# Separation of variables
# in the geodesic Hamilton-Jacobi equation

S.Benenti

*Dedicated to Jean-Marie Souriau*

## 1. Introduction

This lecture is devoted to the basic notions of the theory of separation of variables for the Hamilton-Jacobi equation. Many of the results presented here are already available in the literature. However, we collect them in a systematic way according to a personal point of view, with a particular emphasis to their geometrical meaning.

Let $Q$ be a differential manifold of dimension $n$. With each system of coordinates $(q^i)$ on $Q$ we associate canonical coordinates $(q^i, p_j)$ on the cotangent bundle $T^*Q$. We call the coordinates $(p_i)$ the **momenta** corresponding to the coordinates $(q^i)$. Let $H$ be a differentiable real function on the cotangent bundle, called the **Hamiltonian**. For each coordinate system on $Q$ the Hamiltonian is locally represented by a real differentiable function $H(q^i, p_j)$ in the $2n$ real variables $(q^i, p_j)$, and it gives rise to a partial differential equation, the **Hamilton-Jacobi equation**

$$H\left(q^i, \frac{\partial W}{\partial q^j}\right) = h, \tag{1.1}$$

where $h$ is a real parameter, called the **energy**. A **complete integral** of this equation is a solution $W(q^i, a_j)$ depending on $n$ real parameters $(a_j)$ such that the $n \times n$ matrix

$$\left(\frac{\partial^2 W}{\partial q^i \partial a_j}\right) \tag{1.2}$$

is everywhere regular. When such a complete integral is known, then, by purely algebraic manipulations, we can find the integral curves of the Hamiltonian dynamical system generated by $H$. This is the classical **Jacobi method** which was originally used for finding the geodesics of an ellipsoid. However, the experience has shown that the Jacobi method can be applied with success for those cases in which it is possible to find a complete integral of the kind

$$W = W_1(q^1, a_j) + W_2(q^2, a_j) + \ldots + W_n(q^n, a_j). \tag{1.3}$$

This property is called **additive separation of variables**, and when this occurs we say that the H-J equation is integrable by **separation of variables** and that the dynamical system generated by the Hamiltonian $H$ is **integrable in the sense of Jacobi** or **Jacobi-integrable**. The coordinates $(q^i)$ for which a complete integral of the kind (1.3) exists are called **separable** (with respect to the Hamiltonian $H$).

Actually, due to the applications to Riemannian geometry and mathematical physics, we deserve the interest to Hamiltonians which are polynomials of second order in the momenta, i.e. of the kind

$$H = \tfrac{1}{2}\, g^{ij} p_i p_j + A^i p_i + V, \tag{1.4}$$

where $g^{ij}$ are the contravariant components of a metric tensor $g$ on $Q$, $A^i$ are the components of a vector field on $Q$ and $V$ is a function on $Q$ , called the **potential** (all fields are assumed to be smooth, i.e. of class $C^\infty$, for simplicity).

We cannot give here a historical perspective on the theory of separation of variables. For an outline of this topic and an essential bibliography we refer to [H] [W] and to the recent book of Kalnins [K]. However, in the following two sections we will present, with proofs, classical fundamental results due to Levi-Civita (1904) and Stäckel (1893), on which we will base our discussion. We will restrict our attention to the separation of the geodesic H-J equation, which is a crucial topic in the theory of separation of variables. In Section 4 we will analyze those transformations of coordinates which preserve the separation and the complete integral. The results will be used in Section 5 for proving the existence of a normal form of the contravariant metric tensor components in separable coordinates, in which the separation is achieved in the simplest way. In Section 6 we will present some fundamental facts concerning the Killing tensors and the orthogonal separation. Section 7 will be finally dedicated to an outline of the geometrical aspects of separation. Some examples are presented for illustrating the theory, without the pretension of a reasonable completeness, as for the bibliography.

## 2. The theorem of Levi-Civita

**Theorem 1.** *The Hamilton-Jacobi equation* (1.1) *has a complete integral of the kind* (1.3) *if and only if the following equations are identically satisfied for each pair* $(i,j)$ *of distinct indices:*

$$\partial^i H \partial^j H \partial_{ij} H + \partial_i H \partial_j H \partial^{ij} H - \partial^i H \partial_j H \partial_i^j H - \partial^j H \partial_i H \partial_j^i H = 0. \tag{2.1}$$

Here we have used the following short notation:

$$\partial_i H = \frac{\partial H}{\partial q^i}, \quad \partial^i H = \frac{\partial H}{\partial p_i}, \quad \text{etc...}$$

*Proof.* The additive separation of variables is equivalent to the condition $\partial_{ij} W = 0$ for $i \neq j$. By differentiating the H-J equation (1.1) with respect to a variable $q^i$, we get

$$\partial_i H + \partial^j H \partial_{ij} W = \partial_i H + \partial^i H \partial_{ii} W = 0 \quad (i \text{ n.s.}).$$

The notation "n.s." means that there is no summation over the repeated index. Thus we are led to consider the following system of partial differential equations in the unknowns $(p_i)$,

$$\partial_i p_i = R_i, \quad \partial_i p_j = 0 \quad (i \neq j), \tag{2.2}$$

where

$$R_i = -\frac{\partial_i H}{\partial^i H}, \tag{2.3}$$

together with equation $p_i = \partial_i W$. The integrability conditions of this system are

$$\partial_i R_j + \partial^i R_j \, R_i = 0 \quad (i \neq j). \tag{2.4}$$

Due to (2.3) they are equivalent to (2.1). ∎

The theorem of Levi-Civita [LC] provides only a criterion to decide whether a given coordinate system is separable or not. It gives no effective method to find separable coordinates for a given Hamiltonian. In fact, this is a hard problem, which has been solved only for particular Hamiltonians (like, for instance, the geodesic Hamiltonian of Riemannian manifolds with constant curvature [K]). However, Levi-Civita theorem can be considered as a starting point for developing the theory of the separation of variables.

If we substitute the polynomial Hamiltonian (1.3) in equations (2.1), then we get equations of fourth order in the momenta $(p_i)$ which must be identically satisfied, so that the coefficients must vanish. These coefficients involve the functions $g^{ij}$, $A^i$, $V$, together with their first and second derivatives, so that we get finally a set of second order PDE's on these functions. For the sake of brevity we do not write them here explicitly; actually, they are very cumbersome. However, as Levi-Civita himself remarked, it can be seen that the coefficients of fourth degree coincide with those coming from the separation conditions (2.1) for the purely geodesic Hamiltonian

$$G = \tfrac{1}{2}\, g^{ij} p_i p_j, \tag{2.5}$$

namely,

$$(\partial_{ij} g^{hk} g^{ir} g^{js} + \tfrac{1}{2}\, g^{ij} \partial_i g^{hk} \partial_j g^{rs} - \partial_i g^{jh} \partial_j g^{kr} g^{is} - \partial_j g^{ih} \partial_i g^{kr} g^{js}) p_h p_k p_r p_s = 0. \tag{2.6}$$

Hence, coordinates which are separable with respect to the Hamiltonian (1.4) are separable with respect to the geodesic Hamiltonian (2.5), and the separation of the geodesic H-J equation becomes the fundamental argument.

## 3. The orthogonal separation and the theorems of Stäckel

The first step for studying the separation of the geodesic H-J equation is to consider **orthogonal** coordinates, i.e. to assume

$$g^{ij} = 0, \quad i \neq j, \tag{3.1}$$

so that $G = \frac{1}{2} g^{ii} (p_i)^2$. The separation conditions (2.6) reduce to equations

$$g^{ii} g^{jj} \partial_{ij} g^{hh} - g^{ii} \partial_i g^{jj} \partial_j g^{hh} - g^{jj} \partial_j g^{ii} \partial_i g^{hh} = 0 \quad (i \neq j,\ \text{n.s.}). \tag{3.2}$$

These equations, which are equivalent to

$$\partial_{ij} g^{hh} - \partial_i \log |g^{jj}|\, \partial_j g^{hh} - \partial_j \log |g^{ii}|\, \partial_i g^{hh} = 0 \quad (i \neq j,\ \text{n.s.}), \tag{3.2'}$$

are fundamental in the theory of the separation of variables. They are similar to the classical Lamé equations (see, for instance, [BI]).

The following two theorems due to Stäckel [ST1] [ST2] are fundamental for the theory of orthogonal separation.

**Theorem 3.1.** *The most general form of the metric tensor in orthogonal separable coordinates is*

$$g^{ii} = \phi^i_{(n)}, \tag{3.3}$$

*where $\phi^i_{(n)}$ is a row (the last one, for instance) of the inverse of a Stäckel matrix $(\phi^{(j)}_i)$ in the coordinates $(q^i)$. The functions*

$$F_j = \tfrac{1}{2} \phi^i_{(j)} (p_i)^2 \tag{3.4}$$

*are geodesic first integrals in involution (the last one is the geodesic Hamiltonian itself):*

$$\{F_j, F_h\} = 0, \quad F_n = G. \tag{3.5}$$

**Definition 3.1.** A **Stäckel matrix** is a regular $n \times n$ matrix $(\phi^{(j)}_i)$ of functions of the $n$ variables $(q^i)$ such that each element depends on the variable corresponding to the lower index only: $\phi^{(j)}_i(q^i)$. We denote by $(\phi^i_{(j)})$ the inverse matrix.

This theorem shows in other words, that the most general functions satisfying equations (3.2) are of the kind (3.3) and, moreover, that the separation of variables is always concomitant with the existence of quadratic first integrals in involution. The very simple proof given by Stäckel can be found in the majority of the textbooks of analytical mechanics. However, we write it here explicitly for further needs.

