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Orthogonal separation of variables on manifolds with constant curvature

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Abstract: Coordinates which allow the integration by separation of variables of the geodesic Hamilton-Jacobi equation are called separable. Particular interest is placed on orthogonal separable coordinates. In this paper it is proved that on a Riemannian manifold with constant curvature and on a Lorentzian manifold with constant positive curvature every system of separable coordinates has an orthogonal equivalent, i.e. that in these manifolds the integration by separation of variables of the geodesic Hamilton-Jacobi equation always occurs in orthogonal coordinates. Proofs of this property concerning strictly-Riemannian manifolds of positive, negative and zero constant curvature (and also for conformally flat manifolds) were firstly given by Kalnins and Miller (1982–1986). The proof presented here is based on elementary properties of Killing vectors of an affine space and on a geometrical characterization of the equivalence classes of separable coordinates.

Keywords: Hamilton-Jacobi equation, geodesic flow, separable coordinates, Riemannian manifolds with constant curvature.

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1. Introduction

The main purpose of the present paper is to prove the following

Theorem. *On a Riemannian manifold with positive metric and constant curvature and on a Lorentzian manifold with constant positive curvature every system of separable coordinates has an orthogonal equivalent.*

Proofs of the existence of orthogonal equivalents of separable systems on the Euclidean spaces $\mathbf{E}_n$, on the spheres $\mathbf{S}_n$, on the hyperboloids $\mathbf{H}_n$ and on the conformally Euclidean spaces $\mathbf{C}_n$ are given in [11, 12, 7]. A first consequence of these interesting results is that the classifications of the orthogonal separable coordinate systems given by

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Eisenhart and Olevskii, for the 3-dimensional Euclidean space [4], the conformally Euclidean 3-spaces [5], and the 3-sphere [14], are in fact exhaustive. Moreover, as Kalnins and Miller and co-workers have done, this classification can be extended to all the above mentioned spaces, for all dimensions [7].

In our approach, the proof of the Theorem is an occasion for exploring the geometry of the separation of variables. By recalling the theorem on the normal form of the metric tensor components in separable coordinates [1,2,9,10], it can be seen that the non-diagonal part of the corresponding matrix is due to the presence of ignorable coordinates (first class coordinates, according to the classification of Levi-Civita). The geometrical counterpart of these coordinates is an abelian sub-algebra of the Lie algebra of Killing vectors, and it is shown that this subalgebra is normal (i.e. its orthogonal distribution is completely integrable) (Section 2). Hence, for proving the existence of an orthogonal equivalent of a separable coordinate system one has to prove that the corresponding subalgebra of Killing vectors has an orthogonal basis. We consider the Killing vector algebras of a Euclidean (positive metric) affine space $\mathbf{E}_n$ and of a Minkowskian (hyperbolic metric) affine space $\mathbf{M}_n$. As for all affine spaces, after fixing a point, the Killing vectors are characterized by pairs $(\boldsymbol{A}, \boldsymbol{u})$, where $\boldsymbol{u}$ is a vector of the underlying vector space and $\boldsymbol{A}$ is a skew-symmetric endomorphism on this space. So, we can relate the differential properties of the Killing vector fields with the algebraic properties of these pairs (Section 2 and 3). We need a preliminary analysis on the canonical form of skew-symmetric 2-tensors in a Euclidean or hyperbolic vector space (Section 3). The central result is the algebraic characterization of a normal abelian subalgebra of Killing vectors (Theorem 1, Section 4); for its proof we used as a tool a particular version of the integrability criterion of a Pfaffian system (Lemma, Section 4). After this characterization, we can immediately derive results concerning the existence of orthogonal equivalents of separable coordinate systems, not only for $\mathbf{E}_n$ and $\mathbf{M}_n$ but also for their hyperquadrics [6], which are manifolds of constant curvature (Section 5).

2. Separation of variables on Riemannian manifolds

Definition 1. A coordinate system (q^i) on a Riemannian manifold (Q_n, g) is called *separable* if the geodesic Hamilton-Jacobi equation

$$\frac{1}{2} g^{ij} \frac{\partial W}{\partial q^i} \frac{\partial W}{\partial q^j} = h \tag{1}$$

has a complete integral $W(q^i, a^k)$ of the form

$$W = W_1(q^1, a^k) + W_2(q^2, a^k) + \cdots + W_n(q^n, a^k). \tag{2}$$

A coordinate system (q^i) is said to be *orthogonal* if $g^{ij} = 0$ for $i \neq j$.

Definition 2. Two separable coordinate systems are said to be *equivalent* if in every domain $U \subset Q$ where both are defined they give rise to the same complete integral, interpreted as a Lagrangian foliation of the cotangent bundle $T^*U \subset T^*Q$.

An equivalent relation is then defined in the set of the separable systems whose domains contain a given point of Q.

The theory of separation of variables is based on a classification of the coordinates due to Levi-Civita [13].

Definition 3. Let (q^i, p_i) be the canonical coordinates on T^*Q corresponding to a coordinate system (q^i) on Q. Let $H = \frac{1}{2} g^{ij} p_i p_j$ be the geodesic Hamiltonian. A coordinate q^i is said to be of *first class* if the function $\partial H / \partial q^i$, which is a polynomial of degree 2 in the momenta, is divisible by $\partial H / \partial p_i$, which is a polynomial of degree 1 in the momenta. Otherwise, the coordinate is said to be of *second class*. A coordinate of first class q^i is said to be *ignorable* if $\partial H / \partial q^i = 0$.

A further definition is needed when the metric g is not positive.

Definition 4. A coordinate of second class q^i is said to be *isotropic* or *null* if $g^{ii} = 0$ (i.e. if the gradient of the coordinate q^i is an isotropic vector)

It can be proved [1,2] that

Proposition 1. (i) *The numbers (m_1, m_2, l) of non-isotropic second class, of isotropic second class and of first class coordinates respectively, are invariant within an equivalence class of separable coordinates.*

(ii) *The second class coordinates are orthogonal and* essential (*in two equivalent separable systems they are related by a separated transformation, i.e. a transformation whose Jacobian is diagonal*).

(iii) *In an equivalence class of separable coordinates there exists a system (q^i) where the first class coordinates (q^α) are ignorable and such that the matrix (g^{ij}) of the contravariant components of the metric tensor has the following* normal form:

$$\begin{pmatrix} \delta^{\hat{a}\hat{b}} g^{\hat{a}\hat{a}} & 0 & 0 \\ 0 & 0 & g^{\bar{a}\beta} \\ 0 & g^{\alpha\bar{a}} & g^{\alpha\beta} \end{pmatrix} \tag{3}$$

We used the following convention for the indices. Latin indices $h, i, j, \dots$ run from 1 to n, the dimension of Q. For indices corresponding to second class coordinates, running from 1 to $m \leqslant n$, we use the first Latin letters $a, b, \dots$. We denote by $\hat{a}, \hat{b}, \dots$ indices corresponding to non-isotropic second class coordinates; they run from 1 to m_1. We denote by $\bar{a}, \bar{b}, \dots$ indices corresponding to isotropic second class coordinates; they run from $m_1 + 1$ to m_2, and $m_1 + m_2 = m$. For indices corresponding to first class coordinates, running from $m + 1$ to n, we use Greek letters $\alpha, \beta, \dots$. We set $l = n - m$.

An immediate consequence of Proposition 1 is the following

Proposition 2. (i) *Separable systems without first class coordinates are orthogonal.*

(ii) *Separable systems with isotropic second class coordinates cannot have an orthogonal equivalent.*

Examples given in [8] as truly non-orthogonal separable coordinates on the Minkowskian space $\mathbf{M}_3$ enter in point (ii) of this proposition.

To state a further consequence of Proposition 1 we need to recall the following

Definition 5. A *Killing vector* X on a Riemannian manifold (Q_n, g) is a vector field on Q such that $d_X g = 0$. (Here d_X denotes the Lie derivative with respect to the field X).

There are other equivalent definitions of Killing vectors, which we do not need to mention here. Killing vectors are in fact infinitesimal isometries. They form a finite-dimensional subalgebra of the Lie algebra of vector fields on Q, which we denote by $\mathcal{K}^1(Q)$.

Definition 6. We say that a subalgebra $D \subset \mathcal{K}^1(Q)$ is *normal* if the distribution $\Delta^\perp$ orthogonal to the distribution Δ generated by D is completely integrable. We say that a subalgebra $D \subset \mathcal{K}^1(Q)$ is *(metrically) degenerate* if the metric tensor reduced to $\Delta_q \subset T_q Q$ is singular for each point $q \in Q$.

