Cocycles of the coadjoint representation of a Lie group interpreted as differential forms

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Summary. *See the introduction.*

Sommario. *In questo lavoro viene data un'interpretazione geometrica degli 1-cocicli della rappresentazione coaggiunta di un gruppo di Lie e della loro relazione con i 2-cocicli a valori reali sulla corrispondente algebra di Lie. Gli 1-cocicli sono interpretati come classi di 1-forme sul gruppo di Lie G soddisfacenti ad una opportuna equazione differenziale. Le immagini di queste 1-forme costituiscono un fogliettamento del fibrato cotangente T*G del gruppo. Questo fogliettamento può anche essere costruito per mezzo di una forma bilineare invariante sul gruppo, antisimmetrica nel caso di 1-cocicli simplettici. La 2-forma è usata per modificare la struttura simplettica canonica di T*G e anche per modificare il rilevamento canonico dei campi vettoriali su G. Non viene usata la trivializzazione naturale di T*G. Vengono invece usate entrambe le algebre di Lie dei campi generatori destri e sinistri e i corrispondenti spazi duali delle 1-forme invarianti a sinistra e a destra. Questo lavoro è una continuazione dell'analisi del fibrato cotangente di un gruppo di Lie iniziata in* [3] *ed è una versione riveduta del lavoro* [4]. *L'interpretazione geometrica degli 1-cocicli e dei 2-cocicli qui data è stata suggerita dall'analisi delle azioni hamiltoniane in termini di sottovarietà coisotrope* [1] [2].

0. - Introduction

In this paper we give a geometrical interpretation of the 1-cocycles of the coadjoint representation of a Lie group and their relation with the 2-cocycles with real values on the corresponding Lie algebra. The 1-cocycles are interpreted as classes of 1-forms on the Lie group G satisfying a suitable differential equation. The images of 1-forms of a class form a foliation of the cotangent bundle T^*G of the group. This foliation can be also constructed by means of an invariant bilinear

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form on the group, or by an invariant closed 2-form in the case of a symplectic cocycle. The 2-form is used to modify the canonical symplectic structure of T^*G and also to modify the canonical lift of vector fields of G to T^*G. We do not use the natural trivialization of T^*G. Instead, we use both Lie algebras of left and right infinitesimal generators and the corresponding dual spaces of right and left-invariant 1-forms. This paper is a continuation of the analysis of the cotangent bundle of a Lie group initiated in [3] and it is a revised version of [4]. The geometrical interpretation of 1-cocycles and 2-cocycles given here was suggested by the analysis of Hamiltonian actions in terms of coisotropic submanifolds [1] [2].

1. - Notation

Let G be a differential manifold. We consider a vector field on G as a section $X : G \to TG$ of the tangent bundle $\tau_G : TG \to G$ and a 1-form on G as a section of the cotangent bundle $\pi_G : T^*G \to G$. The spaces of smooth vector fields and k-forms on G are denoted by $\mathscr{X}(G)$ and $\Phi_k(G)$ respectively. Symbols $[X, Y]$, d_X and i_X denote the Lie brackets of vector fields X and Y, the Lie derivative and the interior product of a form with respect to a vector field X.

We denote by θ_G the Liouville 1-form on T^*G. The 2-form $d\theta_G$ is the canonical symplectic form on T^*G.

The *canonical lift of a diffeomorphism* $\psi : G \to G$ is the diffeomorphism $\hat{\psi} : T^*G \to T^*G$ characterized by the following properties:

$$(1.1) \qquad \hat{\psi}^* \theta_G = \theta_G \ , \quad \pi_G \cdot \hat{\psi} = \psi \cdot \pi_G \ .$$

The canonical lift is defined by the equation

$$(1.2) \qquad \langle v, \hat{\psi}(h) \rangle = \langle T\psi^{-1}(v), h \rangle \qquad (h \in T^*_g G,\ v \in T_{\psi(g)} G)$$

where T is the tangent functor, or by the equation

$$(1.3) \qquad \hat{\psi} \cdot \psi^* \mu = \mu \cdot \psi \qquad (\mu \in \Phi_1(G)) ,$$

where $\psi^* : \Phi_1(G) \to \Phi_1(G)$ is the pull-back by ψ.

The *canonical lift of a vector field* X on G is the vector field $\hat{X}$ on T^*G characterized by the following properties:

$$(1.4) \qquad d_{\hat{X}} \theta_G = 0 \ , \qquad T\pi_G \cdot \hat{X} = X \cdot \pi_G \ .$$

The canonical lift is defined by the equation

(1.5) $$i_{\hat{X}}\, d\,\theta_G = - dE_X \ ,$$

where

(1.6) $$E_X : T^*G \to R : h \to \langle X, h \rangle \, .$$

A vector field Z on T^*G is said to be π_G*-projectable* to a vector field X on G if $T\pi_G \cdot Z = X \cdot \pi_G$. This condition is equivalent to

(1.7) $$i_Z\ \theta_G = E_X \ .$$

The vector field $\hat{X}$ is π_G-projectable to X.

We consider the natural left actions of G on itself:

(1.8) $$\lambda : G \times G \to G : (g, g') \to \lambda_g(g') = gg' \ ,$$

$$\rho : G \times G \to G : (g, g') \to \rho_g(g') = g'g^{-1} \ .$$

We denote by l_G and r_G the Lie algebras of the infinitesimal generators of the actions λ and ρ respectively. The linear dual spaces l_G^* and r_G^* are the spaces of *right-invariant* and *left-invariant* 1-forms on G respectively:

(1.9) $$\nu \in l_G^* \iff \rho_g^* \nu = \nu, \quad g \in G \iff d_Y \nu = 0, \quad \forall Y \in r_G \, ,$$

$$\mu \in r_G^* \iff \lambda_g^* \mu = \mu, \quad g \in G \iff d_X \mu = 0, \ \forall X \in l_G \, .$$

Since the two actions commute, $[X, Y] = 0$ and $d_X d_Y - d_Y d_X = d_{[X,Y]} = 0$ for all $X \in l_G$, $Y \in r_G$. The lifted actions

(1.10) $$\hat{\lambda} : G \times T^*G \to T^*G : (g, h) \to \hat{\lambda}_g(h),$$

$$\hat{\rho} : G \times T^*G \to T^*G : (g, h) \to \hat{\rho}_g(h),$$

are generated by the canonical lifts $\hat{X}$ and $\hat{Y}$ of the vector fields $X \in l_G$ and $Y \in r_G$ respectively. The images of the left-invariant (resp. right-invariant) 1-forms are the orbits of $\hat{\lambda}$ (resp. of $\hat{\rho}$).

We have two representations of G in the vector space $\mathfrak{X}_1(G)$:

(1.11) $$\lambda^* : G \times \Phi_1(G) \to \Phi_1(G) : (g, \nu) \to \lambda_g^{*-1} \nu ,$$

$$\rho^* : G \times \Phi_1(G) \to \Phi_1(G) : (g, \mu) \to \rho_g^{*-1} \mu .$$

When restricted to l_G^* and to r_G^* these actions provide the *left* and the *right coadjoint representations* of G respectively.

