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HOMOGENEOUS FORMULATION OF HAMILTONIAN GROUP ACTIONS

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1. The purpose of this lecture is to illustrate a geometrical approach to the momentum mapping theory as a generalization of the classical homogeneous formulation of Hamiltonian mechanics. We recall basic definitions and results concerning Hamiltonian actions and momentum mappings. For details, applications to mechanics and bibliography we can refer to the original works by Smale [1] and Souriau [2] and to Abraham and Marsden [3], Marsden and Weinstein [4], Marle [5,6], Liberman and Marle [7].

Let G be a Lie group and (P,ω) a symplectic manifold (we assume that G and P are connected). A smooth action $\varphi: G \times P \to P$ of G on P is said to be <u>symplectic</u> if for each $g \in G$ the diffeomorphism $\varphi_g : P \to P : p \mapsto \varphi(g,p)$ is symplectic, i.e. $\varphi_g^* \omega = \omega$ (we assume that the action is a left action: $\varphi_g \circ \varphi_{g'} = \varphi_{gg'}$). Let $\mathfrak{g}$ be the Lie algebra of the group G. It is the algebra of left (or right) invariant vector fields on G, usually identified with the tangent space $T_e(G)$ at the identity e of G. Let us denote by K_X the infinitesimal generator of φ corresponding to the element $X \in \mathfrak{g}$ in such a way that $[K_X,K_Y] = K_{[X,Y]}$. The generators of φ are symplectic (i.e. locally Hamiltonian) vector fields. If all generators are globally Hamiltonian we say that φ is a Hamiltonian action. Let us denote by $H_X:P \to R$ the Hamiltonian function of K_X: $i_{K_X}\omega = -dH_X$. The mapping $J:P \to \mathfrak{g}^*$ defined by $\langle X,J(p)\rangle = H_X(p)$ is called the momentum mapping of the Hamiltonian action φ.

The momentum mapping corresponding to a Hamiltonian action is only defined up to a constant element of $\mathfrak{g}^*$. What is uniquely determined by a Hamiltonian action is, first of all, a cohomology class of degree 2 of the cohomology of the Lie algebra $\mathfrak{g}$ with respect to the trivial representation in R. A 2-cocycle B is indeed defined by $\langle X \wedge Y,B\rangle = H_{[X,Y]} - \{H_X,H_Y\}$, since the right hand side is actually a constant function on P. The change $J \mapsto J + A$ with $A \in \mathfrak{g}^*$ produces the change $B \mapsto B + dA$, being $\langle X \wedge Y,dA\rangle = -\langle [X,Y],A\rangle$. Hence we remain in the cohomology class [B] determined by B. If this class is zero, then the action is said to be strongly Hamiltonian and there is a momentum mapping for which $\{H_X,H_Y\} = H_{[X,Y]}$.

A Hamiltonian action generates also a cohomology class of degree 1 in the cohomology of the Lie group G with respect to the coadjoint representation in $\mathfrak{g}^*$. A 1-cocycle θ is indeed defined by $\theta(g) = J \circ \varphi_g(p) - Ad^{*-1}_g \circ J(p)$, since the right hand side does not depend on the choice of $p \in P$. The substitution $J \mapsto J + A$ produces the change $\theta \mapsto \theta + A - Ad^{*-1}_g(A)$ and we remain in the cohomology class $[\theta]$ determined by θ. The relation between the cocycles B and θ is given by $\langle Y,T_e\theta(X)\rangle = \langle X \wedge Y,B\rangle$, where T is the tangent functor and e the identity of G. If the cohomolgy class is zero, then there is a coadjoint equivariant momentum mapping J, i.e. a momentum mapping making the following diagram commutative:

(1)
$$\begin{array}{ccc} P & \xrightarrow{\varphi_g} & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \mathfrak{g}^* & \xrightarrow{Ad^{*-1}_g} & \mathfrak{g}^* \end{array}$$

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2. The simplest example of a Hamiltonian action is represented by a complete globally Hamiltonian vector field on a symplectic manifold (case $G = R$).

Let T denote the time axis ($T = R$) and let us consider the manifold $\tilde{P} = T^*(T) \times P \simeq R \times R \times P$ with the symplectic form $\tilde{\omega} = pr_2^* \omega - du \wedge dt$ where $pr_2 : \tilde{P} \to P$, $t : \tilde{P} \to R$ (first factor, $R = T$), $u : \tilde{P} \to R$ (second factor) are the canonical projections. The dynamics of a system whose phase space is P is given by a submanifold $M \subset \tilde{P}$ which is the image of a section $m : T \times P \to \tilde{P}$ of the canonical projection $\varepsilon : \tilde{P} \to T \times P$. This submanifold is represented by a function $H : T \times P \to R$ (the Hamiltonian) as follows: $M = \{(a,b,p) \in \tilde{P};\ b = -H(a,p)\}$. Since it is of codimension 1, M is necessarily coisotropic. The characteristic distribution $D' = T^{\S}(M) \subset T(M)$ (we use the symbol $\S$ for the symplectic polar operator) is locally described by the equations [8]

$$\dot{q}^i = \dot{t}\,\frac{\partial H}{\partial p_i}\ , \quad \dot{p}_i = -\,\dot{t}\,\frac{\partial H}{\partial q^i}\ , \quad \dot{u} = \dot{t}\,\frac{\partial H}{\partial t}\ , \quad u = H.$$

where (q^i, p_i) are canonical coordinates on P and $(q^i, p_i, \dot{q}^i, \dot{p}_i)$ are the corresponding coordinates on $T(P)$. These are the equations of motion in the homogeneous formulation of Hamiltonian mechanics.