Here, a distribution Δ is intended as a subbundle of the tangent bundle TQ. Hence we have to exclude the closed subset of Q made of those points in which the vector fields of the differential system D span a space whose dimension is less than the dimension of the subalgebra D. (For questions concerning differential systems and distribution we refer to [15]).

Proposition 3. *Let $U \subset Q$ be the open domain of definition of an equivalence class of separable coordinates with $l > 0$ first class coordinates. Then there exists a normal abelian subalgebra $D \subset \mathcal{K}^1(U)$ of dimension l. This subalgebra is degenerate if and only if there are isotropic second class coordinates. The integral manifolds of the corresponding distribution Δ are defined by equations $q^a =$ const., where (q^a) are the second class coordinates.*

Proof. Let (q^i) be separable coordinates for which the matrix (g^{ij}) takes the normal form (3) (they are called *normal separable coordinates*). Let us consider the vector fields $X_i = \partial/\partial q^i$. All these vectors commute, $[X_i, X_j] = 0$. Since the coordinates (q^α) are ignorable, the vectors (X_α) are Killing vectors. They are independent everywhere in the domain U of the coordinates, so that they define an abelian subalgebra of $\mathcal{K}^1(U)$ of dimension l and a distribution Δ. Let us consider the orthogonal distribution $\Delta^\perp$.
(i) Assume that there are no isotropic second class coordinates. The normal form (3) reduces to

$$\begin{pmatrix} \delta^{ab}\, g^{aa} & 0 \\ 0 & g^{\alpha\beta} \end{pmatrix}. \tag{4}$$

We see that the vectors X_a generate $\Delta^\perp$. Since they commute, this distribution is completely integrable. Thus D is normal. Since $\Delta \cap \Delta^\perp = 0$, both distributions are not degenerate. (ii) Assume that isotropic second class coordinates are present. Let us consider the dual vector fields $X^i = g^{ij} X_j$, so that $X^i \cdot X_j = \delta^i_j$ (here $\cdot$ denotes the scalar product). From the normal form (3) it follows in particular that

$$X^{\hat{a}} = g^{\hat{a}\hat{a}} X_{\hat{a}}, \quad X^{\bar{a}} = g^{\bar{a}\alpha} X_\alpha, \quad X^{\bar{a}} \cdot X^{\bar{a}} = 0.$$

Hence the isotropic vector fields $(X^{\bar{a}})$ span an isotropic distribution I which is contained in the distribution Δ generated by (X_α), and which is also orthogonal to Δ, since $X^{\bar{a}} \cdot X_\alpha = 0$. This proves that Δ is degenerate. Since we have also $X^{\hat{a}} \cdot X_\alpha = X_{\hat{a}} \cdot X_\alpha = 0$, the orthogonal distribution $\Delta^\perp$ is generated by the vector fields $(X_{\hat{a}}, X^{\bar{a}})$. It is obvious that $[X_{\hat{a}}, X_{\hat{b}}] = 0$ and $[X_\alpha, X_{\hat{a}}] = 0$. Moreover, $[X^{\bar{a}}, X^{\bar{b}}] = [g^{\bar{a}\alpha} X_\alpha, g^{\bar{b}\beta} X_\beta] = 0$ and $[X_\alpha, X^{\bar{a}}] = [X_\alpha, g^{\bar{a}\beta} X_\beta] = 0$ since X_α are Killing vectors. Finally, $[X_{\hat{a}}, X^{\bar{a}}] = [X_{\hat{a}}, g^{\bar{a}\alpha} X_\alpha] = X_{\hat{a}} g^{\bar{a}\alpha} X_\alpha$. However, it is known from the theory of the separation of variables [2] that $X_{\hat{a}} g^{\bar{a}\alpha} = g^{\bar{a}\alpha} f^{\bar{a}}_{\hat{a}}$ where $f^{\bar{a}}_{\hat{a}}$ is a factor which does not depend on the index α. Thus the last expression of the Lie brackets becomes $= f^{\bar{a}}_{\hat{a}} g^{\bar{a}\alpha} X_\alpha = f^{\bar{a}}_{\hat{a}} X^{\bar{a}}$. These complete the proof that $\Delta^\perp$ is completely integrable. □

Proposition 4. *A separable system without isotropic second class coordinates has an orthogonal equivalent if and only if the normal abelian subalgebra D has an orthogonal basis.*

Proof. According to the general theory of separation, a linear transformation with constant coefficients of ignorable first class coordinates (q^α) produces an equivalent system of separable coordinates. Such a transformation is equivalent to a linear transformation of the vectors X_α. The normal form (4) shows that the matrix (g^{ij}) is diagonal when $g^{\alpha\beta} = 0$ for $\alpha \neq \beta$, i.e. $g_{\alpha\beta} = X_\alpha \cdot X_\beta = 0$ for $\alpha \neq \beta$. □

This last proposition shows that the proof that on a manifold Q every separable system (without isotropic second class coordinates) has an orthogonal equivalent, is equivalent to proving that all normal non-degenerate abelian subalgebras of $\mathcal{K}^1(Q)$ have an orthogonal basis.

3. Killing vectors in an affine space

Let (Q,g) be a Euclidean or pseudo-Euclidean affine space. We choose a point $O \in Q$ (the origin) so that the affine space is identified with the underlying Euclidean or pseudo-Euclidean vector space (E,g): points of Q are vectors of E.

Let us denote by $\boldsymbol{x} \cdot \boldsymbol{y}$ the scalar product of two vectors of E: $\boldsymbol{x} \cdot \boldsymbol{y} = g(\boldsymbol{x}, \boldsymbol{y})$ and set $||\boldsymbol{x}|| = \boldsymbol{x} \cdot \boldsymbol{x}$. Let us denote by $\boldsymbol{A} \cdot \boldsymbol{B}$ the composition of two linear endomorphisms on E and set $[\boldsymbol{A}, \boldsymbol{B}] = \boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{A}$; they *commute* iff $[\boldsymbol{A}, \boldsymbol{B}] = 0$. Let us denote by $\boldsymbol{x} \cdot \boldsymbol{A}$ the image of the vector $\boldsymbol{x} \in E$ by $\boldsymbol{A}$. The *rank* of an endomorphism $\boldsymbol{A}$ is the dimension of the image space $E \cdot \boldsymbol{A}$. There is a natural identification between linear endomorphisms and bilinear forms on E, defined by equation $\boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{y}$. The metric tensor g corresponds to the identity. Linear endomorphisms or bilinear forms will be simply called 2-*tensors*. A 2-tensor $\boldsymbol{A}$ is *skew-symmetric* if $\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{y} = -\boldsymbol{y} \cdot \boldsymbol{A} \cdot \boldsymbol{x}$. The *exterior product* of two vectors $(\boldsymbol{u}, \boldsymbol{v})$ is the skew-symmetric 2-tensor $\boldsymbol{u} \wedge \boldsymbol{v}$ such that

$$\boldsymbol{x} \cdot (\boldsymbol{u} \wedge \boldsymbol{v}) = \boldsymbol{x} \cdot \boldsymbol{u}\, \boldsymbol{v} - \boldsymbol{x} \cdot \boldsymbol{v}\, \boldsymbol{u}.$$

A skew-symmetric 2-tensor is said to be *simple* if its rank is 2 or, equivalently, if it can be represented (in a non-unique way) as an exterior product of two vectors.

