

# Integrable Lagrangian systems on quadrics with additional symmetries

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**Abstract.** *The family of the integrable natural Lagrangian systems on quadrics having additional symmetries is constructed by the classical Hamilton-Jacobi method.*

## 1. Construction of the elliptic coordinates on quadrics with additional symmetries

In the classical work by Jacobi [1] the elliptic coordinates were introduced for the proof of integrability of geodesics on two-dimensional ellipsoids. Jacobi's constructions were generalized by Moser [2] for the  $n$ -dimensional ellipsoid with different semi-axes,  $n$ -dimensional sphere and Euclidean space.

The present work is devoted to the construction of the Lagrange equations, integrable by the Hamilton-Jacobi method on quadrics with additional symmetries. We consider the natural Lagrangians

$$(1.1) \quad L = \frac{1}{2} \sum_{\nu=0}^n (\dot{x}_\nu)^2 - V(x_0, \dots, x_n)$$

on the quadrics determined by the equations

$$(1.2) \quad \sum_{\nu=0}^n \frac{x_\nu^2}{a_\nu} = 1$$

where some of the coefficients  $a_\nu$  are equal. Two important cases when all coefficients  $a_\nu$  are equal and when they all are different were studied by Jacobi [1] and Moser [2].

Let us suppose that the coefficients  $(a_0, \dots, a_n)$  be divided into  $k + 1$  groups of equal coefficients,

$$(1.3) \quad \begin{aligned} a_0 &= \dots = a_{i_1} = b_0, \\ a_{i_1+1} &= \dots = a_{i_2} = b_1, \\ &\dots, \\ a_{i_{k+1}+1} &= \dots = a_n = b_k \end{aligned}$$

In this case the quadric (1.2) possesses a large group of isometries

$$(1.4) \quad G = \prod_{m=0}^k SO(h_m), \quad h_m = i_{m+1} - i_m, \quad i_0 = 0, \quad i_{k+1} = n.$$

We use the following substitution

$$(1.5) \quad x_{i_m+j} = r_m y_{mj}, \quad m = 0, \dots, k, \quad j = 1, \dots, h_m,$$

in the  $k + 1$  groups of coordinates  $x_\nu$ , corresponding to the decomposition of the coefficients (1.3). Here variables  $y_{mj}$  satisfy the equation of  $k_m$ -dimensional sphere

$$(1.6) \quad y_{m1}^2 + \dots + y_{mk_m}^2 = 1, \quad k_m = h_m - 1.$$

Therefore equation (1.2) after the substitution (1.5) takes the form

$$(1.7) \quad \sum_{j=0}^k \frac{r_j^2}{b_j} = 1,$$

where all coefficients  $b_j$  are different. We define the elliptic coordinates  $(u_{m1}, \dots, u_{mk_m})$  on each sphere  $\rightarrow^{k_m}$  (1.6) by the equation [1, 2]

$$(1.8) \quad \sum_{i=1}^{h_m} \frac{y_{mi}^2}{z - c_{mi}} = \frac{U_m(z)}{A_m(z)},$$

where  $z$  is an independent variable, and

$$(1.9) \quad U_m(z) = \prod_{i=1}^{k_m} (z - u_{mi}), \quad C_m(z) = \prod_{i=1}^{h_m} (z - c_{mi}).$$

Here  $c_{mi}$  are some non-equal constants. The residues of the both-hand-sides of (1.8) in the poles  $z = c_{mi}$  lead to the relations

$$(1.10) \quad y_{mi}^2 = \frac{U_m(c_{mi})}{C'_m(c_{mi})}$$

between the Cartesian coordinates  $y_{mi}$  and the elliptic coordinates  $u_{mi}$ .