With each vector field X on T we associate a T-dependent Hamiltonian vector field K_X on P defined by the equation $i_{K_X(t,p)}\,\omega = -\,dH_{X,t}(p)$, where $H_{X,t} : P \to R : p \mapsto X(t) \cdot H(t,p)$ ($X(t) \in R$). The group R acts on T by translations. This action can be transferred to P if for each generator $X \in \mathfrak{g} \simeq R$: (i) the vector field K_X is T-independent, (ii) K_X is complete. The action $\varphi : R \times P \to P$ generated by the vector fields K_X is symplectic because these vectors are Hamiltonian. Condition (i) is in particular satisfied when the Hamiltonian function H does not depend on T. In this case the function H, interpreted as a mapping from P to $\mathfrak{g}^* = R$, is a momentum mapping. Since R is commutative the action is of course strongly Hamiltonian. If we add to H any function of time, then condition (i) is still satisfied, but we have not a momentum mapping in the usual sense.

The action φ can be constructed as follows. It can be seen that the characteristics of M (i.e. the maximal connected integral manifolds of the characteristic distribution) are projected by ε to local sections of the natural projection $T \times P \to T$ (by "local section" we understand "union of images of smooth local sections"). The completeness condition (ii) is actually equivalent to the fact that these sections are global sections. In this case, for any choice of two elements $t_1, t_2 \in T$ we can define a relation $D_{t_2,t_1} \subset P \times P$ by

(2) $D_{t_2,t_1} = \{(p_2,p_1) \in P \times P$; (t_1,p_1) and (t_2,p_2) belong to the image by the projection ε of the same characteristic of $M\}$.

It is evident that, under the assumption of completeness, this relation is the graph of a diffeomorphism $\varphi_{t_2,t_1} : P \to P$. It turns out that this relation is a Lagrangian submanifold of the product $(P,-\omega) \times (P,\omega)$, so that the diffeomorphism is symplectic. If φ_{t_2,t_1} depends only on the difference $s = t_2 - t_1$, then we can define a symplectic action $\varphi : R \times P \to P$ by setting $\varphi_s(p) = \varphi_{t_0+s,t_0}(p)$ with $t_0 \in T$ arbitrarily chosen. This invariance property is actually equivalent to condition (i). Finally, one can check that this action is the action generated by the vector fields K_X.

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3. The homogeneous formulation of Hamiltonian mechanics admits a natural generalization to Hamiltonian actions, which provides a geometrical approach to the momentum mapping theory [8] [9]. Let T be a manifold, (P, ω) a symplectic manifold and M a submanifold of the symplectic manifold $(\tilde{P},\tilde{\omega}) = (T^*(T) \times P, pr_1^* \omega_T + pr_2^* \omega)$, where $pr_1:\tilde{P} \to T^*(T)$ and $pr_2:\tilde{P} \to P$ are the canonical projections, and $\omega_T = d\,\theta_T$ where θ_T is the Liouville 1-form on $T^*(T)$. We assume that:

(A.1) M is the image of a section $m:T \times P \to \tilde{P}$ of the projection $\varepsilon = \pi_T \times id_P:\tilde{P} \to T \times P$ ($\pi_T:T^*(T) \to T$ is the cotangent bundle projection).

(A.2) M is coisotropic in $(\tilde{P},\tilde{\omega})$.

Assumption (A.1) means that M can be described by a mapping $H:T \times P \to T^*(T)$ such that for each $p \in P$ the mapping

$$H_p:T \to T^*(T):t \mapsto H(t,p)$$

is a section of π_T (i.e. a 1-form on T):

$$M = \{(h,p) \in P;\ h = -H(\pi_T(h),p)\}.$$

As a consequence we have a mapping

$$J:P \to \Phi_1(T):p \mapsto H_p$$

(we denote by $\Phi_1(Q)$ the space of 1-forms on a manifold Q). We call H and J the <u>Hamiltonian</u> and the <u>momentum</u> <u>mapping</u> associated with M respectively. For each vector $u \in T(T)$ we define the function

$$H_u:P \to R:p \mapsto \langle u,H_p\rangle.$$

The Hamiltonian mapping H can be identified with the following function (we can use the same symbol without danger of confusion):

$$H: T(T) \times P \to R: (u,p) \mapsto H_u(p).$$

This function is linear when restricted to each fibre of T(T). From this point of view H can also be interpreted as a 1-form on T with values in the space $C^\infty(T;R)$ of smooth functions on T. We denote by $\underline{d}H$ the corresponding differential (it is the differential "with respect to T").

With each vector field $X \in \mathcal{X}(T)$ (we denote by $\mathcal{X}(Q)$ the space of smooth vector fields on a manifold Q) we associate the following functions:

$$(3) \quad \begin{cases} H_X = \langle X,H\rangle : T \times P \to R : (t,p) \mapsto \langle X(t), H(t,p)\rangle, \\ E_X : T^*(T) \to R : h \mapsto \langle X,h\rangle, \\ \tilde{H}_X = (pr_1^* E_X + \varepsilon^* H_X) : \tilde{P} \to R. \end{cases}$$

We recognize that $H_X = H^*E_X = E_X \circ H$ and that $m^*\tilde{H}_X = 0$, so that M is the set characterized by the vanishing of all functions $\tilde{H}_X$. We also recall that the function E_X is the Hamiltonian of the canonical lift $\hat{X}$ to $T^*(T)$ of the vector field X: $i_{\hat{X}}\,\omega_T = -\,dE_X$.