*Proof.* Let us differentiate the geodesic H-J equation

$$\tfrac{1}{2}g^{ii}(\partial_i W)^2 = h \tag{3.6}$$

with respect to the parameters $(a_j)$ entering in a complete integral $W$ and set

$$\phi_i^{(j)} = \frac{\partial W}{\partial q^i}\frac{\partial^2 W}{\partial q^i \partial a_j}, \quad c^j = \frac{\partial h}{\partial a_j}.$$

We get an equation of the kind

$$g^{ii}\phi_i^{(j)} = c^j.$$

The matrix $(\phi_i^{(j)})$ is regular provided that no $p_i = \partial_i W$ is zero, and it is a Stäckel matrix if the coordinates are separable. By reversing these equations we find

$$g^{ii} = c^j \phi^i_{(j)}.$$

However, there is no loss of generality in assuming that one of the parameters coincides with the energy, say $a_n = h$. We have $c^j = \delta^j_n$, so that the metric components have the form (3.3). Once (3.3) is proved, we can write the most general separable Hamilton-Jacobi equation for the geodesics:

$$\tfrac{1}{2}\phi^i_{(n)}(\partial_i W)^2 = h.$$

Then the separation of variables can been achieved by writing the whole system of equations associated with the matrix $(\phi^i_{(j)})$,

$$\tfrac{1}{2}\phi^i_{(j)}(\partial_i W)^2 = a_j, \tag{3.7}$$

by introducing $n$ real parameters $(a_j)$, with $a_n = h$. So that we get, by an inversion, the following system of separated ordinary differential equations,

$$\left(\frac{dW_i}{dq^i}\right)^2 = 2a_j\phi_i^{(j)}.$$

It follows that $W = \sum_{i=1}^n W_i(q^i, a_j)$ is a separated complete integral. Moreover, by the Jacobi theorem we know that the real parameters entering in a complete integral correspond to first integrals in involution. Hence, from (3.7) we deduce that the functions (3.4) are first integrals in involution. ∎

Let us consider a potential function $V$ added to the geodesic Hamiltonian:

$$H = \tfrac{1}{2}g^{ii}(p_i)^2 + V.$$

The separability equations of Levi-Civita for this Hamiltonian give rise to the system of equations (3.2), involving the metric coefficients only, together with the following additional conditions on the potential $V$:

$$g^{ii}g^{jj}\partial_{ij}V - g^{ii}\partial_i g^{jj}\partial_j V - g^{jj}\partial_j g^{ii}\partial_i V = 0 \quad (i \neq j,\ \text{n.s.}), \tag{3.8}$$

which can also be written

$$\partial_{ij}V - \partial_i \log|g^{jj}|\,\partial_j V - \partial_j \log|g^{ii}|\,\partial_i V = 0 \quad (i \neq j,\ \text{n.s.}). \tag{3.8'}$$

**Theorem 3.2.** *The most general potential compatible with orthogonal separation is of the kind*

$$V = \eta_i \, g^{ii}, \tag{3.9}$$

*where $(\eta_i)$ are functions of the variable corresponding to the index only.*

*Proof.* Since $\partial_i W$ is a function of the variable $q^i$ only, the H-J equation

$$\tfrac{1}{2}\, g^{ii}(\partial_i W)^2 + V = h$$

shows that in separable coordinates $V$ is a function of kind (3.9) up to an inessential additive constant. ∎

Furthermore, following the same way of the proof of Theorem 3.1, it can be seen that the functions

$$F_j = \tfrac{1}{2}\, \phi^i_{(j)} (p_i)^2 + \eta_j$$

are first integrals in involution.

There is another way to prove all the above results which does not involve the theorem of Jacobi.

Let us consider a linear connection $\Gamma$ on the manifold $Q$. We can think of invariant 1-forms $\phi$ and vector fields $X$ and write the following two differential systems:

$$\partial_i \phi_j - \Gamma^h_{ij} \phi_h = 0, \quad \partial_i X^j + \Gamma^j_{ih} X^h = 0. \tag{3.10}$$

If $\phi$ and $X$ are two solutions then

$$\langle X, \phi \rangle = X^i \phi_i = \text{ const.} \tag{3.11}$$

Both systems (3.10) are completely integrable if and only if the connection is locally flat. Then the general solution of the first system (3.10) has the form

$$\phi_i = c_j \phi^{(j)}_i, \quad (c_j) \in \mathbb{R}^n, \tag{3.12}$$

where $(\phi^{(j)}_i;\, j = 1, \ldots, n)$ are independent solutions: $\det(\phi^{(j)}_i) \neq 0$. The inverse matrix $\phi^i_{(j)}$ gives the general solution of the second system (3.10):

$$X^i = c^j \phi^i_{(j)}, \quad (c^j) \in \mathbb{R}^n, \tag{3.13}$$

If the connection $\Gamma$ is symmetric, then two invariant vector fields (solutions of $(3.10)_2$) commute. This is equivalent to

$$\phi^i_{(h)} \partial_i \phi^j_{(k)} - \phi^i_{(k)} \partial_i \phi^j_{(h)} = 0. \tag{3.14}$$

We say that the connection $\Gamma$ is **separable** with respect to the coordinates $(q^i)$ (or, conversely, that the coordinates $(q^i)$ are separable with respect to the connection $\Gamma$) if the coefficients of $\Gamma$ in these coordinates are such that

$$\Gamma^h_{ij} = 0, \quad \text{for } \ i \neq j. \tag{3.15}$$

In this case equations (3.10) become

$$\partial_i \phi_j = \delta_{ij} B^h_i \phi_h, \quad \partial_i X^j = -B^j_i X^i, \quad \text{where } \ B^j_i = \Gamma^j_{ii}. \tag{3.16}$$

The integrability conditions are:

$$\partial_i B^h_j + B^i_j B^h_i = 0, \quad i \neq j \ \text{ n.s..} \tag{3.17}$$

For every non trivial solution of equations $(3.16)_2$, we have

$$B_i^j = -\frac{1}{X^i}\partial_i X^j. \tag{3.18}$$

If we substitute this expression in the integrability conditions (3.17), then we get equations

$$X^i X^j \partial_{ij} X^h - X^i \partial_i X^j \partial_j X^h - X^j \partial_j X^i \partial_i X^h = 0 \quad (i \neq j, \text{ n.s.}). \tag{3.19}$$

On the other hand, equations $(3.16)_1$ show that a set of independent solutions form a Stäckel matrix $(\phi_i^{(k)})$, since $\partial_i \phi_j = 0$ for $i \neq j$. Hence, we have proved that

**Theorem 3.3.** *The most general solution of equations* (3.19) *is of the form* (3.13) *where* $(\phi^i_{(j)})$ *is the inverse of a Stäckel matrix, or more simply, of the form*

$$X^i = \phi^i_{(n)}, \tag{3.20}$$

*where* $(\phi^i_{(n)})$ *is a line of the inverse of a Stäckel matrix.*

This is in fact the Stäckel theorem 3.1, provided we set

$$X^i = g^{ii}, \quad \phi_i = (p_i)^2 = (\partial_i W)^2$$

and look at equation (3.11) as the H-J equation. Moreover, equations (3.14) are equivalent to the commutation equations (3.5).

**Remark 3.1.** Equations (3.19) are the Levi-Civita separation conditions (2.1) for a linear Hamiltonian $H = X^i p_i$. For a Hamiltonian of the kind $H = X^i p_i + V$ the separation conditions are still equations (3.19) together with equations

$$X^i X^j \partial_{ij} V - X^i \partial_i X^j \partial_j V - X^j \partial_j X^i \partial_i V = 0, \quad (i \neq j, \text{ n.s.}). \tag{3.21}$$

Then we can re-state Theorem 3.2 as follows:

**Theorem 3.4.** *The most general solution of equations* (3.21)*, where* $(X^i)$ *are solutions of equations* (3.19)*, are of the kind* $V = \eta_i X^i$*, where* $(\eta_i)$ *are functions of the variable corresponding to the index only.*

**Remark 3.2.** Since the components $(g^{ii})$ are a particular solution of system $(3.16)_2$, if we consider equations (3.18) for $X^i = g^{ii}$ we get equations

$$\frac{1}{X^i}\partial_i X^j = \frac{1}{g^{ii}}\partial_i g^{jj},$$

which can be interpreted as a differential system in the unknown functions $(X^i)$:

$$\partial_i X^j = \frac{1}{g^{ii}}\partial_i g^{jj} X^i. \tag{3.22}$$

According to the preceding remarks, this system is completely integrable if and only if equations (3.2) hold. One solution is given by $(g^{ii})$. A set of independent solutions $(X^i = \phi^i_{(j)})$ gives the inverse of a Stäckel matrix. Furthermore, if we introduce the functions

$$\rho_i = \frac{X^i}{g^{ii}},$$

then system (3.22) is equivalent to

$$\partial_i \rho_j = (\rho_i - \rho_j)\partial_i \log |g^{ii}|. \tag{3.23}$$

The integrability conditions of this new differential system are still equations (3.2). We will go back to this remark in the sequel.

**Remark 3.3.** Transformations of the metric components of the kind

$$g^{ii} \longmapsto \zeta_i\, g^{ii},$$

where $(\zeta_i)$ are functions of the variable corresponding to the index only, preserve the separation of variables. This can be seen straightly from the H-J equation (3.6).

**Remark 3.4.** Assume that a Hamiltonian $H$ satisfies the Levi-Civita separation conditions (2.1). Let us consider a "conformal" Hamiltonian $fH$, where $f$ is a real smooth function of the coordinates only. This new Hamiltonian satisfies equations (2.1) if and only if, for each pair of distinct indices:

$$\begin{aligned}
&f^2\, H\, [\partial^i H \partial^j H \partial_{ij} f - \partial^i H \partial_i^j H \partial_j f - \partial^j H \partial_j^i H \partial_i f] - \\
&2\, f\, H\, \partial^i H \partial^j H \partial_i f \partial_j f + \\
&f^2\, \partial^{ij} H\, [H^2 \partial_i f \partial_j f + f H (\partial_i f \partial_j H + \partial_j f \partial_i H)] = 0.
\end{aligned}$$

If we set $V = f^{-1}$ (where $f \neq 0$), then we get simpler equations:

$$\begin{aligned}
&\partial^i H \partial^j H \partial_{ij} V - \partial^i H \partial_i^j H \partial_j V - \partial^j H \partial_j^i H \partial_i V + \\
&V^{-2}\, \partial^{ij} H\, [H \partial_i V \partial_j V + V (\partial_i V \partial_j H + \partial_j V \partial_i H)] = 0.
\end{aligned}$$

These are the characteristic equations of a **conformal cofactor** $V$ preserving the separability of a Hamiltonian $H$. When $\partial^{ij} H = 0$ for $i \neq j$, they simplify to

$$\partial^i H \partial^j H \partial_{ij} V - \partial^i H \partial_i^j H \partial_j V - \partial^j H \partial_j^i H \partial_i V = 0,$$

and become similar to (3.21) and (3.8). This is actually the case of a linear Hamiltonian and of an orthogonal geodesic Hamiltonian. For this reason solutions of equations (3.21) or (3.8) are called **Stäckel multipliers** [KM3]: a conformal transformation of the kind $g_{ij} \longmapsto V g_{ij}$ preserves the separability of an orthogonal metric. The fact that a potential compatible with the (orthogonal) separation has the same form as a conformal cofactor preserving the (orthogonal) separation is rather a consequence of the Maupertuis principle of mechanics: the dynamical trajectories of fixed energy $h$ of a mechanical system, whose kinetic energy and potential energy are respectively $\frac{1}{2} g_{ij} v^i v^j$ and $V$, are geodesics of the conformal metric $(V - h) g_{ij}$.

