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Le Colloque de Balaruc de Géométrie Symplectique et Mécanique a été organisé par le groupe de géométrie et topologie différentielle de Montpellier dans le cadre du "Séminaire sud-rhodanien de géométrie", qui regroupe les équipes de géométrie des universités d'Avignon, Lyon, Marseille et Montpellier. Il a pris place parmi les Journées de la Société Mathématique de France, à la suite de la rencontre de Lyon de juin 1983.

Le Département de Mathématiques et la Présidence de l'Université des Sciences et Techniques du Languedoc ont participé à son organisation et le Conseil général de l'Hérault lui a apporté son aide.

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1. Introduction

Time dependent Hamiltonian dynamics finds an elegant interpretation as a homogeneous system determined by a coisotropic submanifold of the cotangent bundle of the space-time manifold. Homogeneous formulation of dynamics is advantageous also in the time independent case since it leads to a geometric interpretation of the Hamilton-Jacobi method [2] [5]. A time independent Hamiltonian system provides the simplest example of a Hamiltonian group action [6]. The group in this case is the group R of real numbers. In order to obtain a geometric framework for Hamiltonian actions of more general Lie groups we generalize the homogeneous formulation of dynamics replacing the time axis by a differential manifold T. We then study coisotropic submanifolds of $T^*(T) \times P$, where P is a symplectic manifold replacing the phase manifold of a Hamiltonian system. Choosing a coisotropic submanifold $M \subset T^*(T) \times P$ which satisfies certain conditions we obtain a generalization of the homogeneous formulation of time dependent Hamiltonian dynamics. Next we assume an action of a Lie group G in the manifold T and obtain a generalization of time independent dynamics postulating certain invariance properties of M with respect to the action of G. At the same time we obtain a Hamiltonian action of G in P represented as a homogeneous system. The manifold M closely related to the momentum mapping [8] is called the <u>momentum relation</u> for the Hamiltonian action.

2. Notation

Differential manifolds are finite dimensional, real, of class C^∞. Differentiable means of class C^∞. Mappings between manifolds are assumed differentiable. A transformation means differentiable automorphism. We denote by:

$\tau_Q : T(Q) \to Q$	the tangent bundle projection of a manifold Q;
$\pi_Q : T^*(Q) \to Q$	the cotangent bundle projection of Q;
$T_q(Q)$	the tangent space at a point $q \in Q$;
$T_q^*(Q)$	the cotangent space at a point $q \in Q$;
θ_Q	the Liouville 1-form on $T^*(Q)$;
$\mathfrak{X}(Q)$	the space of differentiable vector field on Q;
$\mathfrak{X}_H(P,\omega)$	the space of differentiable Hamiltonian vector fields on a symplectic manifold (P,ω);
$\Phi_k(Q)$	the space of k-forms on Q ($k = 0,1,2,\ldots.$);
$T\alpha : T(Q) \to T(Q')$	the tangent mapping of a mapping $\alpha : Q \to Q'$;
$\alpha^* : \Phi_k(Q') \to \Phi_k(Q)$	the pull back by α;
$d : \Phi_k(Q) \to \Phi_{k+1}(Q)$	the exterior differential;
$id_Q : Q \to Q$	the identity mapping;
$\langle , \rangle$	the evaluation between vectors (or vector fields) and covectors (or forms);
i_v	the interior product by a vector v (or vector field);
$[,]$	Lie brackets of vector fields;
$\{ , \}$	Poisson brackets of functions;
$\hat{\xi} : T^*(Q) \to T^*(Q)$	the canonical lift of a transformation ξ of Q;
$\hat{X} : T^*(Q) \to T(T^*(Q))$	the canonical lift of a vector field X on Q;
$\S$	the symplectic polar operator acting on subspaces of symplectic spaces.

3. The submanifold M

Let T be a connected differential manifold, (P,ω) a connected symplectic manifold, and $\tilde{P} = T^*(T) \times P$. Let $pr_T : T \times P \to T$, $pr_P : T \times P \to P$, $pr_1 : \tilde{P} \to T^*(T)$, $pr_2 : \tilde{P} \to P$ be the canonical projections and $\varepsilon = \pi_T \times id_P : \tilde{P} \to T \times P$. Let $\omega_T = d\,\theta_T$ be the canonical symplectic form on $T^*(T)$ and $\tilde{\omega} = pr_1^*(\omega_T) + pr_2^*(\omega)$. Then $(\tilde{P}, \tilde{\omega})$ is a symplectic manifold.

Let M be a submanifold of $\tilde{P}$. We assume that:

(A.1) M is the image of a differentiable section $\mu : T \times P \rightarrow \tilde{P}$ of the projection $\varepsilon = \pi_T \times id_P$.

Let $H : T \times P \rightarrow T^*(T)$ be the mapping defined by $H = -pr_1 \circ \mu$. Then

$$M = \{(h,p) \in \tilde{P};\ h = -H(\pi_T(h),p)\}.$$

Conversely, a function $H : T \times P \rightarrow T^*(T)$ such that $\pi_T \circ H = pr_T$ generates a submanifold $M \subset \tilde{P}$ satisfying (A.1).

For each $t \in T$ and $p \in P$ we introduce the mappings

$$H_t : P \rightarrow T_t^*(T) : p \mapsto H(t,p);$$
$$H_p : T \rightarrow T^*(T) : t \mapsto H(t,p).$$

H_p is a section of π_T, i.e. a 1-form on T. We can interpreted H as a P-dependent family of 1-forms on T.

The mapping H can be identified with the function

$$H : T(T) \times P \rightarrow R : (u,p) \mapsto H(u,p) = \langle u, H(\tau_T(u),p)\rangle$$

which is linear on the fibers of T(T). Hence,

$$M = \{(h,p) \in \tilde{P};\ \langle u,h\rangle = -H(u,p),\quad \forall\, u \in T_t(T),\ t = \pi_T(h)\}.$$

From this point of view H can be interpreted as a 1-form on T with values in the space $\Phi_0(P)$. We denote by $\underset{\sim}{d}H$ the corresponding differential, which is a 2-form on T with values in $\Phi_0(P)$. For each $u \in T(T)$ and $X \in \mathfrak{X}(T)$ we introduce the functions

$$H_u = \langle u,H\rangle : P \rightarrow R : p \rightarrow H(u,p),$$
$$H_X = \langle X,H\rangle : T \times P \rightarrow R : (t,p) \rightarrow \langle X(t),H\rangle.$$

H_X can be interpreted as a 0-form on T with values in $\Phi_0(P)$. We denote by $\underset{\sim}{d}H_X$ the corresponding differential.

