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A GEOMETRICAL INTERPRETATION OF THE 1-COCYCLES OF A LIE GROUP

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0.- Introduction.

In this paper we give a geometrical interpretation of the 1-cocycles of a Lie group and their relation with the 2-cocycles on the correponding Lie algebra. 1-cocycles are interpreted as classes of 1-forms on the Lie group G satisfying a suitable differential equation. The images of 1-forms of a class give rise to a foliation of the cotangent bundle T*G of the group. This foliation can be also constructed by means of an invariant closed 2-form on the group, representing a 2-cocycle. This form is used to correct the canonical symplectic structure of T*G and also to correct the canonical lift of vector fields of G to T*G. In this approach the structure of the cotangent bundle of G plays an essential role. We do not use the natural trivialization of T*G. Instead, we use both Lie algebras of left and right infinitesimal generators and the corresponding dual spaces of right and left-invariant 1-forms. This paper is a continuation of the analysis of the cotangent bundle of a Lie group initiated in [3]. The geometrical interpretation of 1-cocycles and 2-cocycles given here has been suggested by results of an analysis concerning the geometrical interpretations of Hamiltonian actions in terms of coisotropic submanifolds [1][2].

1.- Notation.

Let G be a Lie group. We consider a vector field on the manifold G as a section $X:G \to TG$ of the tangent bundle $\tau_G:TG \to G$ and a 1-form on G as a section of the cotangent bundle $\pi_G:T^*G \to G$. The spaces of smooth vectorfields and k-forms on G are denoted by $\mathcal{X}(G)$ and $\Phi_k(G)$ respectively. Symbols $[X,Y]$, d_X and i_X denote the Lie brackets of vector fields X and Y, the Lie derivative and the interior product with respect to a vector field X. We consider the left and the right translations on G:

$$\lambda:G \times G \to G:(g,g') \mapsto \lambda_g(g') = gg',$$
$$\rho:G \times G \to G:(g,g') \mapsto \rho_g(g') = g'g^{-1}.$$

They are smooth left actions of G onto itself. We denote by ℓ_G and τ_G the Lie algebras of the infinitesimal generators (vector fields) of the actions λ and ρ respectively. The linear dual spaces ℓ_G^* and τ_G^* are identified with the spaces of right-ivariant and left-invariant 1-forms on G:

$$\mu \in \tau_G^* \iff \lambda_g^* \mu = \mu \ , \ \forall g \in G \iff d_X \mu = 0, \ \forall X \in \ell_G,$$
$$\nu \in \ell_G^* \iff \rho_g^* \nu = \nu \ , \ \forall g \in G \iff d_Y \nu = 0, \ \forall Y \in \tau_G.$$

Since the two actions commute, we have $[X,Y] = 0$ for each $X \in \ell_G$ and $Y \in \tau_G$. Consequently: $d_X d_Y - d_Y d_X = d_{[X,Y]} = 0$.

We denote by θ_G the Liouville 1-form on T^*G. The 2-form $d\theta_G$ is the canonical symplectic form on T^*G.

The canonical lift of a diffeomorphism $\psi:G \to G$ is the unique diffeomorphism $\hat{\psi}:T^*G \to T^*G$ such that

$$\text{(1.1)} \quad \hat{\psi}^* \theta_G = \theta_G \quad , \quad \pi_G \circ \hat{\psi} = \psi \circ \pi_G.$$

It can be also defined by equation

(1.2) $\langle v, \hat{\psi}(k)\rangle = \langle T\psi^{-1}(v), k\rangle \qquad (k \in T^*_gG,\ v \in T_{\psi(g)}G),$

where T denotes the tangent functor. We use the following formula relating $\hat{\psi}$ with the pull-back $\psi^*: \Phi_1(G) \to \Phi_1(G)$:

(1.3) $(\psi^*\mu)(g) = \hat{\psi}^{-1}(\mu(\psi(g))) \qquad (g \in G,\ \mu \in \Phi_1(G)).$

The canonical lift of a vector field X on G is the unique vector field $\hat{X}$ on T*G such that

(1.4) $d_{\hat{X}}\theta_G = 0 \quad , \quad T\pi_G \circ \hat{X} = X \circ \pi_G.$

It can be also defined by equation

(1.5) $i_X d\theta_G = -dE_X,$

where

(1.6) $E_X: T^*G \to R: k \mapsto \langle X \circ \pi_G(k), k\rangle.$

The lifted actions

$$\hat{\lambda}: G \times T^*G \to T^*G: (g,k) \mapsto \hat{\lambda}_g(k),$$
$$\hat{\rho}: G \times T^*G \to T^*G: (g,k) \mapsto \hat{\rho}_g(k),$$

are generated by the canonical lifts $\hat{X}$ and $\hat{Y}$ of the vector fields $X \in \ell_G$ an $Y \in r_G$ respectively. The images of the left-invariant (resp. right invariant) 1-forms are the orbits of $\hat{\lambda}$ (resp. of $\hat{\rho}$).

We have two representations of G on the vector space $\Phi_1(G)$:

$$\lambda^*: G \times \Phi_1(G) \to \Phi_1(G): (g,\nu) \mapsto \lambda_g^{*-1}\nu,$$
$$\rho^*: G \times \Phi_1(G) \to \Phi_1(G): (g,\mu) \mapsto \rho_g^{*-1}\mu.$$

When restricted to ℓ_G^* and to r_G^* we have the <u>left</u> and the <u>right</u> <u>coadjoint</u> <u>representations</u> of G respectively.

2.- The 1-cocycle differential equation.

