}{2w^2}\, \boldsymbol{w} - \boldsymbol{L}_Q \cdot \boldsymbol{w}\right).$$

This defines a unique point O. Furthermore, let us consider the points $P = O + k\, \boldsymbol{w}$ where $k \in \mathbb{R}$. According to (3.8),

$$\boldsymbol{L}_P = \boldsymbol{L}_O + 2k\, \boldsymbol{w} \otimes \boldsymbol{w},$$

thus

$$\boldsymbol{L}_P \cdot \boldsymbol{w} = 2kw^2\, \boldsymbol{w},$$

since $\boldsymbol{L}_O \cdot \boldsymbol{w} = 0$. ∎

According to formula (5.1) and Proposition 5.1 we can write

$$\boldsymbol{L} = \boldsymbol{L}_O + \boldsymbol{r} \otimes \boldsymbol{w} + \boldsymbol{w} \otimes \boldsymbol{r} \qquad (\boldsymbol{r} = OP), \qquad \boldsymbol{L}_O \cdot \boldsymbol{w} = 0. \tag{5.2}$$

Let (x_α) be Cartesian coordinates with origin at the point O defined by the eigendirections of $\boldsymbol{L}_O$. Let $x_1 = x$ be the coordinate corresponding to the eigenvector $\boldsymbol{w}$. Let (a_α) be the eigenvalues of $\boldsymbol{L}_O$ and (u^i) be the eigenvalues of $\boldsymbol{L}$ at a generic point. Equation (5.1) is equivalent to

$$L^{\alpha\beta} = a_\alpha\, \delta_{\alpha\beta} + x_\alpha\, w_\beta + x_\beta\, w_\alpha, \tag{5.3}$$

where, according to Proposition 5.1,

$$a_1 = 0, \qquad w_\alpha = \delta_{1\alpha}\, w. \tag{5.4}$$

Let $(\boldsymbol{X}_\alpha)$ be the unitary constant vector fields corresponding to the Cartesian coordinates (x_α), with $\boldsymbol{X}_1 = \boldsymbol{X}$.

Proposition 5.2. *If $a_2 < a_3 < \ldots < a_n$, then the eigenvalues (u^i) of $\boldsymbol{L}$ coincide with the parabolic coordinates and the tensor $\boldsymbol{L}$ generates a basis of the parabolic Stäckel system according to Proposition 2.1.*

Proof. Let us consider the vector fields

$$\boldsymbol{E}_i = \tfrac{1}{2}\left(\frac{1}{w}\, \boldsymbol{X} + \sum_{\alpha=2}^{n} \frac{x_\alpha}{u^i - a_\alpha}\, \boldsymbol{X}_\alpha\right), \tag{5.5}$$

where (u^i) are the roots of equation

$$\sum_{\alpha=2}^{n} \frac{x_\alpha^2}{\lambda - a_\alpha} = \frac{1}{w^2}\big(\lambda - 2w\, x\big). \tag{5.6}$$

It follows that

$$\begin{aligned}(\boldsymbol{L}\cdot\boldsymbol{E}_i)^1 &= L^{11}E_i^1+\sum_{\gamma=2}^{n}L^{1\gamma}E_i^\gamma\\ &= x+\frac{w}{2}\sum_{\gamma=2}^{n}\frac{x_\gamma^2}{u^i-a_\gamma}=x+\frac{1}{2w}(u^i-2wx)=\frac{1}{2w}\,u^i,\\ (\boldsymbol{L}\cdot\boldsymbol{E}_i)^\alpha &= L^{\alpha 1}E_i^1+\sum_{\gamma=2}^{n}L^{\alpha\gamma}E_i^\gamma\\ &= \tfrac{1}{2}\,x_\alpha+\tfrac{1}{2}\,a_\alpha\frac{x_\alpha}{u^i-a_\alpha}=\tfrac{1}{2}\,\frac{x_\alpha}{u^i-a_\alpha}\,u^i\end{aligned}$$

that is $\boldsymbol{L}\cdot\boldsymbol{E}_i=u^i\,\boldsymbol{E}_i$. This shows that the parabolic coordinates (u^i), which are defined by equation (5.6) (usually with $w=1$), and are separated by the constants (a_α),

$$u^1<a_2<u^2<\ldots<a_n<u^n,$$

are eigenvalues of the conformal Killing tensor $\boldsymbol{L}$. Furthermore, the eigenvectors $(\boldsymbol{E}_i)$ form the natural frame corresponding to these coordinates. Indeed equation (5.6) is equivalent to equation

$$\sum_{\alpha=2}^{n}\frac{x_\alpha^2}{\lambda-a_\alpha}-\frac{1}{w^2}(\lambda-2w\,x)=-\frac{1}{w^2}\,\frac{U(\lambda)}{A_0(\lambda)}. \tag{5.7}$$

where

$$U(\lambda)=\prod_{i=1}^{n}(\lambda-u^i),\qquad A_0(\lambda)=\prod_{\alpha=2}^{n}(\lambda-a_\alpha). \tag{5.8}$$