For each $X \in \mathfrak{X}(T)$ we introduce the functions

$$E_X : T^*(T) \to R : h \mapsto \langle X,h\rangle;$$
$$\tilde{H}_X = pr_1^*(E_X) + \varepsilon^*(H_X) : \tilde{P} \to R.$$

We remark that E_X is the Hamiltonian of the canonical lift $\hat{X} \in \mathfrak{X}(T^*(T))$ of X and that $H_X = E_X \circ H = H^*(E_X)$.

The calculation: $\mu^*(\tilde{H}_X) = \mu^* \circ pr_1^*(E_X) + \mu^* \circ \varepsilon^*(H_X) = (pr_1 \circ \mu)^*(E_X) + (\varepsilon \circ \mu)^*(H_X) = - H^*(E_X) + H_X = 0$, shows that

$$M = \{(h,p) \in \tilde{P};\ \tilde{H}_X(h,p) = 0,\ \forall\ X \in \mathfrak{X}(T)\}.$$

The mapping $\tilde{H}: \mathfrak{X}(T) \to \Phi_0(\tilde{P}) : X \mapsto \tilde{H}_X$ is a linear monomorphism and provides a further description of M.

4. The symplectic polar of T(M)

The symplectic polar $T^{\S}(M)$ of $T(M)$ is defined by

$$T^{\S}(M) = \{(v,w) \in T(\tilde{P});\ (h,p) = (\tau_{T^*(T)}(v), \tau_P(w)) \in M,$$
$$\langle (v,w) \wedge (v',w'), \tilde{\omega}\rangle = 0,\ \ \forall (v',w') \in T_{(h,p)}(M)\}.$$

Since

$$T_{(h,p)}(M) = \{(v',w') \in T_{(h,p)}(\tilde{P});\ (v',w') = T\mu(u',w'),$$
$$w' \in T_p(P),\ u' \in T_t(T),\ t = \pi_T(h)\}$$
$$= \{(v',w') \in T_{(h,p)}(\tilde{P});\ v' = TH(u',w'),$$
$$w' \in T_p(P),\ u' \in T_t(T),\ t = \pi_T(h)\}$$
$$= \{(v',w') \in T_{(h,p)}(\tilde{P});\ v' = TH_p(u') + TH_t(w'),$$
$$w' \in T_p(P),\ u' \in T_t(T),\ t = \pi_T(h)\},$$

we have:

$$T_{(h,p)}(M) = \{(v,w) \in T_{(h,p)}(\tilde{P});\ \langle v \wedge (TH_p(u') + TH_t(w')), \omega_T\rangle +$$
$$+ \langle w \wedge w', \omega\rangle = 0,\ \forall\ w' \in T_p(P),\ u' \in T_t(T),\ t = \pi_T(h)\},$$

The choice $w' = 0$ implies $\langle v \wedge TH_p(u'), \omega_T\rangle = 0$, $\forall\ u' \in T_t(T)$, i.e. $v \in (TH_p(T_t(T))^{\S}$; hence, $\langle v \wedge TH_t(w'), \omega_T\rangle + \langle w \wedge w', \omega\rangle = \langle TH_t(w'), i_v\omega_T\rangle + \langle w', i_w\omega\rangle = 0$. It follows that:

$$T^{\S}(M) = \{(v,w) \in T(\tilde{P});\ h = -H(t,p),\ h = \tau_{T^*(T)}(v),\ t = \pi_T(h),$$
$$p = \tau_P(w),\ v \in (TH_p(T_t(T)))^{\S},\ i_w\omega + (T_pH_t)^*(i_v\,\omega_T) = 0\}.$$

Here we denote by $(T_pH_t)^*$ the linear dual mapping of $T_pH_t = TH_t|T_p(P)$.

Remarks. (i) From the fact that the fibers of $T^*(T)$ are Lagrangian it follows that the restriction of $T\pi_T$ to the subspace $(TH_p(T_t(T))^{\S} \subset T_h(T^*(T))$ is an isomorphism. (ii) If $(v,w) \in T^{\S}_{(h,p)}(M)$ then w is uniquely determined by v and by p. From (i) and (ii) it follows that: (iii) $T^{\S}(M)$ is the image of a section $\sigma: T(T) \times P \to T(\tilde{P})$ of the projection $T\pi_T \times \tau_P$. This section is clearly differentiable. (iv) The restriction of the projection $T\varepsilon$ to the space $T^{\S}_{(h,p)}(M)$ is a linear monomorphism into $T_{t,p}(T \times P)$, where $t = \pi_T(h)$. (v) The restriction of Tpr_T to the space $T\varepsilon\,(T^{\S}_{(h,p)}(M))$ is an isomorphism onto $T_t(T)$.

5. The coisotropy of M

In order to characaterize the coisotropy of M ($T^{\S}(M) \subset T(M)$), we use the natural Poisson structures on $T^*(T)$, P, $T \times P$ and $\tilde{P}$ induced by ω_T, ω and $\tilde{\omega}$. We use the same symbol $\{,\}$ for all Poisson brackets.