**Remark 3.5.** When for a given coordinate system the metric tensor components satisfy equations (3.2), then by the Stäckel theorem 3.1 we only conclude that a Stäckel matrix exists such that (3.3) holds and, in order to perform the separation and to construct the first integrals, we need to know explicitly such a matrix (it is not uniquely determined), or its inverse; we know only the last line of the inverse. Actually, in practice the separation can be usually achieved by a direct inspection of the H-J equation and by extracting from it equations involving single coordinates and the so called **separation constants** (see the examples in Section 5). However, the problem of finding a Stäckel matrix, when only one line of its inverse is known, can be solved through algebraic manipulations by comparison with some **canonical form** of the inverse of a Stäckel matrix. Such canonical forms depend on the dimension $n$. For instance, when $n = 2$ the inverse of a Stäckel matrix has the following canonical form:

$$(\phi^i_{(j)}) = \begin{pmatrix} \phi^1_{(1)} & \phi^2_{(1)} \\ \phi^1_{(2)} & \phi^2_{(2)} \end{pmatrix} = \begin{pmatrix} \dfrac{\phi_2 \psi_1}{\phi_1 + \phi_2} & \dfrac{-\phi_1 \psi_2}{\phi_1 + \phi_2} \\ \dfrac{\psi_1}{\phi_1 + \phi_2} & \dfrac{\psi_2}{\phi_1 + \phi_2} \end{pmatrix}, \tag{3.24}$$

where $\phi_i$ and $\psi_i$ are functions of the variable corresponding to the index only (or constant). Hence, if we know the second row (i.e. the components of the metric) we can construct the first row (i.e. the associated first integral). The Stäckel matrix corresponding to (3.24) is

$$(\phi_i^{(j)}) = \begin{pmatrix} \phi_1^{(1)} & \phi_1^{(2)} \\ \phi_2^{(1)} & \phi_2^{(2)} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\psi_1} & \dfrac{\phi_1}{\psi_1} \\ -\dfrac{1}{\psi_2} & \dfrac{\phi_2}{\psi_2} \end{pmatrix}. \tag{3.25}$$

For $n = 3$ a canonical form is

$$\begin{cases} \phi^i_{(1)} = \dfrac{\psi_i}{\phi}(\mu_{i+2}\nu_{i+1} - \mu_{i+1}\nu_{i+2}), \\ \phi^i_{(2)} = \dfrac{\psi_i}{\phi}(\nu_{i+2} - \nu_{i+1}), \\ \phi^i_{(3)} = \dfrac{\psi_i}{\phi}(\mu_{i+1} - \mu_{i+2}), \end{cases} \tag{3.26}$$

where the functions depend on the variable corresponding to the index and $\phi$ is the determinant

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix}.$$

Canonical forms of Stäckel matrices for $n = 2, 3, 4$ are discussed in [BF1].

## 4. Equivalent separable coordinates

The discussion of the Levi-Civita separation conditions (2.6) for non-orthogonal variables seems to be much more difficult than the orthogonal case. In order to simplify this problem, one way is to perform suitable transformations of coordinates which preserve the separation.

First of all we remark that every coordinate transformation which involves separately each one of the coordinates (i.e. whose Jacobian is diagonal) is always allowed: it preserves the separation property. It also preserves the coordinate surfaces. We will call such a coordinate transformation a **separated transformation**. However, one can ask if these transformations are the only ones compatible with the separation. We can give an answer to this question, following again a suggestion of Levi-Civita who, in the case of a purely geodesic Hamiltonian, distinguished the separable coordinates in two **classes** [LC]. We can extend this classification to a generic Hamiltonian as follows.

**Definition 4.1.** Let $(q^i)$ be a system of separable coordinates. We say that a coordinate $q^i$ is of **first class** if the function $R_i$ (defined in (2.3)) is a linear function in the momenta, i.e. of the kind $R_i = B_i^j(q^h)\,p_j$. Otherwise, we say that $q^i$ is of **second class**. A coordinate $q^i$ is called **ignorable** if $R_i = 0$, i.e. if $\partial_i H = 0$ (an ignorable coordinate is of first class).

It is known that a complete integral is geometrically represented by a local Lagrangian foliation of the cotangent bundle $T^*Q$, transversal to the fibers, such that the Hamiltonian $H$ is constant on each leaf. With a given Hamiltonian we can have different separable systems of coordinates corresponding to different foliations. One can consider, as a simple example, the case of the geodesic Hamiltonian in the Euclidean plane. Cartesian coordinates and polar coordinates are both separable, but they give rise to different complete integrals. We say that they are <u>n</u>ot equivalent accordingly to the following

**Definition 4.2.** Two separable coordinate systems are said to be **equivalent** if in every domain in which they are both defined they give rise to the same complete integral, interpreted as a Lagrangian foliation of the cotangent bundle.

In order to simplify the discussion it is convenient to adopt the following notation for the indices [B1].

**Notation.** Latin indices $h, i, j, \ldots$ always run from 1 to $n$, the dimension of $Q$. For indices corresponding to coordinates of second class we use the first Latin letters $a, b, c \ldots$, which run from 1 to $m \leq n$. For indices corresponding to coordinates of first class we use the first Greek letters $\alpha, \beta, \gamma, \ldots$, which run from $m+1$ to $n$. We set $r = n - m$, the number of coordinates of first class.

**Theorem 4.1.** *Two equivalent separable systems have the same number of first class coordinates (hence, the same number of second class coordinates).*

*Proof.* Let $(q^i)$ and $(q^{i'})$ be two equivalent systems of separable coordinates. The corresponding momenta are related by equations

$$p_i = A_i^{i'} p_{i'}, \quad p_{i'} = A_{i'}^i p_i, \tag{4.1}$$

where

$$A_i^{i'} = \frac{\partial q^{i'}}{\partial q^i}, \quad A_{i'}^i = \frac{\partial q^i}{\partial q^{i'}}.$$

When a complete integral $W$ is given, we have $p_i = \partial_i W$ and $p_{i'} = \partial_{i'} W$. By applying the partial derivative $\partial_{j'} = A_{j'}^j \partial_j$ to the first set of equations (4.1) we get

$$A_{j'}^j \partial_j p_i = A_{j'}^j \partial_j A_i^{i'} p_{i'} + A_i^{i'} \partial_{j'} p_{i'}.$$

On the other hand the complete integral is a solution of the following two differential systems (see (2.2)),

$$\partial_j p_i = \delta_{ji} R_i, \quad \partial_{j'} p_{i'} = \delta_{j'i'} R_{i'}.$$

Thus the following equations must be identically satisfied:

$$A_{j'}^i R_i = A_{j'}^j \partial_j A_i^{i'} p_{i'} + A_i^{j'} R_{j'} \quad (i, j' \text{ n.s.}).$$

Let us consider the case $i = \alpha$ (index of first class) and $j' = a'$ (index of second class):

$$A_{j'}^\alpha R_\alpha = A_{a'}^j \partial_j A_\alpha^{i'} p_{i'} + A_\alpha^{a'} R_{a'} \quad (\alpha, a' \text{ n.s.}).$$

By definition of coordinate of first class, the term $R_\alpha$ is linear in the momenta $(p_j)$, thus it is linear in the momenta $(p_{j'})$, and this equation shows that also $R_{a'}$ is linear, against our assumption, unless $A_\alpha^{a'} = 0$. Hence, by symmetry,

$$A_\alpha^{a'} = 0, \quad A_{\alpha'}^a = 0.$$

This means that the second class coordinates $(q^a)$ depend on the second class coordinates $(q^{a'})$ only, and viceversa. Hence, the number of second class coordinates in both systems is the same. ∎

**Theorem 4.2.** *Every separable system is equivalent to a separable system in which all first class coordinates are ignorable.*

*Proof.* Accordingly to the distinction in two classes of the coordinates, we can split system (2.2) into the following equations:

$$\partial_\alpha p_\alpha = B_\alpha^i p_i, \quad \partial_i p_j = 0 \quad (i \neq j), \quad \partial_a p_a = R_a. \tag{4.2}$$

The integrability conditions of the whole system (2.4), with a particular choice of the indices, give

$$\partial_a B_\alpha^i p_i + B_\alpha^a R_a = 0 \quad (a \text{ n.s.}).$$

This equation shows that $R_a$ is linear in the momenta, which is against the assumptions, unless $B_\alpha^a = 0$, so that

$$B_\alpha^a = 0, \quad \partial_a B_\alpha^\beta = 0. \tag{4.3}$$

Hence, the system (4.2) contains an autonomous sub-system in the first coordinates only,

$$\partial_\alpha p_\beta = 0 \quad (\alpha \neq \beta), \quad \partial_\alpha p_\alpha = B^\gamma_{\alpha} p_\gamma.$$

This system is integrable and it is linear in the momenta $(p_\alpha)$. Hence, locally there exist $r = n - m$ independent solutions $(\phi^{(\beta)}_\alpha)$, such that the general solution is

$$p_\alpha = c_\beta \, \phi^{(\beta)}_\alpha, \quad (c_\beta) \in \mathbb{R}^n. \tag{4.4}$$

Let us consider the new coordinate system $(x^i)$ defined by

$$dx^a = dq^a, \quad dx^\alpha = \phi^{(\alpha)}_\beta dq^\beta. \tag{4.5}$$

Equations (4.4) shows that the momenta $(p_{(\alpha)})$ associated with these new coordinates, evaluated on the original complete integral $W$, are constant: $p_{(\alpha)} = c_\alpha$. Hence, the local expression of the Hamiltonian $H$ in these coordinates cannot depend on the coordinates $(x^\alpha)$, or, in other words, the coordinates $(x^\alpha)$ are ignorable. Since the second class coordinates are unchanged, the new system $(x^i)$ is separable. ■

The matrix $(\phi^{(\beta)}_\alpha)$ in the preceding proof is a Stäckel matrix in the $r$ variables $(q^\alpha)$.

**Remark 4.1.** When the Hamiltonian is linear, $H = X^i p_i$, all separable coordinates are of first class, so that we can always reduce to the case in which all separable coordinates are ignorable, and the complete integral takes the form $W = a_i q^i$.

The preceding theorems hold for whatever Hamiltonian. From now on we will consider the case of a geodesic Hamiltonian (the following two statements can be extended to polynomial Hamiltonians of second degree (1.4), but with longer proofs).