2. The 1-cocycle differential equation

We consider local 1-forms $\gamma : U \to T^*G$, $U \subset G$, satisfying the system of differential equations

(2.1) $$d_X d_Y \gamma = 0 , \quad X \in l_G, \; Y \in r_G .$$

These equations are linear in γ and symmetric with respect to X and Y. Sums of left and right-invariant 1-forms are called *trivial solutions* of (2.1). A solution γ in a neighbourhood of the identity e of the group G such that $\gamma(e) = 0$ is called a *normal solution.* Each trivial normal solution is of the form $\gamma = \mu - \nu$ with $\mu \in r_G^*$, $\nu \in l_G^*$ and $\mu(e) = \nu(e)$. The space of solutions of (2.1) is invariant under addition of left and right-invariant 1-forms and the pull-back actions λ^* and ρ^*.

For a simpler discussion, in the present section we assume that all local solutions can be extended to global 1-forms on G. The solutions form a linear subspace $S^1(G)$ of $\Phi_1(G)$. We denote by $T^1(G)$ the subspace of trivial solutions, i.e. $T^1(G) = l_G^* + r_G^*$. The quotient space $H^1(G) = S^1(G)/T^1(G)$ is the space of the equivalence classes of the following equivalence relation in $S^1(G)$:

(2.2) $$\gamma \sim \gamma' \Leftrightarrow \gamma' = \gamma + \mu + \nu ,$$

for some $\mu \in r_G^*$ and $\nu \in l_G^*$. We denote by $[\gamma]$ the class of a solution γ.

PROPOSITION 2.1 - A 1-form γ on G is a solution of (2.1) if and only if for each $g \in G$ the 1-form $\gamma - \lambda_g^* \gamma$ is right-invariant (or $\gamma - \rho_g^* \gamma$ is left-invariant).

Proof. - Equations (2.1) imply that $d_Y \gamma$ is a left-invariant 1-form for each $Y \in r_G$. Thus $\lambda_g{}^* d_Y \gamma = d_Y \gamma$ for each $g \in G$. Since Y is left-invariant, we have $\lambda_g{}^* d_Y \gamma = d_Y \lambda_g{}^* \gamma$. It follows that $d_Y(\gamma - \lambda_g{}^*\gamma) = 0$, for each $Y \in r_G$, thus $\gamma - \lambda_g{}^*\gamma$ is right-invariant. Reversing this reasoning we show that γ is a solution if $\gamma - \lambda_g{}^*\gamma$ is invariant for each $g \in G$. (Q.E.D.)

PROPOSITION 2.2 - Let γ be a 1-form on G. The mapping $\sigma : G \to \Phi_1(G)$ defined by

$$\sigma(g) = \gamma - \lambda^*_{g^{-1}}\, \gamma \tag{2.3}$$

satisfies

$$\sigma(gg') = \lambda^*_{g^{-1}}\, \sigma(g') + \sigma(g)\,, \tag{2.4}$$

for all g and g' in G.

Proof. -
$$\begin{aligned} \sigma(gg') &= \gamma - (\lambda^*_{gg'})^{-1}\, \gamma \\ &= \gamma - \lambda^*_{g^{-1}}\, (\lambda^*_{g'^{-1}}\, \gamma) \\ &= \gamma - \lambda^*_{g^{-1}}\, (\gamma - \sigma(g')) \\ &= \gamma - \lambda^*_{g^{-1}}\, \gamma + \lambda^*_{g^{-1}}\, \sigma(g') \\ &= \sigma(g) + \lambda^*_{g^{-1}}\, \sigma(g') \end{aligned}$$
(Q.E.D.)

REMARK 2.1 - The mapping σ is a 1-coboundary in the representation λ^* of G on $\Phi_1(G)$ (definition (2.3)). Hence, it is a 1-cocycle (formula (2.4)). Proposition 2.1 shows that if γ is a solution of (2.1), then $\sigma(g) \in l_G^*$ and $\sigma : G \to l_G^*$ is a 1-cocycle in the coadjoint representation. If γ is a trivial solution of (2.1), then σ is a 1-coboundary. Indeed, if $\gamma = \mu + \nu$, $\mu \in r_G^*$, $\nu \in l_G^*$, then $\sigma(g) = \gamma - \lambda^*_{g^{-1}}\, \gamma = \mu + \nu - \lambda^*_{g^{-1}}\, (\mu + \nu) = \nu - \lambda^*_{g^{-1}}\, \nu$.

Let $Z^1(G, l_G^*)$ and $B^1(G, l_G^*)$ be the spaces of the 1-cocycles and 1-coboundaries in the left coadjoint representation, and let $H^1(G, l_G^*) = Z^1(G, l_G^*) / B^1(G, l_G^*)$. From the Remark 2.1 it follows that the linear mapping

$$S^1(G) \to Z^1(G, l_G^*) \; : \gamma \to \sigma \tag{2.5}$$

defined by (2.3) induces linear mappings

$$(2.6) \qquad S^1(G)/\overset{*}{r_G} \to Z^1(G, \overset{*}{l_G}) :: [\gamma]_l \to [\sigma],$$

$$H^1(G) \to H^1(G, \overset{*}{l_G}) : [\gamma] \to [\sigma],$$

where $[\gamma]_l$ denotes the class of a solution γ in the equivalence relation on $S^1(G)$

$$(2.7) \qquad \gamma \sim_l \gamma' \Leftrightarrow \gamma - \gamma' \in \overset{*}{r_G} \ ,$$

and $[\sigma]$ is the class of the cocycle σ in $H^1(G, \overset{*}{l_G})$. The class $[\gamma]_l$ is the inverse image of σ in the mapping (2.5). Proposition 2.3 below shows that the mapping (2.5) is surjective and that mappings (2.6) are isomorphisms. For these reasons we can call (2.1) the *1-cocycle differential equation.*

For the mapping $\sigma : G \to \Phi_1(G)$ defined in (2.3) we have (see formula (1.3)):

$$(2.8) \qquad \sigma(g)(g) = \gamma(g) - \hat{\lambda}_g(\gamma(e)) .$$

PROPOSITION 2.3 - The equation

$$(2.9) \qquad \gamma(g) = \theta(g)(g), \ g \in G ,$$

defines an isomorphism between the space $\Phi_1(G)$ of 1-forms γ and the space of 1-cochaines $\theta : G \to \overset{*}{l_G}$. The 1-form γ is a normal (resp. a trivial normal) solutions of (2.1) if and only if θ is a 1-cocycle (resp. a 1-coboundary).

