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THE HAMILTON-JACOBI EQUATION FOR A HAMILTONIAN ACTION

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0. – Introduction.

The purpose of this lecture is to construct a coisotropic submanifold suitable for a homogeneous representation of a Hamiltonian action $\varphi : G \times P \to P$ on a symplectic manifold (P, ω), and to write the corresponding Hamilton-Jacobi equation. Following a suggestion by W.M. Tulczyjew, we work on the symplectic manifold $(T^*G, \omega_G) \times (P, \omega)$ where ω_G is the canonical symplectic form of T^*G. The discussion is not restricted to Abelian actions. Our approach is based on the geometrical version of the Hamilton-Jacobi theory, connected with the theory of symplectic reductions ([2], [3], [4]), which is partially summarized in Section 1.

All manifolds and mapping are tacitly assumed to be of class C^∞. We use the following notation:

$\langle v, p \rangle$	evaluation of a covector p on a vector v.
$\langle v, \theta \rangle$	evaluation of a form θ on a vector v.
$T_q Q$	the tangent space at a point q of a manifold Q.
$T_q^* Q$	the cotangent space at a point q of a manifold Q.
$T\alpha : TQ \to TM$	the tangent mapping of a mapping $\alpha : Q \to M$.
$\tau_Q : TQ \to Q$	the tangent bundle projection of a manifold Q.
$\pi_Q : T^*Q \to Q$	the cotangent bundle projection of a manifold Q.
θ_Q	the Liouville 1-form of T^*Q.
$\omega_Q = d\theta_Q$	the canonical symplectic form on T^*Q.
$d\theta$	the exterior differential of a form θ.
$i_X \theta$	the interior product of a form θ by a vector field X.
$L_X \theta$	the Lie derivative of a form θ by a vector field X.
$\alpha^* \theta$	the pull-back of a form θ by a mapping α.
$\theta \mid K$	the pull-back of a form θ on a submanifold K.

1. – Homogeneous Systems.

We call *homogeneous system* a triple $(P, \omega; C)$ where (P, ω) is a symplectic manifold and C is a coisotropic submanifold of (P, ω). We call *characteristic distribution* of C the integrable subbundle

$$\dot{D} = \cup_{x \in C} (T_x C)^{\xi},$$

where ξ denotes the symplectic orthogonal, and *characteristic* a maximal integral connected integral submanifold of D. The relation

$$D = \{(x, y) \in C \times C; x \text{ and } y \text{ belong to the same characteristic } \}$$

is a Lagrangian submanifold (may be immersed) of the product $(P, -\omega) \times (P, \omega)$; hence a symplectic relation on (P, ω). It is a symmetric relation whose image is the submanifold C. We call D the *Hamiltonian relation* associated with the homogeneous system $(P, \omega; C)$. We denote by $P_{[C]}$ the quotient set of the caracteristic distribution of C, i.e. the set of the characteristics of C. The relation

$$[C] = \{(x, c) \in P \times P_{[C]}; x \in c\}$$

composed with its transpose $[C]^t$ gives D:

$$D = [C]^t \circ [C].$$

When the quotient $P_{[C]}$ has a differentiable structure such that the natural projection from C onto $P_{[C]}$ is a submersion, then C is called *simplecticly regular* and (P, ω) is said to be *globally reducible* by C. It is well known that in this case on $P_{[C]}$ a symplectic form is defined in a canonical way. We denote it by $\omega_{[C]}$ and we call $(P_{[C]}, \omega_{[C]})$ the *reduced symplectic manifold*. It turns out that $[C]$ is a Lagrangian submanifold of $(P, -\omega) \times (P_{[C]}, \omega_{[C]})$ i.e. a symplectic relation from (P, ω) onto the reduced symplectic manifold. We call *reduction* any differentiable relation which is the graph of a surjective submersion. Hence $[C]$ is a *symplectic reduction.*

Let us assume that (P, ω) is a cotangent bundle: $(P, \omega) = (T^*Q, \omega_Q)$. A *regular solution* (resp. a *local regular solution*) of the Haminton-Jacobi equation corresponding to the homogenous system $(T^*Q, \omega_Q; C)$ is a function $S : Q \to \mathbb{R}$ (resp. $S : U \to \mathbb{R}$, with U open subset of Q) such that $dS(Q) \subset C$ (resp. $dS(U) \subset C$). We can define non-regular solution (which represent Lagrangian submanifolds of C which are not images of section of T^*Q) by using Morse families (see [2]).