The elliptic coordinates  $(w_1, \dots, w_k)$  are determined on the quadric (1.7) by the equation [1, 2]

$$(1.11) \quad \sum_{i=0}^k \frac{r_i^2}{z - b_i} + 1 = \frac{zW(z)}{B(z)},$$

where  $z$  is an independent parameter and

$$(1.12) \quad W(z) = \prod_{j=1}^k (z - w_j), \quad B(z) = \prod_{i=0}^k (z - b_i).$$

Due to the equality of the residues of both hand-sides of (1.11) in the poles  $z = b_j$  we obtain the expressions

$$(1.13) \quad r_i^2 = \frac{b_i W(b_i)}{B'(b_i)}$$

for the Cartesian coordinates  $r_i$  in terms of the elliptic coordinates  $w_j$ . The expressions for the Cartesian coordinates  $x_{i_m+j}$  (1.5) follow from (1.10) and (1.13)

$$(1.14) \quad x_{i_m+j} = r_m y_{mj} = \left( \frac{b_m W(b_m) U_m(c_{mj})}{B'(b_m) C'_m(c_{mj})} \right)^{\frac{1}{2}}$$

## 2. The Hamilton-Jacobi equation in elliptic coordinates

Lagrangian (1.1) in the coordinates  $(r_m, y_{mj})$  takes the form

$$(2.1) \quad L = \frac{1}{2} \sum_{i=0}^k \dot{r}_i^2 + \frac{1}{2} \sum_{m=0}^k r_m^2 (\dot{y}_{m1}^2 + \dots + \dot{y}_{mk_m+1}^2) - V(x_i),$$

while the standard kinetic energy on the sphere (1.6) in the elliptic coordinates  $(u_{m1}, \dots, u_{mk_m})$  has the form [1, 2]:

$$(2.2) \quad \dot{y}_{m1}^2 + \dots + \dot{y}_{mh_m}^2 = \sum_{i=1}^{k_m} f_{mi} \dot{u}_{ki}^2, \quad f_{mi} = -\frac{U'_m(u_{mi})}{4C_m(u_{mi})}.$$

The standard kinetic energy on the quadric (1.7) has the known expression [1, 2] in the elliptic coordinates  $(w_1, \dots, w_k)$ :

$$(2.3) \quad \dot{r}_0^2 + \dots + \dot{r}_k^2 = \sum_{i=1}^k g_i \dot{v}_i^2, \quad g_i = \frac{w_i W'(w_i)}{4B(w_i)}.$$

By using these formulae and the expressions (1.13) we get the form of the Lagrangian (2.1) in elliptic coordinates,

$$(2.4) \quad L = \frac{1}{2} \sum_{i=1}^k \frac{w_i W'(w_i)}{4B(w_i)} \dot{w}_i^2 - \frac{1}{2} \sum_{m=0}^k \frac{b_m W(b_m)}{B'(b_m)} \sum_{i=1}^{k_m} \frac{U'_m(u_{mi})}{4C_m(u_{mi})} \dot{u}_{mi}^2 - V(w_i, u_{mj}).$$

The corresponding Lagrange system after the Legendre transformation turns into the Hamiltonian system

$$(2.5) \quad \dot{p}_i = -\frac{\partial H}{\partial u_i}, \quad \dot{u}_i = \frac{\partial H}{\partial p_i},$$

with the Hamiltonian

$$(2.6) \quad H = \sum_{i=1}^k \frac{2B(w_i)}{w_i W'(w_i)} p_i^2 - \sum_{m=0}^k \frac{2B'(b_m)}{b_m W(b_m)} \sum_{i=1}^{k_m} \frac{C_m(u_{mi})}{U'_m(u_{mi})} p_{mi}^2 + V(w_i, u_{mj}).$$

Therefore the Hamilton-Jacobi equation

$$(2.7) \quad H\left(\frac{\partial S}{\partial u_i}, u_i\right) = \eta_1$$

for the generating function  $S(w_i, u_{mj}, \eta_1, \dots, \eta_n)$  assumes the following explicit form

$$(2.8) \quad \sum_{i=1}^k \frac{2B(w_i)}{w_i W'(w_i)} \left(\frac{\partial S}{\partial v_i}\right)^2 - \sum_{m=0}^k \frac{2B'(b_m)}{b_m W(b_m)} \sum_{i=1}^{k_m} \frac{C_m(u_{mi})}{U'_m(u_{mi})} \left(\frac{\partial S}{\partial u_{mi}}\right)^2 + V(w_i, u_{mj}) = \eta_1$$

in the elliptic coordinates  $(v_i, u_{mj})$ .