Proof. - If $\gamma \in \Phi_1(G)$ is given, then for each $g \in G$ there exists a unique 1-form $\theta(g) \in \overset{*}{l_G}$ whose value at g is equal to $\gamma(g)$, because the images of right-invariant 1-forms form a foliation of T^*G. Conversely, if a mapping $\theta : G \to \overset{*}{l_G}$ is given, then (2.9) defines a unique section $\gamma : G \to T^*G$ of π_G, i.e. a 1-form on G. The correspondence so defined is linear and bijective, hence, it is an isomorphism. Let γ be a normal (resp. trivial normal) solution of (2.1). Because of Prop. 2.2 the mapping $\sigma : G \to \overset{*}{l_G}$ defined by $\sigma(g) = \gamma - \lambda^*_{g^{-1}} \gamma$ is a 1-cocycle (resp. a 1-coboundary). Since $\gamma(e) = 0$, this mapping coincides with the mapping θ defined in (2.9) (see (2.8)). Conversely, let θ be a 1-cocycle and γ be the 1-form defined by (2.9). Since θ is a cocycle (see (2.4), $\theta(e)=0$ thus $\gamma(e)=0$. By the substitution $g \to g^{-1} g'$ in the equation (2.4) applied to θ we obtain the identity

$$(2.10) \qquad \lambda^*_{g^{-1}} \, \theta(g^{-1} g') = \theta(g') - \theta(g) .$$

The calculation

$$\begin{aligned}(\gamma - \lambda^*_{g^{-1}} \gamma)(g') &= \theta(g')(g') - \hat{\lambda}_g(\gamma(g^{-1} g')) \\ &= \theta(g')(g') - \hat{\lambda}_g(\theta(g^{-1} g')(g^{-1} g')) \\ &= \theta(g')(g') - (\lambda^*_{g^{-1}} \theta(g^{-1} g'))(g') \\ &= \theta(g')(g') - (\theta(g') - \theta(g))(g') \\ &= \theta(g)(g') ,\end{aligned}$$

shows that $\theta(g) = \gamma - \lambda^*_{g^{-1}} \gamma$, hence, $\gamma - \lambda^*_{g^{-1}} \gamma$ is a right-invariant 1-form for each $g \in G$. From Proposition 2.1 it follows that γ is a solution of equation (2.1). If θ is a coboundary, then $\theta(g) = \nu - \lambda^*_{g^{-1}} \nu$, where $\nu \in l^*_G$. It follows that $\gamma - \nu = \lambda^*_{g^{-1}}(\gamma - \nu)$, i.e. that $\gamma - \nu \in r^*_G$. Hence, γ is a trivial solution. (Q.E.D.)

Results of this section remain valid with the interchange of the action λ, the algebra l_G and left invariance with the action ρ, the algebra r_G and right-invariance. The same holds for the following sections.

3. - The lift of a vector field by means of a 2-form

Let G be a differential manifold and let B be a bilinear form (i.e. a covariant tensor field of rank 2) on G. Let us consider on T^*G the bilinear form

$$(3.1) \qquad \omega = d\theta_G + \pi_G{}^* B .$$

PROPOSITION 3.1 - The bilinear form ω is non-degenerate.

Proof. - Let $p \in T^*G$ and $v \in T_p T^*G$. Assume that, for each $u \in T_p T^*G$, $\langle u \otimes v, \omega \rangle = 0$, i.e. $\langle u \wedge v, d\theta_G \rangle - \langle T\pi_G(u) \otimes T\pi_G(v), B \rangle = 0$. We have in particular $\langle u \wedge v, d\theta_G \rangle = 0$ for each vertical vector

$u\,(T\pi_G(u)=0)$. Since the fibres of T^*G are Lagrangian submanifolds and the vertical vectors are tangent to the fibres, it follows that v is vertical. Hence, $\langle u \wedge v, d\theta_G\rangle = 0$ for each vector u. It follows that $v=0$ since $d\theta_G$ is non-degenerate. (Q.E.D.)

Let X be a vector field on G. Two vector fields $\overline{X}$ and $\widetilde{X}$ on T^*G are defined by equations

$$i_{\widetilde{X}} d\theta_G = -\pi_G{}^* i_X B\,, \tag{3.2}$$

$$i_{\overline{X}}\,\omega = -dE_X\,. \tag{3.3}$$

REMARK 3.1. - If B is a closed 2-form (i.e. skew-symmetric and such that $dB=0$), then the 2-form ω is a symplectic form on T^*G, the vector field $\overline{X}$ is the Hamiltonian vector field generated by the function E_X with respect to ω. It follows that

$$d_X\,\omega = 0\,. \tag{3.4}$$

PROPOSITION 3.2 - (i) The vector field X is vertical, i.e. $T\pi_G \cdot \widetilde{X} = 0$. (ii) $\overline{X} = \hat{X} + \widetilde{X}$, where $\hat{X}$ is the canonical lift of X. (iii) The vector field X is π_G-projectable to X, i.e. $T\pi_G \cdot \overline{X} = X \cdot \pi_G$.

Proof. - (i) If $v \in T_p\,T^*G$ and $T\pi_G(v)=0$, then $\langle v, i_{\widetilde{X}}\,d\theta_G\rangle = -\langle T\pi_G(v), i_X B\rangle = 0$. Since v is tangent to a fibre of π_G and the fibres are Lagrangian submanifolds, it follows that also $X(p)$ is vertical. (ii) Since X is vertical, $i_X\pi_G{}^*B=0$. Hence, $i_{\hat{X}+\widetilde{X}}(d\theta_G + \pi_G{}^*B) = = i_{\hat{X}}\,d\theta_G + i_{\hat{X}}\,\pi_G{}^*B + i_{\widetilde{X}}\,d\theta_G = -dE_X + \pi_G{}^* i_X B + i_{\widetilde{X}}\,d\theta_G = -dE_X$. (iii) is a direct consequence of (i) and (ii). (Q.E.D.)

PROPOSITION 3.3 - For each pair (X_1, X_2) of vector fields on G the following identity holds:

$$i_{[\overline{X}_1, \overline{X}_2]} d\theta_G = -dE_{[X_1, X_2]} + \pi_G{}^*(d_{X_2}\, i_{X_1} B - d_{X_1}\, i_{X_2} B). \tag{3.5}$$

Proof. - The definition (3.3) can be written in the form

$$i_{\overline{X}}\,d\theta_G = -dE_X - \pi_G{}^*\,i_X B. \tag{3.3'}$$

It follows that $d_{\overline{X}}\,\theta_G = -dE_X - \pi_G{}^*\,i_X B$, i.e. that

$$d_{\overline{X}}\theta_G = -\pi_G{}^* i_X B\,. \tag{3.6}$$

For two vector fields X_1 and X_2 we have the identities

(3.7) $$i_{\hat{X}_1} i_{\hat{X}_2} d\theta_G = - i_{\hat{X}_1} dE_{X_2} = - E_{[X_1, X_2]} .$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} d_{\bar{X}_1} E_{X_2} &= i_{\bar{X}_1} dE_{X_2} \\ &= - i_{\bar{X}_1} d_{\hat{X}_2} d\theta_G = i_{\hat{X}_2} i_{\bar{X}_1} d\theta_G \\ &= - i_{\hat{X}_2} (dE_{X_1} + \pi_G {}^* i_{X_1} B) \\ &= E_{[X_1, X_2]} - \pi_G {}^* i_{X_2} i_{X_1} B , \end{aligned}$$

i.e.