A *regular complete solution* is a differentiable function $S : A \times Q \to \mathbb{R}$ on the product of a manifold A with the manifold Q such that the family of Lagrangian submanifold of T^*Q

$$\{L_a = dS_a(Q); a \in A\}$$

form a foliation covering C and the natural projection from C onto the quotient A is differentiable. We remark that dim A = dim Q − codim C. We have the following version of the *Jacobi theorem: If* $S : A \times Q \to \mathbb{R}$ *is a regular complete solution of the Hamilton-Jacobi associated with the homogenous system* $(T^*Q, \omega_Q ; C)$*, then the relation*

$$R = \{ (p, b) \in T^*Q \times T^*A ; (p, -b) = dS(q, a), (q, a) = (\pi_Q \times \pi_A)(p, b) \}$$

is a symplectic reduction from (P, ω) *onto* (T^*A, ω_A)*. If the fibers of* R *are connected, then* R *is isomorphic to the reduction* $[C]$*, i.e. there exists a symplectomorphism* $\gamma : (P_{[C]}, \omega_{[C]}) \to (T^*A, \omega_A)$ *such that* $R = \Gamma \circ [C]$*, where* $\Gamma =$ graph γ.
It follows that

$$D = R^t \circ R.$$

The definition of *local* regular complete solution can be given in a natural way. Let C be locally defined by independent equations

$$C^a (q^i, p_j) = 0$$

($a = 1, \ldots, r; r = \operatorname{codim} C; i, j = 1, \ldots, n; n = \dim Q$), where (q^i) are local coordinates on Q and (q^i, p_j) are the corresponding canonical coordinates on T^*Q. A local complete solution has a local representative $S(a^k, q^i)$, where (a^k) are $n - r$ parameters, satisfying the Hamilton-Jacobi equation

$$C^a \left(q^i, \frac{\partial S}{\partial q^j} \right) = 0$$

and the condition

$$\operatorname{rank} \left(\frac{\partial^2 s}{\partial a^k \, \partial q^i} \right) = n - r,$$

which means that the Lagrangian submanifolds L_a form a foliation.

The classical homogeneous formulation of Hamiltonian dynamics represent a basic example of homogeneous system (see for instance [2], [4]).

2. – Group Actions.

If $\varphi : G \times P \to P$ is an action of a Lie group G on a manifold P then for each $(g, p) \in G \times P$ we set:

$$\varphi_g : P \to P : p \mapsto \varphi(g, p),$$
$$\varphi^p : G \to P : g \mapsto \varphi(g, p).$$

We denote by X^φ the infinitesimal generator of the action φ corresponding to an element X of the space $T_e G$, where e is the identity in G, identified with the

Lie algebra $\mathscr{T}$ of the left invariant vector field on G. The vector field X^φ is defined by:

$$X^\varphi : P \to TP : p \mapsto T\varphi^p(X).$$

In particular we denote by

$$\lambda : G \times G \to G : (g, g') \mapsto gg',$$

$$\rho : G \times G \to G : (g, g') \mapsto g'g$$

the left and the right action of G onto itself respectively. The Lie algebra structure on $\mathscr{T}$ is defined by the equation

$$[X, Y]^\lambda = -[X^\lambda, Y^\lambda].$$

For any action $\varphi : G \times P \to P$ we have

$$[X, Y]^\varphi = \mp [X^\varphi, Y^\varphi],$$

with $-$ (resp $+$) sign if φ is a left (resp. right) action, i.e. if $\varphi_g \circ \varphi_{g'} = \varphi_{gg'}$ (resp. $\varphi_g \circ \varphi_{g'} = \varphi_{g'g}$ for each $g, g' \in G$. *In the following discussion the group G is assumed to be connected.*

3. – Hamiltonian Actions.