Proposition 5.1. M is a coisotropic submanifold of (P,ω) if and only if one of the following equations holds:

(a) $\{H_u,H_v\} + \langle u \wedge v, \underset{\sim}{d}H\rangle = 0, \quad \forall\ u,v \in T(T):\ \tau_T(u) = \tau_T(v).$

(b) $\{H_X,H_Y\} = H_{[X,Y]} - \langle X,\underset{\sim}{d}H_Y\rangle + \langle Y,\underset{\sim}{d}H_X\rangle, \ \forall\ X,Y \in \mathfrak{X}(T).$

(c) $\{\tilde{H}_X,\tilde{H}_Y\} = \tilde{H}_{[X,Y]}, \ \forall\ X,Y \in \mathfrak{X}(T).$

Proof. Projections $pr_1: \tilde{P} \to T^*(T)$ and $\varepsilon = \pi_T \times id_P: \tilde{P} \to T \times P$ are Poisson mappings. If Z_A is the Hamiltonian vector field on $(T^*(T), \omega_T)$ generated by the function $A: T^*(T) \to \mathbf{R}$, then the vector field $Z_A \times 0$ on $\tilde{P}$ is the Hamiltonian vector field generated by $pr_1^*(A) = A \circ pr_1$. It follows that $\{\varepsilon^*(B), pr_1^*(A)\} = \langle Z_A \times 0, \varepsilon^*(dB)\rangle$ for each function $B: T \times P \to \mathbf{R}$. In particular we have: $\{\varepsilon^*(H_X), pr_1^*(E_Y)\} = \langle \hat{Y} \times 0, \varepsilon^*(dH_X)\rangle = \varepsilon^*\langle Y\ \ 0, dH_X\rangle = \varepsilon^*\langle Y, \underset{\sim}{d}H_X\rangle$, where $X,Y \in \mathfrak{X}(T)$. Hence,

$$\begin{aligned}\{\tilde{H}_X,\tilde{H}_Y\} &= \{pr_1^*(E_X) + \varepsilon^*(H_X),\ pr_1^*(E_Y) + \varepsilon^*(H_Y)\} \\ &= \{pr_1^*(E_X), pr_1^*(E_Y)\} + \{\varepsilon^*(H_X), \varepsilon^*(H_Y)\} + \\ &\quad + \{\varepsilon^*(H_X), pr_1^*(E_Y)\} - \{\varepsilon^*(H_Y), pr_1^*(E_X)\}\end{aligned}$$

$$= pr_1^*\{E_X,E_Y\} + \varepsilon^*\{H_X,H_Y\} + \varepsilon^*(\langle Y,\underset{\sim}{d}H_X\rangle - \langle X,\underset{\sim}{d}H_Y\rangle)$$
$$= pr_1^*(E_{[X,Y]}) + \varepsilon^*(H_{[X,Y]}) + \varepsilon^*(\{H_X,H_Y\} - H_{[X,Y]} + \langle Y,\underset{\sim}{d}H_X\rangle - \langle X,\underset{\sim}{d}H_Y\rangle),$$

i.e.

(d) $\{\widetilde{H}_X,\widetilde{H}_Y\} = \widetilde{H}_{[X,Y]} + \varepsilon^*(\{H_X,H_Y\} - H_{[X,Y]} + \langle Y,\underset{\sim}{d}H_X\rangle - \langle X,\underset{\sim}{d}H_Y\rangle)$.

It follows that

(e) $\mu^*\{H_X,H_Y\} = \{H_X,H_Y\} - H_{[X,Y]} + \langle Y,\underset{\sim}{d}H_X\rangle - \langle X,\underset{\sim}{d}H_Y\rangle$.

Since M is characterized by the equations $\mu^*(H_X) = 0$, $\forall X \in \mathfrak{X}(T)$, M is coisotropic if and only if $\mu^*\{H_X,H_Y\} = 0$, $\forall X,Y \in \mathfrak{X}(T)$. Because of (e) and (d), this condition is equivalent to (b), thus also equivalent to (c). For a 1-form on T with values in any vector space the identity

(f) $\langle X \wedge Y,\underset{\sim}{d}H\rangle = \langle X,\underset{\sim}{d}\langle Y,H\rangle\rangle - \langle Y,\underset{\sim}{d}\langle X,H\rangle\rangle - \langle [X,Y],H\rangle$.

holds. Then (b) is equivalent to

$$\{H_X,H_Y\} + \langle X \wedge Y,\underset{\sim}{d}H\rangle = 0, \quad \forall X,Y \in \mathfrak{X}(T),$$

thus it is also equivalent to (a). (Q.E.D.)

6. A generalization of the homogeneous formulation of dynamics

Let us assume that

(A.2) M is a coisotropic submanifold of (P,ω).

The triple $(P,\omega;M)$ forms a <u>homogeneous</u> <u>system</u> [2] [5]. It is well known that the characteristic distribution $T^{\S}(M)$ of M, which we simply denote by D', is completely integrable [1] [8] [12]. We call <u>characteristic</u> a maximal connected integral manifold of D'. We introduce the relation $D = \{h,p,h',p') \in \widetilde{P} \times \widetilde{P}$; (h,p) and (h',p') belong to the same characteristic of M$\}$.

From the theory of homogeneous systems [2] [5] we know that:

(i) D' is an <u>infinitesimal</u> <u>symplectic</u> <u>relation</u> on $(\tilde{P},\tilde{\omega})$, i.e. a Lagrangian submanifold of the symplectic manifold $(T(\tilde{P}), d_T\tilde{\omega})$, where d_T is the derivation operator defined in [9].

(ii) D is a <u>symplectic</u> <u>relation</u> on $(\tilde{P},\tilde{\omega})$, i.e. a Lagrangian submanifold (may be immersed) of the symplectic manifold $(\tilde{P},\tilde{\omega}) \times (\tilde{P},-\tilde{\omega})$.

Let $\alpha : T(T^*(T)) \to T^*(T(T))$ be the diffeomorphism characterized by the conditions [10] [2] [3]: $\pi_{T(T)} \circ \alpha = T\,\pi_T$, $d_T\,\theta_T = \alpha^*(\theta_{T(T)})$. Let $\beta : T(P) \to T^*(P)$ be the vector bundle isomorphism define by the symplectic form ω: $\beta(w) = i_w\omega$. In the following discussion we identify the manifold $T(\tilde{P}) = T(T^*(T) \times P)$ with the manifold $T(T^*(T)) \times T(P)$ and the manifold $T^*(T(T) \times P)$ with the manifold $T^*(T(T)) \times T^*(P)$. The diffeomorphism $\iota : T^*(T(T)) \times T^*(P) \to T^*(T(T) \times P)$ is defined by the equation $\langle(a,w),(b,c)\rangle = \langle a,b\rangle + \langle w,c\rangle$, where $a \in T_u(T(T))$, $w \in T_p(P)$, $b \in T^*_u(T(T))$, $c \in T^*_p(P)$.