It is known that a vector field X on the affine space (Q, g) is a Killing vector if and only if

$$X(\boldsymbol{x}) = \boldsymbol{x} \cdot A + \boldsymbol{u}, \quad \forall \boldsymbol{x} \in E, \tag{1}$$

where $\boldsymbol{A}$ is a skew-symmetric 2-tensor and $\boldsymbol{u} \in E$. We say that X is a *rotation* (around the origin) if $\boldsymbol{u} = 0$, a *translation* if $\boldsymbol{A} = 0$. Equation (1) will be written

$$X = (\boldsymbol{A}, \boldsymbol{u}). \tag{2}$$

The elements $(\boldsymbol{A}, \boldsymbol{u})$ will be called the *generators* of the Killing vector X. The following identities hold for Killing vectors in an affine space:

$$\begin{aligned}
&a(\boldsymbol{A}, \boldsymbol{u}) = (a\,\boldsymbol{A}, a\,\boldsymbol{u}) \qquad (a \in \mathbb{R}),\\
&(\boldsymbol{A}, \boldsymbol{u}) + (\boldsymbol{B}, \boldsymbol{v}) = (\boldsymbol{A} + \boldsymbol{B}, \boldsymbol{u} + \boldsymbol{v}),\\
&[(\boldsymbol{A}, \boldsymbol{u}), (\boldsymbol{B}, \boldsymbol{v})] = ([\boldsymbol{A}, \boldsymbol{B}], \boldsymbol{u} \cdot \boldsymbol{B} - \boldsymbol{v} \cdot \boldsymbol{A}),\\
&d(\boldsymbol{A}, \boldsymbol{u}) = \boldsymbol{A},\\
&(\boldsymbol{A}, \boldsymbol{u})(\boldsymbol{x}) \cdot (\boldsymbol{B}, \boldsymbol{v})(\boldsymbol{x}) = -\boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{x} + \boldsymbol{x} \cdot \boldsymbol{A} \cdot \boldsymbol{v} + \boldsymbol{x} \cdot \boldsymbol{B} \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{v}.
\end{aligned} \tag{3}$$

In $(3)_4$ and in the following discussion we use the natural indentification between vector fields and exterior 1-forms, so that the differential dX of a vector field makes sense. The square brackets at the right hand side of $(3)_3$ are the Lie brackets of vector fields. Identities $(3)_{1,2,3}$ give a linear representation of $\mathcal{K}^1(Q)$ in the vector space $\Omega^2(E) \oplus E$, where $\Omega^2(E)$ is the space of skew-symmetric 2-tensors over E.

From $(3)_3$ and $(3)_5$ it follows that

Proposition 1. (i) *Two Killing vector fields* $(\boldsymbol{A}, \boldsymbol{u})$ *and* $(\boldsymbol{B}, \boldsymbol{v})$ *commute if and only if*

$$[\boldsymbol{A}, \boldsymbol{B}] = 0, \quad \boldsymbol{u} \cdot \boldsymbol{B} = \boldsymbol{v} \cdot \boldsymbol{A}.$$

(ii) *Two Killing vector fields* $(\boldsymbol{A}, \boldsymbol{u})$ *and* $(\boldsymbol{B}, \boldsymbol{v})$ *are everywhere orthogonal if and only if*

$$\boldsymbol{A} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{A} + \boldsymbol{u} \cdot \boldsymbol{B} = 0, \quad \boldsymbol{u} \cdot \boldsymbol{v} = 0.$$

Our aim is to relate the differential properties of sets of Killing vectors with the algebraic properties of their generators (like Proposition 2 (i)). For our purposes we restrict our attention to Euclidean (i.e. elliptic) or Minkowskian (i.e. hyperbolic) affine spaces, i.e. to the cases $Q = \mathbf{E}_n$ or $Q = \mathbf{M}_n$. The signature of $\mathbf{M}_n$ is assumed to be $(- + + \ldots +)$.

To this end a preliminary analysis of skew-symmetric 2-tensors is needed. It can be shown that (see [3] for a discussion on this topic):

Proposition 2. *A skew-symmetric* 2*-tensor* $\boldsymbol{A}$ *admits one of the following canonical representations*:

$$\begin{aligned} &\boldsymbol{A} = \mu^i\, \boldsymbol{E}_i \qquad (i = 1, \dots, m), \\ &\boldsymbol{A} = \mu\, \boldsymbol{H} + \mu^i\, \boldsymbol{E}_i, \\ &\boldsymbol{A} = \boldsymbol{P} + \mu^i\, \boldsymbol{E}_i, \end{aligned} \tag{4}$$

with

$$\begin{aligned} &\boldsymbol{E}_i = \boldsymbol{a}_i \wedge \boldsymbol{b}_i, \\ &\boldsymbol{H} = \boldsymbol{a} \wedge \boldsymbol{b}, \\ &\boldsymbol{P} = \boldsymbol{c} \wedge \boldsymbol{d}, \end{aligned} \tag{5}$$

where the $2m$ *vectors* $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ *form a canonical basis of a Euclidean subspace* E_{2m} (*they are spacelike, orthogonal, and unitary*), *the* 2 *vectors* $(\boldsymbol{a}, \boldsymbol{b})$ *form a canonical basis of a hyperbolic* 2*-space* H_2 *orthogonal to* E_{2m}, *the* 2 *vectors* $(\boldsymbol{c}, \boldsymbol{d})$ *are independent and orthogonal, they belong to the orthogonal space* $E_{2m}^{\perp}$ *and* $\boldsymbol{c}$ *is isotropic* (*thus* $\boldsymbol{d}$ *is spacelike*).

In case $(4)_1$ $\boldsymbol{A}$ is said to be *elliptic* and the vectors $\boldsymbol{b}_i \pm i\, \boldsymbol{a}_i$ are complex eigenvectors corresponding to the eigenvalues $\pm \imath \mu^i$. In case $(4)_2$ $\boldsymbol{A}$ is said to be *hyperbolic*, the vectors $\boldsymbol{a}_1 \mp \boldsymbol{b}_1$ are isotropic eigenvectors corresponding to the eigenvalues $\mp \mu^1$; for $i = 2, \dots, m$ the vectors $\boldsymbol{b}_i \pm i\, \boldsymbol{a}_i$ are complex eigenvectors corresponding to the eigenvalues $\pm\, i\, \mu^i$. In case $(4)_3$ $\boldsymbol{A}$ is said to be *parabolic*, the vector $\boldsymbol{c}$ is an eigenvector corresponding to the eigenvalue 0, the vectors $\boldsymbol{b}_i \pm i\, \boldsymbol{a}_i$ are complex eigenvectors corresponding to the eigenvalues $\pm \imath \mu^i$.

We recall that on a hyperbolic vector space: the maximum dimension of an isotropic subspace is 1; a subspace S is degenerate if and only if it is orthogonal to an isotropic subspace I, which is contained in S; the vectors of $S = I^{\perp}$ not belonging to I are spacelike; a subspace S is non-degenerate if and only if $S \cap S^{\perp} = 0$; the subspace $\boldsymbol{u}^{\perp}$ orthogonal to a timelike vector $\boldsymbol{u}$ is spacelike (i.e. elliptic).

According to this classification we have three kinds of Killing vectors: *elliptic*, *hyperbolic* and *parabolic*. In a Euclidean space all Killing vectors are elliptic.

It can be proved (see [3] for details) that

Proposition 3. *A set* $(\boldsymbol{A}_\alpha)$ *of skew-symmetric endomorphisms commute if and only if they assume the canonical form*

$$\boldsymbol{A}_\alpha = \varepsilon_\alpha\, \boldsymbol{P}_\alpha + \mu_\alpha^i\, \boldsymbol{S}_i, \qquad i = 1, \dots, s. \tag{6}$$

with

$$\boldsymbol{P}_\alpha = \boldsymbol{c} \wedge \boldsymbol{d}_\alpha, \qquad \boldsymbol{S}_i = \boldsymbol{a}_i \wedge \boldsymbol{b}_i, \tag{7}$$

where the pairs $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ *are independent and span non-degenerate bidimensional orthogonal subspaces*, (S_i), $(\boldsymbol{c}, \boldsymbol{d}_\alpha)$ *are orthogonal to all spaces* (S_i), $\boldsymbol{c}$ *is isotropic and* $(\boldsymbol{d}_\alpha)$ *are orthogonal to* $\boldsymbol{c}$.

Remark 1. We can choose the vectors $(\boldsymbol{a}_i, \boldsymbol{b}_i)$ to be unitary and orthogonal, and $(\boldsymbol{d}_\alpha)$ to be unitary. If at least one of the coefficients $\mu_\alpha \neq 0$, then all these vectors are necessarily spacelike.