The following general identity holds:

$$(4) \qquad \{\tilde{H}_X,\tilde{H}_Y\} = \tilde{H}_{[X,Y]} + \varepsilon^*(\{H_X,H_Y\} - H_{[X,Y]} - \langle Y,\underline{d}H_X\rangle + \langle X,\underline{d}H_Y\rangle),$$

for each $X,Y \in \mathcal{X}(T)$, where $\underline{d}H_X$ denotes the differential of H_X interpreted as a 0-form on T with values on $C^\infty(P,R)$, and {,} are the Poisson brackets on P and on $T \times P$ induced by $\tilde{\omega}$ and ω respectively. Since $\varepsilon \circ \mu = id_{T \times P}$, one can see that the coisotropy of M (assumption (A.2)) is characterized by one of the following two equivalent conditions:

$$(5) \qquad \{\tilde{H}_X,\tilde{H}_Y\} = \tilde{H}_{[X,Y]},$$

$$(6) \qquad \{H_X,H_Y\} = H_{[X,Y]} - \langle Y,\underline{d}H_X\rangle + \langle X,\underline{d}H_Y\rangle,$$

for each $X,Y \in \mathcal{X}(T)$. Because of the identity

$$(7) \qquad \langle X \wedge Y, \underline{d}H \rangle = \langle X, \underline{d}H_Y \rangle - \langle Y, \underline{d}H_X \rangle - H_{[X,Y]}$$

equation (6) is equivalent to

$$(8) \qquad \{H_X, H_Y\} + \langle X \wedge Y, \underline{d}H \rangle = 0.$$

Condition (5) means that the mapping

$$\mathcal{X}(T) \to C^{\infty}(\tilde{P}, R) : X \mapsto \tilde{H}_X$$

is a representation (i.e. a homomorphism) of the Lie algebra $\mathcal{X}(T)$ in the Poisson algebra on $(\tilde{P}, \tilde{\omega})$.

Arguments of linear symplectic algebra show that the first component v of a vector $(v,w) \in T_{(h,p)}(M)$ belongs to the space $(TH_p(T_t(T)))^{\S}$, where $t = \pi_T(h)$, i.e.

$$\langle v \wedge v', \omega_T \rangle = 0 \quad \text{for each } v' \in TH_p((T_t(T)).$$

The second component w is then uniquely determined by v through the equation

$$(9) \qquad i_w \omega = - (T_p H_t)^* (i_v \, \omega_T)$$

(i denotes the interior product of a form by a vector, $*$ denotes the linear dual functor, and $H_t : P \to T_t^*(T) : p \mapsto H(t,p)$).

It is remarkable the fact that, under assumption of coisotropy (A.2), the Hamiltonian H (interpreted as a function on $T(T) \times P$) is the generating function of the infinitesimal symplectic relation $T^{\S}(M) \subset T(M)$ associated with M (see [10]). This implies in particular that instead of (9) we have the following simpler expression:

(10) $i_W \omega = - dH_u(p), \quad u = T\pi_T(v).$

Further consequences:

(11) The image by the projection ε of a characteristic of M is a "local section" of the canonical projection $T \times P \to T$.

(12) The image by the projection $T^*(T) \times P \to T^*(T)$ of a characteristic of M is a "local section" of π_T (i.e. a "local 1-form").

(We recall that a characteristic of M is a maximal connected integral manifold of the characteristic distribution T (M), and that by "local section" we understand "union of images of smooth local sections".)

These projections could not be "global sections" since, for any two fixed points t_1 and t_2 of T, a characteristic of M could not have or could have many intersections with both fibres $T^*_{t_1}(T) \times P$ and $T^*_{t_2}(T) \times P$.

We assume that the following completeness condition is satisfied:

(A.3) The images by the projection ε of the characteristics of M are "global sections" of the canonical projection $T \times P \to T$.

As a consequence, the image by the projection $T^*(T) \times P \to T^*(T)$ of the characteristics of M are images of 1-forms on T. Moreover, the relation D_{t_2,t_1} defined as in (2) for each pair $(t_1,t_2) \in T \times T$ is the graph of a diffeomorphism $\varphi_{t_2,t_1} : P \to P$. On the other hand, the relation D_{t_2,t_1} is the reduced set of the symplectic relation

$$D = \{(h,p;h'p') \in \tilde{P} \times \tilde{P};\ (h,p) \text{ and } (h',p') \text{ belong to the same characteristic of } M\}$$

with respect to the transversal coisotropic submanifold of $(\tilde{P},-\tilde{\omega}) \times (\tilde{P},\tilde{\omega})$ obtained by fixing the fibres over t_1 and t_2. It follows that D_{t_2,t_1} is a symplectic relation, thus φ_{t_2,t_1} is a symplectomorphism (see [10]; for the process of reduction see [11] [12] [13] [14]).

We call <u>momentum relation</u> a submanifold $M \subset T^*(T) \times P$ satisfying assumptions (A.1), (A.2) and (A.3). If a momentum relation is given, then: (i) with each vector $u \in T(T)$ we associate a symplectic vector field K_u on (P, ω)(the vector field whose Hamiltonian is H_u); (ii) with each pair $(t_1, t_2) \in T \times T$ we associate a symplectomorphism φ_{t_2, t_1} on (P, ω); (I) with each vector field $X \in \mathcal{X}(T)$ we associate a T-dependent Hamiltonian vector field $K_X : T \times P \to T(P)$ (defined by $K_X(t,p) = K_{X(t)}(p)$); (II) with each diffeomorphism $\xi : T \to T$ we associate a T-dependent family of symplectomorphisms on (P, ω), $\varphi_{(\xi, t)} = \varphi_{\xi(t), t}$.