Proposition 6.1. The infinitesimal symplectic relation D' is generated by the function $H : T(T) \times P \to R$ with respect to the symplectomorphism $\alpha \times \beta : T(\tilde{P}) \to T^*(T(T) \times P)$, i.e.

$$\begin{aligned} D' &= (\alpha \times \beta)^{-1} \circ (-dH)(T(T) \times P) \\ &= \{(v,w) \in T(\tilde{P});\ v = -\ \alpha^{-1}(dH_p(u)),\ i_w\omega = -\ dH_u(p), \\ &\quad u = T\,\pi_T(v),\ p = \tau_P(w)\}. \end{aligned}$$

Proof. We know (Section 4) that D' is the image of a section of the projection $T\,\pi_T \times \tau_P$. We must prove that

$$(\alpha \times \beta) \circ \sigma = -\ dH.$$

Let $\beta' : T(T^*(T))) \to T^*(T^*(T))$ and $\tilde{\beta} : T(\tilde{P}) \to T^*(\tilde{P})$ be the vector bundle isomorphisms defined by the symplectic forms $d\,\theta_{T^*(T)}$ and $\tilde{\omega}$ respectively. We know that [2] [11]

$$\beta'^*(\theta_{T^*(T)}) - \alpha^*(\theta_{T(T)}) = dW,$$

where $W : T(T^*(T)) \to R : v \to \langle v, \theta_T\rangle$, and that the pull back to D' of the 1-form $\tilde{\theta} = \tilde{\beta}^*(\theta_{\tilde{P}})$ is zero. Then $\sigma^*(\tilde{\theta}) = 0$. Since $\tilde{\beta} = \beta' \times \beta$, we can write:

$$\tilde{\theta} = \pi_1^* \times \beta'^*(\theta_{T^*(T)}) + \pi_2^* \circ \beta^*(\theta_P),$$

where $\pi_1 : T(\tilde{P}) \to T(T^*(T))$ and $\pi_2 : T(\tilde{P}) \to T(P)$ are the canonical projections. Let us introduce the 1-form

$$\tilde{\lambda} = \pi_1^* \circ \alpha^*(\theta_{T(T)}) + \pi_2^* \circ \beta^*(\theta_P) = (\alpha \times \beta)^*(\theta_{T(T) \times P}).$$

We have $\tilde{\lambda} - \tilde{\theta} = d\tilde{W}$, where $\tilde{W} = \pi_1^*(W) : T(\tilde{P}) \to R : (v,w) \mapsto W(v)$. From $\sigma^*(\tilde{\theta}) = 0$ it follows that $\sigma^*(\tilde{\lambda}) = d(\sigma^*(\tilde{W}))$. For the left hand side of this equality we have, by definition of Liouville form, $\sigma^*(\tilde{\lambda}) = \sigma^* \circ (\alpha \times \beta)^*(\theta_{T(T) \times P}) = ((\alpha \times \beta) \circ \sigma)^*(\theta_{T(T) \times P}) = (\alpha \times \beta) \circ \sigma$. For the right hand side we observe that if $(v,w) = \sigma(u,p)$, then $\tilde{W}(v,w) = W(v) = \langle v, \theta_T \rangle = \langle u,h \rangle$, where $h = \tau_{T^*(T)}(v) = -H(\tau_T(u),p) = -H(u,p)$. This shows that $\sigma^*(\tilde{W}) = -H$. (Q.E.D.)

7. The reduction of D' by vectors tangent to T

From Remarks (iv) and (v) at the end of Section 4 it follows that: (i) The set $T\varepsilon(D') \subset T(T \times P)$ is a completely integrable distribution on $T \times P$ whose leaves (maximal connected integral manifolds) are the images by ε of the characteristics of M. (ii) For each $(u,p) \in T(T) \times P$ there is a unique vector $w \in T_p(P)$ such that $(u,w) \in T\varepsilon(D')$; this vector is defined by (Proposition 6.1): $i_w \omega = -dH_u(p)$. As a consequence we have:

Proposition 7.1. For each $u \in T(T)$ the set $D'_u = \{w \in T(P);\ (u,w) \in T\varepsilon(D')\}$ is the image of a Hamiltonian vector field $K_u : P \to T(P)$ generated by the function $H_u : P \to R : p \mapsto H(u,p)$, i.e.: $i_{K_u}\omega = -dH_u$.

This result has the following symplectic interpretation. D'_u is the image of D' by the symplectic reduction [3] determined by the coisotropic submanifold $C_u = \{(v,w) \in T(\tilde{P});\ T\pi_T(v) = u\} = (T\pi_T \times \tau_P)^{-1}(\{u\} \times P)$. Since D' and C_u are transverse [12], the reduced set D'_u is a Lagrangian submanifold of (P,ω) generated by the restriction of the generating function H of D' to the submanifold $\{u\} \times P$ [3], i.e. by the function H_u.