We consider the differential equation

(2.1) $d_X d_Y \gamma = 0 \quad , \quad \forall\, X \in \ell_G,\ Y \in r_G,$

where $\gamma : U \to T^*G$ is a 1-form on an open set $U \subset G$. This equation is linear in γ and symmetric with respect to X and Y. Sums of left and right-invariant 1-forms are <u>trivial solutions</u> of (2.1). We call <u>normal solution</u> a solution γ in the neighborhood of the identity e of the group G such that $\gamma(e) = 0$. A trivial normal solution is of the kind $\gamma = \mu - \nu$ with $\mu \in r^*_G$, $\nu \in \ell^*_G$, and $\mu(e) = \nu(e)$. The space of solutions of (2.1) is invariant under addition of left and right-invariant 1-forms and the pull-back actions λ^* and ρ^*.

By extending local solutions we can obtain global solutions of (2.1). A <u>global solution</u> is a maximal connected submanifold of T*G obtained as union of images of local solutions. The existence and the geometrical meaning of global solutions will be discussed in the next sections. In the present section we assume that all global solutions are 1-forms defined on all of G. They form a subspace $S^1(G)$ of $\Phi_1(G)$. We denote by $T^1(G)$ the subspace of trivial solutions, i.e. $T^1(G) = \ell^*_G + r^*_G$. The quotient space $H^1(G) = S^1(G)/T^1(G)$ is the space of the equivalence classes of the following equivalence relation in $S^1(G)$:

$$\gamma \sim \gamma' \iff \gamma' = \gamma + \mu + \nu\ , \quad \mu \in r^*_G,\ \nu \in \ell^*_G.$$

We denote by $[\gamma]$ the class represented by the solution γ. For each $g \in G$ let us consider the subspaces $S^1_g(G) = \{\gamma \in S^1(G);\ \gamma(g) = 0\}$, $T^1_g(G) = \{\gamma \in T^1(G);\ \gamma(g) = 0\} \subset S^1_g(G)$. In particular, $S^1_e(G)$ is the space of the normal solutions and $T^1_e(G)$ is the space of the trivial normal solutions. We have natural isomorphisms

$$S^1_g(G) \simeq S^1_{g'}(G) \quad , \quad T^1_g(G) \simeq T^1_{g'}(G),$$

defined by means of the representations λ^* and ρ^*, and also natural isomorphisms

$$S^1(G)/T^1(G) \simeq S^1_g(G)/T^1_g(G).$$

In particular:

$$S^1(G)/T^1(G) \simeq S^1_e(G)/T^1_e(G).$$

PROPOSITION 2.1.- A 1-form $\gamma \in \Phi_1(G)$ is a solution of equation (2.1) if and only if for each $g \in G$ the 1-form $\gamma - \lambda^*_g\gamma$ is right-invariant (resp. $\gamma - \rho^*_g\gamma$ is left-invariant).

Proof.- Equation (2.1) means that $d_Y\gamma$ is a left-invariant 1-form for each $Y \in \mathfrak{r}_G$. Thus $\lambda^*_g d_Y\gamma = d_Y\gamma$ for each $g \in G$. Since Y is left-invariant, we have $\lambda^*_g d_Y\gamma = d_Y \lambda^*_g\gamma$. It follows that $d_Y(\gamma - \lambda^*_g\gamma) = 0$, for each $Y \in \mathfrak{r}_G$, hence that $\gamma - \lambda^*_g\gamma$ is right-invariant. The reasoning is reversible. (Q.E.D.)

PROPOSITION 2.2.- Let γ be a solution of equation (2.1). The mapping

$$(2.2) \quad \theta : G \to \ell^*_G : g \mapsto \gamma - \lambda^*_{g^{-1}}\gamma$$

(resp. $\theta : G \to \mathfrak{r}^*_G : g \mapsto \gamma - \rho^*_{g^{-1}}\gamma$) satisfies equation

$$(2.3) \quad \theta(gg') = \lambda^*_{g^{-1}}\theta(g') + \theta(g)$$

(resp. $\theta(gg') = \rho^*_{g^{-1}}\theta(g') + \theta(g)$) for each $g,g' \in G$, i.e. it is a 1-cocycle of G with respect to the left coadjoint representation on ℓ^*_G (resp. right coadjoint representation on $\mathfrak{r}^*_G$). If γ is a trivial solution, then θ is a 1-coboundary.

Proof.- Because of Prop. 2.1 the mapping θ is well defined. Moreover, $\theta(gg') = \gamma - (\lambda^*_{gg'})^{-1}\gamma = \gamma - \lambda^*_{g^{-1}}(\lambda^*_{g'^{-1}}\gamma) = \gamma - \lambda^*_{g^{-1}}\theta(g') - \lambda^*_{g^{-1}}\gamma = \theta(g) - \lambda^*_{g^{-1}}\theta(g')$. This shows that θ is a cocycle. If $\gamma = \mu + \nu$, where $\mu \in z^*_G$ and $\nu \in \ell^*_G$, then $\theta(g) = \gamma - \lambda^*_{g^{-1}}\gamma = \mu + \nu - \lambda^*_{g^{-1}}(\mu + \nu) = \mu + \nu - \lambda^*_{g^{-1}}\nu - \mu = \nu - \lambda^*_{g^{-1}}\nu$. This shows that θ is a coboundary. (Q.E.D.)

REMARK 2.3.- If a mapping $\theta : G \to \Phi_1(G)$ is defined as in (2.2), then for each $g \in G$:

$$(2.4) \quad \theta(g)(g) = \gamma(g) - \hat{\lambda}_g(\gamma(e))$$

(we use identity (1.3)).