The following identities can be proved:

(13) $[K_X, K_Y] = K_{[X,Y]} - \langle X, \underline{d}K_Y \rangle + \langle Y, \underline{d}K_X \rangle$, for each $X, Y \in \mathcal{X}(T)$,

where $\underline{d}K_X$ is the differential of K_X interpreted as a 0-form on T with values in the space of vector fields on P, and

(14) $\varphi_{(id_T, t)} = id_P$, $\varphi_{(\xi \circ \eta, t)} = \varphi_{(\xi, \eta(t))} \circ \varphi_{(\eta, t)}$,

for each pair of diffeomorphisms ξ, η on T.

Finally we emphasize the following fact:

(15) The momentum relation M defines two subsets of $\Phi_1(T)$ which we denote by $\Gamma(M)$ and $\Delta(M)$: $\Gamma(M)$ is the set of 1-forms whose images are the projections to $T^*(T)$ of the characteristics of M, $\Delta(M)$ is the set of the forms H_p, $p \in P$.

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4. Let $\ell \subset \mathcal{X}(T)$ be a Lie algebra of vector fields on T. Formula (13) shows that if $\underline{d}K_X = 0$ for each $X \in \ell$ (i.e. if K_X is a vector field on P) then the mapping $\ell \to \mathcal{X}_H(P, \omega) : X \mapsto K_X$ is a representation of the Lie algebra ℓ in the Lie algebra $\mathcal{X}_H(P, \omega)$ of Hamiltonian vector fields on P.

Let us call condition

(WI) $\underline{d}K_X = 0 \quad , \quad \forall X \in \mathcal{L}$,

the weak invariance property of M with respect to $\mathcal{L}$, and

(SI) $\underline{d}H_X = 0 \quad , \quad \forall X \in \mathcal{L}$,

the strong invariance property of M with respect to $\mathcal{L}$.

It is evident that (SI) implies (WI). The inverse is not true in general: the case of T-independent Hamiltonian vector field defined by a T-dependent Hamiltonian is an example.

We remark that if (SI) holds, then (from (6))

(16) $\{H_{X_1}, H_{X_2}\} = H_{[X_1,X_2]}$, for each $X_1, X_2 \in \mathcal{L}$,

i.e. the mapping

$$\mathcal{L} \to C^{\infty}(P,\omega): X \mapsto H_X$$

is a representation of the Lie algebra $\mathcal{L}$ in the Poisson algebra $C^{\infty}(P,\omega)$.

We assume from the beginning that the Lie algebra $\mathcal{L}$ is transitive (at each point $t \in T$ the vectors of $\mathcal{L}$ span the whole tangent space $T_t(T)$) and free ($X \in \mathcal{L}$, $X(t) = 0$ imply $X = 0$). We have $\dim(\mathcal{L}) = \dim(T)$.

For each $X \in \mathcal{L}$ we define a vector field $\hat{K}_X$ on $\tilde{P}$ by

(17) $\hat{K}_X(h,p) = (\hat{X}(h), K_X(t,p))$

where $t = \pi_T(h)$ and $\hat{X}$ is the canonical lift of X. The vector field $\hat{K}_X$ differs from the Hamiltonian vectorfield $\tilde{K}_X$ generated by $\tilde{H}_X$ by a vector vertical with respect to the projection ε . More precisely, for each vector (v,w) tangent to $\tilde{P}$ at a point (h,p), we have:

$$\langle (v,w) \wedge (\tilde{K}_X - \hat{K}_X), \tilde{\omega} \rangle = \langle u, \underline{d}H_X \rangle (p)$$

where $u = T\pi_T(v)$. From this formula we can see that (SI) holds if and only if $\tilde{K}_X = \hat{K}_X$ for each $X \in \ell$, i.e. if and only if for each $X \in \ell$ the vector field $\hat{K}_X$ is tangent to M. Since $\tilde{K}_X$ is tangent to the characteristics of M (because $\tilde{H}_X$ is constant (= 0) on M), it follows that:

(18) (SI) holds if and only if each vector field $\hat{X}$ with $X \in \ell$ is tangent to the images of the 1-forms $\Gamma(M)$, i.e. $d_X\mu = 0$ for each $\mu \in \Gamma(M)$ and $X \in \ell$ (we say that the 1-form μ is ℓ-invariant).

It can be shown that:

(19) The set of vector fields $\tau \in \mathfrak{X}(T)$ which commute with each element of the transitive and free subalgebra $\ell \in \mathfrak{X}(T)$ is a Lie subalgebra isomorphic to ℓ, transitive and free.

This remarkable fact yields the following further characterization of (SI) which is more effective then (18) since it does not require the knowledge of the characteristics of M (see [10]):

(20) (SI) holds if and only if for each $Y \in \tau$ the vector field $(\widehat{Y \times 0})$ is tangent to M, or, equivalently, $d_Y H_p = 0$ for each $Y \in \tau$ and $p \in P$ (the forms of $\Delta(M)$ are τ-invariant).

Because of the identity $d_Y = di_Y + i_Y d$, from the last proposition we see that (SI) is also equivalent to

(21) $\underline{d}H_Y + i_Y\underline{d}H = 0$, for each $Y \in \tau$.

From (7) and (8) it follows that

(22) $\{H_{Y_1}, H_{Y_2}\} + H_{[Y_1,Y_2]} = 0,$ for each $Y_1, Y_2 \in \mathcal{r}$.