Let us introduce the differentiable mapping

$$K : T(T) \times P \to T(P) : (u,p) \mapsto K_u(p).$$

Since $K_{ru+sv} = rK_u + sK_v$, for each $r,s \in R$ and for each $u,v \in T(T)$ such that $\tau_T(u) = \tau_T(v)$, the mapping K can be interpreted as a 1-form on T with values on the space $\mathfrak{X}_H(P,\omega)$ of the Hamiltonian vector fields on (P,ω). We denote by $\underset{\sim}{d}K$ the corresponding differential and we define

$$K_u = \langle u,K\rangle : P \rightarrow T(P) : p \mapsto K_u(p).$$

For each vector field $X \in \mathfrak{X}(T)$ we introduce the mapping

$$K_X = \langle X,K\rangle : T \times P \rightarrow T(P) : (t,p) \mapsto K_{X(t)}(p),$$

which is a T-dependent Hamiltonian vector field on P. We denote by $\underset{\sim}{d}K_X$ the differential of the mapping K_X interpreted as a 0-form on T with values on $\mathfrak{X}_H(P,\omega)$. We also define the vector fields

$$\bar{K}_X \in \mathfrak{X}(T \times P), \qquad \hat{K}_X \in \mathfrak{X}(\tilde{P}), \qquad \tilde{K}_X \in \mathfrak{X}(\tilde{P})$$

by:

$$\bar{K}_X(t,p) = (X(t),K_X(t,p)),\ \hat{K}_X(h,p) = (\hat{X}(h),K_X(t,p)),\ i_{\tilde{K}_X}\tilde{\omega} = -\,d\tilde{H}_X.$$

Proposition 7.2. The following identities hold:

(a) $[K_u,K_v] + \langle u \wedge v,\underset{\sim}{d}K\rangle = 0, \quad \forall\, u,v \in T(T):\ \tau_T(u) = \tau_T(v);$

(b) $[K_X,K_Y] = K_{[X,Y]} - \langle X,\underset{\sim}{d}K_Y\rangle + \langle Y,\underset{\sim}{d}K_X\rangle, \quad \forall\, X,Y \in \mathfrak{X}(T);$

(c) $[\tilde{K}_X,\tilde{K}_Y] = \tilde{K}_{[X,Y]}, \quad \forall\, X,Y \in \mathfrak{X}(T).$

Proof. Since K is a 1-form on T with values in $\mathfrak{X}_H(P,\omega)$, $\beta \circ K$ is a 1-form with values in $\Phi_1(P)$ and $\underset{\sim}{d}(\beta \circ K) = \beta \circ \underset{\sim}{d}K$. If δH is the 1-form on T with values on $\Phi_1(P)$ defined by $\langle u, \delta H\rangle = dH_u$, then $\beta \circ K = -\,\delta H$ because of Proposition 7.1, and $\langle u \wedge v,\underset{\sim}{d}\,\delta H\rangle = d\langle u \wedge v,\underset{\sim}{d}H\rangle$. Hence, because of Proposition 5.1, (a), $\beta \circ [K_u,K_v] = -\,d\{H_u,H_v\} = d\langle u \wedge v,\underset{\sim}{d}H\rangle = \langle u \wedge v,\underset{\sim}{d}\,\delta H\rangle = -\,\beta \circ \langle u \wedge v,\underset{\sim}{d}K\rangle$, and identity (a) is proved. (a) is equivalent to

(d) $[K_X, K_Y] + \langle X \wedge Y, \underset{\sim}{d}K\rangle = 0, \quad \forall X, Y \in \mathfrak{X}(T)$.

Then equivalence between (a) and (b) follows from the identity (f) of Section 5 applied to K. (c) follows from the fact that $\widetilde{K}_X$ is the Hamiltonian vector field generated by $\widetilde{H}_X$ and from Proposition 5.1, (c). (Q.E.D.)

Proposition 7.3. The following properties hold:

(i) $\widetilde{K}_X$ is a characteristic vector field, i.e. $\widetilde{K}_X(M) \subset D'$.

(ii) $\widetilde{K}_X$ is ε-projectable on $\bar{K}_X$, i.e. $\bar{K}_X \circ \varepsilon = T\varepsilon \circ \widetilde{K}_X$.

(iii) $\hat{K}_X$ is ε-projectable on $\bar{K}_X$, i.e. $\bar{K}_X \circ \varepsilon = T\varepsilon \circ \hat{K}_X$.

(iv) The vector fields $\bar{K}_X$ span the distribution $T\varepsilon\,(D')$.

(v) $\hat{K}_X$ is tangent to $M \iff \hat{K}_X = \widetilde{K}_X \iff \underset{\sim}{d}H_X = 0$.

Proof. (iii) follows directly from the definition of $\hat{K}_X$. (i) is a consequence of the fact that the generating function $\widetilde{H}_X$ of $\widetilde{K}_X$ is constant (= 0) on M. (iv) follows from (i) and (ii). Let $(v,w) \in D'$, $u = T\pi_T(v)$, $t = \tau_T(u)$, $p = \tau_P(w)$. The calculation

$$\begin{aligned}
\langle (v,w) \wedge \widetilde{K}_X, \tilde{\omega}\rangle &= \langle (v,w), d\widetilde{H}_X\rangle \\
&= \langle (v,w), pr_1^*(dE_X) + \varepsilon^*(dH_X)\rangle \\
&= \langle v, dE_X\rangle + \langle (u,w), dH_X\rangle \\
&= \langle v \wedge \hat{X}, \omega_T\rangle + \langle w \wedge K_{X(t)}, \omega\rangle + \langle u, \underset{\sim}{d}H_X\rangle(p) \\
&= \langle (v,w) \wedge \hat{K}_X, \tilde{\omega}\rangle + \langle u, \underset{\sim}{d}H_X\rangle(p)
\end{aligned}$$

shows that $\widetilde{K}_X(h,p)$ differs from $\hat{K}_X(h,p)$ by a vector $(z,0)$ such that $T\pi_T(z) = 0$ and proves (ii) and (v). (Q.E.D.)

8. The reduction of D by pairs of points of T

From Remarks (iv) and (v) at the end of Section 4 it follows that the leaves of the distribution $T\varepsilon\,(D')$ intersect transversally the fibers of pr_T. Hence, each leaf of $T\varepsilon\,(D')$ is the union of images of local sections of pr_T. For a simpler discussion we postulate the following <u>completeness condition</u>:

(A.3) The leaves of the distribution $T\varepsilon\,(D')$ are images of global sections of the projection $pr_T : T \times P \to T$.

Proposition 8.1. For each $t_1, t_2 \in T$ the set $D_{t_2,t_1} = \{(p_2, p_1) \in P \times P;\ (t_1, p_1)$ and (t_2, p_2) belong to the same leaf of $T\varepsilon(D')\}$ is the graph of a symplectomorphism φ_{t_2,t_1} on (P, ω).