**Theorem 4.3.** *In two equivalent separable systems the second class coordinates are related by a separated transformation (i.e. they generate the same coordinate surfaces).*

*Proof.* With the same procedure of the proof of Theorem 4.1, let us apply the partial derivative $\partial_{j'}$ to the second set of equations (4.1). By assuming that $j' \neq i'$, we get

$$\partial_{j'} A^i_{i'} \, p_i + A^i_{j'} A^i_{i'} \, R_i = 0. \tag{4.6}$$

Let us consider in particular two indices of second class, $a' \neq b'$:

$$\partial_{a'} A^i_{b'} \, p_i + A^a_{a'} A^a_{b'} \, R_a + A^\alpha_{a'} A^\alpha_{b'} \, R_\alpha = 0.$$

We remark that the first and the last terms in this equation are linear in the momenta. If we assume that there is no linear combination of the kind $f^a R_a$, where $f^a$ are functions of the coordinates $(q^i)$ only, which reduces to a linear form in the momenta $(p_i)$, then the above formula cannot hold, unless

$$A^a_{a'} A^a_{b'} = 0 \quad (a' \neq b', a \text{ n.s.})$$

for each index $a$ of second class. This means that for each index of second class $a$ there exists one and only one index of second class $a'$ such that $A^a_{a'} \neq 0$. Indeed, if two such indices $(a', b')$ exist, the equation above cannot be satisfied; and, on the contrary, if no such indices exist, in the whole matrix $(A^i_{i'})$ the line $i = a$ is made of zeros: absurd, since this matrix is regular. The fact that only one $A^a_{a'}$ does not vanish, proves that $q^a$ depends on one second class coordinate only. It remains to prove that no sum of the kind $f^a R_a$ can be linear in the momenta. We have

$$R_a = \tfrac{1}{2} \frac{\partial_a g^{ij} p_i p_j}{g^{ak} p_k} = -\frac{1}{2v^a} \partial_a g_{ij} v^i v^j,$$

being $(v^i)$ new variables replacing the momenta:

$$v^i = g^{ij} p_j, \quad p_i = g_{ij} v^j.$$

Let us assume that there is a combination such that $f^a R_a = L_i v^i$, where $(L_i)$ are functions of the coordinates only. With a change of notation this equation can be written as follows:

$$\sum_a \frac{1}{2v^a} P_{aij} v^i v^j = L_i v^i. \tag{4.7}$$

Let us differentiate equation (4.7) with respect to $v^b$:

$$-\frac{1}{2(v^b)^2} P_{bij} v^i v^j + \sum_a \frac{1}{v^a} P_{abi} v^i = L_b.$$

And again, twice with respect to $v^c$, with $c \neq b$:

$$-\frac{1}{(v^b)^2} P_{bci} v^i - \frac{1}{(v^c)^2} P_{cbi} v^i + \sum_a \frac{1}{v^a} P_{abc} = 0.$$

$$-\frac{1}{(v^b)^2} P_{bcc} + \frac{2}{(v^c)^3} \sum_{i \neq c,b} P_{cbi} v^i = 0.$$

Just like the preceding ones, this last equation must be identically satisfied for all admissible values of $(v^i)$. Hence, $P_{bcc} = 0$ and $P_{cbi} = 0$ for $b \neq c, c \neq i$. Let us differentiate (4.7) twice with respect to $v^\alpha$ and $v^\beta$ (indices of first class). We get:

$$\sum_a \frac{1}{v^a} P_{a\alpha\beta} = 0.$$

This proves that $P_{a\alpha\beta} = 0$. These results show that the polynomial $P_{aij} v^i v^j$ is divisible by $v^a$, against the assumption that $a$ is an index of second class. ■

**Theorem 4.4.** *Let $(q^i) = (q^a, q^\alpha)$ be a separable system such that all coordinates of first class $(q^\alpha)$ are ignorable. Then every equivalent separable system $(q^{i'}) = (q^{a'}, q^{\alpha'})$ is related to this one, modulo a separated transformation, by equations of the kind*

$$dq^a = dq^{a'}, \quad dq^\alpha = A^\alpha_{i'}\, dq^{i'}, \tag{4.8}$$

*where $A^\alpha_{i'}$ are functions depending on the variable corresponding to the lower index only.*

*Proof.* We have already seen that coordinates of second class are invariant up to a separated transformation. Since $(q^\alpha)$ are ignorable coordinates, we have $R_\alpha = 0$ and equation (4.6) gives $\partial_{j'} A^\alpha_{i'} = 0$ for $i' \neq j'$. ■

## 5. The normal form of the metric tensor components in separable coordinates

As we have said in the last part of the preceding section, we are considering the purely geodesic Hamiltonian: $H = G = \frac{1}{2} g^{ij} p_i p_j$.

**Theorem 5.1.** *In a separable system $(q^i) = (q^a, q^\alpha)$, the coordinates of second class $(q^a)$ are orthogonal: $g^{ab} = 0$ for $a \neq b$.*

*Proof.* The Levi-Civita separation conditions (2.2) for two indices of second class $a \neq b$ can be written as follows,

$$H^a (H^b H_{ab} - H_b H^b_a) = H_a (H^b H^a_b - H_b H^{ab}), \tag{5.1}$$

with the further simplifying convention $H^i = \partial^i H$, $H_i = \partial_i H$, etc.. Under our assumptions we have $H^i = v^i$ and $H^{ij} = g^{ij}$. By definition of second class coordinate the ratio $R_a = H_a/H^a$ is not a homogeneous first degree polynomial in the momenta, i.e. the first degree homogeneous polynomial $H^a$ is not a divisor of the second degree homogeneous polynomial $H_a$. Equation (5.1) shows that it must be a divisor of the second degree polynomial $H^b H^a_b - H_b g^{ab}$, i.e. that there exists a first degree homogenous polynomial

$$L_{ba} = L^i_{ba} p_i = L_{bai} v^i$$

such that

$$H^b H^a_b - H_b g^{ab} = H^a L_{ba}. \tag{5.2}$$

From (5.1) it follows that

$$H^b H_{ab} - H_b H^b_a = H_a L_{ba}. \tag{5.3}$$

Our aim is to prove that condition $g^{ab} \neq 0$ implies $L_{ba} = 0$ so that from (5.2) $H^b$ would be a divisor of $H_b$, which is against the assumption that also $b$ is an index of second class. For this purpose, let us differentiate equation (5.2) twice with respect to $p_a$:

$$H^b g^{aa}_b = g^{aa} L_{ba} + H^a L^a_{ba}, \tag{5.4}$$

$$g^{ab} g^{aa}_b = 2 g^{aa} L^a_{ba}. \tag{5.5}$$

The explicit form of equation (5.4) in the variables $(v^i)$ is

$$v^b g^{aa}_b = g^{aa} L_{bai} v^i + v^a L^a_{ba}.$$

As a polynomial identity in $(v^i)$, this equation implies:

$$g^{aa}_b = g^{aa} L_{bab}, \tag{5.6}$$

$$g^{aa} L_{baa} + L^a_{ba} = 0, \tag{5.7}$$

$$g^{aa} L_{bai} = 0 \quad (i \neq b, a). \tag{5.8}$$

Let us differentiate equation (5.2) by $q^a$ and equation (5.3) by $p_a$:

$$H^b_a H^a_b + H^b H^a_{ba} - H_{ba} g^{ab} - H_b g^{ab}_a = H^a_a L_{ba} + H^a L_{ia,a},$$

$$g^{ab} H_{ab} + H^b H^a_{ab} - H^a_b H^b_a - H_b g^{ab}_a = H_a L^a_{ba} + H^a_a L_{ba},$$

where $L_{ba,a} = \partial_a L_{ba}$. By subtracting term by term, we get equation

$$2\, H^b_a H^a_b - 2\, H_{ba} g^{ba} = H^a L_{ba,a} - H_a L^a_{ba}.$$

On the other hand, equations (5.2) and (5.3) give respectively

$$H^a_b = \frac{1}{H^b}(H^a L_{ba} + H_b g^{ab}), \quad H_{ab} = H_{ba} = \frac{1}{H^b}(H_a L_{ba} + H_b H^b_a),$$

so that from the last equation we get finally:

$$H^a(2\,H^b_a L_{ba} - H^b L_{ba,a}) = H_a(2\,g^{ba} L_{ba} - H^b L^b_{ba}).$$

This equation shows that $H^a = v^a$ must be a divisor of the polynomial

$$H^b L^a_{ba} - 2g^{ab} L_{ba} = v^b L^a_{ba} - 2\,g^{ab} L_{bai} v^i,$$

so that

$$L^a_{ba} = 2\,g^{ab} L_{bab}, \tag{5.9}$$

$$g^{ab} L_{bai} = 0 \quad (i \neq a, b). \tag{5.10}$$

Now, we make use of all equations (5.5)-(5.10). From (5.6), (5.5) and (5.9) it follows that

$$g^{ab} g^{aa} L_{bab} = g^{ab} g^{aa}_b = 2\,g^{aa} L^a_{ba} = 4g^{ab} g^{aa} L_{bab}.$$

Hence,

$$g^{ab} g^{aa} L_{bab} = 0, \quad g^{aa} L^a_{ba} = 0. \tag{5.11}$$

Let us assume that $g^{ab} \neq 0$. We have two cases: (i) $g^{aa} \neq 0$, (ii) $g^{aa} = 0$. (i) We have: $L_{bab}$ from (5.11), $L_{bai} = 0\,(i \neq a, b)$ from (5.8) or from (5.10), $L^a_{ba} = 0$ from (5.11), $L_{baa} = 0$ from (5.7). Thus we have proved $L_{ba} = 0$: absurd. (ii) If $g^{bb} \neq 0$ we find case (i) by symmetry. Thus let us assume $g^{aa} = g^{bb} = 0$. We have: $L_{bai} = 0\,(i \neq a, b)$ from (5.10), $L^a_{ba} = 0$ from (5.7) and $L_{bab} = 0$ from (5.9). This means that

$$L_{ba} = L_{baa} v^a = L_{baa} g^{ak} p_k. \tag{5.12}$$

Let us differentiate equation (5.2) twice with respect to $p_b$. Under the assumption $g^{bb} = 0$ we get $g^{ab} L^b_{ba} = 0$, thus $L^b_{ba} = 0$. However, due to equation (5.12), $L^b_{ba} = \partial^b L_{ba} = L_{baa} g^{ab}$. This shows that $L_{baa} = 0$, too. We have proved that $L_{ba} = 0$: absurd. ∎

This proof, which is taken from [B2], is inspired to a proof given by Dall'Acqua in 1912 [DA] under the assumption $g^{aa} \neq 0$, which is always fulfilled by a positive-definite metric (see also [CN]). This proof can be extended to Hamiltonians of the kind (1.4), through suitable modifications of the classification of the coordinates. This proof suggests a further distinction between the coordinates of second class.