The vector fields $\{\hat{X}; X \in \ell\}$ and $\{\hat{Y}; Y \in \mathcal{r}\}$ span two integrable distributions on $T^*(T)$ of rank equal to $\dim(T)$. We denote them by L and R respectively. Their integral manifolds are images of the ℓ-invariant and $\mathcal{r}$-invariant 1-forms respectively. Since ℓ and $\mathcal{r}$ commute, from the identities $0 = E_{[X,Y]} = \{E_X, E_Y\} = -\langle X, dE_Y\rangle = \langle Y, dE_X\rangle = \langle X \wedge Y, \omega_T\rangle$ we see that:

(23) The space of ℓ-invariant (resp. $\mathcal{r}$-invariant) forms is the linear dual $\mathcal{r}^*$ of the space $\mathcal{r}$ (resp. the linear dual ℓ^* of $\mathcal{r}$).

(24) $L^{\S} = R$, $R^{\S} = L$: the distributions L and R (and the corresponding integral manifolds) are simplecticly dual.

We can express (18) and (20) as follows:

(25) (SI) $\Longleftrightarrow$ $\Gamma(M) \subset \mathcal{r}^*$ $\Longleftrightarrow$ $\Delta(M) \subset \ell^*$.

We remark that from the last condition it follows that $H_p \in \ell^*$, $\forall\ p \in P$, i.e. that the momentum mapping J takes its values in $\ell^* \subset \Phi_1(T)$.

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5. Let $\lambda: G \times T \to T$ be an action of a Lie group G on the manifold T. For each $g \in G$ we denote by λ_g the diffeomorphism $T \to T: t \mapsto \lambda(g,t)$. We assume from the beginning, for the sake of simplicity, that this action is transitive and free. Let us consider the following property:

(FWI) $\varphi_{\lambda_g(t_1),t_1} = \varphi_{\lambda_g(t_2),t_2}$ for each $g \in G$ and $t_1, t_2 \in T$.

Identities (14) show that the mapping

$$\varphi : G \times P \to P : (g,p) \mapsto \varphi_{\lambda_g(t_o),t_o}(p),$$

where t_o is an arbitrarily chosen element of T, is a symplectic action of G on (P,ω). Going back to the definition of the vector fields K_X and the symplectomorphisms φ_{t_2,t_1}, one can see that if $\ell \subset \mathcal{X}(T)$ is the Lie algebra of generators of λ, then the vector fields K_X with $X \in \ell$ are the generators of φ. Hence (FWI) is the finite counterpart of the infinitesimal weak invariance property (WI).

A definition of finite strong invariance should be given in terms of "generating functions" of the symplectomorphisms φ_{t_2,t_1}. We adopt here a different approach. Let us consider the mapping $\hat{\varphi} : G \times \tilde{P} \to \tilde{P}$ defined by

(26) $\hat{\varphi}(g;h,p) = (\hat{\lambda}_g(p), \varphi_{\lambda_g(t),t}(p))$

where $t = \pi_T(h)$ and $\hat{\lambda}_g$ is the canonical lift of $\lambda_g : T \to T$. This mapping is an action of G on $\tilde{P}$ because of (14), and for each $X \in \ell$ the vector field $\hat{K}_X$ defined in (17) is a generator of $\hat{\varphi}$. We consider the property

(FSI) M is invariant under the action $\hat{\varphi}$.

After the discussion which leads to (18) we see that this property is the finite counterpart of (SI) where ℓ is the Lie algebra of generators of the action λ. There is a characterization of (FSI) which involves the momentum mapping $J : P \to \Phi_1(T)$.

(27) (FSI) holds if and only if (FWI) holds and the diagram

(28)
$$\begin{array}{ccc} P & \xrightarrow{\varphi_g} & P \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle J} \\ \Phi_1(T) & \xrightarrow{\lambda_g^{*-1}} & \Phi_1(T) \end{array}$$

is commutative.

Now we use the fact that the action λ is free and we can prove a proposition analogous to (19):

(29) The set of transformations of T which commute with λ is a Lie group E isomorphic to G. The action of E is transitive and free.

We define an element $\rho_{t_2,t_1} \in E$ for each pair $(t_1,t_2) \in T \times T$ by

$$\rho_{t_2,t_1}(t) = \lambda(\gamma(t,t_1),t_2)$$

where $\gamma: T \times T \to G$ is the differentiable mapping defined by the condition $g = \gamma(t',t)$ if $t' = \lambda(g,t)$. It can be shown that $\rho_{t_2,t_1} = \rho_{\lambda_g(t_2),\lambda_g(t_1)}$ for each $g \in G$. Hence there is exactly one element of E for each orbit of the product action of G on $T \times T$. We can fix a point $t_0 \in T$ and set

$$\rho: G \times T \to T: (g,t) \mapsto \rho_{\lambda_g(t_0),t_0}(t).$$

It follows that ρ is a transitive and free action commuting with λ. The Lie algebra $\tau \subset \mathcal{X}(T)$ of generators of this action does not depend on the choice of t_0 (while the definition of ρ does). It is clear that τ coincides with the Lie algebra associated with the Lie algebra ℓ of generators of λ according to proposition (19). Finally, instead of proposition (16) we have:

(30) (FSI) holds if and only if M is invariant under the action of the group $\hat{E} \times \{id_P\}$, where $\hat{E}$ is the group of the canonical lifts of the elements of E.

Since the action λ is transitive and free, the manifold T can be identified with the group G by fixing an element t_0 of T. The action λ

becomes equivalent to the left translation on G and the action ρ to the right one. The use of the manifold T instead of G shows in particular that the above considerations does not involve the distinguished identity element $e \in G$.