PROPOSITION 2.4.- Equation

$$(2.5) \quad \gamma(g) = \theta(g)(g) \quad , \ \forall g \in G,$$

defines an isomorphism of the space of 1-forms $\gamma \in \Phi_1(G)$ and the space of mappings $\theta : G \to \ell^*_G$ (or $\theta : G \to z^*_G$). **The 1-form γ is a normal (resp. a trivial normal) solution of equation (2.1) if and only if θ is a 1-cocycle (resp. a 1-coboundary).**

Proof.- If $\gamma \in \Phi_1(G)$ is given, then for each $g \in G$ there exists a unique 1-form $\theta(g) \in \ell^*_G$ whose value at g itself is equal to $\gamma(g)$, because the images of the right-invaraiant 1-forms span a foliation of T^*G. Thus mapping $\theta : G \to \ell^*_G$ is well defined by (2.5). Conversely, if a mapping $\theta : G \to \ell^*_G$ is given, then (2.5) defines a unique section $\gamma : G \to T^*G$ of π_G, i.e. a 1-form on G. The correspondence so defined is linear, hence it is an isomorphism. Let γ be a normal (resp. trivial normal) solution of (2.1). Because of Prop. 2.2 the mapping defined by $\theta(g) = \gamma - \lambda^{*-1}_g\gamma$ is a 1-cocylce (resp. a coboundary). Because of Remark 2.3 this mapping coincides with that one defined in 2.5 (since $\gamma(e) = 0$). Conversely, let

$\theta : G \to \ell^*_G$ be a 1-cocylce and $\gamma : G \to T^*G$ be the 1-form defined by (2.5). We remark first of all that, because of (2.3), $\theta(e) = 0$; hence $\gamma(e) = 0$. In the following calculation we use identity (1.3) and also identity $\lambda^*_{g^{-1}}\theta(g^{-1}g') = \theta(g') - \theta(g)$ which comes from (2.3) by substituting g' with $g^{-1}g'$:

$$\begin{aligned}(\gamma - \lambda^*_{g^{-1}}\gamma)(g') &= \theta(g')(g') - \hat{\lambda}_g(\gamma(g^{-1}g')) \\ &= \theta(g')(g') - \hat{\lambda}_g(\theta(g^{-1}g')(g^{-1}g')) \\ &= \theta(g')(g') - (\lambda^*_{g^{-1}}\theta(g^{-1}g'))(g') \\ &= \theta(g')(g') - (\theta(g') - \theta(g))(g') \\ &= \theta(g)(g').\end{aligned}$$

This shows that $\gamma - \lambda^*_{g^{-1}}\gamma = \theta(g) \in \ell^*_G$, hence that $\gamma - \lambda^*_{g^{-1}}\gamma$ is a right-invariant 1-form for each $g \in G$. From Prop. 2.1 it follows that γ is a solution of equation (2.1). If θ is a coboundary (for instance in the coadjoint representation in ℓ^*_G), then $\theta(g) = \nu - \lambda^*_{g^{-1}}\nu$ where $\nu \in \ell^*_G$. It follows that $\gamma - \nu = \lambda^*_{g^{-1}}(\gamma - \nu)$, i.e. that $\gamma - \nu \in \mathfrak{r}^*_G$, which means that γ is a trivial solution. (Q.E.D.)

CONCLUSION.- The quotient space S^1_e/T^1_e, which is isomorphic to the quotient space $H^1(G) = S^1(G)/T^1(G)$, is isomorphic to the first cohomology group of G with respect to the left or right coadjoint representation.

An equivalent interpretation of the mapping $\gamma : G \to T^*G$ representing a 1-cocycle as a Lie group homomorphism is due to Marle (private communication).

3.- The lift of a vector field by means of a 2-form.

Let B a 2-form on G. With each vector field X on G we associate a vector field $\bar{X}$ on T*G defined by equation

(3.1) $i_{\bar{X}}\,\bar{\omega} = -\,dE_X$,

where E_X is defined in (1.6), and

$$\bar{\omega} = d\,\theta_G + \pi_G^* B,$$

and a vector field $\tilde{X}$ on T*G defined by equation

(3.2) $i_{\tilde{X}} d\,\theta_G = -\,\pi_G^* i_X B$.

PROPOSITION 3.1.- The vector field $\tilde{X}$ is vertical with respect the projection π_G, i.e. $T\pi_G \circ \tilde{X} = 0$.

Proof.- If $v \in T_k T^*G$ and $T\pi_G(v) = 0$, then $i_v i_{\tilde{X}} d\,\theta_G = \langle T\pi_G(v) \wedge X, B\rangle = 0$. Since v is tangent to the fibre of π_G at $k \in T^*G$ and the fibre is a Lagrangian submanifold, it follows that also X(k) is tangent to the fibre, hence $T\pi_G(\tilde{X}(k)) = 0$. (Q.E.D.)

PROPOSITION 3.2.- For each vector field X on G we have $\bar{X} = \hat{X} + \tilde{X}$ where $\hat{X}$ is the canonical lift of X. The vector field $\bar{X}$ is π_G-projectable on to X, i.e. $T\pi_G \circ \bar{X} = X \circ \pi_G$.

Proof.- Since $\tilde{X}$ is vertical, $i_X \pi_G^* B = 0$. Hence, $i_{\hat{X} + \tilde{X}}(d\,\theta_G + \pi_G^* B) = i_{\hat{X}} d\,\theta_G + i_{\hat{X}} \pi_G^* B + i_{\tilde{X}} d\,\theta_G = -\,dE_X + i_X \pi_G^* B + i_{\tilde{X}} d\,\theta_G = -\,dE_X$. (Q.E.D.)