Let (P, ω) be a symplectic manifold. An action $\varphi : G \times P \to P$ is said to be a *symplectic action* if $\varphi_g^* \omega = \omega$ for each $g \in G$. A *momentum mapping* corresponding to a symplectic action $\varphi : G \times P \to P$ is a differentiable mapping $H : P \to \mathscr{T}^*$ such that, for each $X \in \mathscr{T}$:

$$i_{X^\varphi}\omega = -dH_X,$$

where

$$H_X : P \to \mathbb{R} : p \mapsto \langle X, H(p) \rangle.$$

A *Hamiltonian action* is a symplectic action $\varphi : G \times P \to P$ on a symplectic manifold (P, ω) which admits a symplectic mapping H such that, for each $X, Y \in \mathscr{T}$:

$$\{H_X, H_Y\} = \mp H_{[X,Y]}, \tag{3.1}$$

with $-$ (resp. $+$) sign if φ is left (resp. right). The Poisson bracket $\{,\}$ is here defined by

$$\{f_1, f_2\} = i_{X_2} i_{X_1} \omega,$$

where X_1 and X_2 are the vector fields defined by $i_{X_k}\omega = df_k$ $(k = 1, 2)$. We call H the *Hamiltonian mapping* of the action φ. If the group G is Abelian H is determined up to a constant value in $\mathscr{T}^*$.

In Section 5 we use the following

LEMMA. — *Let $H : P \to \mathscr{T}^*$ be the Hamiltonian mapping of a Hamiltonian action $\varphi : G \times P \to P$ on a symplectic manifold (P, ω). If $0 \in \mathscr{T}^*$ is a regular value of H, then $C = H^{-1}(0) = \{p \in P; H(p) = 0\}$ is a coisotropic submanifold of (P, ω) invariant under the action φ. The characteristics of C (i.e. the maximal connected integral manifolds of the characteristic distribution of C) are orbits of φ restricted to C.*

Proof. — By definition of regular value the mapping $T_x H : T_x P \to T_0 \mathscr{T}^* \simeq \mathscr{T}^*$ is a surjective for each $x \in C$. It is known that in this case C is a submanifold of P and that codim $C = \dim G$. If (X_α) $(\alpha = 1, \ldots, m; m = \dim G)$ is a basis of $\mathscr{T}$ and (μ^α) is the corresponding dual basis of $\mathscr{T}^*$, then $H(p) = H_{X_\alpha}(p)\mu^\alpha$ and the functions (H_{X_α}) have independent differentials at each point of C. The submanifold C is defined by the independent equations

$$H_{X_\alpha} = 0 \; (\alpha = 1, \ldots, m)$$

and it is characterized by the following conditions:

$$p \in C \Longleftrightarrow H_X(p) = 0, \quad \forall X \in \mathscr{T}.$$

Since

$$\{H_{X_\alpha}, H_{X_p}\} \mid C = H_{[X_\alpha, X_\beta]} \mid C = 0,$$

C is coisotropic. Furthermore, for each $X, Y \in \mathscr{T}$ we have: $L_{X^\varphi} H_Y = i_{X^\varphi} dH =$ $= - i_{X^\varphi} i_{Y^\varphi} \omega = \{H_X, H_Y\} = H_{[X;Y]}$. Hence $L_{X^\varphi} H_Y \mid C = 0$. This shows that each infinitesimal generator X^φ is tangent to C; thus C is invariant. In particular we have:

$$i_{X^\varphi} \omega \mid C = - dH_X \mid C = 0.$$

This shows that the vectors (X^φ) span the caracteristic distribution of K. In particular the vector fields (X_α^φ) are independent at each point of C. A characteristic of C is a set of points which can be joined by a finite sequence of integral paths of vector fields (X^φ). On the other hand it is known that, for any action of a connected group G on a manifold P, a point $\overline{p} \in P$ belongs to the orbit $\varphi(G, p)$ of a point $\overline{p} \in P$ if and only if $\overline{p}$ can be joined to p by a finite sequence of paths which are images of integral curves of the infinitesimal generators X^φ. It follows that an orbit coincides with a characteristic. (Q.E.D.).

4. – Canonical Lift of Actions.