Equations

$$x=\frac{1}{2w}\biggl(\sum_{i=1}^{n}u^i-\sum_{\alpha=2}^{n}a_\alpha\biggr),\qquad x_\alpha^2=-\frac{1}{w^2}\,\frac{U(a_\alpha)}{A_0'(a_\alpha)}\qquad(\alpha=2,\ldots,n), \tag{5.9}$$

relate the Cartesian coordinates (x_α) with the parabolic coordinates (u^i). Equations $(5.9)_2$ follow from (5.7) multiplied by $\lambda-a_\alpha$ and evaluated for $\lambda=a_\alpha$. Equation $(5.9)_1$ is a particular case of formula (5.11) below (Remark 5.1), and it follows from the expression of the coefficient of λ^{n-1} in the algebraic equation equivalent to (5.6). From equations (5.7) it follows that

$$\frac{\partial x}{\partial u^i}=\frac{1}{2w},\qquad\frac{\partial x_\alpha}{\partial u^i}=\tfrac{1}{2}\,\frac{x_\alpha}{u^i-a_\alpha}\qquad(\alpha=2,\ldots,n). \tag{5.10}$$

so that we get the natural frame (5.5). Hence, the tensor field $\boldsymbol{L}$ satisfies the requirements of Proposition 2.1. ∎

Remark 5.1. As a consequence of Proposition 5.2, since the roots of equation (5.6) are the eigenvalues of $\boldsymbol{L}$, equation (5.6) is equivalent to the characteristic equation of $\boldsymbol{L}$, so that the elementary symmetric functions (σ_a) of the roots (u^i) are the coefficients of this equation and coincide with the principal invariants of $\boldsymbol{L}$. The comparison between the characteristic equation of $\boldsymbol{L}$ and equation (5.6) shows that

$$\boxed{\sigma_a=\varsigma_a+2\,\varsigma_{a-1}\,w\,x-w^2\sum_{\gamma=2}^{n}\varsigma_{a-2}^{\gamma}\,x_\gamma^2}\qquad a=0,1,\ldots,n, \tag{5.11}$$

where (ς_a) are the elementary symmetric functions of the variables (a_α) and

$$\sigma_0 = \varsigma_0 = 1, \qquad \varsigma_{-1} = \varsigma_{-2} = 0.$$

Note that, due to (5.4), $\varsigma_a = \varsigma_a^1$, $\varsigma_a^\alpha = \varsigma_a^{1\alpha}$, and so on, and that for $a = 1$ we find formula $(5.7)_1$.

Remark 5.2. The definition of the parabolic coordinates (u^i) depends on the differences $a_\alpha - a_\beta$ only, i.e. it is invariant under a translation over the real line of these constants (as for the elliptic coordinates). However, it is convenient to assume that $a_\alpha \neq 0$ for all $\alpha > 1$, so that all the constants a_α are different (since $a_1 = 0$).

Proposition 5.3. *The parabolic Stäckel system corresponding to the distinct constants* $(a_2, \dots, a_n)$ *is generated by the Killing tensors* $(\boldsymbol{K}_a)$ $(a = 0, 1, \dots, n-1)$ *whose Cartesian components are*

$$\text{(5.12)} \qquad \boxed{\begin{aligned} K_a^{11} &= \varsigma_a \\ K_a^{1\alpha} &= -\, w\, \varsigma_{a-1}^{\alpha}\, x_\alpha \\ K_a^{\alpha\alpha} &= \varsigma_a^\alpha + 2\, w\, \varsigma_{a-1}^{\alpha}\, x - w^2 \sum_{\gamma=2}^{n} \varsigma_{a-2}^{\alpha\gamma}\, x_\gamma^2 \\ K_a^{\alpha\beta} &= w^2\, \varsigma_{a-2}^{\alpha\beta}\, x_\alpha\, x_\beta \end{aligned}} \qquad \alpha \neq \beta, \quad \alpha, \beta \neq 1.$$

where (ς_a) *are the elementary symmetric functions of the* n *constants* $(a_\alpha) = (0, a_2, \dots, a_n)$.