(3.8) $$d_{X_1} E_{X_2} = E_{[X_1, X_2]} - \pi_G {}^* i_{X_2} i_{X_1} B .$$

It follows from (3.6) and (3.8) that

$$\begin{aligned} i_{[\bar{X}_1, \bar{X}_2]} d\theta_G &= d_{\bar{X}_1} i_{\bar{X}_2} d\theta_G - i_{\bar{X}_2} d_{\bar{X}_1} d\theta_G \\ &= - d_{\bar{X}_1} (dE_{X_2} + \pi_G {}^* i_{X_2} B) - i_{\bar{X}_2} dd_{\bar{X}_1} \theta_G \\ &= - dd_{X_1} E_{X_2} - \pi_G {}^* d_{X_1} i_{X_2} B + i_{\bar{X}_2} \pi_G {}^* di_{X_1} B \\ &= - dE_{[X_1, X_2]} + \pi_G {}^* (di_{X_2} i_{X_1} B - \\ &\quad - d_{X_1} i_{X_2} B + i_{X_2} di_{X_1} B) \\ &= - dE_{[X_1, X_2]} + \pi_G {}^* (d_{X_2} i_{X_1} B - d_{X_1} i_{X_2} B). \end{aligned}$$

(Q.E.D.)

PROPOSITION 3.4. - The vector field $[\bar{X}_1, \bar{X}_2]$ is π_G-projectable to the vector field $[X_1, X_2]$.

Proof. - It follows from (3.6) and (3.8) that

$$\begin{aligned} i_{[\bar{X}_1, \bar{X}_2]} \theta_G &= d_{\bar{X}_1} i_{\bar{X}_2} \theta_G - i_{\bar{X}_2} d_{\bar{X}_1} \theta_G \\ &= d_{\bar{X}_1} E_{X_2} + i_{\bar{X}_2} \pi_G {}^* i_{X_1} B \\ &= E_{[X_1, X_2]} , \end{aligned}$$

i.e.

(3.9) $$i_{[\bar{X}_1, \bar{X}_2]} \theta_G = E_{[X_1, X_2]} \,. \qquad \text{(Q.E.D.)}$$

PROPOSITION 3.5. - Equality $[\bar{X}_1, \bar{X}_2] = [X_1, X_2]^-$ holds if and only if

(3.10) $$i_{[X_1, X_2]} B + d_{X_2} i_{X_1} B - d_{X_1} i_{X_2} B =$$

Proof. - From (3.5) and from (3.3) applied to the vector field $[X_1, X_2]$ it follows that

$$i_{([\bar{X}_1, \bar{X}_2] - [X_1, X_2]^-)} d\theta_G =$$

$$= \pi_G{}^* (i_{[X_1, X_2]} B + d_{X_2} i_{X_1} B - d_{X_1} i_{X_2} B) \,.$$

(Q.E.D.)

REMARK 3.2. - If B is a 2-form, then condition (3.10) is equivalent to

(3.11) $$d i_{X_2} i_{X_1} B - i_{X_2} i_{X_1} dB = 0 \,.$$

Indeed, for a 2-form B we have $i_{[X_1, X_2]} B = d_{X_1} i_{X_2} B - i_{X_2} d_{X_1} B$, so that the left side of (3.10) is equal to: $d_{X_2} i_{X_1} B - i_{X_2} d_{X_1} B = = d i_{X_2} i_{X_1} B + i_{X_2} d i_{X_1} B - i_{X_2} d i_{X_1} B - i_{X_2} i_{X_1} dB$.

Now we assume that G is a Lie group.

PROPOSITION 2.6 - Let $X \in l_G$. The vector field $\bar{X}$ defined by (3.3) is characterized by the following properties: (i) $\bar{X}$ is π_G-projectable to X and (ii)

(3.12) $$i_{\hat{Y}} i_{\bar{X}} \omega = 0 \,, \quad \text{for each } Y \in r_G \,.$$

Proof. - Let $\bar{X}$ be defined by (3.3). Since $i_{\hat{Y}} i_{\hat{X}} d\theta_G = \langle \hat{X} \wedge \hat{Y}, d\theta_G \rangle$, it follows from Proposition 3.2 that $i_Y i_X \omega = i_Y i_X \pi_G{}^* B + i_{\hat{Y}} i_{\tilde{X}} d\theta_G = = i_{\hat{Y}} (\pi_G{}^* i_X B + i_{\tilde{X}} d\theta_G)$. Thus (3.12) follows form (3.2). Conversely, if $\bar{X}$ is π_G-projectable to X and we use the decomposition $\bar{X} = \hat{X} + \check{X}$ where $\check{X}$ is a vertical vector field, then the calculation above shows that (3.12) implies $i_{\hat{Y}} \zeta_X = 0$, where $\zeta_X = \pi_G{}^* i_X B + + i_{\tilde{X}} d\theta_G$. Since the 1-form ζ_X is vertical (i.e. $\langle v, \zeta_X \rangle = 0$ when $T\pi_G(v) = 0$) and r_G is transitive, it follows that $\zeta_X = 0$. (Q.E.D.)

Let us consider the set of vector fields

$$\overline{l}_G = \{\overline{X};\, X \in l_G\}$$

and the corresponding distribution

$$\overline{L}_G = \{v \in T(T^*G); \quad X \in l_G : X(h) = v,\, h = \tau_{T^*G}(v)\} .$$

This distribution is transversal to π_G.

PROPOSITION 2.7. - The set $\overline{l}_G$ is a Lie sub-algebra of $\mathscr{X}(T^*G)$ (i.e. $\overline{L}_G$ is completely integrable) if and only if (3.10) holds for each X_1 and X_2 in l_G.

Proof. - The set $\overline{l}_G$ is clearly a linear sub-space. If (3.10) holds, then from Proposition 3.5 it follows that $\overline{l}_G$ is a sub-algebra. If $\overline{l}_G$ is a sub-algebra, then $[\overline{X}_1, \overline{X}_2]$ is a linear combination of elements of $\overline{l}_G$. However, $[\overline{X}_1, \overline{X}_2]$ is π_G-projectable to $[X_1, X_2]$ (Proposition 3.4). Hence, $[\overline{X}_1, X_2] = [X_1, X_2]^-$ and (3.10) holds because of Proposition 3.5. (Q.E.D.)

Now we consider the case of a 2-form B.

PROPOSITION 3.8. - If B is a 2-form, then the integrability condition (3.10) is equivalent to

$$(3.13) \qquad di_Y B = d_Y B - i_Y dB = 0 ,$$

for each $Y \in r_G$.

Proof. - We apply the interior product i_Y to the left side of (3.11) which is equivalent to (3.10) (see Remark 3.2). Since Y commutes with X_1 and X_2, we have $i_Y di_{X_2} i_{X_1} B - i_Y i_{X_2} i_{X_1} dB = i_{X_2} i_{X_1} (d_Y B - i_Y dB)$. Since r_G is transitive, equations (3.11) and (3.13) are equivalent. (Q.E.D.)

