Integrable Lagrangian systems on quadrics with additional symmetries

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Abstract. *The family of the integrable natural Lagrangian systems on quadrics having additional symmetries is constructed by the classical Hamilton-Jacobi method.*

1. Construction of the elliptic coordinates on quadrics with additional symmetries

In the classical work by Jacobi [1] the elliptic coordinates were introduced for the proof of integrability of geodesics on two-dimensional ellipsoids. Jacobi's constructions were generalized by Moser [2] for the n-dimensional ellipsoid with different semi-axes, n-dimensional sphere and Euclidean space.

The present work is devoted to the construction of the Lagrange equations, integrable by the Hamilton-Jacobi method on quadrics with additional symmetries. We consider the natural Lagrangians

$$L = \tfrac{1}{2} \sum_{\nu=0}^{n} (\dot{x}_\nu)^2 - V(x_0, \dots, x_n) \tag{1.1}$$

on the quadrics determined by the equations

$$\sum_{\nu=0}^{n} \frac{x_\nu^2}{a_\nu} = 1 \tag{1.2}$$

where some of the coefficients a_ν are equal. Two important cases when all coefficients a_ν are equal and when they all are different were studied by Jacobi [1] and Moser [2].

Let us suppose that the coefficients $(a_0, \dots, a_n)$ be divided into $k+1$ groups of equal coefficients,

$$\begin{aligned} &a_0 = \ldots = a_{i_1} = b_0, \\ &a_{i_1+1} = \ldots = a_{i_2} = b_1, \\ &\cdots, \\ &a_{i_k+1} = \ldots = a_n = b_k \end{aligned} \tag{1.3}$$

In this case the quadric (1.2) possesses a large group of isometries

$$G = \prod_{m=0}^{k} SO(h_m), \qquad h_m = i_{m+1} - i_m, \qquad i_0 = 0,\ i_{k+1} = n. \tag{1.4}$$

We use the following substitution

$$x_{i_m+j} = r_m y_{mj}, \qquad m = 0, \dots, k, \qquad j = 1, \dots, h_m, \tag{1.5}$$

in the $k+1$ groups of coordinates x_ν, corresponding to the decomposition of the coefficients (1.3). Here variables y_{mj} satisfy the equation of k_m-dimensional sphere

$$y_{m1}^2 + \ldots + y_{mh_m}^2 = 1, \qquad k_m = h_m - 1. \tag{1.6}$$

Therefore equation (1.2) after the substitution (1.5) takes the form

$$\sum_{j=0}^{k} \frac{r_j^2}{b_j} = 1, \tag{1.7}$$

where all coefficients b_j are different. We define the elliptic coordinates $(u_{m1}, \dots, u_{mk_m})$ on each sphere $\to^{k_m}$ (1.6) by the equation [1, 2]

$$\sum_{i=1}^{h_m} \frac{y_{mi}^2}{z - c_{mi}} = \frac{U_m(z)}{A_m(z)}, \tag{1.8}$$

where z is an independent variable, and

$$U_m(z) = \prod_{i=1}^{k_m} (z - u_{mi}), \qquad C_m(z) = \prod_{i=1}^{h_m} (z - c_{mi}). \tag{1.9}$$

Here c_{mi} are some non-equal constants. The residues of the both-hand-sides of (1.8) in the poles $z = c_{mi}$ lead to the relations

$$y_{mi}^2 = \frac{U_m(c_{mi})}{C'_m(c_{mi})} \tag{1.10}$$

between the Cartesian coordinates y_{mi} and the elliptic coordinates u_{mi}.
The elliptic coordinates $(w_1, \dots, w_k)$ are determined on the quadric (1.7) by the equation [1, 2]

$$\sum_{i=0}^{k} \frac{r_i^2}{z - b_i} + 1 = \frac{z\, W(z)}{B(z)}, \tag{1.11}$$

where z is an independent parameter and

$$W(z) = \prod_{j=1}^{k} (z - w_j), \qquad B(z) = \prod_{i=0}^{k} (z - b_i). \tag{1.12}$$