PROPOSITION 3.3.- For each pair (X_1,X_2) of vector fields on G the following identity holds:

$$(3.3)\quad i_{[\bar{X}_1,\bar{X}_2]}\bar{\omega} = -\, dE_{[X_1,X_2]} + \pi_G^*(di_{X_2}i_{X_1}B + i_{X_1}i_{X_2}dB).$$

Proof.- We use identities $i_{\hat{X}_1}i_{\hat{X}_2}d\theta_G = -\,i_{\hat{X}_1}dE_{X_2} = -\,E_{[X_1,X_2]}$.

$$\begin{aligned} i_{\bar{X}_1}i_{\bar{X}_2}\bar{\omega} &= i_{\hat{X}_1+\tilde{X}_1}i_{\bar{X}_2}\bar{\omega} \\ &= -\,i_{\hat{X}_1}dE_{X_2} + i_{\tilde{X}_1}i_{\hat{X}_2}d\theta_G \\ &= -\,E_{[X_1,X_2]} + i_{\hat{X}_2}\pi_G^*i_{X_1}B \\ &= -\,E_{[X_1,X_2]} + \pi_G^*i_{X_2}i_{X_1}B. \end{aligned}$$

$$\begin{aligned} i_{[\bar{X}_1,\bar{X}_2]}\bar{\omega} &= (d_{\bar{X}_1}i_{\bar{X}_2} - i_{\bar{X}_2}d_{\bar{X}_1})\bar{\omega} \\ &= di_{\bar{X}_1}i_{\bar{X}_2}\bar{\omega} - i_{\bar{X}_2}i_{\bar{X}_1}\pi_G^*dB \\ &= -\,dE_{[X_1,X_2]} + \pi_G^*(di_{X_2}i_{X_1}B + i_{X_1}i_{X_2}dB) \end{aligned}$$

(Q.E.D.)

REMARK 3.4.- Identity (3.4) ca also be written as follows:

$$(3.4)\quad i_{[\bar{X}_1,\bar{X}_2] - \overline{[X_1,X_2]}}\bar{\omega} = \pi_G^*(di_{X_2}i_{X_1}B + i_{X_1}i_{X_2}dB).$$

In the discussion above we use only the differential manifold structure of G. In the next discussion we use the Lie group structure of G. Analogous results hold by exchanging λ with ρ and ℓ_G with r_G.

REMARK 3.5.- If B is closed $(dB = 0)$ then the 2-form $\bar{\omega}$ is a symplectic form on T*G and (3.1) shows that the vector field $\bar{X}$ is the Hamiltonian vector field generated by the function E_X with reference to this symplectic form. In particular from (3.1) it follows that

$$(3.5)\quad d_{\bar{X}}\bar{\omega} = 0.$$

PROPOSITION 3.6.- Let $X \in \ell_G$. The vector field $\bar{X}$ defined by (3.1) has the following characteristic properties: it is π_G-projectable onto X and

$$(3.6) \quad \langle \bar{X} \wedge \hat{Y}, \bar{\omega} \rangle = 0 \quad , \quad \forall\, Y \in r_G.$$

Proof.- If $\bar{X}$ is defined by (3.1), then because of Props. 3.1 and 3.2 we have: $\langle \bar{X} \wedge \hat{Y}, \bar{\omega} \rangle = \langle \hat{X} \wedge \hat{Y}, \pi_G^{*}B \rangle + \langle \tilde{X} \wedge \hat{Y}, d\theta_G \rangle = i_{\hat{Y}}(\pi_G^{*} i_X B + i_{\tilde{X}} d\theta_G)$. Thus (3.6) follows from (3.2). Conversely, if $\bar{X}$ is π_G-projectable onto X and we use the decomposition $\bar{X} = \hat{X} + \tilde{X}$ where $\tilde{X}$ is a vertical vector field, then the above calculation shows that from (3.6) it follows that $i_{\hat{Y}} \xi_X = 0$ where $\xi_X \doteq \pi_G^{*} i_X B + i_{\tilde{X}} d\theta_G$. The 1-form ξ_X is such that $\langle v, \xi_X \rangle = 0$ when $T\pi_G(v) = 0$. Since r_G is transitive, it follows that $\xi_X = 0$. (Q.E.D.)

REMARK 3.7.- If B is closed and left-invariant ($d_X B = 0$, for each $X \in \ell_G$), then the canonical lift $\hat{X}$ of $X \in \ell_G$ is a symplectic vector field with respect to the symplectic form $\bar{\omega}$, i.e.:

$$(3.7) \quad d_{\hat{X}} \bar{\omega} = 0.$$

This follows from equation (3.1), which can be written $i_{\hat{X}} \bar{\omega} = - dE_X + \pi_G^{*} i_X B$, and the fact that $i_X B$ is closed: $d i_X B = d_X B - i_X dB = 0$.

REMARK 3.8.- If B is closed and right-invariant then

$$(3.8) \quad [\bar{X}, \hat{Y}] = 0,$$

for each $X \in \ell_G$ and $Y \in r_G$. Both vector fields X and Y are indeed Hamiltonian vector fields with respect to the symplectic structure $\bar{\omega}$. Hence the Lie bracket [X,Y] is the globally Hamiltonian vector field generated by the function $\langle X \wedge Y, \bar{\omega} \rangle$, which is zero because of Prop. 3.6.