REMARK 3.3 - The integrability condition (3.13) is satisfied if B is closed $(dB = 0)$ and right-invariant $(d_Y B = 0,$ for all $Y \in r_G)$.

PROPOSITION 3.9. - Let B be a closed and right-invariant 2-form. The canonical lift $\hat{Y}$ of $Y \in r_G$ is a symplectic vector field with respect to the symplectic form ω, i.e.

(3.14) $$d_Y \omega = 0 ,$$

and

(3.15) $$[\bar{X}, \hat{Y}] = 0 ,$$

for each $X \in l_G$.

Proof. - It follows from equations (3.3) and (3.2) applied to Y, that $i_Y \omega = - dE_Y + \pi_G{}^* i_Y B$. Hence, $d_Y \omega = \pi_G{}^* d i_Y B = d_Y B - - i_Y dB = 0$. Both vector fields $\bar{X}$ and $\hat{Y}$ are Hamiltonian vector fields with respect to the symplectic form ω. Hence, the Lie bracket $[X, Y]$ is the globally Hamiltonian vector field generated by the function $\langle \bar{X} \wedge \hat{Y}, \omega \rangle$, which is zero because of Proposition 3.6. (Q.E.D.)

4. - Global solutions of the 1-cocycle equation.

PROPOSITION 4.1. - Let B be a bilinear form on a Lie group G. (i) The distribution $\bar{L}_G$ corresponding to B is completely integrable if the system of differential equations

(4.1) $$i_X B + d_X \gamma = 0 , \ X \in l_G ,$$

is locally integrable, i.e. for each $g \in G$ there exists a neighbourhood U of g and a local section $\gamma : U \to T^* G$ of π_G satisfying the systems (4.1). (ii) If the system (4.1) is locally integrable, then the image $\gamma(U)$ of a local solution is an integral manifold of $\bar{L}_G$. (iii) If the system (4.1) is locally integrable, then the local solutions satisfy the 1-cocycle equation if and only if B is right-invariant.

To prove this proposition we use the following Lemmas.

LEMMA 4.1. - Let G be a manifold. A vector field Z on $T^* G$ is tangent to the image $\gamma(G) \subset T^* G$ of a 1-form γ on G if and only if

(4.2) $$\gamma^* d_Z (\theta_G - \pi_G{}^* \gamma) = 0.$$

Proof. - Let $S = \gamma(G)$ and $\xi = \theta_G - \pi_G{}^* \gamma$. We have $\langle v, \xi \rangle = \langle v, \theta_G \rangle - - \langle T\pi_G(v), \gamma \rangle = \langle u, p - \gamma(g) \rangle$, with $v \in T_p(T^* G)$, $p \in T_g^* G$, $u = T\pi_G(v)$. It follows that $p \in S$ if and only if $\xi(p) = 0$. Let us consider the symplectic form $d\xi = d\theta_G - \pi_G{}^* d\gamma$ on $T^* G$ (Proposition 3.1).

The calculation $\gamma^* d\xi = \gamma^* d\theta_G - \gamma^* \pi_G{}^* d\gamma = d\gamma^*\theta_G - (\pi_G \cdot \gamma)^* d\gamma = d\gamma - d\gamma = 0$ shows that S is a lagrangian submanifold with respect to $d\xi$. Equation $(i_Z \xi)|S = 0$ holds for any arbitrary vector field Z on T^*G. This equation is equivalent to $\gamma^* i_Z \xi = 0$. It follows that

$$\begin{aligned}\gamma^* d_Z \xi &= \gamma^* (di_Z \xi + i_Z d\xi) \\ &= d\gamma^* i_Z \xi + \gamma^* i_Z d\xi \\ &= \gamma^* i_Z d\xi .\end{aligned}$$

Since S is lagrangian, the vector $Z(p)$ is tangent to S at a point $p \in S$ if and only if $\langle v \wedge Z(p), d\xi\rangle = 0$ for each $v \in T_p S$, i.e. if and only if $(i_Z d\xi)|S = 0$. This condition is equivalent to $\gamma^* i_Z d\xi = 0$, i.e. to equation $\gamma^* d_Z \xi = 0$. (Q.E.D.)

LEMMA 4.2. - Let B be a bilinear form on a manifold G and X a vector field on G. The vector field $\bar{X}$ defined by (3.3) is tangent to the image $\gamma(G)$ of a 1-form γ on G if and only if $i_X B + d_X \gamma = 0$.

Proof. - We apply Lemma 4.1 to the vector field $\bar{X}$. We have $\bar{X} = \hat{X} + \tilde{X}$ (Proposition 3.2, (ii)). From

$$d_{\hat{X}} \theta_G = 0,$$

$$d_{\tilde{X}} \theta_G = i_{\tilde{X}} d\theta_G + di_{\tilde{X}} \theta_G = i_{\tilde{X}} d\theta_G = -\pi_G{}^* i_X B,$$

$$d_{\hat{X}} \pi_G{}^* \gamma = \pi_G{}^* d_X \gamma ,$$

$$d_{\tilde{X}} \pi_G{}^* \gamma = 0,$$

it follows that $\gamma^* d_{\bar{X}} (\theta_G - \pi_G{}^* \gamma) = -\gamma^* \pi_G^* (i_X B + d_X \gamma) = -(i_X B + d_X \gamma)$. (Q.E.D.)

Proof of Proposition 4.1. - (i) (ii) Let $\bar{L}_G$ be completely integrable. Since the generating vector fields $\bar{X}$ are transverse to the fibres, integral manifolds of $\bar{L}_G$ are images of local sections $\gamma : U \to T^*G$ of π_G which satisfy equations (4.1) (Lemma 4.2). Conversely, if (4.1) is locally integrable, then $i_X B = -d_X \gamma$ for each $X \in l_G$. Hence, for each X_1 and X_2 in l_G we have:

$$i_{[X_1,X_2]}B + d_{X_2}\, i_{X_1} B - d_{X_1}\, i_{X_2} B =$$

$$= - d_{[X_1,X_2]} B - d_{X_2}\, d_{X_1}\, \gamma + d_{X_1}\, d_{X_2}\, \gamma = 0.$$

It follows from Proposition 3.7 that $\overline{L}_G$ is completely integrable. (iii) If (4.1) is locally integrable, then from $i_X B = -d_X \gamma$ it follows that $d_Y d_X \gamma = -d_Y\, i_X B = -i_X\, d_Y B$, for each $Y \in r_G$. The 1-cocycle equation (2.1) is satisfied if and only if $i_X d_Y B = 0$. Since l_G is transitive, this equation is equivalent to $d_Y B = 0$. (Q.E.D.)

It follows from Proposition 4.1, (i), (iii) and from the integrability condition (4.13) that:

PROPOSITION 4.2 - Let B be a 2-form on G. The system (4.1) is locally integrable and local solutions satisfy the 1-cocycle equation (2.1) if and only if B is closed and right-invariant.