Due to the equality of the residues of both hand-sides of (1.11) in the poles $z = b_j$ we obtain the expressions

$$r_i^2 = \frac{b_i W(b_i)}{B'(b_i)} \tag{1.13}$$

for the Cartesian coordinates r_i in terms of the elliptic coordinates w_j. The expressions for the Cartesian coordinates x_{i_m+j} (1.5) follow from (1.10) and (1.13)

$$x_{i_m+j} = r_m y_{mj} = \left(\frac{b_m W(b_m) U_m(c_{mj})}{B'(b_m) C'_m(c_{mj})} \right)^{\frac{1}{2}} \tag{1.14}$$

2. The Hamilton-Jacobi equation in elliptic coordinates

Lagrangian (1.1) in the coordinates (r_m, y_{mj}) takes the form

$$L = \tfrac{1}{2}\sum_{i=0}^{k} \dot r_i^2 + \tfrac{1}{2}\sum_{m=0}^{k} r_m^2\left(\dot y_{m1}^2 + \ldots + \dot y_{mk_m+1}^2\right) - V(x_i), \tag{2.1}$$

while the standard kinetic energy on the sphere (1.6) in the elliptic coordinates $(u_{m1}, \ldots, u_{mk_m})$ has the form [1, 2]:

$$\dot y_{m1}^2 + \ldots + \dot y_{mh_m}^2 = \sum_{i=1}^{k_m} f_{mi}\dot u_{ki}^2, \qquad f_{mi} = -\frac{U_m'(u_{mi})}{4C_m(u_{mi})}. \tag{2.2}$$

The standard kinetic energy on the quadric (1.7) has the known expression [1, 2] in the elliptic coordinates $(w_1, \ldots, w_k)$:

$$\dot r_0^2 + \ldots + \dot r_k^2 = \sum_{i=1}^{k} g_i\,\dot v_i^2, \qquad g_i = \frac{w_i W'(w_i)}{4B(w_i)}. \tag{2.3}$$

By using these formulae and the expressions (1.13) we get the form of the Lagrangian (2.1) in elliptic coordinates,

$$L = \tfrac{1}{2}\sum_{i=1}^{k}\frac{w_i W'(w_i)}{4B(w_i)}\,\dot w_i^2 - \tfrac{1}{2}\sum_{m=0}^{k}\frac{b_m W(b_m)}{B'(b_m)}\sum_{i=1}^{k_m}\frac{U_m'(u_{mi})}{4C_m(u_{mi})}\,\dot u_{mi}^2 - V(w_i, u_{mj}). \tag{2.4}$$

The corresponding Lagrange system after the Legendre transformation turns into the Hamiltonian system

$$\dot p_i = -\frac{\partial H}{\partial u_i}, \qquad \dot u_i = \frac{\partial H}{\partial p_i}, \tag{2.5}$$

with the Hamiltonian

$$H = \sum_{i=1}^{k}\frac{2B(w_i)}{w_i W'(w_i)}p_i^2 - \sum_{m=0}^{k}\frac{2B'(b_m)}{b_m W(b_m)}\sum_{i=1}^{k_m}\frac{C_m(u_{mi})}{U_m'(u_{mi})}\,p_{mi}^2 + V(w_i, u_{mj}). \tag{2.6}$$

Therefore the Hamilton-Jacobi equation

$$H\left(\frac{\partial S}{\partial u_i}, u_i\right) = \eta_1 \tag{2.7}$$

for the generating function $S(w_i, u_{mj}, \eta_1, \ldots, \eta_n)$ assumes the following explicit form

$$\sum_{i=1}^{k}\frac{2B(w_i)}{w_i W'(w_i)}\left(\frac{\partial S}{\partial v_i}\right)^2 - \sum_{m=0}^{k}\frac{2B'(b_m)}{b_m W(b_m)}\sum_{i=1}^{k_m}\frac{C_m(u_{mi})}{U_m'(u_{mi})}\left(\frac{\partial S}{\partial u_{mi}}\right)^2 + V(w_i, u_{mj}) = \eta_1 \tag{2.8}$$

in the elliptic coordinates (v_i, u_{mj}).

3. The family of the integrable Lagrangian systems

Theorem. *The Lagrange system with the Lagrangian* (1.1) *on a quadric* (1.2) *with an additional symmetry* (1.4) *is integrable by quadratures if the potential $V(x_0, \ldots, x_n)$ in the elliptic coordinates (w_i, u_{mj}) has the form*

$$V(w_i, u_{mj}) = \sum_{i=1}^{k}\frac{F_i(w_i)}{W'(w_i)} + \sum_{m=0}^{k}\frac{1}{W(b_m)}\sum_{i=1}^{k_m}\frac{F_{mi}(u_{mi})}{U_m'(u_{mi})}, \tag{3.1}$$

where $F_i(v_i)$ and $F_{mi}(u_{mi})$ are arbitrary smooth functions of one variable.