PROPOSITION 3.9.- The space of vector fields $\overline{\ell}_G = \{\overline{X};\ X \in \ell_G\}$ is a Lie sub-algebra of $\mathfrak{X}(T^*G)$ if and only if

$$(3.9)\quad di_Y B = d_Y B - i_Y dB = 0, \quad \text{for each } Y \in r_G.$$

Proof.- Let $X_1, X_2 \in \ell_G$ and $Y \in r_G$. Because of (3.4) and Prop.3.5, we have:

$$\begin{aligned}
\langle [\overline{X}_1,\overline{X}_2] \wedge \hat{Y}, \varpi\rangle &= i_{\hat{Y}} i_{[\overline{X}_1,\overline{X}_2]} \varpi \\
&= i_{\hat{Y}} i_{\overline{[X_1,X_2]}} \varpi + i_{\hat{Y}} \pi_G^*(di_{X_2} i_{X_1} B + i_{X_1} i_{X_2} dB) \\
&= 0 + \pi_G^*(i_Y di_{X_2} i_{X_1} B + i_Y i_{X_1} i_{X_2} dB) \\
&= \pi_G^* i_{X_2} i_{X_1} (d_Y B - i_Y dB).
\end{aligned}$$

If (3.9) holds, then $[\overline{X}_1,\overline{X}_2]$ is π_G-projectable onto $[X_1,X_2]$ and (3.6) holds. Thus $[\overline{X}_1,\overline{X}_2] = \overline{[X_1,X_2]}$. Conversely, if $\overline{\ell}_G$ is a sub-algebra, then $[\overline{X}_1,\overline{X}_2]$ is a linear combination of elements of $\overline{\ell}_G$. Because of (3.6), we have $i_{X_2} i_{X_1}(d_Y B - i_Y dB) = 0$, for each $X_1, X_2 \in \ell_G$ and $Y \in r_G$. Since ℓ_G is transitive, (3.9) follows. (Q.E.D.)

CONCLUSION.- With each 2-form B on G we associate a distribution $\overline{L}_G$ on T^*G of rank equal to the dimension of G. This distribution is determined by the vector fields $\overline{X} \in \overline{\ell}_G$ defined as in (3.1), where $X \in \ell_G$. The distribution is completely integrable if and only if (3.9) holds. In this case the mapping

$$\ell_G \rightarrow \mathfrak{X}(T^*G) : X \mapsto \overline{X}$$

is a Lie algebra homomorphism.

4.- Global solutions of the cocycle equation.

PROPOSITION 4.1.- The distribution $\bar{L}_G$ corresponding to a 2-form B on G is completely integrable if and only if the equation

$$(4.1) \quad i_X B + d_X \gamma = 0 \quad , \ \forall\, X \in \ell_G,$$

where γ is a 1-form on G, is locally integrable, i.e. for each $g \in G$ there exists a local section $\gamma : U \to T^*G$ of π_G satisfying (4.1) such that $g \in U$. The image $\gamma(U)$ of a local solution of (4.1) is an integral manifold of L_G. Equation (4.1) is locally integrable and the local solutions satisfy the 1-cocycle equation (2.1) if and only if B is closed and right-invariant, i.e. $dB = 0$ and $d_Y B = 0$ for each $Y \in \mathfrak{r}_G$.

LEMMA 4.2.- Let B a 2-form on G and X a vector field on G. The lift $\bar{X}$ of X (definition (3.1)) is tangent to the image $\gamma(G)$ of a 1-form γ on G if and only if $i_X B + d_X \gamma = 0$.

Proof.- From $\gamma^*(\theta_G - \pi_G^*\gamma) = \gamma - (\pi_G \circ \gamma)^*\gamma = 0$ it follows that a vector field $\bar{X}$ on T^*G is tangent to the image of the 1-form γ if and only if

$$\gamma^* d_{\bar{X}}(\theta_G - \pi_G^*\gamma) = 0.$$

If $\bar{X} = \hat{X} + \tilde{X}$, where X is any vector field on G, then:

$$d_{\hat{X}}\theta_G = 0;$$
$$d_{\tilde{X}}\theta_G = i_{\tilde{X}} d\theta_G + d i_{\tilde{X}}\theta_G = i_{\tilde{X}} d\theta_G = -\pi_G^* i_X B;$$
$$d_{\hat{X}}\pi_G^*\gamma = \pi_G^* d_X \gamma;$$
$$d_{\tilde{X}}\pi_G^*\gamma = 0.$$

Hence, $\gamma^* d_{\bar{X}}(\theta_G - \pi_G^*\gamma) = -\gamma^* \circ \pi_G^*(i_X B + d_X\gamma) = -(i_X B + d_X\gamma)$. (Q.E.D.)

LEMMA 4.3.- Equation (4.1) is equivalent to equation

$$(4.2)\quad i_Y B - d i_Y \gamma = 0, \quad \forall\, Y \in \mathcal{r}_G.$$

Proof.- Since X and Y commute, $i_X(d i_Y \gamma - i_Y B) = d_X i_Y \gamma - i_X i_Y B = i_Y(d_X \gamma + i_X B)$. (Q.E.D.)

Proof of Prop. 4.1.- Let $\bar{L}_G$ be completely integrable. Since the generating vector fields $\bar{X}$ are transverse to the fibres, integral manifolds of L_G are images of local sections $\gamma : U \to T^*G$ of π_G. They satisfy equation (4.1) because of Lemma 4.2. Conversely, if (4.1) is locally integrable, then from (4.2) it follows that $d i_Y B = 0$, $\forall\, Y \in \mathcal{r}_G$, i.e. the condition of complete integrability of $\bar{L}_G$ (Prop. 3.9). By applying the Lie derivative d_Y to equation (4.1) with $Y \in \mathcal{r}_G$, since $[X,Y] = 0$ we find:

$$(4.3)\quad i_X d_Y B + d_X d_Y \gamma = 0.$$

If $dB = 0$ and $d_Y B = 0$ for each $Y \in \mathcal{r}_G$, then $d i_Y B = 0$ and $\bar{L}_G$ is completely integrable. From (4.3) it follows that $d_X d_Y \gamma = 0$. Conversely, if $\bar{L}_G$ is completely integrable and $d_X d_Y \gamma = 0$, from (4.3) we derive $i_X d_Y B = 0$, hence: $d_Y B = 0$. From integrability condition (3.9) we derive also $dB = 0$. (Q.E.D.)