**Definition 5.1.** A coordinate $q^a$ is said to be **isotropic** if $g^{aa} = 0$.

This means that the gradient of the coordinate $q^a$ is an **isotropic** (or **null**) vector and the coordinate surface $q^a =$ const. is a **coisotropic** submanifold (a submanifold $S \subset Q$ of a Riemannian manifold is called coisotropic if $(T_q S)^\perp \subset T_q S$ for each point $q \in S$). Hence, isotropic coordinates cannot occur in a strictly-Riemannian manifold. Let $m_1$ and $m_2$ be the number of non-isotropic and of isotropic coordinates of second class respectively: $m_1 + m_2 = m = n - r$. Also these numbers are invariant for equivalent systems of separable coordinates, because of Theorem 4.3. Let us denote by $\tilde{a}, \tilde{b}, \dots$ the indices of second class corresponding to non-isotropic coordinates and running from 1 to $m_1$, and by $\bar{a}, \bar{b}, \dots$ the indices of second class corresponding to the isotropic coordinates, running from $m_1 + 1$ to $m = n - r$.

Now we can state the main theorem of our discussion (see [B1]).

**Theorem 5.2.** *In an equivalence class of separable coordinates there exists a coordinate system $(q^i)$ such that the first class coordinates $(q^\alpha)$ are ignorable and such that the $n \times n$ matrix $(g^{ij})$ has the following form*

$$(g^{ij}) = \begin{array}{c} \\ m_1 \\ m_2 \\ r \end{array} \begin{array}{c} \begin{array}{ccc} m_1 & m_2 & r \end{array} \\ \begin{pmatrix} \delta^{\tilde{a}\tilde{b}} g^{\tilde{a}\tilde{a}} & 0 & 0 \\ 0 & 0 & g^{\bar{a}\beta} \\ 0 & g^{\alpha\bar{a}} & g^{\alpha\beta} \end{pmatrix} \end{array}. \tag{5.13}$$

*Proof.* We have already proved the existence of equivalent systems where the first class coordinates are ignorable (Theorem 4.2) and that the second class coordinates are orthogonal (Theorem 5.1). It remains to prove that we can find equivalent separable coordinates for which $g^{\tilde{a}\alpha} = 0$, for each non-isotropic second class index $\tilde{a}$. Equations (5.1) reduce to

$$H^a(H^b H_{ab} - H_b H^b_a) = H_a H^b H^a_b,$$

so that $H^a$ must be a divisor of $H^b H^a_b$. Since $H^a = v^a$ is not a divisor of $H^b = v^b$, it must be a divisor of $H^a_b$, so that there exists a functions $f^{ab}$ of the coordinates only such that $H^a_b = f^{ab} H^a$, i.e. $\partial_b g^{ai} = f^{ab} g^{ai}$. A straightforward discussion shows that this equation implies

(5.14) $$g^{aj}\partial_b g^{ai} = g^{ai}\partial_b g^{aj} \quad (a \neq b),$$

for every pair of indices $(i, j)$. By choosing $(i, j) = (a, \alpha)$ such that $g^{aa} \neq 0$, we get equation [DA]

(5.15) $$\partial_b \left(\frac{g^{a\alpha}}{g^{aa}}\right) = 0, \quad \text{for } a \neq b.$$

This means that

$$\theta^\alpha_a = \frac{g^{a\alpha}}{g^{aa}}$$

is a function of the variable $q^a$ only (the coordinates of first class are ignorable, thus they do not appear in the metric components). The above considerations apply to all non-isotropic coordinates of second class $(q^{\tilde{a}})$: there exist functions $(\theta^\alpha_{\tilde{a}})$ depending on the variable corresponding to the lower index only, such that $g^{\tilde{a}\alpha} = \theta^\alpha_{\tilde{a}} g^{\tilde{a}\alpha}$. Let us consider a new coordinate system $(q^{a'}, q^{\alpha'})$ defined by equations

$$dq^{a'} = dq^a, \quad dq^{\alpha'} = dq^\alpha - \theta^\alpha_{\tilde{a}} dq^{\tilde{a}}.$$

This system is equivalent to the old one (see Theorem 4.4). The coordinates of first class $(q^{\alpha'})$ are still ignorable and moreover,

$$g^{\tilde{a}'\alpha'} = A^{\tilde{a}'}_i A^{\alpha'}_j g^{ij} = g^{\tilde{a}j} A^{\alpha'}_j = g^{\tilde{a}\beta} A^{\alpha'}_\beta + g^{\tilde{a}\tilde{c}} A^{\alpha'}_{\tilde{c}} = g^{\tilde{a}\alpha} - g^{\tilde{a}\tilde{a}}\theta^\alpha_{\tilde{a}} = 0.$$

Apart from the notation, this is a coordinate system which we were looking for. ∎

**Definition 5.2.** We call (5.13) the **normal form** of a metric tensor in separable coordinates. The separable coordinates for which such a normal form holds are called **normal separable coordinates**.

**Remark 5.1.** For the number $m_2$ of second class isotropic coordinates we have the limits

$$m_2 \leq r, \quad m_2 \leq \min(p, q),$$

where $(p, q)$ is the signature of the metric. Indeed, from the normal form (5.13) we see that $m_2 > r$ implies $\det(g^{ij}) = 0$, and the second limit follows from the fact that the maximal dimension of an isotropic subspace of a space of signature $(p, q)$ is $\min(p, q)$, and the tangent subspaces spanned by the gradients of the coordinates $(q^{\bar{a}})$ are isotropic.

**Theorem 5.4.** *The most general form of the metric tensor components in normal separable coordinates is the following:*

(5.16) $$\begin{cases} g^{ab} = 0 \quad \textit{for} \quad a \neq b \quad \textit{and} \quad a, b = 1, \dots, m \leq n, \\ g^{\tilde{a}\tilde{a}} = \phi^{\tilde{a}}_{(m)} \quad \textit{for} \quad \tilde{a} = 1, \dots, m_1 \leq m, \\ g^{\bar{a}\bar{a}} = 0 \quad \textit{for} \quad \bar{a} = m_1 + 1, \dots, m\,, \\ g^{\tilde{a}\alpha} = 0 \quad \textit{for} \quad \alpha = m+1, \dots, n \quad , \\ g^{\bar{a}\alpha} = \theta^\alpha_{\bar{a}}\, \phi^{\bar{a}}_{(m)} \quad (\bar{a}\ \textit{n.s.}), \\ g^{\alpha\beta} = \eta^{\alpha\beta}_a \phi^a_{(m)}, \end{cases}$$

*where $\phi^a_{(m)}$ is the last row of the inverse of a $m\times m$ Stäckel matrix in the second class coordinates $(q^a)$, and $\theta^\alpha_{\bar a}$ and $\eta^{\alpha\beta}_a$ are functions depending on the coordinates corresponding to the lower index only.*

*Proof.* We have the metric tensor components in the normal form (5.13). The first class coordinates are all ignorable. Let us consider equation (5.14) in the particular case

$$g^{\bar a\alpha}\partial_b g^{\bar a\beta} = g^{\bar a\beta}\partial_b g^{\bar a\alpha}.$$

For a fixed isotropic index of second class $\bar a$ there is at least one index of first class $\beta$ such that $g^{\bar a\beta}\neq 0$ (otherwise the matrix (5.13) would be singular), so that the preceding equation can be written

$$\partial_b\left(\frac{g^{\bar a\alpha}}{g^{\bar a\beta}}\right)=0.$$

This means that for each index $\bar a$ there is a function $X^{\bar a}$ and functions $(\theta^\alpha_{\bar a})$ depending on the variable corresponding to the lower index only, such that for each index $\alpha$

$$g^{\bar a\alpha}=\theta^\alpha_{\bar a}X^{\bar a}. \tag{5.17}$$

Now, let us consider the Hamiltonian in normal separable coordinates,

$$H=\tfrac{1}{2}\,g^{\tilde a\tilde a}(p_{\tilde a})^2+g^{\bar a\alpha}p_{\bar a}p_\alpha+\frac{1}{2}g^{\alpha\beta}p_\alpha p_\beta.$$

Since the first class coordinates are ignorable, the corresponding momenta can be considered as constants of integration of the H-J equation. The Hamiltonian can be written

$$H=X^{\tilde a}\phi_{\tilde a}+X^{\bar a}\phi_{\bar a}+V,$$

by setting

$$X^{\tilde a}=\tfrac{1}{2}\,g^{\tilde a\tilde a},\quad X^{\bar a}=g^{\bar a\alpha}p_\alpha,\quad V=\tfrac{1}{2}\,g^{\alpha\beta}p_\alpha p_\beta,\quad \phi_{\tilde a}=(p_{\tilde a})^2,\quad \phi_{\bar a}=p_{\bar a}. \tag{5.18}$$

We can apply to this case the discussion of Section 3. The $m$ functions $(X^a)=(X^{\tilde a},X^{\bar a})$ will be solutions of integrability conditions of the kind (3.19), and the "potential" $V$ will be a solution of an equation of the kind (3.21). This means that there exists a $m\times m$ Stäckel matrix $(\phi^a_{(b)})$ such that $X^a=\phi^a_{(m)}$ and $V=\eta_a\phi^a_{(m)}$, with $(\eta_a)$ function of the variable corresponding to the index only. Due to (5.17) and (5.18), the theorem is proved. ∎

**Remark 5.2.** Since we know the general form of the coordinate transformation from a separable system with ignorable first class coordinates (like a normal separable system) to a generic equivalent separable system (Theorem 4.4, formulae (4.8)), we can perform the corresponding tensor transformation on the components (5.16). The result will be the general form of the contravariant metric tensor components in separable coordinates. By a suitable adaptation of the notation, this form turns out to be the following:

$$\begin{cases} g^{ab}=0 \quad \text{for} \quad a\neq b \quad \text{and} \quad a,b=1,\dots,m\leq n,\\ g^{\tilde a\tilde a}=\phi^{\tilde a}_{(m)} \quad \text{for} \quad \tilde a=1,\dots,m_1\leq m,\\ g^{\bar a\bar a}=0 \quad \text{for} \quad \bar a=m_1+1,\dots,m\,,\\ g^{a\alpha}=\phi^\alpha_{(\beta)}\theta^\beta_a\,\phi^a_{(m)} \quad (a \text{ n.s.}),\\ g^{\alpha\beta}=\phi^\alpha_{(\gamma)}\phi^\beta_{(\delta)}\eta^{\gamma\delta}_a\phi^a_{(m)}, \end{cases} \tag{5.19}$$

where $(\phi^a_{(m)})$ is the last row of the inverse of a $m\times m$ Stäckel matrix in the second class coordinates $(q^a)$, $(\phi^\alpha_{(\beta)})$ is the inverse of a $r\times r$ Stäckel matrix in the first class coordinates $(q^\alpha)$, and the other functions depending on the coordinate corresponding to the lower index only (compare with [II] [CN]).