Proof. Let us apply the iterative formula (2.19). For $a = 0$ we get from (5.12) $K_0^{11} = K_0^{\alpha\alpha} = 1$ and $K_0^{1\alpha} = K_0^{\alpha\beta} = 0$, that is $\boldsymbol{K}_0 = \boldsymbol{g}$, in agreement with (2.19). Assume that the expressions (5.12) hold for the index $a - 1$ and let us check if they hold for the following index a. According to (2.19) we have,

$$\begin{aligned} K_a^{11} &= \sigma_a - \sum_\gamma K_{a-1}^{1\gamma}\, L^{\gamma 1} \\ &= \sigma_a - K_{a-1}^{11}\, L^{11} - \sum_{\gamma \neq 1} K_{a-1}^{1\gamma}\, L^{\gamma 1} \\ &= \sigma_a - 2\, w\, \varsigma_{a-1}\, x + w^2 \sum_{\gamma \neq 1} \varsigma_{a-2}^{\gamma}\, x_\gamma^2 \\ &= \varsigma_a, \end{aligned}$$

due to (11). So that $(5.12)_1$ is proved. The remaining expressions in (5.12) are proved in a similar way, by using the fundamental identities for the symmetric functions (ς_a). ■

Remark 5.3. In the Cartesian frame $(\boldsymbol{X}_\alpha)$ the Killing tensors (5.12) can be written as follows,

$$\text{(5.13)} \qquad \boxed{\boldsymbol{K}_a = \sum_{\alpha=1}^{n} \varsigma_a^\alpha\, \boldsymbol{X}_\alpha \otimes \boldsymbol{X}_\alpha + w \sum_{\alpha=2}^{n} \varsigma_{a-1}^{\alpha}\, \boldsymbol{S}_\alpha \cap \boldsymbol{X}_\alpha + \tfrac{1}{2}\, w^2 \sum_{\alpha,\beta=1}^{n} \varsigma_{a-2}^{\alpha\beta}\, \boldsymbol{R}_{\alpha\beta}}$$

where, for $\alpha \neq 1$,

$$\text{(5.14)} \qquad \boldsymbol{S}_\alpha = x\, \boldsymbol{X}_\alpha - x_\alpha\, \boldsymbol{X} \qquad (x = x_1,\ \boldsymbol{X} = \boldsymbol{X}_1)$$

is a unitary rotation in the (x, x_α) plane and $\boldsymbol{R}_{\alpha\beta}$ is defined in (4.13).

6. Spherical-elliptic Stäckel systems

With a set of n real numbers (a_α) such that

$$a_1 < a_2 < \ldots < a_n$$

we associate a family of cones

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\lambda - a_\alpha} = 0 \tag{6.1}$$

parametrized by $\lambda \in \mathbf{R}$. For every point not belonging to the coordinate planes equation (6.1) has $n-1$ real roots $(u^s) = (u^1, \ldots, u^{n-1})$ such that

$$a_1 < u^1 < a_2 < u^2 < \ldots < u^{n-1} < a_n.$$

To these roots we add the distance from the origin

$$u^n = r.$$

The functions $(u^i) = (u^s, r)$ are called **spherical-elliptic coordinates** or **conical coordinates**. It is known that they are separable.

The spherical-elliptic coordinates can be considered as the limit for $m \to \infty$ of the elliptic coordinates. Indeed, equation (4.4) reduces to equation (6.1). Moreover equation (4.4) can be written

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\mu - \dfrac{a_\alpha}{m}} = 1, \tag{6.2}$$

where

$$\mu = \frac{\lambda}{m}.$$

Keeping μ fixed, when $m \to \infty$ the ellipsoids in equation (6.2) become spheres (with $r^2 = \mu$).

A basis $(\boldsymbol{H}_a)$ of the spherical-elliptic Stäckel system can be directly derived from the elliptic one $(\boldsymbol{K}_a)$ defined in (4.11) by considering only the terms proportional to the parameter m (this means that we consider the elliptic coordinates with large values of m). The parameter m does not appear in $\boldsymbol{K}_0 = \boldsymbol{g}$, so that we take $\boldsymbol{H}_0 = \boldsymbol{g}$. The other Killing tensors are

$$\boldsymbol{H}_a = \boldsymbol{K}_a - \boldsymbol{K}_a(O) \qquad (a = 1, \ldots, n-1).$$

Thus,

Proposition 6.1. *The spherical-elliptic Stäckel system is generated by the independent Killing tensors*

$$\boxed{\boldsymbol{H}_0 = \boldsymbol{g}, \qquad \boldsymbol{H}_a = \tfrac{1}{2} \sum_{\alpha,\beta=1}^{n} \varsigma_{a-1}^{\alpha\beta}\, \boldsymbol{R}_{\alpha\beta}} \quad (a = 1, \ldots, n-1). \tag{6.3}$$

whose Cartesian components are (for $a \neq 0$)

$$\boxed{H_a^{\alpha\alpha} = \sum_{\gamma=1}^{n} \varsigma_{a-1}^{\alpha\gamma}\, x_\gamma^2 \qquad H_a^{\alpha\beta} = -\,\varsigma_{a-1}^{\alpha\beta}\, x_\alpha\, x_\beta} \qquad (\alpha \neq \beta) \tag{6.4}$$