If the distribution $\overline{L}_G$ associated with the bilinear form B is completely integrable, then we call *global solution* of equation (4.1) a maximal connected integral manifold Γ of $\overline{L}_G$. If B si right-invariant (Proposition 4.1, (iii)), then a global solution Γ of (4.1) is also a global solution of the 1-cocycle equation (2.1), i.e. the union of images of local sections satisfying (2.1). Since the distribution $\overline{L}_G$ is transverse to π_G, π_G restricted to a global solution Γ form a covering of G. It follows that if G is connected and simply connected, then global solutions are (images of) 1-forms on G. If Γ is a global solution, then the set

$$\Gamma + \mu = \{k \in T^*G\,;\, k = k' + \mu(\pi_G(k')),\, k' \in \Gamma\}\,, \qquad \mu \in r_G^*\,,$$

is a global solution. All global solutions corresponding to B are obtained from Γ in this way. There is a unique global solution which contains the zero convector $0 \in T_e^*G$ at the identity e of G, i.e. a unique *normal global solution* (see Section 1).

We have seen how to construct 1-cocycles (i.e. 1-forms γ satisfying the 1-cocycle equation) starting from bilinear forms B on G. The following proposition shows how to construct bilinear forms starting from 1-cocycles.

PROPOSITION 4.3. - Let γ be a 1-form satisfying the 1-cocycle equation (2.1). A right-invariant bilinear form B satisfying the system (4.1) is defined by equation

(4.3) $\quad i_{X_2} i_{X_1} B = - i_{X_2} d_{X_1} \gamma, \; X_1$ and X_2 in l_G ,

Proof. - We apply the Lie derivative d_Y with $Y \in r_G$ to both sides of (4.3). We obtain

$$\begin{aligned} d_Y i_{X_2} i_{X_1} B &= - d_Y i_{X_2} d_{X_1} \gamma \\ &= - i_{X_2} d_{X_1} d_Y \gamma \\ &= 0 \,. \end{aligned}$$

Since r_G is transitive, it follows that $i_{X_2} i_{X_1} B = \text{const.}$ (on the connected components of G). This means that (4.3) defines a right-invariant bilinear form. Equation (4.3) can be written in the form $i_{X_2} (i_{X_1} B + d_{X_1} \gamma) = 0$. Since l_G is transitive, we have $i_{X_1} B + d_{X_1} \gamma = 0$ for each $X_1 \in l_G$. (Q.E.D.)

If the bilinear form B defined by (4.3) is skew-symmetric (i.e. if B is a 2-form), then the 1-cocycle corresponding to γ is said to be a *symplectic cocycle* (Souriau [8]). Symplectic cocycles arise in connection with symplectic group actions. It is well known that right-invariant closed 2-forms can be interpreted as 2-cocycles on l_G with real values. Let $[B]$ be the cohomology class of the 2-cocycle B; it is the space of 2-forms $B' = B + dA$ where $A \in l_G^*$. If γ is a local solution of (4.1) corresponding to B, then $\gamma + \mu$, $\mu \in r_G^*$, is a local solution corresponding to B (since $d_X \mu = 0$), and $\gamma' = \gamma - A$ is a local solution corresponding to $B' = B + dA$. Equations (4.1) and (4.3) give the relationships between 2-cocycles $B \in Z_1^1(G, l_G^*)$ and 1-cocycles $\theta \in Z^1(G, l_G^*)$ through the representative 1-forms γ of the equivalence class $[\gamma]_l \in S^1(G)/r_G^*$ (Section 2).

REMARK 4.1 - Let B^T be the *transpose* of a bilinear form B. Since l_G and r_G commute, for each $X \in l_G$ and $Y \in r_G$ we have $i_Y (d_X \gamma + i_X B) = d_X i_Y \gamma + i_Y i_X B = i_X (d i_Y \gamma + i_Y B^T)$. Since l_G and r_G are transitive, it follows that the system (4.1) is equivalent to the system

(4.4) $\quad i_Y B^T + d i_Y \gamma = 0, \; Y \in r_G$,

i.e. to the system

(4.4′) $\quad i_Y B - d i_Y \gamma = 0, \; Y \in r_G$, in the case of a skew-symmetric bilinear form ($B^T = -B$). It follows from Proposition 3.2 (ii) and definitions (3.2) and (3.3) that if γ is a local solution of (4.1) corresponding to B, then the function

$$E_Y = E_Y - \pi_G * i_Y \gamma$$

is a local Hamiltonian of $\hat{Y}$ with respect to the symplectic form ω defined by (3.1), i.e. $i_{\hat{Y}} \omega = - dE_Y$. Hence, the vector fields Y are globally Hamiltonian if equations (4.1) have global solutions which are 1-forms.

5. - 2-cocycles and actions on T^*G

If a foliation of T^*G made of images of 1-forms is given, then any action on G can be lifted to an action on T^*G in a natural way. Each infinitesimal generator X of the action on G can be lifted to an infinitesimal generator $\overline{X}$ of the lifted action. The vector field $\overline{X}$ is uniquely defined by the following conditions: (i) it is tangent to the leaves of the foliation, (ii) it is π_G-projectable to the vector field $\overline{X}$. For example, the actions λ and ρ of a Lie group onto itself are lifted to the canonical actions $\hat{\lambda}$ and $\hat{\rho}$ by means of the foliations determined by the left-invariant and right-invariant forms respectively.

PROPOSITION 5.1. - Let B be a 2-cocycle (i.e. a closed and right-invariant 2-form) on a Lie group G such that the corresponding equation has 1-forms as global solutions. Let $\theta \in Z^1(G, l_G^*)$ be the 1-cocycle corresponding to B and defined by $\theta(g) = \gamma - \lambda^*_{g^{-1}} \gamma$ (see (2.3)), where γ is a solution of (4.1). The vector fields $\{\overline{X}; X \in l_G\}$ of $\overline{l}_G$ defined in (3.3) are the infinitesimal generators of the action $\overline{\lambda}: G \times T^*G \to T^*G$ defined by

$$\overline{\lambda}_g(k') = \hat{\lambda}_g(k') + \theta(g)(gg'), \quad g' = \pi_G(k') \quad . \tag{5.1}$$

for each $g \in G$ and $k' \in T^*G$.

Proof. - Let $\overline{L}_G$ be the distribution spanned by the vector fields $\overline{X}$. This distribution is completely integrable (Proposition 3.8) and the maximal connected integral manifolds are images of 1-forms satsfying (4.1) (Proposition 4.1). Let $\overline{\lambda}$ be the action of G on T^*G obtained by lifting the action λ on G through this foliation. This means that

$\overline{\lambda}_g(k')$ is the covector belonging to $T^*_{\lambda_g(g')}G = T^*_{gg'}G$, with $g' = \pi_G(k')$, and to the integral manifold containing k', i.e.:

$$\overline{\lambda}_g(k') = \gamma(gg')\,,$$

where γ is the solution of (4.1) such that $k' = \gamma(g')$. The vector fields $\overline{X}$ are the infinitesimal generators of this action, since each $\overline{X}$ is a lift of a generator of the action λ and it is tangent to the foliation. On the other hand (see (1.3) and definition (2.3)):

$$\begin{aligned}\hat{\lambda}_g(k') + \theta(g)(gg') &= \hat{\lambda}_g(\gamma(g')) + \theta(g)(gg')\\ &= (\lambda_g^{*^{-1}}\gamma + \theta(g))(gg')\\ &= \gamma(gg')\,,\end{aligned}$$

and (5.1) follows. (Q.E.D.)