4. Normal abelian subalgebras of Killing vectors in Euclidean or Minkowskian affine spaces

Theorem 1. *On* E_n *or* M_n *a subalgebra of Killing vectors of dimension* l *is abelian and normal (Definition* 6, *Section* 1) *if and only if it admits a basis*

$$\begin{pmatrix} \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho & \boldsymbol{u}_\rho \\ \boldsymbol{c} \wedge \boldsymbol{d}_\sigma & \boldsymbol{v}_\sigma \\ 0 & \boldsymbol{w}_\tau \end{pmatrix} \qquad \begin{array}{r} \rho = 1, \dots, p \\ \sigma = 1, \dots, q \\ p+q+r=l, \quad \tau = 1, \dots, r \end{array} \tag{1}$$

where vectors $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{c}, \boldsymbol{d}_\sigma)$ *are independent,* $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho)$ *are orthogonal and non-isotropic,* $\boldsymbol{c} \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho\}^\perp$ *is isotropic,* $\boldsymbol{d}_\sigma \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{c}\}^\perp$, $\boldsymbol{u}_\rho \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho\}$, $\boldsymbol{w}_\tau \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{c}, \boldsymbol{d}_\sigma\}^\perp$, *and moreover, if the space* $\{\boldsymbol{w}_\tau\}$ *is a non-degenerate subspace or zero* $(r = 0)$, $\boldsymbol{v}_\sigma \in \{\boldsymbol{c}, \boldsymbol{d}_1, \dots, \boldsymbol{d}_q\}$, $\boldsymbol{v}_\sigma \cdot \boldsymbol{d}_{\sigma'} = \boldsymbol{v}_{\sigma'} \cdot \boldsymbol{d}_\sigma$.

Here, we denote by $\{\boldsymbol{a}, \boldsymbol{b}, \dots\}$ the space generated by the vectors $(\boldsymbol{a}, \boldsymbol{b}, \dots)$, and $\{\boldsymbol{a}, \boldsymbol{b}, \dots\}^\perp$ is the orthogonal subspace.

Remark 1. If the space $\{\boldsymbol{w}_\tau\}$ is degenerate, we have no conditions over the vectors $\boldsymbol{v}_\sigma$ and, moreover, we can always choose $\boldsymbol{w}_1 = \boldsymbol{c}$. We can choose the vectors $(\boldsymbol{d}_\sigma)$ to be orthogonal to each other. We can choose the vectors $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{d}_\sigma)$ to be unitary. We can choose the vectors $(\boldsymbol{w}_\tau)$ to be unitary, except for $\boldsymbol{w}_1 = \boldsymbol{c}$ in the degenerate case. Some of the integers (p, q, r) may be zero.

To prove this proposition we need the following version of the complete integrability criterion of a Pfaffian system:

Lemma. *Let* $(\theta_\alpha) = (\theta_1, \theta_2, \dots, \theta_l)$ *be (local) characteristic* 1*-forms of a regular distribution* $\Theta \subset TQ$: $\Theta = \{v \in TQ \mid \langle v, \theta_\alpha \rangle = 0,\ \alpha = 1, 2, \dots, l\}$. *Then the distribution is completely integrable if and only if*

$$\theta_1 \wedge \theta_2 \wedge \dots \wedge \theta_l \wedge d\theta_\alpha = 0 \qquad (\alpha = 1, 2, \dots, l). \tag{2}$$

Proof. The usual integrability criterion for a distribution characterized by independent differential 1-forms (θ_α) is that 1-forms (ω_α^β) exist so that

$$d\theta_\alpha = \omega_\alpha^\beta \wedge \theta_\beta. \tag{3}$$

Equations (3) imply (2). To prove the converse, let us consider a local frame $(\theta_i) = (\theta_\alpha, \theta_a)$, $i = 1, \dots, n$, $a = l+1, \dots, n$, extending the given system (θ_α). Let us represent the differentials $d\theta_\alpha$ in this frame:

$$d\theta_\alpha = f_\alpha^{ij}\, \theta_i \wedge \theta_j = f_\alpha^{\beta\gamma}\, \theta_\beta \wedge \theta_\gamma + f_\alpha^{ab}\, \theta_a \wedge \theta_b + 2 f_\alpha^{\beta a}\, \theta_\beta \wedge \theta_a.$$

From equation (2), from equation $\theta_1 \wedge \ldots \wedge \theta_l \wedge d\theta_1 = 0$ for instance, it follows that $f_1^{ab}\,\theta_a \wedge \theta_b \wedge \theta_1 \wedge \ldots \wedge \theta_l = 0$, i.e. $f_1^{ab} = 0$, since $(\theta_a, \theta_b, \theta_1, \ldots, \theta_l)$ are independent (when $a \neq b$). Hence, $f_\alpha^{ab} = 0$ and

$$d\theta_\alpha = f_\alpha^{\beta\gamma}\,\theta_\beta \wedge \theta_\gamma + 2f_\alpha^{\beta a}\,\theta_\beta \wedge \theta_a.$$

This is an expression of (3). □

Proof of Theorem 1. Let $(X_\alpha = (\boldsymbol{A}_\alpha, \boldsymbol{v}_\alpha))$, $\alpha = 1, \ldots, l$, be a basis of a subalgebra $D \subset \mathcal{K}^1(Q)$ of dimension l. Let us assume that D is abelian. Since all X_α commute, according to Proposition 1 (i), Section 3, all $\boldsymbol{A}_\alpha$ commute, and according to Proposition 3, Section 3, we can always write

$$\boldsymbol{A}_\alpha = \mu_\alpha\, \boldsymbol{c} \wedge \boldsymbol{d}_\alpha + \mu_\alpha^i\, \boldsymbol{a}_i \wedge \boldsymbol{b}_i, \qquad i = 1, \ldots s,$$

where, for each α the $2s + 2$ vectors $(\boldsymbol{c}, \boldsymbol{d}_\alpha, \boldsymbol{a}_i, \boldsymbol{b}_i)$ are independent and orthogonal and $\boldsymbol{c}$ is isotropic. With the exception of $\boldsymbol{c}$, we can always assume that these vectors are unitary. After linear transformations involving the parabolic terms we can reduce to a system

$$\begin{pmatrix} \mu_\iota^i\, \boldsymbol{a}_i \wedge \boldsymbol{b}_i & \boldsymbol{v}_\iota \\ \boldsymbol{c} \wedge \boldsymbol{d}_\sigma + \mu_\sigma^i\, \boldsymbol{a}_i \wedge \boldsymbol{b}_i & \boldsymbol{v}_\sigma \end{pmatrix} \quad \begin{matrix} \iota = 1, \ldots, k \\ \sigma = k+1, \ldots, m \end{matrix}$$

where all $\boldsymbol{d}_\sigma$ are independent and orthogonal. Let p be the dimension of the space generated by $\boldsymbol{A}_\iota = \mu_\iota^i\, \boldsymbol{a}_i \wedge \boldsymbol{b}_i$. After linear transformations involving a basis of this space we reduce to a system

$$\begin{pmatrix} \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho + \mu_\rho^\kappa\, \boldsymbol{a}_\kappa \wedge \boldsymbol{b}_\kappa & \boldsymbol{v}_\rho \\ 0 & \boldsymbol{w}_\tau \\ \boldsymbol{c} \wedge \boldsymbol{d}_\sigma + \mu_\sigma^i\, \boldsymbol{a}_i \wedge \boldsymbol{b}_i & \boldsymbol{v}_\sigma \end{pmatrix} \quad \begin{matrix} \rho = 1, \ldots, p\,;\ \kappa = p+1, \ldots, s \\ \tau = p+1, \ldots, k \\ \sigma = k+1, \ldots, m \end{matrix}$$

By subtracting from the last $q = m - k$ elements the first p elements we finally get, after a suitable reordering, a system

$$\begin{pmatrix} X_\rho \\ Y_\sigma \\ Z_\tau \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho + \mu_\rho^\kappa\, \boldsymbol{a}_\kappa \wedge \boldsymbol{b}_\kappa & \boldsymbol{u}_\rho \\ \boldsymbol{c} \wedge \boldsymbol{d}_\sigma + \nu_\sigma^\kappa\, \boldsymbol{a}_\kappa \wedge \boldsymbol{b}_\kappa & \boldsymbol{v}_\sigma \\ 0 & \boldsymbol{w}_\tau \end{pmatrix} \quad \begin{matrix} \rho = 1, \ldots, p \\ \sigma = 1, \ldots, q \\ \tau = 1, \ldots, r \end{matrix} \tag{5}$$

All the vector fields (5) commute. According to Proposition 1 (i), Section 3, this implies in particular

$$\boldsymbol{w}_\tau \in \{\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{c}, \boldsymbol{d}_\sigma\}^\perp. \tag{6}$$