### 3. The family of the integrable Lagrangian systems

**Theorem.** *The Lagrange system with the Lagrangian (1.1) on a quadric (1.2) with an additional symmetry (1.4) is integrable by quadratures if the potential  $V(x_0, \dots, x_n)$  in the elliptic coordinates  $(w_i, u_{mj})$  has the form*

$$(3.1) \quad V(w_i, u_{mj}) = \sum_{i=1}^k \frac{F_i(w_i)}{W'(w_i)} + \sum_{m=0}^k \frac{1}{W(b_m)} \sum_{i=1}^{k_m} \frac{F_{mi}(u_{mi})}{U'_m(u_{mi})},$$

where  $F_i(v_i)$  and  $F_{mi}(u_{mi})$  are arbitrary smooth functions of one variable.

*Proof.* We use the classical Hamilton-Jacobi method for the Hamiltonian system (2.5), (2.6), corresponding to the Lagrangian (1.1). To construct the solution of the Hamilton-Jacobi equation (2.8) we represent the generating function  $S(u_i, \eta_j)$  in the form

$$(3.2) \quad \begin{aligned} S(w_i, u_{mj}, \eta_1, \dots, \eta_n) &= G(w_1, \dots, w_k, \eta_1, \dots, \eta_k, \eta_{m1}) + \\ &+ \sum_{m=0}^k G_m(u_{m1}, \dots, u_{mk_m}, \eta_{m1}, \dots, \eta_{mk_m}). \end{aligned}$$

The Hamilton-Jacobi equation (2.8) after substituting the expressions (3.1) and (3.2) is split into  $k + 1$  independent equations for the functions  $(G_m)$ ,

$$(3.3) \quad \sum_{i=1}^{k_m} \frac{1}{U'_m(u_{mi})} \left( F_{mi}(u_{mi}) - \frac{2}{b_m} B'(b_m) C_m(u_{mi}) \left( \frac{\partial G_m}{\partial u_{mi}} \right)^2 \right) = \eta_{m1}$$

and one independent equation for the function  $G$ ,

$$(3.4) \quad \sum_{i=1}^k \frac{1}{U'(w_i)} \left( F_i(w_i) + \frac{2}{w_i} B(w_i) \left( \frac{\partial G}{\partial w_i} \right)^2 \right) + \sum_{m=0}^k \frac{\eta_{m1}}{W(b_m)} = \eta_1.$$

Let us introduce  $k + 1$  polynomials

$$(3.5) \quad P_m(z) = \eta_{m1} z^{k_m-1} + \eta_{m2} z^{k_m-2} + \dots + \eta_{mk_m}$$

and one polynomial  $P(z)$

$$(3.6) \quad P(z) = \eta_1 z^{k-1} + \eta_2 z^{k-2} + \dots + \eta_k,$$

where the coefficients  $\eta_{mi}$  and  $\eta_j$  are independent constants. From the Cauchy theorem about the residues the known identities follow

$$(3.7) \quad \eta_{m1} = \frac{1}{2\pi i} \oint \frac{P_m(z)}{U_m(z)} dz = \sum_{i=1}^{k_m} \frac{P_m(u_{mi})}{U'_m(u_{mi})},$$

$$(3.8) \quad \eta_1 = \frac{1}{2\pi i} \oint \frac{P(z)}{W(z)} dz = \sum_{i=1}^k \frac{P(w_i)}{W'(w_i)}.$$

By substituting the expression (3.7) into the equation (3.3) we obtain the explicit solution

$$(3.9) \quad \begin{aligned} G_m(u_{m1}, \dots, u_{mk_m}, \eta_{m1}, \dots, \eta_{mk_m}) &= \\ &= \sum_{i=1}^{k_m} \int_0^{u_{mi}} \left( \frac{b_m(F_{mi}(z) - P_m(z))}{2B'(b_m)C_m(z)} \right)^{\frac{1}{2}} dz. \end{aligned}$$

To solve equation (3.4) we prove the following statement.