The following proposition is a consequence of Proposition 3.9.

PROPOSITION 5.2. - The actions $\hat{\lambda}$ and $\hat{\rho}$ commute and are simplectic with respect to $\omega = d\theta_G + \pi_G{}^*B$.

For the pair of actions $(\overline{\lambda}, \hat{\rho})$ properties analogous to those considered in [3] for $(\hat{\lambda}, \hat{\rho})$ hold, with respect to the modified symplectic form ω on T^*G. In particular, the orbits of the composed action $(G \times G) \times T^*G \to T^*G : (g_1, g_2, k) \to \overline{\lambda}_{g_1}(\hat{\rho}_{g_2}(k))$ form a (generalized) coisotropic foliations of T^*G and the corresponding reduced symplectic manifolds can be identified with the orbits of the affine action on l_G^* corresponding to the 1-cocycle θ [8]. We also mention that, in a different approach, actions $\overline{\lambda}$ and $\overline{\rho}$ were already considered in [5] and [6].

6. - Central extensions and symplectic reductions

Let us consider a central extension of the Lie group G by the group $\mathbf{R}$, i.e. an exact sequence of homomorphism of Lie groups:

$$1 \to \mathbf{R} \overset{\epsilon}{\to} F \overset{\eta}{\to} G \to 1\,.$$

The homomorphism $\eta : F \to G$ is a principal fibre bundle with structural group $\mathbf{R}$ whose action δ on F is defined by $\delta(r, f) = \delta_r(f) = \epsilon(r)f$.

Let V be the corresponding infinitesimal generator (the fundamental vector field). This vector field belongs to the centers of the Lie algebras l_F and $r_F : [V, Z] = 0$, for each $Z \in l_F \cap r_F$.

PROPOSITION 6.1. - Each element $Z \in l_F$ (resp. $Z \in r_F$) is η-projectable to an element $X \in l_G$ (resp. $X \in r_G$) : $T\eta \cdot Z = X \cdot \eta$.

Proof. - Since Z and V commute, $T\delta_r \cdot Z = Z \cdot \delta_r$ for each $r \in \mathbf{R}$. It follows from $\eta \cdot \delta_r = \eta$ that $T\eta \cdot Z \cdot \delta_r = T\eta \cdot T\delta_r \cdot Z = T\eta \cdot Z$. Hence, equation $T\eta(Z(f)) = X(\eta(f))$, $f \in F$, defines a vector field X on G. We can denote by λ and ρ the actions on F defined as in (1.8). Each element $Z \in l_F$ is right-invariant: $T\rho_f \cdot Z = Z \cdot \rho_f$. Since η is a homomorphism, $\eta \cdot \rho_f = \rho_{\eta(f)} \cdot \eta$. If $g = \eta(f)$, then:

$$\begin{aligned} T\rho_g \cdot X \cdot \eta &= T\rho_g \cdot T\eta \cdot Z \\ &= T\eta \cdot T\rho_f \cdot Z \\ &= T\eta \cdot Z \cdot \rho_f \\ &= X \cdot \eta \cdot \rho_f \\ &= X \cdot \rho_g \cdot \eta \, . \end{aligned}$$

This shows that $T\rho_g \cdot X = X \cdot \rho_g$, i.e. that X is also right-invariant, thus $X \in l_G$. (Q.E.D.)

A *connection* of η is a 1-form $\alpha \in \Phi_1(F)$ such that $i_V \alpha = 1$ and $d_V \alpha = 0$. Since $i_V d\alpha = 0$ and $d_V d\alpha = 0$, there exists a unique 2-form B on G such that $d\alpha = \eta^* B$. The closed 2-form B is the *curvature* of α. If α' is another connection and B' is the corresponding curvature, then $\alpha' - \alpha = \eta^* A$, where $A \in \Phi_1(G)$, and $B' - B = dA$. Hence, with a central extension of G by $\mathbf{R}$ we associate a distinguished de-Rham cohomology class [B] of degree 2. It can be easily shown that two equivalent central extensions give rise to the same cohomology class. A central extension $1 \to \mathbf{R} \xrightarrow{\epsilon'} F' \xrightarrow{\eta'} G \to 1$ is equivalent to $1 \to \mathbf{R} \xrightarrow{\epsilon} F \xrightarrow{\eta} G \to 1$ if there exists an isomorphism $\iota : F \to F'$ such that the diagram

$$1 \to \mathbf{R} \begin{array}{c} \nearrow^{\epsilon} \; F \; \searrow^{\eta} \\ \quad\;\; \downarrow \iota \\ \searrow_{\epsilon'} \; F' \; \nearrow_{\eta'} \end{array} G \to \mathbf{R} \to 1$$

is commutative.

PROPOSITION 6.1. - If the connection α is right-invariant ($\rho_f{}^*\alpha = \alpha$ for all $f \in F$), then the curvature B is also right-invariant ($\rho_g{}^*B = B$ for all $g \in G$). If α and α' are right-invariant, then the 1-form A such that $\alpha' - \alpha = \eta^*A$ is also right-invariant.

Proof. - From $\rho_f{}^*\alpha = \alpha$ it follows that $\eta^*B = d\alpha = \rho_f{}^*d\alpha = \rho_f{}^*\eta^*B = = \eta^*\rho_g{}^*B$, where $g = \eta(f)$. Hence $B = \rho_g{}^*B$. The proof of the invariance of A is analogous. (Q.E.D.)

PROPOSITION 6.3. - Let S be a subspace of l_F of codimension 1 not containing V. A right-invariant connection α is defined by equations $i_V\alpha = 1$ and $i_Z\alpha = 0$, for each $Z \in S$. If S is a subalgebra, then the connection α is flat: $d\alpha = 0$.

Proof. - The evaluation of α on an element of l_F is a constant function. Hence, α is an element of l_F^*. Consequently, α is right-invariant. Since $[V, Z] = 0$ for each vector field Z in S, the vector fields $Z \in S$ are δ-invariant, so that also α is δ-invariant and $d_V\alpha = 0$. Thus α is a right-invariant connection. From the general identity

$$\langle Z_1 \wedge Z_2, d\alpha\rangle = d_{Z_1} i_{Z_2}\alpha - d_{Z_2} i_{Z_1}\alpha - i_{[Z_1, Z_2]}\alpha,$$

it follows that

$$\langle Z \wedge V, d\alpha\rangle = 0, \quad Z \in S,$$

and

$$\langle Z_1 \wedge Z_2, d\alpha\rangle = -i_{[Z_1, Z_2]}\alpha, \quad Z_1, Z_2 \in S.$$

This shows that if S is a sub-algebra, then $d\alpha = 0$. (Q.E.D.)