Starting from the definitions of momentum relation and in discussing the invariance properties with respect to a transitive and free Lie algebra or a transitive and free action, we have found the peculiar properties of the momentum mapping in the case of a strongly Hamiltonian action. We remark that in this geometrical approach both Lie algebras of left and right invariant vector fields and both spaces of left and right invariant 1-forms of the group are involved. Although they are isomorphic, they play a different role.

It can be shown that conversely (see [9]), starting from a strongly Hamiltonian action, a momentum relation satisfying the invariance properties and representing the given symplectic action can be constructed. Hence, momentum relations provide the homogeneous model for strongly Hamiltonian actions.

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6. We construct now a model for Hamiltonian actions including weakly Hamiltonian actions. We consider a principal fiber bundle $\pi : S \to T$ with structural group R (S and T are assumed to be connected). We denote by $\zeta : R \times S \to S$ the action of the group R on S and by Z its infinitesimal generator. A <u>connection</u> for such a principal fibre bundle is a 1-form $\alpha \in \Phi_1(S)$ satisfying equations

$$i_Z\alpha = 1 \quad , \quad d_Z\alpha = 0 \; .$$

All connection forms are given by

(31) $\alpha' = \alpha + \pi^* A$

where $A \in \Phi_1(T)$. Since $i_Z d\alpha = 0$, $d_Z d\alpha = 0$, there exists a closed 2-form B on T such that $d\alpha = \pi^* B$. The 2-form B is the curvature of the connection α. The curvature B' of the connection form $\alpha' = \alpha + \pi^* A$ is:

(32) $B' = B + dA$.

Let us consider the coisotropic submanifold C of $T^*(S)$ defined by

$$C = \{k \in T^*(S);\ \langle Z, k\rangle = 1\}$$

(C is coisotropic because of codimension 1). The characteristics of C are orbits of the canonical lift $\hat{\xi}$ of the action ξ. The coisotropic submanifold C defines a symplectic reduction. It turns out that the reduced manifold, i.e. the set of the characteristics, is isomorphic to $T^*(T)$.

With each connection form α we associate a mapping $\varkappa : C \to T^*(T)$ defined by

(33) $\langle v, \varkappa(k)\rangle = \langle w, k\rangle$

where $k \in C$, $v \in T_t(T)$, $t = \pi(s)$, $s = \pi_S(k)$ and w is the horizontal lift of v, i.e. the vector defined by equations $\langle w, \alpha\rangle = 0$, $T\pi(w) = v$.

It can be shown that (see [15]):

(34) The mapping $\varkappa$ is a surjective submersion whose fibres are characteristics of C,

and

(35) $\varkappa^*(\omega_T + \pi^*{}_T B) = \omega_S | C$

where $\omega_S = d\,\theta_S$ is the canonical symplectic form on $T^*(S)$ (θ_S is the Liouville form).

Hence, the mapping $\varkappa$ defines a symplectic reduction isomorphic to the reduction associated with C: the reduced symplectic manifold of $(T^*(S), \omega_S)$ by C is symplectomorphic to $(T^*(T), \sigma)$ where

$$\sigma = \omega_T + \pi^*_T B.$$

We see that the canonical symplectic form ω_T is "varied" by the term $\pi^*_T B$.

The analogous mapping $\varkappa'$ corresponding to the choice $\alpha' = \alpha + \pi^* A$ of the connection form is

(36) $\varkappa' = \varkappa + A \circ \pi$.

Let us choose a connection α and take the corresponding reduction mapping $\varkappa$. Every connection form α' can be "reduced" according to the following property:

(37) If α' is a connection, then the image by $\varkappa$ of the set $\alpha'(S)$ is the image of the 1-form A on T defined by (31).

Indeed we have (see the definition of $\varkappa$): $\langle v, \varkappa(\alpha'(s))\rangle = \langle w, \alpha'(s)\rangle = \langle w, \alpha(s) + \pi^* A(s)\rangle = \langle v, A\rangle$ since $\langle w, \alpha\rangle = 0$.

We say that A in the <u>reduced form of</u> α' <u>by the connection</u> α.
We note that the reduced form of α by itself is the zero form.

Let $\check{M} \subset T^*(S) \times P$ be a momentum relation: assumptions (A.1), (A.2) and (A.3) (Section 4) hold for $\check{M}$. We make the following further assumption:

(A.2') $\check{M}$ is contained in the coisotropic submanifold $C \times P$.

The characteristics of $C \times P$ are pairs of characteristics of C and points of P. The characteristics of $\check{M}$ are union of characteristics of C. The vector field $\hat{Z} \times 0$ is then tangent to the characteristics of M. Let us denote by $\Gamma(\check{M})$ the set of 1-forms defined by projecting to $T^*(S)$ the characteristics of $\check{M}$ (see (15)).

(37) The 1-forms of $\Gamma(\check{M})$ are connection forms of the principal fibre bundle $\pi : S \to T$.

Condition $i_Z \alpha = 1$ is a consequence of (A.2') while $d_Z \alpha = 0$ (i.e. $\zeta^* \alpha = \alpha$) is a consequence of the fact that $\hat{Z}$ is tangent to $\alpha(S)$. Property (37) is equivalent to (A.2').

Let $\check{H}$ be the Hamiltonian function of $\check{M}$ and let H_p be the 1-form corresponding to the point $p \in P$. Let $\Delta(\check{M})$ be the set of these 1-forms (see (15)). It can be shown that

(38) The 1-forms $\check{H}_p$ are connection forms of the principal fibre bundle $\pi : S \to T$.

Hence with respect to the chosen connection α, for each $p \in P$ we have the following decomposition of H_p:

(39) $\check{H}_p = \alpha + \pi^* H_p$,

where H_p is the reduced form of $\check{H}_p$.