Let us assume that D is normal. The distribution orthogonal to the vector fields (5) is completely integrable. Then, according to the Lemma, the following conditions hold:

$$X_1 \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge Y_q \wedge Z_1 \wedge \ldots \wedge Z_r \wedge dW = 0$$

where W is any one of the vector fields $(X_\rho, Y_\sigma, Z_\tau)$. Indeed, these vector fields, interpreted as 1-forms, are the characteristic forms of the orthogonal distribution. However, $Z_\tau = \boldsymbol{w}_\tau$, $dZ_\tau = 0$, so that the only significant conditions are

$$\begin{aligned} &X_1 \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge Y_q \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r \wedge dX_\rho = 0, \\ &X_1 \wedge \ldots \wedge X_p \wedge Y_1 \wedge \ldots \wedge Y_q \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r \wedge dY_\sigma = 0, \end{aligned} \tag{7}$$

where:

$$\begin{aligned} &X_\rho = \boldsymbol{x} \cdot \boldsymbol{a}_\rho\, \boldsymbol{b}_\rho - \boldsymbol{x} \cdot \boldsymbol{b}_\rho\, \boldsymbol{a}_\rho + \mu_\rho^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa + \boldsymbol{u}_\rho, \\ &Y_\sigma = \boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_\sigma - \boldsymbol{x} \cdot \boldsymbol{d}_\sigma\, \boldsymbol{c} + \nu_\sigma^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa + \boldsymbol{v}_\sigma, \\ &\boldsymbol{B}_\kappa = \boldsymbol{a}_\kappa \wedge \boldsymbol{b}_\kappa, \\ &dX_\rho = \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho + \mu_\rho^\kappa\, \boldsymbol{B}_\kappa, \\ &dY_\sigma = \boldsymbol{c} \wedge \boldsymbol{d}_\sigma + \nu_\sigma^\kappa\, \boldsymbol{B}_\kappa. \end{aligned}$$

The left hand sides of equations (7) are polynomials of degree $p+q$ in the variables $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$. Let us consider the coefficient of maximal degree for $\sigma = 1$. We get the equation

$$\begin{aligned} &(\boldsymbol{x} \cdot \boldsymbol{a}_1\, \boldsymbol{b}_1 - \boldsymbol{x} \cdot \boldsymbol{b}_1\, \boldsymbol{a}_1 + \mu_1^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \wedge \ldots \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p + \mu_p^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_1 - \boldsymbol{x} \cdot \boldsymbol{d}_1\, \boldsymbol{c} + \nu_1^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \wedge \ldots \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_q - \boldsymbol{x} \cdot \boldsymbol{d}_q\, \boldsymbol{c} + \nu_q^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \\ &\quad \wedge (\boldsymbol{c} \wedge \boldsymbol{d}_1 + \nu_1^\kappa \boldsymbol{B}_\kappa) \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r = 0. \end{aligned}$$

By developing the exterior product, in the resulting sum we get only one term

$$\begin{aligned} &(\boldsymbol{x} \cdot \boldsymbol{a}_1\, \boldsymbol{b}_1 - \boldsymbol{x} \cdot \boldsymbol{b}_1\, \boldsymbol{a}_1 + \mu_1^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \wedge \ldots \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p + \mu_p^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{c})^p\, \boldsymbol{d}_1 \wedge \ldots \wedge \boldsymbol{d}_q \wedge (\nu_1^\kappa\, \boldsymbol{B}_\kappa) \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r, \end{aligned}$$

so that this term must vanish. According to the commutability condition (6) the vectors $\boldsymbol{w}_\tau$ are orthogonal to the space $\{\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{d}_\sigma\}$, which is a regular subspace, since all generators are orthogonal and non-isotropic. Thus the vectors $(\boldsymbol{a}_i, \boldsymbol{b}_i, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau)$ are independent. Hence, the term we have considered can vanish if and only if $\nu_1^\kappa = 0$ for each value of the index κ. This proves that $\nu_\sigma^\kappa = 0$. Now we can consider the coefficient of maximal degree for $\rho = 1$. We get the equation

$$\begin{aligned} &(\boldsymbol{x} \cdot \boldsymbol{a}_1\, \boldsymbol{b}_1 - \boldsymbol{x} \cdot \boldsymbol{b}_1\, \boldsymbol{a}_1 + \mu_1^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \wedge \ldots \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p + \mu_p^\kappa\, \boldsymbol{x} \cdot \boldsymbol{B}_\kappa) \\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_1 - \boldsymbol{x} \cdot \boldsymbol{d}_1\, \boldsymbol{c}) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_q - \boldsymbol{x} \cdot \boldsymbol{d}_q\, \boldsymbol{c}) \\ &\quad \wedge (\boldsymbol{a}_1 \wedge \boldsymbol{b}_1 + \mu_1^\kappa \boldsymbol{B}_\kappa) \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r = 0. \end{aligned}$$

We get only one term

$$
\begin{aligned}
&(\boldsymbol{x}\cdot\boldsymbol{a}_1\,\boldsymbol{b}_1-\boldsymbol{x}\cdot\boldsymbol{b}_1\,\boldsymbol{a}_1+\mu_1^\kappa\,\boldsymbol{x}\cdot\boldsymbol{B}_\kappa)\wedge\ldots\\
&\quad\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_p\,\boldsymbol{b}_p-\boldsymbol{x}\cdot\boldsymbol{b}_p\,\boldsymbol{a}_p+\mu_p^\kappa\,\boldsymbol{x}\cdot\boldsymbol{B}_\kappa)\\
&\quad\wedge\boldsymbol{d}_1\wedge\ldots\wedge\boldsymbol{d}_r\wedge\ldots\wedge(\boldsymbol{a}_1\wedge\boldsymbol{b}_1+\mu_1^\kappa\boldsymbol{B}_\kappa)\wedge\boldsymbol{w}_1\wedge\ldots\wedge\boldsymbol{w}_r,
\end{aligned}
$$

and thus only one term

$$
\begin{aligned}
&(\mu_1^\kappa\,\boldsymbol{x}\cdot\boldsymbol{B}_\kappa)\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_2\,\boldsymbol{b}_2-\boldsymbol{x}\cdot\boldsymbol{b}_2\,\boldsymbol{a}_2)\wedge\ldots\\
&\quad\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_p\,\boldsymbol{b}_p-\boldsymbol{x}\cdot\boldsymbol{b}_p\,\boldsymbol{a}_p)\\
&\quad\wedge\boldsymbol{d}_1\wedge\ldots\wedge\boldsymbol{d}_r\wedge\ldots\wedge\boldsymbol{a}_1\wedge\boldsymbol{b}_1\wedge\boldsymbol{w}_1\wedge\ldots\wedge\boldsymbol{w}_r.
\end{aligned}
$$

Again all vectors involved are independent, so that this term vanishes if and only if $\mu_1^\kappa=0$. This proves $\mu_\rho^\kappa=0$. Hence, system (5) reduces to

$$
\begin{pmatrix} X_\rho\\ Y_\sigma\\ Z_\tau\end{pmatrix}=\begin{pmatrix}\boldsymbol{a}_\rho\wedge\boldsymbol{b}_\rho & \boldsymbol{u}_\rho\\ \boldsymbol{c}\wedge\boldsymbol{d}_\sigma & \boldsymbol{v}_\sigma\\ 0 & \boldsymbol{w}_\tau\end{pmatrix}\qquad \begin{aligned}\rho&=1,\ldots,p\\ \sigma&=1,\ldots,q\\ \tau&=1,\ldots,r\end{aligned}\tag{8}
$$

According to Proposition 1 (i), Section 3, the commutation relations now imply

$$
\begin{aligned}
&\boldsymbol{w}_\tau\in\{\boldsymbol{a}_\rho,\boldsymbol{b}_\rho,\boldsymbol{c},\boldsymbol{d}_\sigma\}^\perp,\\
&\boldsymbol{v}_\sigma\in\{\boldsymbol{a}_\rho,\boldsymbol{b}_\rho,\boldsymbol{c}\}^\perp,\qquad\qquad \boldsymbol{v}_\sigma\cdot\boldsymbol{d}_{\sigma'}=\boldsymbol{v}_{\sigma'}\cdot\boldsymbol{d}_\sigma,\\
&\boldsymbol{u}_\rho\in\{(\boldsymbol{a}_{\hat\rho},\boldsymbol{b}_{\hat\rho}),\boldsymbol{c},\boldsymbol{d}_\sigma\}^\perp,
\end{aligned}\tag{9}
$$

where the diacritic ˆ over an index means the exclusion of that index. Let us write the integrability conditions (7) in this simplified situation:

$$
\begin{aligned}
&(\boldsymbol{x}\cdot\boldsymbol{a}_1\,\boldsymbol{b}_1-\boldsymbol{x}\cdot\boldsymbol{b}_1\,\boldsymbol{a}_1+\boldsymbol{u}_1)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_p\,\boldsymbol{b}_p-\boldsymbol{x}\cdot\boldsymbol{b}_p\,\boldsymbol{a}_p+\boldsymbol{u}_p)\\
&\quad\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_1-\boldsymbol{x}\cdot\boldsymbol{d}_1\,\boldsymbol{c}+\boldsymbol{v}_1)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_q-\boldsymbol{x}\cdot\boldsymbol{d}_q\,\boldsymbol{c}+\boldsymbol{v}_q)\\
&\quad\wedge\boldsymbol{w}_1\wedge\ldots\wedge\boldsymbol{w}_r\wedge\boldsymbol{a}_\rho\wedge\boldsymbol{b}_\rho=0.
\end{aligned}\tag{10}
$$

$$
\begin{aligned}
&(\boldsymbol{x}\cdot\boldsymbol{a}_1\,\boldsymbol{b}_1-\boldsymbol{x}\cdot\boldsymbol{b}_1\,\boldsymbol{a}_1+\boldsymbol{u}_1)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_p\,\boldsymbol{b}_p-\boldsymbol{x}\cdot\boldsymbol{b}_p\,\boldsymbol{a}_p+\boldsymbol{u}_p)\\
&\quad\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_1-\boldsymbol{x}\cdot\boldsymbol{d}_1\,\boldsymbol{c}+\boldsymbol{v}_1)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_q-\boldsymbol{x}\cdot\boldsymbol{d}_q\,\boldsymbol{c}+\boldsymbol{v}_q)\\
&\quad\wedge\boldsymbol{w}_1\wedge\ldots\wedge\boldsymbol{w}_r\wedge\boldsymbol{c}\wedge\boldsymbol{d}_\sigma=0.
\end{aligned}\tag{11}
$$

Let us consider equation (10) for $\rho=1$. It reduces to

$$
\begin{aligned}
&\boldsymbol{u}_1\wedge\boldsymbol{a}_1\wedge\boldsymbol{b}_1\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_2\,\boldsymbol{b}_2-\boldsymbol{x}\cdot\boldsymbol{b}_2\,\boldsymbol{a}_2+\boldsymbol{u}_2)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{a}_p\,\boldsymbol{b}_p-\boldsymbol{x}\cdot\boldsymbol{b}_p\,\boldsymbol{a}_p+\boldsymbol{u}_p)\\
&\quad\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_1-\boldsymbol{x}\cdot\boldsymbol{d}_1\,\boldsymbol{c}+\boldsymbol{v}_1)\wedge\ldots\wedge(\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{d}_q-\boldsymbol{x}\cdot\boldsymbol{d}_q\,\boldsymbol{c}+\boldsymbol{v}_q)\\
&\quad\wedge\boldsymbol{w}_1\wedge\ldots\wedge\boldsymbol{w}_r=0.
\end{aligned}
$$

The left hand side is a polynomial of degree $p+q-1$ in $\boldsymbol{x}$. Let us take the part of maximal degree:

$$\begin{aligned}&\boldsymbol{u}_1 \wedge \boldsymbol{a}_1 \wedge \boldsymbol{b}_1 \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_2\, \boldsymbol{b}_2 - \boldsymbol{x} \cdot \boldsymbol{b}_2\, \boldsymbol{a}_2) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p)\\ &\quad \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_1 - \boldsymbol{x} \cdot \boldsymbol{d}_1\, \boldsymbol{c}) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_q - \boldsymbol{x} \cdot \boldsymbol{d}_q\, \boldsymbol{c})\\ &\quad \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r = 0.\end{aligned}$$

This equation produces in particular the equation

$$\begin{aligned}&\boldsymbol{u}_1 \wedge \boldsymbol{a}_1 \wedge \boldsymbol{b}_1 \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_2\, \boldsymbol{b}_2 - \boldsymbol{x} \cdot \boldsymbol{b}_2\, \boldsymbol{a}_2) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p)\\ &\quad \wedge \boldsymbol{d}_1 \wedge \ldots \wedge \boldsymbol{d}_q \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r = 0.\end{aligned}$$

This means that

$$\boldsymbol{u}_1 \in \{\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \ldots, \boldsymbol{e}_p, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau\},$$

where $\boldsymbol{e}_\rho$ for $\rho = 2, \ldots, p$ means one of the two elements $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho)$. This implies

$$\boldsymbol{u}_1 \in \{\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau\},$$

However $\boldsymbol{u}_1$ is orthogonal to all $\boldsymbol{d}_\sigma$ due to the commutability conditions $(9)_3$, so that we finally get: $\boldsymbol{u}_1 \in \{\boldsymbol{a}_1, \boldsymbol{b}_1, \boldsymbol{w}_\tau\}$. By adding suitable combinations of the independent vectors $\boldsymbol{w}_\tau$ we can reduce to the case $\boldsymbol{u}_1 \in \{\boldsymbol{a}_1, \boldsymbol{b}_1\}$. Hence we have proved that conditions $(9)_3$ for the system (8) are equivalent to

$$\boldsymbol{u}_\rho \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho\}. \tag{12}$$

Let us consider equation (11) for $\sigma = 1$. It reduces to

$$\begin{aligned}&(\boldsymbol{x} \cdot \boldsymbol{a}_1\, \boldsymbol{b}_1 - \boldsymbol{x} \cdot \boldsymbol{b}_1\, \boldsymbol{a}_1 + \boldsymbol{u}_1) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p + \boldsymbol{u}_p)\\ &\quad \wedge \boldsymbol{v}_1 \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_2 + \boldsymbol{v}_2) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{c}\, \boldsymbol{d}_q + \boldsymbol{v}_q)\\ &\quad \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r \wedge \boldsymbol{c} \wedge \boldsymbol{d}_1 = 0.\end{aligned}$$

The left hand side is a polynomial of degree $p + q - 1$ in $\boldsymbol{x}$. Let us take the part of maximal degree:

$$\begin{aligned}&(\boldsymbol{x} \cdot \boldsymbol{a}_1\, \boldsymbol{b}_1 - \boldsymbol{x} \cdot \boldsymbol{b}_1\, \boldsymbol{a}_1 + \boldsymbol{u}_1) \wedge \ldots \wedge (\boldsymbol{x} \cdot \boldsymbol{a}_p\, \boldsymbol{b}_p - \boldsymbol{x} \cdot \boldsymbol{b}_p\, \boldsymbol{a}_p + \boldsymbol{u}_p)\\ &\quad \wedge \boldsymbol{v}_1 \wedge \boldsymbol{d}_2 \wedge \ldots \wedge \boldsymbol{d}_q \wedge \boldsymbol{w}_1 \wedge \ldots \wedge \boldsymbol{w}_r \wedge \boldsymbol{c} \wedge \boldsymbol{d}_1 = 0.\end{aligned} \tag{13}$$

Now we have two cases: (i) the space $F = \{\boldsymbol{w}_\tau\}$ is regular, including the case $F = 0$; (ii) the space F is not regular. In case (i), due to conditions $(9)_1$, all vectors $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{c}, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau)$ are independent, so that the previous equation implies that

$$\boldsymbol{v}_1 \in \{\boldsymbol{e}_\rho, \boldsymbol{c}, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau\},$$

thus

$$\boldsymbol{v}_1 \in \{\boldsymbol{c}, \boldsymbol{d}_\sigma, \boldsymbol{w}_\tau\}.$$

Again, this condition can be reduced to $\boldsymbol{v}_1 \in \{\boldsymbol{c}, \boldsymbol{d}_\sigma\}$ by adding linear combinations of $\boldsymbol{w}_\tau$. Hence we have proved that conditions $(9)_2$ for the system (8) are equivalent to

$$\boldsymbol{v}_\sigma \in S, \quad \boldsymbol{v}_\sigma \cdot \boldsymbol{d}_{\sigma'} = \boldsymbol{v}_{\sigma'} \cdot \boldsymbol{d}_\sigma, \tag{14}$$

where $S = \{\boldsymbol{c}, \boldsymbol{d}_\sigma\}$. Conversely, if conditions (14) and $(12)_1$ hold, we see that both integrability conditions (10) and (11) are identically satisfied. In case (ii) the space F must be orthogonal to an isotropic vector. Due to the commutability conditions $(9)_1$, this vector is necessarily the vector $\boldsymbol{c}$. Since $\boldsymbol{c} \in F$, we can always rearrange the vectors $\boldsymbol{w}_\tau$ in such a way that $\boldsymbol{w}_1 = \boldsymbol{c}$, and equation (13) is identically satisfied. In this case, however, also the whole integrability condition (11) is satisfied. □

5. The existence of orthogonal separable coordinates

Theorem 1 in the preceding section gives a necessary and sufficient condition for a subalgebra D of Killing vectors on $\mathbf{E}_n$ or $\mathbf{M}_n$ to be abelian and normal. Let us consider some remarkable consequences.