**Lemma.** *The function  $U(z)$  in (1.12) satisfies the identity*

$$(3.10) \quad \frac{1}{W(z)} = \sum_{i=1}^k \frac{1}{W'(w_i)} \cdot \frac{1}{z - w_i}.$$

*Proof.* The residues of two meromorphic functions in both hand-sides of (3.10) in the poles  $(w_1, \dots, w_k)$  coincide. Both functions tend to zero for  $|z| \rightarrow \infty$ . Therefore these two meromorphic functions coincide. ■

As a consequence of this Lemma we get the useful identity

$$(3.11) \quad \frac{1}{W(b_m)} = \sum_{i=1}^k \frac{1}{W'(w_i)} \cdot \frac{1}{b_m - w_i}.$$

Equation (3.4) after the substitution of the expressions (3.8) and (3.11) takes the form

$$(3.12) \quad \sum_{i=1}^k \frac{1}{W'(w_i)} \left( F_i(w_i) - P(w_i) + \sum_{m=0}^k \frac{\eta_{m1}}{b_m - w_i} + \frac{2}{w_i} B(w_i) \left( \frac{\partial G}{\partial w_i} \right)^2 \right) = 0.$$

The explicit solution of this equation is defined by the formula

$$(3.13) \quad \begin{aligned} G(w_1, \dots, w_k, \eta_1, \dots, \eta_k, \eta_{m1}) = \\ = \sum_{i=1}^k \int_0^{w_i} \left( \frac{z}{2B(z)} \left( P(z) + \sum_{m=0}^k \frac{\eta_{m1}}{z - b_m} - F_i(z) \right) \right)^{\frac{1}{2}} dz. \end{aligned}$$

Therefore the generating function  $S(w_i, u_{mj}, \eta_1, \dots, \eta_n)$  (3.2) is completely determined by the formulae (3.9) and (3.13). The generating function  $S(w_i, u_{mj}, \eta_i, \eta_{mj})$  (3.2) defines the canonical transformation [3] to new phase coordinates

$$(3.14) \quad \xi_i, \xi_{mj}, \eta_i, \eta_{mj}, \quad \begin{cases} i = 1, \dots, k; \\ m = 0, 1, \dots, k; \\ j = 1, \dots, k_m \end{cases}$$

by the formulae

$$(3.15) \quad \begin{aligned} \xi_i &= -\frac{\partial S}{\partial \eta_i}, & \xi_{mj} &= -\frac{\partial S}{\partial \eta_{mj}}, \\ p_i &= \frac{\partial S}{\partial w_i}, & p_{mj} &= \frac{\partial S}{\partial u_{mj}}. \end{aligned}$$

In view of the Hamilton-Jacobi equation (2.7) the Hamiltonian  $H$  (2.6) takes in coordinates (3.14) the simplest form,

$$(3.16) \quad H(\xi_i, \xi_{mj}, \eta_i, \eta_{mj}) = \eta_1.$$

Therefore the Hamiltonian equations in coordinates (3.14) have the form

$$(3.17) \quad \dot{\xi}_i = -\delta_{1i}, \quad \dot{\xi}_{mj} = 0, \quad \dot{\eta}_i = 0, \quad \dot{\eta}_{mj} = 0.$$

So we get  $\xi_1 = c_1 - t$  and all other coordinates are constant.

Dynamics of trajectories of the system (2.5), (2.6) is integrable by quadratures, because it is obtained from the simplest dynamics (3.17) by a canonical transformation, inverse to (3.15). The generating function  $S$  (3.2) is determined by the quadratures (3.9), (3.13). Hence, the theorem is proved. ■

**Remark 1.** The constructed family of integrable Lagrange systems has applications in the rigid body dynamics. This family leads by the methods of [4] to the integrable cases of an axially-symmetric rigid body rotation around fixed point in the fields with the special potentials  $V(x_1, x_2, x_3)$  (3.1).

**Remark 2.** In [5] the "degeneration" of elliptic coordinates in the Euclidean space  $\mathbb{R}^n$  was considered, corresponding to the limit  $a_1 \rightarrow a_2$ , and separable Hamiltonian systems in the case  $a_1 = a_2$  having "invariance with respect to rotations in the  $(q_1, q_2)$  plane" (see [5], p. 1343). Our Theorem, which can be extended to the Euclidean space  $\mathbb{R}^n$ , describes more complicated separable Hamiltonian systems without any rotational symmetry, and without angular momentum-type first integrals.

## References

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