PROPOSITION 4.4.- Let γ be a 1-form satisfying the 1-cocycle equation (2.1). Then equation

$$(4.4)\quad \langle X_1 \wedge X_2, B\rangle = i_{X_1} d_{X_2} \gamma, \quad \forall\, X_1, X_2 \in \ell_G,$$

defines a closed and right-invariant 2-form B satisfying equation (4.1).

Proof.- The right hand side of (4.4) is bi-linear in X_1 and X_2. Let us apply the Lie derivative d_Y, with $Y \in \mathcal{r}_G$, to both sides of (4.1). We

LEMMA 4.3.- Equation (4.1) is equivalent to equation

$$(4.2)\quad i_Y B - d i_Y \gamma = 0, \quad \forall Y \in \mathcal{r}_G.$$

Proof.- Since X and Y commute, $i_X(di_Y\gamma - i_YB) = d_Xi_Y\gamma - i_Xi_YB = i_Y(d_X\gamma + i_XB)$. (Q.E.D.)

Proof of Prop. 4.1.- Let $\bar{L}_G$ be completely integrable. Since the generating vector fields $\bar{X}$ are transverse to the fibres, integral manifolds of L_G are images of local sections $\gamma:U \to T^*G$ of π_G. They satisfy equation (4.1) because of Lemma 4.2. Conversely, if (4.1) is locally integrable, then from (4.2) it follows that $di_YB = 0$, $\forall Y \in \mathcal{r}_G$, i.e. the condition of complete integrability of $\bar{L}_G$ (Prop. 3.9). By applying the Lie derivative d_Y to equation (4.1) with $Y \in \mathcal{r}_G$, since $[X,Y] = 0$ we find:

$$(4.3)\quad i_X d_Y B + d_X d_Y \gamma = 0.$$

If $dB = 0$ and $d_YB = 0$ for each $Y \in \mathcal{r}_G$, then $di_YB = 0$ and $\bar{L}_G$ is completely integrable. From (4.3) it follows that $d_Xd_Y\gamma = 0$. Conversely, if $\bar{L}_G$ is completely integrable and $d_Xd_Y\gamma = 0$, from (4.3) we derive $i_Xd_YB = 0$, hence: $d_YB = 0$. From integrability condition (3.9) we derive also $dB = 0$. (Q.E.D.)

PROPOSITION 4.4.- Let γ be a 1-form satisfying the 1-cocycle equation (2.1). Then equation

$$(4.4)\quad \langle X_1 \wedge X_2, B\rangle = i_{X_1} d_{X_2}\gamma, \quad \forall X_1, X_2 \in \ell_G,$$

defines a closed and right-invariant 2-form B satisfying equation (4.1).

Proof.- The right hand side of (4.4) is bi-linear in X_1 and X_2. Let us apply the Lie derivative d_Y, with $Y \in \mathcal{r}_G$, to both sides of (4.1). We obtain: $d_Y i_{X_1} d_{X_2}\gamma = i_{X_1} d_{X_2} d_Y \gamma = 0$, because of (2.1). Hence: $\langle X_1 \wedge X_2, B\rangle = -i_{X_1} i_{X_2} B$ = const., and B is right-invariant 2-form on G satisfying equation $i_{X_1}(i_{X_2} B + d_{X_2}\gamma) = 0$. It follows that B satisfies also equation (4.3). B is closed because of the last part of Prop. 4.1. (Q.E.D.)

REMARKS.

(a) A leaf (maximal connected integral manifold) of the integrable distribution $\bar{L}_G$ associated with a 2-form B represents a global solution of equation (4.1). In general a leaf Γ form a covering of G with respect to the projection π_G restricted to Γ. It follows that if G is connected and simply conected then global solutions are 1-forms on G. The set of global solutions is invariant under addition of left-invariant 1-forms and the action $\hat{\rho}$. This means that if Γ is a leaf, then the sets

(4.5) $\Gamma + \mu = \{k \in T^*G;\ k = k' + \mu \circ \pi_G(k'),\ k' \in \Gamma\}$

and $\hat{\rho}_g(\Gamma)$ are also leaves for each $\mu \in r^*_G$ and $g \in G$. It follows that there is a unique global solution Γ which contains the zero covector $0 \in T^*_e G$ at the identity e of the group, i.e. a unique normal solution (see Section 1).

(b) Closed and right-invariant 2-forms on G are 2-cocycles on ℓ_G with respect to the trivial representation of G on ℓ_G. Let us denote by $[B]_\ell$ the cohomology class determined by B: it is the space of 2-forms $B' = B + dA$ where $A \in \ell^*_G$. If Γ is a global solution of (4.1) where B is a 2-cocycle, then Γ is a global solution of the 1-cocycle equation (2.1). A global solution corresponding to $B' = B + dA$ is given by $\Gamma - A$ (see definition (4.5)). The set of all global solutions corresponding to the class $[B]_\ell$ is then given by the class $[\Gamma]$ of the global solutions of the type $\Gamma - A + \mu$, where $\mu \in r^*_G$ and $A \in \ell^*_G$. Conversely, if Γ is a global solution of the 1-cocylce equation (2.1), then we can define a 2-cocycle B through formula (4.4), by any local solution γ representing Γ, for instance in the neighborhood of the identity e of G.