Proposition 1. *An abelian normal subalgebra $D \subset \mathcal{K}^1(\mathbf{E}_n)$ has an orthogonal basis.*

Proof. Since parabolic Killing vectors do not exist, due to Theorem 1, Section 4, there is a basis of D of kind

$$\begin{pmatrix} X_\rho \\ Z_\tau \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho & \boldsymbol{u}_\rho \\ 0 & \boldsymbol{w}_\tau \end{pmatrix} \qquad \begin{matrix} \rho = 1, \dots, p \\ \tau = 1, \dots, r \end{matrix} \tag{1}$$

where $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho)$ are independent and orthogonal, $\boldsymbol{u}_\rho \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho\}$ and $\boldsymbol{w}_\tau \in \{\boldsymbol{a}_\rho, \boldsymbol{b}_\rho\}^\perp$. Due to Proposition 1, Section 3, the vector fields (X_ρ, Z_τ) are orthogonal. □

Hence, because of Proposition 4, Section 2:

Proposition 2. *Every separable coordinate system on a Euclidean affine space $\mathbf{E}_n$ has an orthogonal equivalent.*

Let us denote by $\mathcal{R}_O(Q)$ the subalgebra of the rotations around the origin of an affine space Q, i.e. the set of all Killing vectors $X = (\boldsymbol{A}, 0)$, i.e. such that $X(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{A}$.

Proposition 3. *An abelian normal subalgebra $D \subset \mathcal{R}_O(\mathbf{M}_n)$ has a non-degenerate orthogonal basis.*

Proof. Due to Theorem 1, Section 4, there is a basis of D

$$\begin{pmatrix} X_\rho \\ Y_\sigma \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho \\ \boldsymbol{c} \wedge \boldsymbol{d}_\sigma \end{pmatrix} \qquad \begin{matrix} \rho = 1, \dots, p \\ \sigma = 1, \dots, q \end{matrix} \tag{2}$$

where $(\boldsymbol{a}_\rho, \boldsymbol{b}_\rho, \boldsymbol{c}, \boldsymbol{d}_\sigma)$ are independent, unitary and orthogonal, and $\boldsymbol{c}$ is isotropic. Due to formula $(3)_5$ or Proposition 1, Section 3, the vector fields (X_ρ, Y_σ) are orthogonal and

$$X_\rho \cdot X_\rho = (\|\boldsymbol{b}_\rho\|\, \boldsymbol{a}_\rho + \|\boldsymbol{a}_\rho\|\, \boldsymbol{b}_\rho) \cdot \boldsymbol{x} = (\boldsymbol{a}_\rho \pm \boldsymbol{b}_\rho) \cdot \boldsymbol{x},$$

$$Y_\sigma \cdot Y_\sigma = \boldsymbol{c} \cdot \boldsymbol{x},$$

where in the first line the sign $-$ corresponds to the case in which the element $\boldsymbol{a}_\rho \wedge \boldsymbol{b}_\rho$ is hyperbolic and $\boldsymbol{a}_\rho$ is timelike (this can happen for only one index and only if there is no parabolic term, i.e. $q = 0$). This shows that, apart from a singular closed set made of hyperplanes, the vectors $(X_\rho(\boldsymbol{x}), Y_\sigma(\boldsymbol{x}))$ form a basis of a non-degenerate subspace. □

As a consequence, due to Proposition 3 and Proposition 4, Section 2:

Proposition 4. *On a Minkowskian affine space* $\mathbf{M}_n$ *every separable coordinate system whose abelian subalgebra* $D \subset \mathcal{K}^1(\mathbf{M}_n)$ *corresponding to the first class coordinates is contained in* $\mathcal{R}_O(\mathbf{M}_n)$ *(i.e. is made of rotations around the same point* O*) has an orthogonal equivalent.*

Furthermore, let us consider the following manifolds:

$$\begin{aligned} \mathbf{S}_n &= \{\boldsymbol{x} \in \mathbf{E}_{n+1} \mid \|\boldsymbol{x}\| = 1\}, \\ \mathbf{H}_n &= \{\boldsymbol{x} \in \mathbf{M}_{n+1} \mid \|\boldsymbol{x}\| = -1\}, \\ \mathbf{L}_n &= \{\boldsymbol{x} \in \mathbf{M}_{n+1} \mid \|\boldsymbol{x}\| = 1\}. \end{aligned}$$

They are called *hyperquadrics* in [6]: $\mathbf{S}_n$ (the n-dimensional sphere) is a Riemannian manifold with positive metric and positive constant curvature; $\mathbf{H}_n$ (the n-dimensional hyperboloid or pseudo-sphere) is a Riemannian manifold with positive metric and negative constant curvature; $\mathbf{L}_n$ is a Lorentzian manifold (signature $(-++\ldots+)$) with positive constant curvature. The rotations of $\mathbf{E}_{n+1}$ and of $\mathbf{M}_{n+1}$ are tangent to $\mathbf{S}_n$ and to $\mathbf{H}_n$, $\mathbf{L}_n$, respectively. So that

$$\begin{aligned} \mathcal{K}^1(\mathbf{S}_n) &= \mathcal{R}_O(\mathbf{E}_{n+1})|\mathbf{S}_n, \\ \mathcal{K}^1(\mathbf{H}_n) &= \mathcal{R}_O(\mathbf{M}_{n+1})|\mathbf{H}_n, \\ \mathcal{K}^1(\mathbf{L}_n) &= \mathcal{R}_O(\mathbf{M}_{n+1})|\mathbf{L}_n, \end{aligned}$$

It follows from Proposition 1 and Proposition 3 that

Proposition 5. *An abelian normal subalgebra of* $\mathcal{K}^1(\mathbf{S}_n)$, $\mathcal{K}^1(\mathbf{H}_n)$, $\mathcal{K}^1(\mathbf{L}_n)$, *has an orthogonal basis. In* $\mathcal{K}^1(\mathbf{L}_n)$ *every orthogonal basis is non-degenerate.*

As a consequence:

Proposition 6. *Every separable coordinate system on the spaces* $\mathbf{S}_n$, $\mathbf{H}_n$, $\mathbf{L}_n$, *has an orthogonal equivalent.*

Since all the manifolds considered in Propositions 6 and 3 locally provide a model of the manifolds with constant curvature considered in the theorem announced in the Introduction, the theorem is proved.

The same property does not hold for Lorentzian manifolds with zero curvature (like the Minkowkian affine spaces) and with constant negative curvature, unless further restrictions are imposed on the algebra D associated with the first class coordinates (as in Proposition 4). Here we give two examples.

Example 1. Let us consider the hyperbolic 3-space $\mathbf{M}_3$ and all possible bi-dimensional normal abelian subalgebras $D_2 \subset \mathcal{K}^1(\mathbf{M}_3)$ (in fact, due to the dimension, every bi-dimensional subalgebra of Killing vectors is normal). According to Theorem 1, Section 4, one of the elements of the canonical basis (1), Section 4, is necessarily a translation. Hence, we have the following cases:

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}\wedge\boldsymbol{b} & \boldsymbol{u} \\ 0 & \boldsymbol{w} \end{pmatrix} \qquad \begin{cases} \|\boldsymbol{a}\| = \pm\|\boldsymbol{b}\| = 1,\ \boldsymbol{a}\cdot\boldsymbol{b} = 0, \\ \boldsymbol{u} \in \{\boldsymbol{a},\boldsymbol{b}\}, \boldsymbol{w} \in \{\boldsymbol{a},\boldsymbol{b}\}^{\perp}, \end{cases} \tag{3}$$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}\wedge\boldsymbol{d} & \boldsymbol{v} \\ 0 & \boldsymbol{w} \end{pmatrix} \qquad \begin{cases} \boldsymbol{c}\cdot\boldsymbol{c} = 0,\ \ \boldsymbol{d}\cdot\boldsymbol{c} = 0, \\ \boldsymbol{v} \in \{\boldsymbol{c},\boldsymbol{d}\},\ \ \boldsymbol{w} \in \{\boldsymbol{c},\boldsymbol{d}\}^{\perp}, \end{cases} \tag{4}$$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}\wedge\boldsymbol{d} & \boldsymbol{v} \\ 0 & \boldsymbol{c} \end{pmatrix} \qquad \boldsymbol{c}\cdot\boldsymbol{c} = 0;\ \ \boldsymbol{d}\cdot\boldsymbol{c} = 0, \tag{5}$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{w}_1 \\ 0 & \boldsymbol{w}_2 \end{pmatrix} \qquad \boldsymbol{w}_1\cdot\boldsymbol{w}_2 = 0 \text{ (non isotropic)}, \tag{6}$$