Proof. D_{t_2,t_1} is not empty because of (A.3). It can also be defined by $D_{t_2,t_1} = \{(p_2,p_1) \in P \times P;\ \exists\ h_1, h_2 \in T^*(T) : (h_2, p_2, h_1, p_1) \in D \cap C_{t_2,t_1}\}$ where $C_{t_2,t_1} = T^*_{t_2}(T) \times P \times T^*_{t_1}(T) \times P$ is a coisotropic submanifold of $(\tilde{P}, \tilde{\omega}) \times (\tilde{P}, -\tilde{\omega})$. We have $T^{\S}(C_{t_2,t_1}) = T^{\S}(T^*_{t_2}(T)) \times 0 \times T^{\S}(T^*_{t_1}(T)) \times 0 = T(T^*_{t_2}(T)) \times 0 \times T(T^*_{t_1}(T)) \times 0$ (here 0 denotes the "zero section" of $T(P)$) because the fibers of $T^*(T)$ are Lagrangian submanifolds. The reduced symplectic manifold of $(\tilde{P}, \tilde{\omega}) \times (\tilde{P}, -\tilde{\omega})$ by C_{t_2,t_1} is canonically symplectomorphic to $(P, \omega) \times (P, -\omega)$. D_{t_2,t_1} is the reduced set of D by C_{t_2,t_1}. From the implication: $(v,0) \in T(M)$, $T\pi_T(v) = 0 \Rightarrow v = 0$ (Section 4), and from $T(D) \subset T(M) \times T(M)$ it follows that $T^{\S}(C_{t_2,t_1}) \cap T(D) \subset T^{\S}(C_{t_2,t_1}) \cap (T(M) \times T(M)) = 0$. This shows that the coisotropic submanifold C_{t_2,t_1} and the Lagrangian submanifold D are transverse. It follows that D_{t_2,t_1} is a Lagrangian submanifold, i.e. a symplectic relation. Assumption (A.3) implies that D_{t_2,t_1} is the graph of a diffeomorphism, denoted by φ_{t_2,t_1}. (Q.E.D.)

We obtain a differentiable mapping

$$\varphi : T \times T \times P \to P : (t_1, t_2, p) \mapsto \varphi_{t_2,t_1}(p)$$

which can be interpreted as a $(T \times T)$-dependent family of symplectomorphisms on (P, ω) and whose infinitesimal counterpart is the mapping K introduced in Section 7.

Proposition 8.2. For each $t, t_1, t_2, t_3 \in T$ we have: $\varphi_{t,t} = id_P$, $\varphi_{t_3,t_2} \circ \varphi_{t_2,t_1} = \varphi_{t_3,t_1}$.

Proof. It is obvious that $D_{t,t}$ is the diagonal of $P \times P$. The composition rule $D_{t_3,t_2} \circ D_{t_2,t_1} = D_{t_3,t_1}$, follows directly from $D \circ D = D$ and the definition of D_{t_2,t_1}. (Q.E.D.)

Let $\gamma : R \to T$ be a curve on T. The lift of γ through the point (t_o, p_o) is defined by $\bar{\gamma} : R \to T \times P : r \mapsto (\gamma(r), \varphi_{\gamma(r),t_o}(p_o))$. From the definition of D_{t,t_o} and D'_u it follows that if $\dot{\bar{\gamma}} : R \to T(T \times P)$ is the tangent curve of $\bar{\gamma}$ and $\dot{\bar{\gamma}}(r) = (u,w)$, then $w \in D'_u$. As a consequence:

Proposition 8.3. Let $\gamma : R \to T$ be a differentiable curve on T and $\dot{\gamma} : R \to T(T)$ be the corresponding tangent curve. Let $K_\gamma : R \times P \to T(P)$ be the R-dependent vector field on P defined by $K_\gamma(r,p) = K_{\dot\gamma(r)}(p)$. Then the mapping $F:R \times R \times P \to P$ defined by $F(r_o, r, p) = \varphi_{\gamma(r),\gamma(r_o)}(p)$ is the flow of K_γ [1].

Proposition 8.4. If $\psi:R \times T \to T$ is the flow of a complete vector field X on T, then the mappings

$$\bar\varphi_\psi : R \times T \times P \to T \times P : (r,t,p) \mapsto (\psi(r,t), \varphi_{\psi(r,t),t}(p)),$$
$$\hat\varphi_\psi : R \times \tilde P \to \tilde P : (r,h,p) \mapsto (\hat\psi_r(h), \varphi_{\psi(r,t),t}(p)),$$

where $\hat\psi_r$ is the canonical lift of $\psi_r : T \to T : t \mapsto \psi(r,t)$, are the flows of the vectorfields $\bar K_X$ and $\hat K_X$ respectively.

Proposition 8.3 provides a direct method of constructing the symplectomorphisms φ_{t,t_o}, through the choice of curves γ and the integration of the corresponding vector fields K_γ.

9. The momentum relation

We call <u>momentum relation</u> a submanifold $M \subset T^*(T) \times P$ satisfying (A.1), (A.2) and (A.3). The corresponding mapping $J:P \to \Phi_1(T)$ defined by $J(p) = H_p$ is called the <u>momentum mapping</u>. With M we associate also the mappings K and φ described in Sections 7 and 8 respectively.

Let $\lambda:G \times T \to T$ a differentiable left action of a Lie group G on T. The momentum relation M transfers the action of G from T to $T \times P$ and to $\tilde P$. These actions are respectively defined by:

$$\bar\varphi_\lambda : G \times T \times P \to T \times P : (g,t,p) \mapsto (\lambda_g(t), \varphi_{\lambda(g,t),t}(p)),$$
$$\hat\varphi_\lambda : G \times \tilde P \to \tilde P : (g,h,p) \mapsto (\hat\lambda_g(h), \varphi_{\lambda(g,t),t}(p)),$$

where $\lambda_g : T \to T$ is the diffeomorphism defined by $\lambda_g(t) = \lambda(g,t)$ and $\hat\lambda_g$ is its canonical lift. These mappings are actions because of Proposition 8.2. Let $\ell \subset \mathfrak{X}(T)$ be the Lie algebra of generators of the action λ. As a direct generalization of Proposition 8.3 we have:

Proposition 9.1. If $X \in \ell$, then $\bar K_X$ is a generator of $\bar\varphi_\lambda$ and $\hat K_X$ is a generator of $\hat\varphi_\lambda$.