Proof. We use the classical Hamilton-Jacobi method for the Hamiltonian system (2.5), (2.6), corresponding to the Lagrangian (1.1). To construct the solution of the Hamilton-Jacobi equation (2.8) we represent the generating function $S(u_i, \eta_j)$ in the form

$$S(w_i, u_{mj}, \eta_1, \dots, \eta_n) = G(w_1, \dots, w_k, \eta_1, \dots, \eta_k, \eta_{m1}) + \\ + \sum_{m=0}^{k} G_m(u_{m1}, \dots, u_{mk_m}, \eta_{m1}, \dots, \eta_{mk_m}). \tag{3.2}$$

The Hamilton-Jacobi equation (2.8) after substituting the expressions (3.1) and (3.2) is split into $k+1$ independent equations for the functions (G_m),

$$\sum_{i=1}^{k_m} \frac{1}{U'_m(u_{mi})} \left(F_{mi}(u_{mi}) - \frac{2}{b_m} B'(b_m) C_m(u_{mi}) \left(\frac{\partial G_m}{\partial u_{mi}} \right)^2 \right) = \eta_{m1} \tag{3.3}$$

and one independent equation for the function G,

$$\sum_{i=1}^{k} \frac{1}{U'(w_i)} \left(F_i(w_i) + \frac{2}{w_i} B(w_i) \left(\frac{\partial G}{\partial w_i} \right)^2 \right) + \sum_{m=0}^{k} \frac{\eta_{m1}}{W(b_m)} = \eta_1. \tag{3.4}$$

Let us introduce $k+1$ polynomials

$$P_m(z) = \eta_{m1} z^{k_m - 1} + \eta_{m2} z^{k_m - 2} + \ldots + \eta_{mk_m} \tag{3.5}$$

and one polynomial $P(z)$

$$P(z) = \eta_1 z^{k-1} + \eta_2 z^{k-2} + \ldots + \eta_k, \tag{3.6}$$

where the coefficients η_{mi} and η_j are independent constants. From the Cauchy theorem about the residues the known identities follow

$$\eta_{m1} = \frac{1}{2\pi i} \oint \frac{P_m(z)}{U_m(z)} dz = \sum_{i=1}^{k_m} \frac{P_m(u_{mi})}{U'_m(u_{mi})}, \tag{3.7}$$

$$\eta_1 = \frac{1}{2\pi i} \oint \frac{P(z)}{W(z)} dz = \sum_{i=1}^{k} \frac{P(w_i)}{W'(w_i)}. \tag{3.8}$$

By substituting the expression (3.7) into the equation (3.3) we obtain the explicit solution

$$G_m(u_{m1}, \dots, u_{mk_m}, \eta_{m1}, \dots, \eta_{mk_m}) = \\ = \sum_{i=1}^{k_m} \int_0^{u_{mi}} \left(\frac{b_m(F_{mi}(z) - P_m(z))}{2B'(b_m) C_m(z)} \right)^{\frac{1}{2}} dz. \tag{3.9}$$

To solve equation (3.4) we prove the following statement.

Lemma. *The function $U(z)$ in (1.12) satisfies the identity*

$$\frac{1}{W(z)} = \sum_{i=1}^{k} \frac{1}{W'(w_i)} \cdot \frac{1}{z - w_i}. \tag{3.10}$$

Proof. The residues of two meromorphic functions in both hand-sides of (3.10) in the poles $(w_1, \ldots, w_k)$ coincide. Both functions tend to zero for $|z| \to \infty$. Therefore these two meromorphic functions coincide. ∎

As a consequence of this Lemma we get the useful identity

$$\frac{1}{W(b_m)} = \sum_{i=1}^{k} \frac{1}{W'(w_i)} \cdot \frac{1}{b_m - w_i}. \tag{3.11}$$

Equation (3.4) after the substitution of the expressions (3.8) and (3.11) takes the form

$$\sum_{i=1}^{k} \frac{1}{W'(w_i)} \left(F_i(w_i) - P(w_i) + \sum_{m=0}^{k} \frac{\eta_{m1}}{b_m - w_i} + \frac{2}{w_i} B(w_i) \left(\frac{\partial G}{\partial w_i} \right)^2 \right) = 0. \tag{3.12}$$