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{c} \\ 0 & \boldsymbol{w} \end{pmatrix} \qquad \boldsymbol{c}\cdot\boldsymbol{c} = 0;\ \ \boldsymbol{c}\cdot\boldsymbol{w} = 0. \tag{7}$$

Actually cases (3) and (4), after a suitable change of the origin O, are equivalent to

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}\wedge\boldsymbol{b} & 0 \\ 0 & \boldsymbol{w} \end{pmatrix} \qquad \boldsymbol{a}\cdot\boldsymbol{b} = 0 \text{ (non isotropic)},\ \ \boldsymbol{w} \in \{\boldsymbol{a},\boldsymbol{b}\}^{\perp}, \tag{3'}$$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \boldsymbol{c}\wedge\boldsymbol{d} & 0 \\ 0 & \boldsymbol{w} \end{pmatrix} \qquad \boldsymbol{c}\cdot\boldsymbol{c} = 0,\ \ \boldsymbol{d}\cdot\boldsymbol{c} = 0,\ \ \boldsymbol{w} \in \{\boldsymbol{c},\boldsymbol{d}\}^{\perp}. \tag{4'}$$

In both cases the pairs of Killing vectors are orthogonal and the corresponding subalgebra is non-degenerate:

$$\begin{aligned} X\cdot X &= (\boldsymbol{x}\cdot\boldsymbol{a})^2\,\|\boldsymbol{b}\| + (\boldsymbol{x}\cdot\boldsymbol{b})^2\,\|\boldsymbol{a}\|, \\ X\cdot Z &= 0, \\ Z\cdot Z &= \|\boldsymbol{w}\| \neq 0, \end{aligned} \tag{3''}$$

$$\begin{aligned} Y\cdot Y &= (\boldsymbol{x}\cdot\boldsymbol{c})^2, \\ Y\cdot Z &= 0, \\ Z\cdot Z &= \|\boldsymbol{w}\| \neq 0. \end{aligned} \tag{4''}$$

Hence, they correspond to orthogonal separable coordinates, two of the first class and one non-isotropic of the second class. In case (5) we have:

$$\begin{aligned} Y\cdot Y &= (\boldsymbol{x}\cdot\boldsymbol{c})^2 + 2\,\boldsymbol{x}\cdot\boldsymbol{c}\,\boldsymbol{v}\cdot\boldsymbol{d} - 2\,\boldsymbol{x}\cdot\boldsymbol{d}\,\boldsymbol{v}\cdot\boldsymbol{c} + \boldsymbol{v}\cdot\boldsymbol{v}, \\ Y\cdot Z &= \boldsymbol{v}\cdot\boldsymbol{c}, \\ Z\cdot Z &= 0, \end{aligned} \tag{5'}$$

and we see that, when $\boldsymbol{v}\cdot\boldsymbol{c} \neq 0$, the subalgebra D given by (Y, Z) cannot have an orthogonal basis and it is non-degenerate. Hence, the corresponding separable coordinates, two of the first class and one non-isotropic of the second class, do not have an orthogonal equivalent. When $\boldsymbol{v} \cdot \boldsymbol{c} = 0$, the vector fields (Y, Z) are orthogonal and the subalgebra is degenerate. The case (6), made of orthogonal translations, gives rise to affine orthogonal coordinates. In case (7) the distribution D is degenerate. Let us consider all the possible one-dimensional normal subalgebras $D_1 \subset \mathcal{K}^1(\mathrm{M}_3)$. They are generated by the following three kinds of Killing vector fields:

$$\begin{aligned} X &= (\boldsymbol{a} \wedge \boldsymbol{b}, 0), \qquad ||\boldsymbol{a}|| = \pm||\boldsymbol{b}|| = 1, \quad \boldsymbol{a} \cdot \boldsymbol{b} = 0, \\ Y &= (\boldsymbol{c} \wedge \boldsymbol{d}, 0), \qquad ||\boldsymbol{c}|| = 0, ||\boldsymbol{d}|| = 1, \quad \boldsymbol{c} \cdot \boldsymbol{d} = 0, \\ Z &= (0, \boldsymbol{w}). \end{aligned}$$

They are, respectively: a Euclidean (+) or hyperbolic (−) rotation, a parabolic rotation and a translation. The only case which can give rise to a degenerate subalgebra is the translation with a $\boldsymbol{w}$ isotropic. In fact, the examples of non-orthogonal separable coordinates considered in [8] are of this kind.

Example 2. Let us consider the pseudo-Euclidean affine space N_4 of signature $(--++)$, and the submanifold

$$\mathbf{L}_3^- = \{\boldsymbol{x} \in \mathrm{N}_4 \mid ||\boldsymbol{x}|| = -1\}.$$

It is a Lorentzian manifold with constant negative curvature. We show that it has a separable coordinate system, without isotropic second class coordinates, which does not admit an orthogonal equivalent. Let us consider a canonical basis $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ of N_4, such that

$$||\boldsymbol{a}|| = ||\boldsymbol{b}|| = 1, \qquad ||\boldsymbol{c}|| = ||\boldsymbol{d}|| = -1.$$

The vectors $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{c}$ and $\boldsymbol{v}' = \boldsymbol{b} + \boldsymbol{d}$ are isotropic and orthogonal: $||\boldsymbol{v}|| = ||\boldsymbol{v}'|| = 0$, $\boldsymbol{v} \cdot \boldsymbol{v}' = 0$. Let us consider the skew-symmetric tensors

$$\boldsymbol{K} = \boldsymbol{v} \wedge \boldsymbol{v}', \qquad \boldsymbol{L} = \boldsymbol{a} \wedge \boldsymbol{c} + \boldsymbol{d} \wedge \boldsymbol{b}.$$

They have the following properties:

$$\boldsymbol{K}^2 = 0, \qquad \boldsymbol{K} \cdot \boldsymbol{L} = \boldsymbol{L} \cdot \boldsymbol{K} = -\boldsymbol{v}' \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{v}', \qquad \boldsymbol{L}^2 = \boldsymbol{g}.$$

Let us consider the two skew-symmetric tensors

$$\boldsymbol{A} = \alpha \boldsymbol{K} + \boldsymbol{L}, \qquad \boldsymbol{B} = \beta \boldsymbol{K} + \boldsymbol{L}, \qquad \alpha \neq \beta.$$

They are independent and commute:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \alpha \boldsymbol{K} \cdot \boldsymbol{L} + \beta \boldsymbol{L} \cdot \boldsymbol{K} + \boldsymbol{L}^2 = \boldsymbol{g} + (\alpha + \beta) \boldsymbol{K} \cdot \boldsymbol{L}.$$

They generate two commuting Killing vectors (X, Y) on N_4. Since they are rotations, they are tangent to $\mathbf{L}_3^-$, so they generate a 2-dimensional abelian subalgebra of $\mathcal{K}^1(\mathbf{L}_3^-)$. A straightforward calculation shows that $X \wedge Y \wedge dX = 0$, and by symmetry $X \wedge Y \wedge dY = 0$. This proves that the subalgebra is normal (as it can be proved, this is in fact a general property of any bi-dimensional subalgebra of rotations in a four-dimensional affine space). Hence, this subalgebra generates a separable coordinate system. We note that the equation

$$(\alpha \boldsymbol{K} + \boldsymbol{L}) \cdot (\gamma \boldsymbol{K} + \delta \boldsymbol{L}) = 0$$

implies

$$\delta \boldsymbol{g} - (\alpha\delta + \gamma)(\boldsymbol{v} \otimes \boldsymbol{v}' + \boldsymbol{v}' \otimes \boldsymbol{v}) = 0,$$

and it can be satisfied only if $\gamma = \delta = 0$. This shows (see Proposition 1, Section 3) that any two linear combinations of (X, Y) cannot be orthogonal, and also that these vector fields span a non-degenerate distribution. Hence, the corresponding separable coordinate system is without isotropic second class coordinates and does not have an orthogonal equivalent.

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