We study the case in which the momentum relation M induces also an action of G on the manifold P. For a simpler discussion we assume that

(A.4) The action λ is transitive: $\forall\, t_1, t_2 \in T,\ \exists\, g \in G:\ t_2 = \lambda(g, t_1)$.

We consider the following T-independence property:

(P.1) For each $g \in G$ and $t, t' \in T$: $\varphi_{\lambda(g,t),t} = \varphi_{\lambda(g,t'),t'}$.

If (P.1) holds, then a symplectic left action of G on (P, ω) is defined by:

(a) $\varphi_\lambda : G \times P \to P : (g,p) \mapsto \varphi_g(p) = \varphi_{\lambda(g,t),t}(p)$

for any choice of $t \in T$.

The action $\bar{\varphi}_\lambda$ is now the product of the two actions λ and φ_λ. For each $X \in \mathcal{l}$ the generator $\bar{K}_X$ must decompose in the product of X and a generator of φ_λ. Since $K_X(t,p) = (X(t), K_X(t,p))$, the vector $K_X(t,p)$ does not depend on $t \in T$; K_X is then a vector field on P. It follows that (see also Proposition 7.2):

Proposition 9.2. If (P.1) holds then: (i) For each $X \in \mathcal{l}$ the vector field K_X is a generator of the symplectic action φ_λ on (P, ω); (ii) $[K_{X_1}, K_{X_2}] = K_{[X_1, X_2]}$, for each $X_1, X_2 \in \mathcal{l}$.

Proposition 9.3. Property (P.1) is equivalent to: (i) For each $X \in \mathcal{l}$, $p \in P$ and $t, t' \in T$: $K_X(t,p) = K_X(t',p)$; i.e. to: (ii) For each $X \in \mathcal{l}$, $\underset{\sim}{d}K_X = 0$.

Let us consider the following strong T-independence property:

(P.2) For each $X \in \mathcal{l}$, $p \in P$ and $t, t' \in T$: $H_X(t,p) = H_X(t',p)$; i.e.: for each $X \in \mathcal{l}$, $\underset{\sim}{d}H_X = 0$.

Since $\underset{\sim}{d}H_X = 0$ implies $\underset{\sim}{d}K_X = 0$, (P.2) implies (P.1). The inverse implication is not true, for instance, in the case of a time independent Hamiltonian vector field (case $T = \mathbb{R}$), which can be generated by a time dependent Hamiltonian: one can add to the time independent Hamiltonian any function of time.

Proposition 9.4. The following four properties are equivalent:

(i) (P.2).

(ii) M is invariant under the action $\hat{\varphi}_\lambda$.

(iii) For each $X \in \mathcal{l}$, $\hat{K}_X$ is tangent to M.

(iv) (P.1) and for each $g \in G$, $J \circ \varphi_g = \lambda_g^{*-1} \circ J$, i.e. the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\ \varphi_g\ } & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \Phi_1(T) & \xrightarrow{\ \lambda_g^{*-1}\ } & \Phi_1(T) \end{array}$$

Proof. Statements (ii) and (iii) are obviously equivalent; (iii) and (P.2) are equivalent because of Proposition 7.2, part (v). If (P.1) holds, then the action φ_λ is well defined and $\hat{\varphi}_\lambda$ is the product of the canonical lift of the action λ and the action φ_λ itself. The second part of (iv) is then equivalent to: $H(\lambda_g(t), \varphi_g(p)) = (\lambda_g^{*-1}(H_p))(\lambda_g(t)) = \hat{\lambda}_g(H(t,p))$, where $g \in G$, $t \in T$, $p \in P$, thus it is equivalent to the invariance of M under $\hat{\varphi}_\lambda$. Hence, (iv) implies (ii) and conversely, (i) implies (iv) since (P.2) implies (P.1). (Q.E.D.)

Proposition 9.5. If (P.2) holds, then: (i) $\hat{\varphi}_\lambda$ is a symplectic action on $(\tilde{P}, \tilde{\omega})$; (ii) for each $X_1, X_2 \in \mathcal{L}$, $\{H_{X_1}, H_{X_2}\} = H_{[X_1, X_2]}$.

Proof. It is a consequence of Propositions 7.2 , 5.1 (b), and 9.4.

10. The case of a free action

A interesting characterization of property (P.2) can be give if we make the following further assumption:

(A.5) The action λ is free: $\lambda(g,t) = t \Rightarrow g = e$ = the identity of G.

Proposition 10.1. The set of transformations of T which commute with λ is a Lie group R isomorphic to G. The action of this group is transitive and free.

Proof. Since λ is transitive and free a differentiable mapping $\gamma: T \times T \to G$ is defined by: $g = \gamma(t', t)$ if $t' = \lambda(g,t)$. We have:

$$\gamma(t,t) = e \quad , \quad \gamma(t_3,t_2)\,\gamma(t_2,t_1) = \gamma(t_3,t_1).$$

Hence,

$$(\gamma(t',t))^{-1} = \gamma(t,t'),$$
$$g\,\gamma(t',t) = \gamma(\lambda(g,t'),t), \quad \gamma(t',t)g^{-1} = \gamma(t',\lambda(g,t)).$$