(c) From the discussion above it follows that if (2.1) has only global solutions which are 1-forms, then there is an isomorphism between cohomology classes $[B]_{\ell}$ and $[\gamma]$.

(d) If G is semi-simple, then the 1-cocycle equation (2.1) has only trivial solutions. Indeed, for each 2-cocycle B we have B = dA with $A \in \ell^*_G$ (Whitehead Lemma); as a consequence, $\gamma' = -A$ is a global solution of (4.1). If $\gamma : U \rightarrow T^*G$ is a local solution of (2.1) and B is the 2-cocycle constructed by means of (4.4), then $\gamma - \gamma'|U = \gamma + A|U$ must be a left-invariant form μ restricted to U. Hence $\gamma = \mu|U - A|U$ and γ is trivial.

(e) From equation (4.2) it follows that if γ is a local solution corresponding to B, then the function

$$\bar{E}_Y = E_Y - \pi^*_G i_Y \gamma$$

is a local Hamiltonian of Y with respect to the symplectic form $\bar{\omega}$. Hence the vector fields $\hat{Y}$ are globally Hamiltonian if equations (4.1) or (4.2) have global solutions which are 1-forms.

5.- The lift of actions by means of a 2-form.

If a foliation of T^*G is assigned, whose leaves are images of sections, then any action of G on itself can be lifted to an action on T^*G in a natural way. Each infinitesimal generator X of the action on G can be lifted to an infinitesimal generator $\bar{X}$ of the lifted action. The vector field $\bar{X}$ is uniquely defined by the following conditions: (i) it is tangent to the leaves of the foliation, (ii) it is π_G-projectable onto the vector field X. For example, the left and right translations λ and ϱ are canonically lifted to the actions $\hat{\lambda}$ and $\hat{\varrho}$ by means of the foliations determined by the left-invariant and right-invariant forms respectively.

Let us consider the case of the foliation on T^*G generated by a 2-cocycle B.

PROPOSITION 5.1.- Let B a 2-cocycle (closed and right-invariant 2-form on G). Let us assume that all global solutions of equation (4.1) are 1-forms and let $\theta : G \to \ell^*_G$ be the mapping defined by (2.2) where γ is a solution of (4.1). Then the vector fields $\{\bar{X};\ X \in \ell_G\}$ defined by (3.1) are the infinitesimal generators of the action $\bar{\lambda} : G \times T^*G \to T^*G$ defined by

$$(5.1) \qquad \bar{\lambda}_g(k') = \hat{\lambda}_g(k') + \theta(g)(gg') \qquad (g' = \pi_G(k'))$$

for each $g \in G$ and $k' \in T^*G$.

Proof.- Let $\bar{\lambda}$ be the lifted action on T^*G of the left translation λ by means of the foliation $\bar{L}_G$ spanned by the vector fields $\bar{X}$. According to the remarks above, the vector fields $\bar{X}$ are in fact infinitesimal generators of $\bar{\lambda}$. Because of Prop.4.1 and by definition of $\bar{\lambda}$ we have:

$$\bar{\lambda}_g(k') = \gamma(gg'),$$

where γ is the solution of equation (4.1) such that $k' = \gamma(g')$. On the other hand (see identity (1.3) and definition (2.2)):

$$\begin{aligned}\hat{\lambda}_g(k') + \theta(g)(gg') &= \hat{\lambda}_g(\gamma(g')) + \theta(g)(gg') \\ &= (\lambda_g^{*-1}\gamma + \theta(g))(gg') \\ &= \gamma(gg'),\end{aligned}$$

and (5.1) follows. (Q.E.D.)

REMARK 5.2.- From Remarks 3.5, 3.7 and 3.8 it follows that the actions $\bar{\lambda}$ and $\hat{\rho}$ are symplectic on $(T^*G, \bar{\omega})$ and commute.

For the pair of actions $(\bar{\lambda}, \hat{\rho})$ properties analogous to those considered in [3] for $(\hat{\lambda}, \hat{\rho})$ hold, with respect to the symplectic structure $\bar{\omega} = d\,\theta_G + \pi_G^*B$ on T^*G. In particular, the orbits of the composed action $(G \times G) \times T^*G \to T^*G:(g_1,g_2,k) \mapsto \bar{\lambda}_{g_1}\hat{\rho}_{g_2}(k)$ form a (generalized) coisotropic foliations of T^*G and the corresponding reduced symplectic manifolds can be identified with the orbits of the affine action on ℓ_G^* corresponding to the 1-cocycle θ [7]. We do not deal with this topic here for the sake of brevity. We mention that, in a different approach, actions $\bar{\lambda}$ and $\bar{\rho}$ have been already considered in [4,5].

6.- Central extensions.

Let us consider a central extension of the Lie group G by the group R, i.e. an exact sequence of homomorphisms of Lie groups:

$$1 \to R \xrightarrow{\varepsilon} F \xrightarrow{\eta} G \to 1.$$

The homomorphism $\eta : F \to G$ is a principal fibre bundle with structural group R whose action on F is defined by $R \times F \to F : (r,f) \mapsto \varepsilon(r)f$. Let V be corresponding infinitesimal generator (the fundamental vector field). This vector field belongs to the center of both Lie algebras ℓ_F and r_F: $[V,Z] = 0$, for each $Z \in \ell_F \cup r_F$.

Each infinitesimal generator $Z \in \ell_F$ (resp. $Z \in r_F$) is η-projectable onto an infinitesimal generator of ℓ_G (resp. of r_G).