Another basis $(\widetilde{\boldsymbol{H}}_\alpha)$ can be obtained by considering the part proportional to m in the tensors $(\widetilde{\boldsymbol{K}}_\alpha)$ defined in (4.18).
Equations

$$x_\alpha^2 = r^2\,\frac{U_0(a_\alpha)}{A'(a_\alpha)}, \tag{6.5}$$

where

$$U_0(\lambda) = \prod_{i=1}^{n-1}(\lambda - u^i), \qquad A(\lambda) = \prod_{\alpha=1}^{n}(\lambda - a_\alpha), \tag{6.6}$$

relate the Cartesian coordinates (x_α) with the spherical-elliptic coordinates (u^i) [MO1]. They follow from the identity

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{\lambda - a_\alpha} = r^2\,\frac{U_0(\lambda)}{A(\lambda)}. \tag{6.7}$$

Equations (6.5) imply

$$\frac{\partial x_\alpha}{\partial u^i} = \tfrac{1}{2}\,\frac{x_\alpha}{u^i - a_\alpha} \quad (i < n), \qquad \frac{\partial x_\alpha}{\partial u^n} = \frac{x_\alpha}{r}. \tag{6.8}$$

Thus the frame $(\boldsymbol{E}_i)$ corresponding to the spherical-elliptic coordinates is

$$\boldsymbol{E}_i = \tfrac{1}{2}\,\frac{x_\alpha}{u^i - a_\alpha}\,\boldsymbol{X}_\alpha \quad (i < n), \qquad \boldsymbol{E}_n = \frac{\boldsymbol{r}}{r}, \tag{6.9}$$

and the non vanishing metric tensor components are

$$g_{ii} = \tfrac{1}{4}\,\frac{U_0'(u^i)}{A(u^i)} \quad (i < n), \qquad g_{nn} = 1. \tag{6.10}$$

It follows that

$$\partial_i \ln g^{jj} = \frac{1}{u^j - u^i} \quad (i \neq j < n), \qquad \partial_i g^{nn} = 0, \qquad \partial_n g^{ii} = 0, \tag{6.11}$$

so that the separability conditions (2.24) are satisfied.

Remark 6.1. For each $a > 0$ the radius vector $\boldsymbol{r}$ is an eigenvector of $\boldsymbol{H}_a$ belonging to the zero eigenvalue,

$$\boldsymbol{H}_a \cdot \boldsymbol{r} = 0. \tag{6.12}$$

Remark 6.2. The restriction of $\boldsymbol{H}_1$ to a sphere centered at O coincides with the induced metric tensor. Indeed, the Cartesian components of $\boldsymbol{H}_1$ are (see (6.4))

$$H_1^{\alpha\alpha} = \sum_{\gamma\neq\alpha} x_\gamma^2 = 1 - x_\alpha^2, \qquad H_1^{\alpha\beta} = -\,x_\alpha x_\beta, \tag{6.13}$$

and a vector $\boldsymbol{v} = (v^\alpha)$ is tangent to the sphere if and only if $\boldsymbol{v}\cdot\boldsymbol{r} = v^\alpha x_\alpha = 0$. Thus

$$\begin{aligned}(\boldsymbol{H}_1\cdot\boldsymbol{v})^\alpha &= (1 - x_\alpha^2)v^\alpha - \sum_{\gamma\neq\alpha} x_\alpha x_\gamma v^\gamma \\ &= (1 - x_\alpha^2)v^\alpha + x_\alpha^2 v^\alpha = v^\alpha\end{aligned}$$

that is, $\boldsymbol{H}_1 \cdot \boldsymbol{v} = \boldsymbol{v}$. We note that the spectrum of the Killing tensor $\boldsymbol{H}_1$ is $(1, \ldots, 1, 0)$.

Remark 6.3. From Remarks 6.1 and 6.2 it follows that the $n-1$ Killing tensors $(\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_{n-1})$ are reducible to every sphere centered at the origin and give rise to a Stäckel system which corresponds to the elliptic coordinates $(u^s) = (u^1, \ldots, u^{n-1})$ originally considered by Neumann [NE] on the sphere.

Remark 6.4. Within the framework present here the following further topics can be developed. (i) The construction of all Stäckel systems in the Euclidean affine spaces. To this end it is necessary to discuss the case in which some of the constants (a_α) coincide. (ii) The Cartesian characterization of all the separable dynamical systems in the Euclidean affine spaces. To this end it is necessary to discuss the geometrical characterization of the orthogonal separation in a Riemannian manifold. This will be done in subsequent papers.

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