We have seen that with each central extension of the group G by $\mathbf{R}$ (or with an equivalence class of such extensions) we can associate a cohomology class $[B] \in H^2(l_G, \mathbf{R})$ represented by the curvatures of the right-invariant connections. If this cohomolgy class is not zero, then a distinguished subspace of dimension 1 of $H^2(l_G, \mathbf{R})$ is formed by elements of the type $m[B]$, with $m \in \mathbf{R}$.

For each $m \in \mathbf{R}$ we introduce the submanifold K_m of T^*F defined by

$$K_m = \{h \in T^*F;\ \langle V, h\rangle = m\}.$$

This submanifold is coisotropic (because it is of codimension 1). Let us choose a connection α. A surjective submersion $\kappa_m : K_m \to T^*G$ is defined by

$$(6.1) \qquad \langle v, \kappa_m(h)\rangle = \langle w, h\rangle ,$$

where $h \in K_m$, $v \in T_g G$, $g = \eta(f)$, $f = \pi_F(h)$, and w is the horizontal lift of v, i.e. the vector defined by equations: $\langle w, \alpha\rangle = 0$, $T\eta(w) = v$. The fibres of κ_m are the orbits of the canonical lift $\hat{\delta}$ of the action δ restricted to K_m. These orbits are characteristics of K_m. (i.e. maximal connected integral manifolds of the characteristic distribution). Moreover, $\kappa_m\ (d\theta_G + m\,\kappa_G^* B) = d\theta_F \,|\, K_m$, where $d\theta_F \,|\, K_m$ is the pull-back of the symplectic form $d\theta_F$ to the submanifold K_m. This means that κ_m defines a symplectic reduction from $(T^*F, d\theta_F)$ to the symplectic manifold (T^*G, ω_m) where $\omega_m = d\theta_G + m\,\pi_G^* B$ (see [7] for details).

Since $V \in l_F \cap r_F$, a right or left-invariant 1-form $\overline{\gamma}$ on F is such that $i_V \overline{\gamma} = \text{const.}$. Condition $i_V \overline{\gamma} = m$ is equivalent to $\overline{\gamma}(F) \subset K_m$.

PROPOSITION 6.4. - Let α be a right-invariant connection. Let $\overline{\gamma}$ be a right-invariant or a left-invariant 1-form on F such taht $i_V \overline{\gamma} = m$. There is a unique 1-form γ on G such that

$$(6.2) \qquad \overline{\gamma} - m\alpha = \eta^*\gamma ,$$

$$(6.3) \qquad \kappa_m\,(\overline{\gamma}(F)) = \gamma(G) .$$

If $\overline{\gamma}$ is right-invariant, then γ is right-invariant. If γ is left-invariant, then γ satisfies equation (4.1) for $mB : m\, i_X B + d_X \gamma = 0$, for each X in l_G.

Proof. - Let $\overline{\gamma}$ be right-invariant: $d_Z \overline{\gamma} = 0$ for each $Z \in l_F$. In particular $d_V \overline{\gamma} = 0$. It follows that $i_V(\overline{\gamma} - m\alpha) = 0$ and $d_V(\overline{\gamma} - m\alpha) = 0$. Hence, there exists a 1-form γ on G such that $\overline{\gamma} - m\alpha = \eta^*\gamma$. From the definition (6.1) of κ_m it follows that $\langle v, \kappa_m(\overline{\gamma}(f))\rangle = \langle w, \overline{\gamma}(f)\rangle = \langle w, m\alpha(f) + \eta^*\gamma(f)\rangle = \langle v, \gamma\rangle$, since $\langle w, \alpha\rangle = 0$. This proves (6.3). The proof is analogous for $\overline{\gamma}$ left-invariant. If $\overline{\gamma}$ is right-invariant, then by applying to (6.2) the Lie derivative d_Z with respect to a vector field $Z \in r_F$ we obtain $0 = d_Z\,\eta^*\gamma = \eta^* d_X \gamma$, where X is the projection of Z. It follows that γ is right-invariant. If $\overline{\gamma}$ is left-invariant, then the same operation with $Z \in l_F$ yields the equation

$-md_Z\,\alpha=\eta^*d_X\gamma$, where $X\in l_G$. Since $i_Z\alpha=$ const. and $d_Z\alpha=$ $=i_Z d\alpha+di_Z\alpha=i_Z\eta^*B=\eta^*i_X B$, we have $-mi_X B=d_X\gamma$ for each $X\in l_G$. (Q.E.D.)

We have seen that a right-invariant connection α associated with the central extension defines a symplectic reduction from $(T^*F, d\theta_F)$ to (T^*G, ω_m), for each $m\in R$, where $\omega_m=d\theta_G+m\pi_G{}^*B$ and B is the curvature of α. We can *reduce* any invariant 1-form $\overline{\gamma}$ on F to a 1-form γ on G. The reduction of a right-invariant form is a right-invariant 1-form, while the reduction of a left-invariant form $\overline{\gamma}$ is a solutions of equation (4.1), i.e. a 1-form representing the 1-cocycle associated with the 2-cocycle mB, where B is the curvature of α and $m=i_V\overline{\gamma}$.

REFERENCES

[1] S. Benenti, *Homogeneous Formulation of Hamiltonian Group Actions*, Proceedings of «4th Meeting on Mathematical Physics», Coimbra, October 15-1984.

[2] S. Benenti, W.M. Tulczyjew, *Momentum Relations for Hamiltonian Actions*, Proceedings of «Geometrie Symplectique et Mécanique», Colloque de la Region Mediterranée de la S.M.F., Montpellier, 14-15 Mai 1984, X.Dufour Ed., Hermann (Paris, 1985).

[3] S. Benenti, W.M. Tulczyjew, *Sur un feuilletage coisotrope du fibré cotangent d'un groupe de Lie*, C.R.A.S. Paris, 300 (1985), 119-122.

[4] S. Benenti, W.M. Tulczyjew, *Geometrodynamics Proceedings 1985,* A. Prastaro Ed., World Scientific Publishing Co. (1985), pp. 3-24.

[5] P. Libermann, C.M. Marle, *Géométrie symplectique base théorique de la mécanique*, Publications Mathématiques de l'Université de Paris VII, (forthcoming).

[6] C.M. MARLE, *Moment de l'action hamiltonienne d'un groupe de Lie; quelques propriétés*, Firenze 1982, Pitagora (Bologna, 1984), 117-133.

[7] M.R. MENZIO, W. M. TULCZYJEW, *Infinitesimal symplectic relations and generalized Hamiltonian dynamics*, Ann. Inst. H. Poincaré, 28 (1978) 349-367.

[8] J. M. SOURIAU, *Structures des systèmes dynamiques*, Dunod (Paris, 1970).