According to Theorem 5.4, the most general separable geodesic Hamiltonian in normal separable coordinates takes the form

$$(5.20) \qquad G = \tfrac{1}{2}\,\phi^{\tilde a}_{(m)}\,(p_{\tilde a})^2 + \phi^{\bar a}_{(m)}\theta^{\alpha}_{\bar a}\,p_{\bar a}p_{\alpha} + \tfrac{1}{2}\,\phi^{a}_{(m)}\eta^{\alpha\beta}_{a}\,p_{\alpha}p_{\beta}.$$

In the general form of the metric tensor components, with respect to normal separable coordinates (5.16), and in the expression of the most general Hamiltonian (5.20), only one line of the inverse of a Stäckel matrix appears (the last one, according to our notation). The analogous formulae obtained by considering any other row $\phi^a_{(b)}$ of this matrix will give the components of a tensor $K^{ij}_{(b)}$, such that the $m$ quadratic functions

$$(5.21) \qquad F_b = \tfrac{1}{2}\,K^{ij}_{(b)}p_ip_j = \tfrac{1}{2}\,\phi^{\tilde a}_{(b)}\,(p_{\tilde a})^2 + \phi^{\bar a}_{(b)}\theta^{\alpha}_{\bar a}\,p_{\bar a}p_{\alpha} + \tfrac{1}{2}\,\phi^{a}_{(b)}\eta^{\alpha\beta}_{a}\,p_{\alpha}p_{\beta},$$

are first integrals in involution. This follows from a discussion similar to that of Section 3, concerning the procedure of separation: we interpret the H-J equation $G = h$ as the last equation of a system $F_b = a_b$ of $m$ equations, which can be solved with respect to the momenta,

$$(5.22) \qquad (p_{\tilde a})^2 = 2a_b\phi^{(b)}_{\tilde a} - \eta^{\alpha\beta}_{\tilde a}a_{\alpha}a_{\beta}, \quad (a_{\alpha}\theta^{\alpha}_{\bar a})\,p_{\bar a} = a_b\phi^{(b)}_{\bar a} - \tfrac{1}{2}\,\eta^{\alpha\beta}_{\bar a}a_{\alpha}a_{\beta},$$

taking into account that the momenta of first class are constant (since the first class coordinates are ignorable):

$$p_{\alpha} = a_{\alpha}.$$

Equations (5.22) are separated equations: they form a system of separated ordinary differential equations in the $m$ unknown functions $W_a(q^a)$ such that $p_a = dW_a/dq^a$. The constants $(a_i) = (a_a, a_{\alpha})$ will appear in the complete integral

$$W = \sum_{a=1}^{m} W_a(q^a, a_i) + a_{\alpha}q^{\alpha}.$$

They are the so called **separation constants**.

We conclude that the separation of variables is concomitant with the existence of $r$ ignorable coordinates $(q^{\alpha})$, which generate $r$ linear first integrals $(p_{\alpha})$, and the existence of a complementary number $m = n - r$ of quadratic first integrals $(F_b)$, one of which is the geodesic Hamiltonian itself. All these first integrals are in involution due to the Jacobi theorem. As we will see in the next sections, in order to discuss the intrinsic characterization of the separation of variables, it is more convenient to think of these first integrals as **Killing vectors** or **Killing tensors**. The Killing vectors are the partial derivatives $(\partial_{\alpha})$ associated with the ignorable coordinates, interpreted as vector fields. The components of the Killing tensors are the coefficients $(K^{ij}_b)$ of the quadratic forms $(F_b)$.

When there are no isotropic second class coordinates (like in the case of a positive definite metric) the normal form (5.13) reduces to

$$(5.23) \qquad (g^{ij}) = \begin{array}{c c} & \begin{array}{cc} m & r \end{array} \\ \begin{array}{c} m \\ \\ r \end{array} & \begin{pmatrix} \delta^{ab}g^{aa} & 0 \\ & \\ 0 & g^{\alpha\beta} \end{pmatrix} \end{array},$$

where

$$(5.24) \qquad g^{aa} = \phi^a_{(m)}, \quad g^{\alpha\beta} = \eta^{\alpha\beta}_a\,g^{aa}.$$

The non vanishing components of the Killing tensors are:

$$(5.25) \qquad K^{aa}_{(b)} = \phi^a_{(b)}, \quad K^{\alpha\beta}_{(b)} = \eta^{\alpha\beta}_a K^{aa}_{(b)}.$$

Several examples of separable metrics are given by exact solutions of Einstein's field equations (see for instance [CR] [W] [KSMH] [BF2]). We present here only three of them, showing how they fit the theory.

**Example 5.1.** The **Reissner-Nordström metric**

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{r^2}{\Delta} dr^2 - \frac{\Delta}{r^2}\, dt^2, \tag{5.26}$$

where $\Delta = r^2 + e^2 - 2mr$, provides a mathematical model for the exterior field of a spherically symmetric body of charge $e$ and mass $m$. For $e = 0$ it reduces to the **Schwarzschild metric**. Coordinates $\phi$ and $t$ are ignorable, so that, according to our notation it is convenient to set

$$q^1 = r, \quad q^2 = \theta, \quad q^3 = \phi, \quad q^4 = t. \tag{5.27}$$

These coordinates are orthogonal and the contravariant components are

$$(g^{ii}) = \left(\frac{\Delta}{r^2}, \frac{1}{r^2}, \frac{1}{r^2 \sin^2\theta}, -\frac{r^2}{\Delta}\right). \tag{5.28}$$

We can see directly from the H-J equation that these coordinates are separable. The separation is achieved by introducing the separation constant

$$c = p_2^2 + \frac{1}{\sin^2\theta} p_3^2,$$

so that the H-J equation reduces to

$$\Delta p_1^2 + c - \frac{r^4}{\Delta} p_4^2 = 2h\, r^2,$$

where $(p_\alpha) = (p_3, p_4)$ are constants due to the ignoration of the coordinates $(q^3, q^4)$. However, the separation can be also recognized by comparing (5.28) with the normal form (5.16). The coordinates $(q^a) = (q^1, q^2)$ are of second class and the components

$$(g^{aa}) = \left(\frac{\Delta}{r^2}, \frac{1}{r^2}\right)$$

fit the condition $g^{aa} = \phi^a_{(2)}$ by setting in the canonical form (3.24),

$$\phi_1 = r^2, \quad \phi_2 = 0, \quad \psi_1 = \Delta, \quad \psi_2 = 1.$$

The condition $g^{\alpha\beta} = \eta_a^{\alpha\beta} g^{aa}$ is fulfilled by setting

$$\eta_1^{33} = 0, \quad \eta_1^{44} = -\frac{r^4}{\Delta^2}, \quad \eta_2^{33} = \frac{1}{\sin^2\theta}, \quad \eta_2^{44} = 0.$$

The conclusion is that the given coordinates are normal separable coordinates. Moreover, by using formula (5.25) and (3.24) we can construct the components of the non trivial Killing tensor $K = K_1$. The result is

$$K^{11} = 0, \quad K^{22} = -1, \quad K^{33} = -\frac{1}{\sin^2\theta}, \quad K^{44} = 0,$$

and, up to the inessential factor $-\frac{1}{2}$, the corresponding first integral coincides with the separation constant $c$. If instead of $t$ we use a new coordinate $u$ defined by equation (see [KSMH])

$$dt = du + \frac{r^2}{\Delta} dr, \tag{5.29}$$

then the metric becomes

$$ds^2 = r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - 2du\, dr - \frac{\Delta}{r^2} du^2.$$

The new system $(r, \theta, \phi, u)$ is separable and equivalent to the old one, since the transformation (5.29) fits the transformation formulae (4.8). However, these coordinates are no more normal, since the matrix $(g^{ij})$ has now the form:

$$(g^{ij}) = \begin{pmatrix} 1 & 0 & 0 & -\dfrac{r^2}{\Delta} \\ 0 & \dfrac{1}{\Delta} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\Delta \sin^2\theta} & 0 \\ -\dfrac{r^2}{\Delta} & 0 & 0 & 0 \end{pmatrix}.$$

**Example 5.2.** The **Kerr-Newmann metric**

$$\begin{aligned} ds^2 = {} & \rho^2 \left( \frac{dr^2}{\Delta} + d\theta^2 \right) + \sin^2\theta \left( r^2 + a^2 - \varepsilon \frac{a^2 \sin^2\theta}{\rho^2} \right) d\phi^2 \\ & - \left( 1 + \frac{\varepsilon}{\rho^2} \right) dt^2 + 2\,\varepsilon a \frac{\sin^2\theta}{\rho^2}\, dt\, d\phi, \end{aligned}$$

where $\Delta = r^2 + a^2 + e^2 - 2mr = r^2 + a^2 + \varepsilon$ and $\rho^2 = r^2 + a^2\cos^2\theta$, is a model for the exterior gravitational field of a rotating source of charge $e$, mass $m$ and angular momentum $a$. For $e = 0$ it reduces to the **Kerr metric**, and for $e = 0$ and $a = 0$ to the Schwarzschild metric. As in (5.27) we set $q^1 = r, \quad q^2 = \theta, \quad q^3 = \phi, \quad q^4 = t$. The contravariant components form the matrix

$$(g^{ij}) = \begin{pmatrix} \dfrac{\Delta}{\rho^2} & 0 & 0 & 0 \\ 0 & \dfrac{1}{\rho^2} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\rho^2}\left( \dfrac{1}{\sin^2\theta} - \dfrac{a^2}{\Delta} \right) & \dfrac{a}{\rho^2}\left( 1 - \dfrac{a^2 + r^2}{\Delta} \right) \\ 0 & 0 & \dfrac{a}{\rho^2}\left( 1 - \dfrac{a^2 + r^2}{\Delta} \right) & \dfrac{1}{\rho^2}\left( a^2\sin^2\theta - \dfrac{(a^2 + r^2)^2}{\Delta} \right) \end{pmatrix}$$

This is a normal form, where $(q^a) = (r, \theta)$ are second class coordinates and $(q^\alpha) = (\phi, t)$ are first class ignorable coordinates. Indeed we have

$$\phi_1 = r^2, \quad \phi_2 = a^2\cos^2\theta, \quad \psi_1 = \Delta, \quad \psi_2 = 1,$$

and

$$(\eta_1^{\alpha\beta}) = -\frac{1}{\Delta^2} \begin{pmatrix} a^2 & a(a^2 + r^2) \\ a(a^2 + r^2) & (a^2 + r^2)^2 \end{pmatrix}, \qquad (\eta_2^{\alpha\beta}) = \begin{pmatrix} \dfrac{1}{\sin^2\theta} & a \\ a & a^2\sin^2\theta \end{pmatrix}.$$

These normal separable coordinates are not orthogonal.