The explicit solution of this equation is defined by the formula

$$\begin{aligned} &G(w_1, \ldots, w_k, \eta_1, \ldots, \eta_k, \eta_{m1}) = \\ &= \sum_{i=1}^{k} \int_0^{w_i} \left(\frac{z}{2B(z)} \left(P(z) + \sum_{m=0}^{k} \frac{\eta_{m1}}{z - b_m} - F_i(z) \right) \right)^{\frac{1}{2}} dz. \end{aligned} \tag{3.13}$$

Therefore the generating function $S(w_i, u_{mj}, \eta_1, \ldots, \eta_n)$ (3.2) is completely determined by the formulae (3.9) and (3.13). The generating function $S(w_i, u_{mj}, \eta_i, \eta_{mj})$ (3.2) defines the canonical transformation [3] to new phase coordinates

$$\xi_i,\ \xi_{mj},\ \eta_i,\ \eta_{mj}, \quad \begin{cases} i = 1, \ldots, k; \\ m = 0, 1, \ldots, k; \\ j = 1, \ldots, k_m \end{cases} \tag{3.14}$$

by the formulae

$$\begin{aligned} \xi_i &= -\frac{\partial S}{\partial \eta_i}, \qquad & \xi_{mj} &= -\frac{\partial S}{\partial \eta_{mj}}, \\ p_i &= \frac{\partial S}{\partial w_i}, \qquad & p_{mj} &= \frac{\partial S}{\partial u_{mj}}. \end{aligned} \tag{3.15}$$

In view of the Hamilton-Jacobi equation (2.7) the Hamiltonian H (2.6) takes in coordinates (3.14) the simplest form,

$$H(\xi_i, \xi_{mj}, \eta_i, \eta_{mj}) = \eta_1. \tag{3.16}$$

Therefore the Hamiltonian equations in coordinates (3.14) have the form

$$\dot{\xi}_i = -\delta_{1i}, \quad \dot{\xi}_{mj} = 0, \quad \dot{\eta}_i = 0, \quad \dot{\eta}_{mj} = 0. \tag{3.17}$$

So we get $\xi_1 = c_1 - t$ and all other coodinates are constant.

Dynamics of trajectories of the system (2.5), (2.6) is integrable by quadratures, because it is obtained from the simplest dynamics (3.17) by a canonical transformation, inverse to (3.15). The generating function S (3.2) is determined by the quadratures (3.9), (3.13). Hence, the theorem is proved. ∎

Remark 1. The constructed family of integrable Lagrange systems has applications in the rigid body dynamics. This family leads by the methods of [4] to the integrable cases of an axially-symmetric rigid body rotation around fixed point in the fields with the special potentials $V(x_1, x_2, x_3)$ (3.1).

Remark 2. In [5] the "degeneration" of elliptic coordinates in the Euclidean space $\mathbb{R}^n$ was considered, corresponding to the limit $a_1 \to a_2$, and separable Hamiltonian systems in the case $a_1 = a_2$ having "invariance with respect to rotations in the (q_1, q_2) plane" (see [5], p. 1343). Our Theorem, which can be extended to the Euclidean space $\mathbb{R}^n$, describes more complicated separable Hamiltonian systems without any rotational symmetry, and without angular momentum-type first integrals.

References

[1] C.G.J. Jacobi, *Vorlesungen Über Dynamik*, Königsberg, 1842-43; Berlin, Reimer, 1866.

[2] J. Moser, *Various Aspects of the Integrable Hamiltonian Systems*, Dynamical Systems (C.I.M.E. Summer School, Bressanone, 1978), Progress in Mathematics, Birkhäuser, 8 (1980), 233-289.

[3] V.I. Arnold, *Mathematical Methods in Classical Mechanics*, Springer 1983.

[4] O.I. Bogoyavlenskij, *Integrable Cases of a Rigid Body Dynamics and Integrable Systems on the Ellipsoids*, Comm. Math. Phys. 103 (1986), 305-322.

[5] I. Marshall, S. Wojciechowski, *When is a Hamiltonian system integrable?*, J. Math. Phys. 29 (1988), 1338-1346.