For each pair $(t_1,t_2) \in T \times T$ we take the differentiable mapping $\rho_{t_2,t_1}:T \to T$ defined by:

$$\rho_{t_2,t_1}(t) = \lambda(\gamma(t,t_1),t_2).$$

From the properties of γ we derive:

$$\rho_{t,t} = id_T, \quad \rho_{t_3,t_2} \circ \rho_{t_2,t_1} = \rho_{t_3,t_1},$$
$$(\rho_{t_2,t_1})^{-1} = \rho_{t_1,t_2},$$
$$\rho_{t_2,t_1}(t_1) = t_2,$$
$$\rho_{t_2,t_1}(t) = t \implies t_1 = t_2,$$
$$\lambda_g \circ \rho_{t_2,t_1} = \rho_{t_2,t_1} \circ \lambda_g,$$
$$\rho_{\lambda(g,t_2),\lambda(g,t_1)} = \rho_{t_2,t_1}.$$

This shows that the mappings ρ_{t_2,t_1} form a group R of transformations of T commuting with the action λ, that the action of R is transitive and free and that there is a bijection between R and the set of orbits of the product action $\lambda \times_G \lambda$ on $T \times T$. Let us take a point $t_o \in T$ and define the transformation $\rho_g:T \to T$ by $\rho_g = \rho_{\lambda(g,t_o),t_o}$. We remark that $\rho_g(t_o) = \lambda(g,t_o)$ and $\rho_e = id_T$. The calculation: $\rho_g \circ \rho_{g'} = \rho_{\lambda(g,t_o),t_o} \circ \rho_{\lambda(g',t_o),t_o} = \rho_{\lambda(g'g,t_o),\lambda(g',t_o)} \circ \rho_{\lambda(g',t_o),t_o} = \rho_{\lambda(g'g,t_o),t_o}$ shows that $\rho_g \circ \rho_{g'} = \rho_{g'g}$. Since $\rho_{t_2,t_1} = \rho_{\lambda(g,t_o),t_o}$, where $g = \gamma(t_o,t_1)\gamma(t_2,t_o)$, we see that the mapping $G \to R:g \mapsto \rho_g{}^{-1}$ is an isomorphism of groups. Let $\xi:T \to T$ be a transformation commuting with the action λ. Let $t_1 = \xi(t_o)$. Then, $\xi(t) = \xi \circ \lambda_{\gamma(t,t_o)}(t_o) = \lambda_{\gamma(t,t_o)} \cdot \xi(t_o) = \lambda_{\gamma(t,t_o)}(t_1) = \rho_{t_1,t_o}(t)$, i.e. $\xi = \rho_{t_1,t_o}$, thus $\xi \in R$. (Q.E.D.)

Remark. (a) With each $t_o \in T$ we associate a differentiable right action on T defined by:

$$\rho:G \times T \to T:(g,t) \mapsto \rho_g(t) = \rho_{\lambda(g,t_o),t_o}.$$

This action is transitive, free and commutes with the action λ. The explicit expression of ρ is:

$$\rho_g(t) = \lambda^{\circ}_{\gamma(t,t_o)g\,\gamma(t_o,t)}(t).$$

If ρ' is the action corresponding to $t'_o \in T$, then

$$\rho'_g = \rho_{g_o g g_o^{-1}} \quad , \quad g_o = \gamma(t_o,t'_o).$$

Let $\tau \subset \mathfrak{X}(T)$ be the Lie algebra of generators of R and $\hat{R}$ the group of the canonical lifts of the elements of R.

Proposition 10.2. The following four properties are equivalent:

(i) (P.2).

(ii) M is invariant under the group $\hat{R} \times \{id_P\}$.

(iii) For each $Y \in \tau$, the vector field $\hat{Y} \times 0$ is tangent to M.

(iv) For each $p \in P$, $\xi \in R$: $\xi^*(H_p) = H_p$ (the 1-forms H_p are R-invariant).

Proof. Statements (ii) and (iii) are obviously equivalent. Equivalence of (ii) and (iv) follows from the identity $\xi^*(H_p)(t) = \hat{\xi}^{-1}(H(\xi(t),p))$ where $p \in P$, $t \in T$. Since the actions are transitive and free the Lie algebras ℓ and τ span separately the tangent space $T_t(T)$ at each point $t \in T$ and $\dim(\ell) = \dim(\tau) = \dim(T)$. Since the actions commute, for each $X \in \ell$, $Y \in \tau$, we have $[X,Y] = 0$ and $\{E_X,E_Y\} = 0$. Now the momentum relation can be defined by: $M = \{(h,p) \in \tilde{P};\ \tilde{H}_X(h,p) = 0,\ \forall X \in \ell\}$. The Hamiltonian vector field $\hat{Y} \times 0$ is generated by $pr_1^*(E_Y)$ and it is tangent to M if and only if $\mu^*\{pr_1^*(E_Y),H_X\} = 0$, for each $X \in \ell$. The calculation

$$\begin{aligned}\{pr_1^*(E_Y),H_X\} &= \{pr_1^*(E_Y),pr_1^*(E_X)\} + \{pr_1^*(E_Y),\ \varepsilon^*(H_X)\}\\ &= pr_1^*\{E_Y,E_X\} + \langle \hat{Y}\times 0,\ \varepsilon^*(dH_X)\rangle\\ &= \varepsilon^*\langle Y,\underset{\sim}{d}H_X\rangle\end{aligned}$$

shows that (iii) is equivalent to: $\langle Y,\underset{\sim}{d}H_X\rangle = 0$, for each $Y \in \tau$, $X \in \ell$. This last equation is clearly equivalent to $\underset{\sim}{d}H_X = 0$, for each $X \in \ell$, i.e. to (P.2). (Q.E.D.)

Remarks. (b) The equivalence of the four properties above does not involve the coisotropy of M. (c) The momentum mapping J has now values in the space of R-invariant 1-forms on T. (d) Because of (A.4) and (A.5), the manifold T is diffeomorphic to G. There is one diffeomorphism for each fixed $t_o \in T$. It is defined by: $G \to T: g \mapsto \lambda(g,t_o)$. Then the action λ becomes equivalent to the left translation on G and the right action φ considered in Remark (a) to the right translation.

Conclusion. With the assumptions (A.1)-(A.5) and

(A.6) M is invariant under the action of the group $\hat{R} \times \{id_P\}$,

the pair (M, λ) gives the model for a Hamiltonian group action on a symplectic manifold (P, ω) [1] [7] [4].

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