**Example 5.3.** The **Friedmann metric**

$$ds^2 = dt^2 - R(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\eta^2 \right),$$

where $k$ is a constant and $R(t)$ an arbitrary smooth function without zeros. The coordinates

$$q^1 = r, \quad q^2 = t, \quad q^3 = \theta, \quad q^4 = \eta,$$

are normal separable coordinates; $(r, t, \theta)$ are of second class, and $\eta$ is ignorable. Indeed, the contravariant components of the metric are

$$(g^{ii}) = \left( \frac{kr^2 - 1}{R(t)}, 1, -\frac{1}{r^2 R(t)}, -\frac{1}{r^2 R(t) \sin^2\theta} \right).$$

The separation can be achieved through the separation constants

$$a_1 = R(\tfrac{1}{2}\, p_2^2 - h), \quad a_2 = p_3^2 + \frac{1}{\sin^2\theta} p_4^2.$$

We can fit the canonical form (3.26) with

$$\psi_1 = \frac{kr^2 - 1}{1 - r^2}, \quad \psi_2 = R(t), \quad \psi_3 = 1,$$

$$\mu_1 = \frac{1}{1 - r^2}, \quad \mu_2 = 0, \quad \mu_3 = 1,$$

$$\nu_1 = 0, \quad \nu_2 = -R(t), \quad \nu_3 = 0.$$

and formula (5.25) with

$$(\eta_a^{44}) = \left(0, 0, \frac{1}{\sin^2\theta}\right).$$

**Example 5.4.** This is an example of normal separable coordinates with an isotropic second class coordinate (it is taken from [KM1], where further examples can be found). The three-dimensional metric

$$ds^2 = y\, dz^2 + 2\, dx\, dy + \tfrac{1}{4}\, \tfrac{z^2}{y} dy^2$$

is a locally flat Lorentzian metric. If we set

$$q^1 = z, \quad q^2 = y, \quad q^3 = x,$$

then the contravariant components are

$$(g^{ij}) = \begin{pmatrix} \frac{1}{y} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{4}\frac{z^2}{y} \end{pmatrix}.$$

The coordinate $q^3$ is ignorable, $q^1$ is of second class and $q^2$ is isotropic of second class. The normal form of the metric (5.16) and the canonical form (3.24) can be fulfilled by setting

$$\phi_1 = 0, \quad \phi_2 = y, \quad \psi_1 = 1, \quad \psi_2 = 1, \quad \theta_2^3 = y, \quad \eta_1^{33} = -\tfrac{1}{4}\, z^2, \quad \eta_2^{33} = 0.$$

## 6. Intrinsic characterization of the orthogonal separation

With each smooth contravariant symmetric tensor field $K$ over a manifold $Q$ we associate a real smooth function $E_K$ on the cotangent bundle $T^*Q$ defined by

$$E_K = \tfrac{1}{k!} K^{i_1 \ldots i_k} p_{i_1} \ldots p_{i_k}, \tag{6.1}$$

where $k$ is the order of $K$. This fact establishes a natural identification between symmetric contravariant tensors on $Q$ and functions on $T^*Q$ which are homogeneous polynomial in the momenta. Due to this identification, the two natural algebraic structures on the space of these functions induce two algebraic structures on the space of contravariant symmetric tensors. The ordinary product of real functions induces a **symmetric product** denoted by $\cap$ and defined by equation:

$$E_{K\cap L} = E_K E_L. \tag{6.2}$$

This product is commutative and associative. The Poisson brackets on $T^*Q$, induces Lie brackets $[K, L]$ defined by equation

$$E_{[K,L]} = \{E_K, E_L\}. \tag{6.3}$$

Moreover, the Lie brackets are a bi-derivation with respect to the symmetric product, i.e.

$$[K, L\cap M] = [K, L]\cap M + [K, M]\cap L. \tag{6.4}$$

We say that symmetric contravariant tensors $K$ and $L$ **commute** or that they **are in involution** when $[K, L] = 0$. If $Q$ is a Riemannian manifold, with metric tensor $g$, then there is a natural identification between contravariant and covariant (symmetric) tensors, and $E_g = G$ is the geodesic Hamiltonian. A **Killing tensor** is a symmetric tensor which commute with the metric: $[K, g] = 0$; i.e. such that the corresponding function $E_K$ is a geodesic first integral: $\{E_K, G\} = 0$.

In the following discussion we will deal only with Killing tensors of order 1, which are called **Killing vectors**, and of order 2, which we will call briefly **Killing tensors**, by understanding the order 2.

A Killing vector $X$ is characterized by the following equivalent properties. (i) The Lie derivative of the metric tensor $g$ with respect to $X$ is zero. (ii) The (local) action on $Q$ generated by $X$ is made of isometries (i.e. of **rigid motions**, according to the old terminology). (iii) In a coordinate system $(q^i)$ such that $\partial_1 = X$, the coordinate $q^1$ is ignorable: $\partial_1 g_{ij} = 0$. (iv) The covariant derivative $\nabla X$ with respect to the Levi-Civita connection is skew-symmetric. (v) As a derivation over the functions, $X = X^i\partial_i$ commute with the Laplacian $\Delta = g^{ij}\nabla_i\partial_j$.

Tensor fields of order 2 can be interpreted as linear operators over the spaces of vector fields and 1-forms. Let us denote by $X\cdot K$ and by $\phi\cdot K$ the value of a 2-tensor field $K$ on a vector field $X$ and a 1-form $\phi$. The tensor is symmetric if and only if $X\cdot K\cdot Y = Y\cdot K\cdot X$. In this sense, we can also consider fields of **eigenvectors**, **eigenforms** and **eigenvalues** of $K$.

It can be proved that

**Theorem 6.1.** *(i)* [E2] *If a symmetric tensor field $K$ of order 2 is diagonalized in a coordinate system $(q^i)$, then it is a Killing tensor if and only if the following equations are satisfied*

$$\partial_i\rho_j = (\rho_i - \rho_j)\,\partial_i \log|g^{jj}|, \tag{6.5}$$

*where $\rho_i$ is the eigenvalue corresponding to the vector fields $\partial_i = \frac{\partial}{\partial q^i}$. (ii) If two Killing tensors and are both diagonalized in a coordinate system $(q^i)$, then they are in involution.*

We remark that equations (6.5) coincide with equations (3.23). This circumstance represents the link between Killing tensors and orthogonal separation. Indeed, as we have seen in Section 3, orthogonal separation of variables gives rise to a set of $n$ independent quadratic first integrals. Interpreted as Killing tensors they have the following characteristic properties [E1] [W] [KM2]:

(a) they are pointwise independent and in involution;
(b) one of them is the metric tensor;
(c) they have $n$ common orthogonal closed eigenforms.

Property (c) follows from the fact that the Killing tensors are all diagonalized in the separable coordinates. This means that the differentials $(dq^i)$ of such coordinates are common orthogonal eigenforms (or, equivalently, that the partial derivatives $(\partial_i)$ with respect to these coordinates, interpreted as vector fields, are common orthogonal eigenvectors). It follows from Theorem 6.1 that the existence of independent Killing tensors satisfying the above requirements is equivalent to the complete integrability of system (6.5). On the other side, as we have seen in Section 3, the integrability conditions of this system coincide with the Levi-Civita separation conditions in the case of a geodesic Hamiltonian in orthogonal coordinates.

**Remark 6.1.** Due to equations (6.5), the characteristic equations (3.8) of a potential compatible with orthogonal separation can be written in the following equivalent form:

$$(\rho_i - \rho_j)\partial_{ij}V = \partial_i\rho_j\,\partial_j V - \partial_j\rho_i\,\partial_i V. \tag{6.1}$$

**Example 6.1. Two-dimensional manifolds**. On Riemannian manifolds of dimension 2, the separation of the H-J equation always occurs in orthogonal coordinates and without second class isotropic coordinates. Indeed, we have three cases. (i) If a separable system has only coordinates of second class, then, due to the normal form (5.13), the coordinates are necessarily orthogonal and non-isotropic. (ii) If a separable system has one coordinate of second class, and this coordinate is not isotropic, then the normal form (5.13) shows again that the coordinates are orthogonal. However, the case of one isotropic coordinate of second class is excluded; indeed, in this case the normal form is

$$(g^{ij}) = \begin{pmatrix} 0 & g^{12} \\ g^{12} & g^{22} \end{pmatrix},$$

with $g^{12} \neq 0$, and $H_1 = \partial_1 g^{ij}p_ip_j$ is divisible by $H^1 = g^{1i}p_i = g^{12}p_2$, which is in contrast with the assumption that $q^1$ is of second class. (iii) If there are two separable coordinates of first class, then they are equivalent to ignorable coordinates. A manifold admitting a system of ignorable coordinates is locally flat, since the components of the metric tensor in these coordinates are constant. So we can always choose orthogonal (ignorable) coordinates (i.e. **orthogonal Cartesian coordinates**).

Due to this property and to the preceding discussion, equivalence classes of orthogonal separable coordinates are then in one-to-one correspondence with equivalence classes of Killing tensors, if we say that two Killing tensors are equivalent if they are related by a linear relation with constant coefficients to the metric tensor $g$. In other words, every Killing tensor $K$ gives rise to an equivalence class of orthogonal separable system; a Killing tensor of the kind $aK + bg$, where $a, b \in \mathbb{R}$, gives rise to the same class. The only requisite is that the Killing tensor is not proportional to the metric tensor and that it has smooth real eigenvalues (this second requisite is automatically fulfilled if the metric is positive-definite). Hence, with every Killing tensor $K$ we associate a **singular set** made of **singular points** in which the two eigenvalues of $K$ coincide or are complex. The separable coordinates are then defined (locally) on the complementary set.

A straightforward discussion based on equations (6.5) shows that [BR]: (i) If the eigenvalues $(\rho_i) = (\rho_1, \rho_2)$ of $K$ are independent functions, then they define a separable system of coordinates: $q^1 = \rho_2$, $q^2 = \rho_1$. More precisely, equations $\rho_i =$ const. define two orthogonal families of unparametrized curves which are the coordinate surfaces of equivalent orthogonal separable systems. (ii) If the eigenvalue $\rho_1$ is constant, then the eigenvector $X_2$ corresponding to the other eigenvalue is, up to a factor, a Killing vector. In case (i), the characteristic equations (6.1) of a separable potential reduce to the single equation

$$(q^1 - q^2)\partial_{12}V = \partial_1 V - \partial_2 V. \tag{6.2}$$

**Example 6.2. The Euclidean plain**. On the Euclidean plain $\mathbb{E}_2$ every Killing tensor is reducible, i.e. it is a linear combination, with constant coefficients, of symmetric product of Killing vectors. This is in fact a property of all manifolds with constant curvature [KL]. In particular it can be shown [BR] that every Killing tensor has the form

$$K = a\,R_P \cap R_Q + b\,g, \quad a, b \in \mathbb{R},$$

where $R_P$ denotes the unitary rotation centered at the point $P \in \mathbb{E}_2$. This is a Killing vector defined by $R_P(x) = \omega \times x$, where $\omega$ is a unitary vector orthogonal to the plane $\mathbb{E}_2$ inbedded in the three-dimensional Euclidean space, and $x$ is the position vector of a generic point of the plain. We do not exclude the case in which the point $P$ goes to the infinity; in this case $P$ is a direction and $R_P$ is a unitary vector field orthogonal to this direction. As a consequence of the preceding discussion (Example 6.1), we have that every equivalence class of orthogonal separable coordinates in the plain is characterized by a Killing tensor of the form

$$K = R_P \cap R_Q.$$

The singular points of $K$ are precisely the points $P$ and $Q$, when they are "true" points (not at the infinity). Hence, we have four possibilities:

(i) The two points $P$ and $Q$ are both at the infinity.
(ii) Only one point (say $Q$) is at the infinity.
(iii) The two points $P$ and $Q$ are true points and $P \neq Q$.
(iv) The two points $P$ and $Q$ are true points and coincide.