A <u>connection</u> of η is a 1-form $\alpha \in \Phi_1(F)$ such that $i_V\alpha = 1$ and $d_V\alpha = 0$. Since $i_V d\alpha = 0$ and $d_V d\alpha = 0$, there exists a unique 2-form $B \in \Phi_2(G)$ such that $d\alpha = \eta^* B$. The closed 2-form B is the <u>curvature</u> of α.

With a subspace $\mathfrak{s}$ of ℓ_F complementar to V we associate a r_F-invariant connection α defined by equations $i_V\alpha = 1$ and $i_Z\alpha = 0$ for each $Z \in \mathfrak{s}$. (r_F-invariant means right-invariant with reference to the group F, etc..) The corresponding curvature B is r_G-invariant. If α' is another r_F-invariant connection and B' is the corresponding curvature, then $\alpha' - \alpha = \eta^* A$, where A is a r_G-invariant 1-form and $B' - B = dA$. Hence, with a central extension of G by R we associate a distinguished cohomology class [B].

The submanifold C of T*F defined by

$$C = \{h \in T^*F;\ \langle V,h \rangle = 1\}$$

is coisotropic (because it is of codimension 1). A surjective submersion

$\varkappa : C \to T^*G$ is defined by

(6.1) $\langle v, \varkappa(h) \rangle = \langle w, h \rangle$,

where $h \in C$, $v \in T_g G$, $g = \eta(f)$, $f = \pi_F(h)$, and w is the horizontal lift of v, i.e. the vector defined by equations: $\langle w, \alpha \rangle = 0$, $T\eta(w) = v$. The fibres of $\varkappa$ coincide with the orbits of the canonical lift $\hat{\eta}$ of the action η [6]. These orbits are characteristics of C (i.e. maximal connected integral submanifolds of the characteristic distribution). Moreover, $\varkappa^*(d\theta_G + \pi_G^* B)$ is equal to the pull-back $d\theta_F | C$ of the symplectic form $d\theta_F$ to the submanifold C. This means that $\varkappa$ defines a symplectic reduction from $(T^*F, d\theta_F)$ to the symplectic manifold $(T^*G, \bar{\omega})$ where $\bar{\omega} = d\theta_G + \pi_G^* B$.

PROPOSITION 6.1.- Let $\bar{\gamma}$ be a r_F-invariant or a ℓ_F-invariant 1-form such that $i_V \bar{\gamma} = 1$. There is a unique 1-form γ on G such that

(6.2) $\bar{\gamma} - \alpha = \eta^* \gamma$,

(6.3) $\varkappa \circ \bar{\gamma}(F) = \gamma(G)$.

If $\bar{\gamma}$ is r_F-invariant, then γ is a r_G-invariant 1-form. If $\bar{\gamma}$ is ℓ_F-invariant, then γ is a 1-form representing the 1- cocycle associated with the curvature B, i.e. satisfying equation (4.1).

Proof.- We note that for a r_F-invariant 1-form $\bar{\gamma}$ we have $i_Z \bar{\gamma}$ = const. for each $Z \in \ell_F$. In particular $i_V \bar{\gamma}$ = const., so that hypothesis $i_V \bar{\gamma} = 1$ is formulated correctly. Since $\bar{\gamma}$ id r_F-invariant we have $d_Z \bar{\gamma} = 0$ for each $Z \in r_F$. In particular $d_V \bar{\gamma} = 0$ ($\bar{\gamma}$ is a connection of the principal fiber bundle η). It follows that $i_V(\bar{\gamma} - \alpha) = 0$ and $d_V(\bar{\gamma} - \alpha) = 0$, hence that there exists a 1-form γ on G such that $\bar{\gamma} - \alpha = \eta^* \gamma$. The same reasoning holds when $\bar{\gamma}$ is ℓ_F-invariant. From the definition (6.1) of $\varkappa$ it follows that $\langle v, \varkappa(\bar{\gamma}(f))\rangle = \langle w, \bar{\gamma}(f)\rangle = \langle w, \alpha(f) + \eta^*\gamma(f)\rangle = \langle v, \gamma\rangle$, since $\langle w, \alpha\rangle = 0$. This proves (6.3). If $\bar{\gamma}$ is τ -invariant, then by applying to (6.2) the Lie derivative d_Z with respect to a vector field $Z \in \tau_F$ we obtain $0 = d_Z \eta^*\gamma = \eta^* d_X \gamma$, where X is the element of τ_G onto which Z projects. It follows that γ is τ_G-invariant. If $\bar{\gamma}$ is ℓ_F-invariant, then the same operation with $Z \in \ell_F$ yields the equation $- d_Z\alpha = \eta^* d_X\gamma$, where $X \in \ell_G$. Since $i_Z\alpha = 1$, $d_Z\alpha = i_Z d\alpha = i_Z \eta^* B = \eta^* i_X B$. Hence, we find $- i_X B = d_X\gamma$ for each $X \in \ell_G$. (Q.E.D.)

CONCLUSION.- A τ_F-invariant connection α associated with the central extension defines a symplectic reduction from $(T^*F, d\,\theta_F)$ to $(T^*G, \bar{\omega})$, where $\bar{\omega}$ is the canonical symplectic form varied by the curvature B of α. We can reduce any connection 1-form $\bar{\gamma}$ on F to a 1-form γ on G. The reduction of the space ℓ^*_F of the τ_F-invariant 1-forms is just the space ℓ^*_G of the τ_G-invariant 1-forms, while the reduction of the space τ^*_F of the ℓ_F-invariant 1-forms is the space of solutions of equation (4.1), i.e. the space of 1-forms representing the 1-cocycles associated with the 2-cocycle B.

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