It is remarkable the fact that

(40) The reduced set $M = (\varkappa \times id_P)(\check{M})$

(i) is the image of a section of the projection $\mathcal{E} = \pi_T \times id_P$;

(ii) is a coisotropic submanifold of $(T^*(T), \sigma) \times (P, \omega)$;

(iii) satisfies the completeness condition.

This means that M is a momentum relation on $T^*(T) \times P$, but with respect to the varied symplectic structure σ on $T^*(T)$. We call M the <u>reduced momentum relation</u> of $\check{M}$ corresponding to the connection α.

The Hamiltonian $H:T \times P \to T^*(T)$ of M is defined by $H(t,p) = H_p(t)$ where H_p is the reduced form of $\check{H}_p$ (see (39)).

M is a coisotropic submanifold since it is the image by a symplectic relation of a coisotropic submanifold. The identity (4) holds also if in the first term the Poisson brackets are those corresponding to the varied symplectic structure σ. Let us denote them by $\{\ ,\ \}_\sigma$. The identity

$$\{\tilde{H}_X,\tilde{H}_Y\}_\sigma = \{\tilde{H}_X,\tilde{H}_Y\} + \varepsilon^* \circ pr_T^*\langle X \wedge Y,B\rangle$$

holds for each $X,Y \in \mathcal{X}(T)$. Consequently, by a reasoning analogous to that of Section 3, we conclude that the coisotropy condition (5) is unchanged while (6) becomes:

(41) $\{H_X,H_Y\} = H_{[X,Y]} - \langle Y,\underline{d}H_X\rangle + \langle X,\underline{d}H_Y\rangle - pr_T^*\langle X \wedge Y,B\rangle,$

or (use (7))

(42) $\{H_X,H_Y\} + \langle X \wedge Y,\underline{d}H\rangle = - pr_T^*\langle X \wedge Y,B\rangle$

for each $X,Y \in \mathcal{X}(T)$.

The reduced momentum relation M generates two sets of 1-forms on T which we denote by $\Gamma(M)$ and $\Delta(M)$ according to (15). It is clear from the previous discussion that:

(43) $\Gamma(M)$ (resp. $\Delta(M)$) is the set of the reduced forms of $\Gamma(\check{M})$ (resp. $\Delta(\check{M})$).

For each pair $(s_1,s_2) \in S \times S$, a symplectic relation D_{s_2,s_1} is defined by considering the characteristics of $\check{M}$. Because of the completeness condition these relations are graphs of symplectomorphisms φ_{s_2,s_1} on (P,ω). The fact that $\check{M}$ is contained in $C \times P$ (assumption (A.2')) implies the following invariance property:

$$D_{\zeta_r(s_2),\,\zeta_{r'}(s_1)} = D_{s_2,s_1}$$

for each $r, r' \in R$. This means that D_{s_2,s_1} depends only on the fibres of π and we can define, for each $(t_1,t_2) \in T \times T$,

(44) $$D_{t_2,t_1} = D_{s_2,s_1}$$

where s_2 are s_1 are arbitrarily chosen in the fibres over t_1 and t_2. It is remarkable the fact that the relation D_{t_2,t_1} so defined coincides with the relation determined by the reduced coisotropic submanifold M as in (2).

As a consequence, the completeness condition (A.3) also holds for M. From the definition (44) of D_{t_2,t_1} we see that these relations (and the corresponding symplectomorphisms) do not depend on the choice of the connection α.

* * *

7. For the <u>reduced momentum relation</u> M remarks analogous to those at the end of Section 3 hold (namely, (i), (ii), (I), (II), and formulae (13), (14)). We consider a free and transitive Lie subalgebra $\ell = \ell_T \subset \mathcal{X}(T)$ and the invariance property (SI) (Section 4): $\underline{d}H_X = 0$, $\forall\, X \in \ell_T$. If (SI) holds, then we have a <u>representation</u> of ℓ_T in the space $\mathcal{X}_H(P,\omega)$ of the Hamiltonian vector fields on (P,ω) (according to (13)). However, in the present case, we have different characterizations of (SI).

Let τ_T be the transitive and free algebra of vector fields on T commuting with ℓ_T according to proposition (15). Let ℓ_S be the subspace of $\mathcal{X}(S)$ over R spanned by the infinitesimal generator Z and the horizontal lifts X^h of the vector fields $X \in \ell_T$ ($T\pi \circ X^h = X \circ \pi$ and $\langle X^h, \alpha\rangle = 1$). From the general identities

(45) $$[Z, X^h] = 0 \quad , \quad [X_1^h, X_2^h] = [X_1, X_2]^h - \pi^*\langle X_1 \wedge X_2, B\rangle\, Z,$$

for each $X, X_1, X_2 \in \mathcal{X}(T)$, we see that ℓ_S is a transitive and free Lie subalgebra of $\mathcal{X}(S)$ if and only if the following invariance property holds:

(CI) The curvature B of the chosen connection α is τ_T-invariant, i.e. $d_Y B = 0, \forall Y \in \tau_T$, i.e. $\langle X_1 \wedge X_2, B\rangle = \text{const.}$, $\forall X_1, X_2 \in \ell_T$.

This condition means that B is a 2-cocycle on the Lie algebra ℓ_T.