It can also be proved that the eigenvalues of $K$ give rise to families of confocal conics, whose focuses are precisely the points $P$ and $Q$. Hence, in correspondence with the above four cases, we have the known four kinds of separable coordinates:

(i) Cartesian;
(ii) parabolic (with focus $P$ and axis $PQ$);
(iii) elliptic-hyperbolic (with focuses $P$ and $Q$);
(iv) polar (with center $P = Q$).

Furthermore, through equations (6.1), it is possible to characterize all possible potentials compatible with the separation. We mention here, as an example, only one result (a detailed discussion is contained in [BR]).

**Theorem 6.1.** The dynamical H-J equation of a material point in the Euclidean plain, submitted to two symmetrical fields centered at two distinct points $P$ and $Q$, is integrable by separation of variables by means of elliptic-hyperbolic coordinates with focuses $P$ and $Q$, if and only if the potential is of the kind

$$V = a(r_P^2 + r_Q^2) + \frac{b}{r_P} + \frac{c}{r_Q} \quad (a, b, c \in \mathbb{R}),$$

where $r_P$ denotes the distance from the point $P$, i.e. if and only if the two fields are a combination of Coulombian fields of any charge (in particular, Newtonian fields) and of elastic or centrifugal fields with the same constant (negative or positive).

**Example 6.3. The sphere**. As far as the Killing vectors are concerned, on the sphere $\mathbb{S}_2$ we have results similar to the Euclidean plain. We consider the unitary sphere centered at a point $O$ of the three-dimensional Euclidean space. Separable coordinates are generated by Killing vectors of the kind

$$K = R_u \cap R_v,$$

where $u$ and $v$ are unitary vectors, and $R_u$ denotes the unitary rotation around the axis $(O, u)$ and defined by $R_u(x) = u \times x$. We have only two possibilities:

(i) the vectors $u$ and $v$ coincide;
(ii) the vectors $u$ and $v$ are distinct.

In case (i) we have two opposite singular points $N$ and $S$, and the corresponding coordinates are the spherical-polar coordinates. In case (ii) we have two pairs of opposite singular points, $(N_1, N_2)$ and $(S_1, S_2)$, and the separable coordinates are those considered by Neumann [NE]: the coordinate curves are two families of confocal spherical conics, one family has the pair of points $(N_1, N_2)$ (or $(S_1, S_2)$) as focuses, the other one has the pair $(N_1, S_2)$ (or $(S_1, N_2)$), being $S_i$ the opposite of $N_i$. We notice that spherical ellipses with focuses $(N_1, N_2)$ are spherical hyperbolae with focuses $(N_1, S_2)$.

**Example 6.4. The asymmetric ellipsoid**. This is a classical subject with a wide literature originated by Jacobi. Through the so called **Jacobian coordinates** it is possible to integrate by separation of variables the geodesic H-J equation. These coordinates are the restriction to the ellipsoid of separable orthogonal coordinates on the three-dimensional Euclidean space generated by three families of confocal quadrics (see [BI]). It can be proved that a Killing tensor generating these coordinates is

$$K = \kappa^{-\frac{3}{4}} B,$$

where $B$ is the second fundamental form and $\kappa$ is the Gaussian curvature, i.e. the product of the two eigenvalues of $B$ (the **main curvatures**). The singular points of $K$ are the umbilical points. Moreover, it can be proved that the separation constant corresponding to $K$ is the **Joachimsthal constant** [BI]: for every geodesic the product of the distance of the center from the tangent plain at a point of the geodesic, times the length of the diameter parallel to the tangent to the geodesic at the same point is constant.

**Example 6.5. Manifolds with constant curvature**. Kalnins [K] has proved that on the Euclidean spaces $\mathbb{E}_n$, on the spheres $\mathbb{S}_n$ and on the pseudo-spheres $\mathbb{H}_n$ every separable system has an orthogonal equivalent. This means that on strictly-Riemannian manifolds with constant curvature the separation always occurs in orthogonal coordinates. In [B3] this property has been extended to Lorentzian manifolds (signature $(+\dots+-)$) with non-negative curvature (for systems without isotropic second class coordinates). The proof is based on the geometrical characterization of separable coordinates in general, illustrated in the next section.

## 7. Intrinsic characterization of the separable systems without second class isotropic coordinates

As we have seen in Section 5, the separation of variables gives rise to $r$ linear first integrals and to a complementary number $m = n-r$ of quadratic first integrals, which are all in involution and independent. If we interpret these first integrals as Killing vectors $(X_\alpha)$ and Killing tensors $(K_a)$, then, in the case in which there are not second class isotropic coordinates, by looking at formulae (5.23), (5.24) and (5.25) (we always refer to normal separable coordinates $(q^a, q^\alpha)$) we can derive for them the following properties:

(a) they are in involution; the Killing vectors are pointwise independent; the Killing tensors are pointwise independent;

(b) one of the Killing tensors is the metric tensor;

(c) the Killing tensors have $m$ common orthogonal closed eigenforms $(\phi^a)$ (or $m$ orthogonal eigenvectors $(X_a)$ in involution: $X_a \cdot X_b = 0$, $[X_a, X_b] = 0$);

(d) The eigenforms are orthogonal to the Killing vectors: $\langle X_\alpha, \phi^a \rangle = 0$ (or, equivalently, $X_\alpha \cdot X_a = 0$).

(e) The eigenforms are invariant with respect to the Killing vectors: $d_{X_\alpha} \phi^a = 0$ (or, equivalently, $[X_\alpha, X_a] = 0$).

(f) The Killing vectors generate a normal distribution.

Indeed, we recognize these conditions by setting

$$\phi^a = dq^a, \quad X_a = \partial_a, \quad X_\alpha = \partial_\alpha.$$

The symbol $d_X$ used in (e) denotes the Lie derivative with respect to the vector field $X$. We remind that, for a closed 1-form $\phi$, $d_X \phi = d\langle X, \phi \rangle$. The last property (f) follows from the fact that the vectors $(\partial_a)$ generate the distribution orthogonal to the Killing vectors. Moreover, since the Killing vectors commute, they generate an integrable distribution whose integral foliation is made of locally flat manifolds. Hence, with a separable system we can associate two mutually orthogonal foliations: a foliations $\mathcal{E}$ made of locally flat submanifolds of dimension $r$, tangent to the Killing vectors and defined by equations $q^a = \text{const.}$, and an orthogonal foliation $\mathcal{S}$ defined by equations $q^\alpha = \text{const.}$, made of submanifolds of dimension $m$ which admits orthogonal systems of separable coordinates.

We can interpret this fact in a slightly different manner. The Killing vectors generate an Abelian free action of isometries on the manifold $Q$, whose fibers are locally flat manifolds which form the foliation

$\mathcal{E}$. Let us denote by $\bar{Q}$ the quotient manifold and by $\pi: Q \to \bar{Q}$ the quotient projection. The metric tensor $g$ is reduced to a metric tensor $\bar{g}$ on $\bar{Q}$ by the projection $\pi$. The restriction of this projection to every manifold of the foliation $\mathcal{S}$ orthogonal to the fibers is an isometry. This is actually a geometrical picture which holds only locally, i.e. for open subset $U \subset Q$. We remark that the orthogonal separable coordinates existing on the reduced manifold $\bar{Q}$, even if they correspond to second class coordinates on the whole manifold $Q$, they are not all necessarily of second class. So that we can perform for $\bar{Q}$ the same procedure of reduction, and so on.

All the properties listed above can be considered as necessary conditions for the existence of separable coordinates. As sufficient conditions they are redundant (see [KM3]). However, there is not a unique or privileged minimal subset of these conditions to be taken as sufficient conditions. We do not enter in this discussion. Instead, we give an example of how the geometrical characterization of separation can be used.

Up to a linear transformation with constant coefficients, we can always reduce the Killing vectors to be orthogonal on any chosen leaf of the locally flat foliation $\mathcal{E}$. These transformations are compatible with the separation: they give rise to equivalent separable systems. However, it is not possible in general to reduce the Killing vectors to be orthogonal everywhere. When this happens, we find orthogonal separable systems. Indeed, the Killing tensors $(K_a, K_\alpha = X_\alpha \cap X_\alpha)$ satisfy the requirements listed in Section 6.

Actually, this always happens in the affine Euclidean spaces $\mathbb{E}_n$ and in the affine Minkowskian spaces $\mathbb{M}_n$. Indeed, in these spaces every system of independent commuting Killing vectors generating a normal distribution (properties (a) and (f)) can be orthogonalized, i.e. it can be transformed into an orthogonal system through a linear transformation with constant coefficients [B3]. In particular this property holds for Killing vectors which are **rotations** around a fixed point $O$ of these spaces. Thus, if we consider the restrictions of these rotations to the **fundamental hyperquadrics** (see [E2]) centered at the point $O$, then we conclude that the same property holds for the Killing vectors on these manifolds. The fundamental hyperquadrics are spheres $\mathbb{S}_{n-1} \subset \mathbb{E}_n$ defined by equation $x{\cdot}x = 1$, hyperboloids $\mathbb{H}_{n-1} \subset \mathbb{M}_n$ defined by equation $x{\cdot}x = -1$, and hyperboloids $\mathbb{L}_{n-1} \subset \mathbb{M}_n$ defined by equation $x{\cdot}x = 1$. The first two are strictly-Riemannian manifolds with constant curvature, positive and negative respectively. The third ones are Lorentzian manifolds with constant positive curvature. The spaces and surfaces so far considered provide a local model for all strictly-Riemannian manifolds with constant curvature and all Lorentzian manifolds with constant non-negative curvature. Hence, as we said at the end of last section, on these manifolds the separation of variables always occurs in orthogonal coordinates.

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