Let $\check{H}_{X^h}: S \times P \to R$ be the function defined as in $(3)_I$. Since $\langle X^h, \alpha\rangle = 0$, we have: $\check{H}_{X^h}(s,p) = \langle X^h(s), \check{H}_p\rangle = \langle X^h(s), \pi^* H_p + \alpha\rangle = \langle X(\pi(s)), H_p\rangle = H_X(\pi(s),p)$. For the function $\check{H}_Z: S \times P \to R$ an analogous calculation shows that $\check{H}_Z(s,p) = 1$, since $\langle Z, \pi^* H_p\rangle = 0$ and $\langle Z, \alpha\rangle = 1$.

We can think of these functions as 0-forms on S with values in $C^\infty(P,R)$ and denote by $\underline{d}H_{X^h}$, $\underline{d}H_Z$ the corrisponding differentials ($\underline{d}$ is essentially the differential with respect to S). The above calculation shows that the invariance property (SI) for the reduced momentum relation M is equivalent to $\underline{d}H_{X^h} = 0$, $\forall X \in \ell_T$ and that $\underline{d}H_Z = 0$ is identically satisfied. Hence (SI) is equivalent to

(SI') $\underline{d}\check{H}_V = 0 \quad , \forall V \in \ell_S$,

i.e. to the strong invariance of M with respect to ℓ_S.

All the results (concerning M and ℓ) of Section 5 hold with reference to $\check{M}$ and ℓ_S. If (CI) holds, then we can introduce the Lie algebra τ_S of vector fields on S commuting with ℓ_S. According to proposition (19), this algebra is transitive, free and isomorphic to ℓ_S. We remark that τ_S is does <u>not</u> coincide in general with the Lie algebra spanned by Z and by the horizontal lifts Y^h of vector fields Y belonging to the algebra τ_T which commutes with ℓ_T.

According to (18) and (20), we have that:

(46) Invariance property (SI) is equivalent to one of the following two conditions:

(i) each form $\bar{\gamma} \in \Gamma(\check{M})$ is ℓ_S-invariant (i.e. $\Gamma(\check{M}) \subset \tau^*_S$).

(ii) (SI) holds and for each $p \in P$ the 1-form $\check{H}_p$ is $\mathcal{r}_S$-invariant (i.e. $\Delta(\check{M}) \subset \ell^*_S$).

Condition (i) is equivalent to $d_{X^h}\bar{\gamma} = 0$, $\forall\, X \in \ell_T$. By applying the Lie derivative d_{X^h} to the equality $\bar{\gamma} = \alpha + \pi^*\gamma$ where $\gamma \in \Gamma(M)$ is the reduced form of $\bar{\gamma}$, we see that

(47) (SI) holds if and only if each reduced form $\gamma \in \Gamma(M)$ satisfies equations

(48) $d_X\gamma + i_X B = 0 \quad , \quad \forall\, X \in \ell_T.$

We have also an effective characterization of (SI), which is similar to (20):

(49) (SI) holds if and only if (CI) holds and $d_Y H_p = 0$ for each $p \in P$ and $Y \in \mathcal{r}_T$ (i.e. H_p is $\mathcal{r}_T$-invariant, i.e. $\Delta(M) \subset \ell^*_T$).

The proof of these propositions is based on the following remarks. Let $\bar{\gamma} = \alpha + \pi^*\gamma$ and $\bar{\nu} = \alpha + \pi^*\nu$ be ℓ_S-invariant and $\mathcal{r}_S$-invariant respectively. The manifolds $\bar{\gamma}(S)$ and $\bar{\nu}(S)$ are symplecticly dual (or "orthogonal"), if they intersect, since ℓ_S and $\mathcal{r}_S$ commute. Since the reduction $\varkappa$ is symplectic, the manifolds $\gamma(T)$ and $\nu(T)$ are orthogonal with respect to the reduced symplectic form σ. Equation (48) means that the vector field $\hat{X} + \tilde{X}$, where $\tilde{X}$ is defined by $i_{\tilde{X}}\omega_T = -\pi^*_T i_X B$, is tangent to $\gamma(T)$. Since $\langle \hat{Y} \wedge (\hat{X} + \tilde{X}), \sigma \rangle = 0$, it follows that $\hat{Y}$ is tangent to $\nu(T)$, hence $d_Y\nu = 0$.

Following the pattern of Section 5 we can now pass to the "finite" analysis by considering an action $\lambda: G \times T \to T$ of a group G on T. We leave this out of the present lecture; however, we remark that in this way we get a geometrical interpretation of the 1-cocycles associated with a Hamiltonian group action. By assuming that λ is transitive and free and by choosing an element of T we obtain an identification $G \simeq T$ and ℓ_T becomes the Lie algebra ℓ_G of generators of the left translation. Equation (48) written for a group G is integrable if and only if B is closed and left invariant. Solutions are in general "local" 1-forms. Assuming that they are global sections, let us take the solution γ for which $\gamma(e) = 0$. It can be seen that the mapping $\theta: G \rightarrow \ell^*$ defined by

$$(50) \qquad \theta(g)(g) = \gamma(g)$$

satisfies the cocycle equation: $\theta(gg') = \lambda^*_{g^{-1}}(\theta(g')) + \theta(g)$. Hence the solutions of equation (48) are the geometrical counterparts of the 1-cocycle on G (with respect to the coadjoint left representation λ^*) associated with the 2-cocycle B. By taking the Lie derivative with respect to a vector field $Y \in r_G$ (i.e. a generator of the right translations $G \times G \rightarrow G: (g,g') \mapsto g'g^{-1}$) we obtain:

$$d_X d_Y \gamma = 0 \quad , \forall X \in \ell_G, \; Y \in r_G.$$

The 1-forms on G solutions of this equation (in particular the solutions such that $\gamma(e) = 0$) represents the 1-cocycles of G, following the definition (50).

* * *

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