The canonical lift of an action $\psi : G \times Q \to Q$ is the action which we denote by

$$\hat{\psi} : G \times T^*Q \to T^*Q,$$

defined by

$$\langle w, \hat{\psi}_g(p) \rangle = \langle T\psi_g^{-1}(w), p \rangle,$$

where $p \in T^*Q$, $w \in T_{\psi(g,q)}Q$ and $q = \pi_Q(p)$. For each $g \in G$ we have

$$\varphi_g^* \theta_Q = \theta_Q .$$

Hence $\hat{\psi}$ is a symplectic homogeneous action on the symplectic manifold (T^*Q, ω_Q). For each $X \in \mathscr{T}$ we introduce the mapping

$$J_X^\psi : T^*Q \to \mathbb{R} : p \mapsto \langle X^\psi, p \rangle. \tag{4.1}$$

It is known that

$$J_X^\psi = i_{X^{\hat{\psi}}} \theta_Q$$

and that

$$-dJ_X^\psi = i_{X^{\hat{\psi}}} \omega_Q .$$

This last equality shows that the mapping

$$J^\psi : T^*Q \to \mathscr{T}^*$$

defined by

$$\langle X, J^\psi(p) \rangle = \langle X^\psi, p \rangle = \langle T\psi^q(X), p \rangle, \tag{4.2}$$

for each $X \in \mathscr{T}$ and $p \in P$, where $q = \pi_Q(p)$, is a momentum mapping of the symplectic action $\hat{\psi}$. We remark that (4.2) implies:

$$J^\psi(p) = (T_e \psi^q)^*(p), \quad q = \pi_Q(q), \tag{4.3}$$

where $(T_e \psi^q)^* : T_q^*G \to T_e^*G$ is the dual of the linear mapping $T_e \psi^q : T_e G \to T_q G$. It can be shown that the following identity holds for each $X, Y \in \mathscr{T}$:

$$\{J_X^\psi, J_Y^\psi\} = \mp J_{[X, Y]}^\psi, \tag{4.4}$$

with the already mentioned choice of the sign. This means that the canonical lift $\hat{\psi}$ of any action ψ is a Hamiltonian action with Hamiltonian mapping J^ψ defined in (4.2).

Now, let us look at the lifting operation of actions from the point of view of the theory of symplectic reductions [3]. We remark that any action $\psi : G \times \times Q \to Q$ is a surjective submersion, hence a reduction. We can apply to ψ the phase functor $\mathbb{P}$. We obtain a reduction.

$$\mathbb{P}\psi : T^*G \times T^*Q \to T^*Q$$

(which is not a mapping) such that the diagram

$$\begin{array}{ccc} T^*G \times T^*G & \xrightarrow{\quad \mathbb{P}\psi \quad} & T^*Q \\ \pi_G \times \pi_Q \Big\downarrow & & \Big\downarrow \pi_Q \\ G \times Q & \xrightarrow{\quad \psi \quad} & Q \end{array}$$

commutes. Moreover, the inverse image C of the reduction $\mathbb{P}\psi$ is a coisotropic submanifold of the symplectic manifold $(T^*(G \times Q), \omega_{G \times Q})$. We use here the natural identification $T^*G \times T^*Q \cong T^*(G \times Q)$. The graph of $\mathbb{P}(\psi)$ is defined as follows (see [3], Section 8):

$$\begin{aligned} \text{graph } \mathbb{P}\psi = \{ (h, p, p') \in T^*G \times T^*Q \times T^*Q ; \varphi(g, q) = q', g = \pi_G(h) \\ q = \pi_Q(p), q' = \pi_Q(p'), \langle (u, v), (h, p) \rangle = \langle T\psi(u, v), p' \rangle, \\ \forall (u, v) \in T_{(g,q)}(G \times Q) \}. \end{aligned} \tag{4.5}$$

We have the following:

THEOREM 1. – *Let $\psi : G \times Q \to Q$ be a left action. The graph of $\mathbb{P}\psi$ is the submanifold*

$$\begin{aligned} \Psi = \{ (h, p, p') \in T^*G \times T^*Q \times T^*Q ; \hat{\psi}(g, p) = p', g = \pi_G(h), \\ J^\rho(h) = J^\psi(p) \}, \end{aligned}$$

where $\hat{\psi} : G \times T^*Q \to T^*Q$ is the canonical lift of ψ and $J^\rho : T^*G \to \mathscr{T}^*$, $J^\psi : T^*Q \to \mathscr{T}^*$ are the Hamiltonian mappings of the canonical lifts of the right action $\rho : G \times G \to G$ and of ψ respectively (see definition (4.2)).

Proof. – In (4.5) we have

$$\langle T\psi(u, v), p' \rangle = \langle T\psi_g(v), p' \rangle + \langle T\psi^q(u), p' \rangle.$$

If we choose $u = 0$ then we have:

$$\langle v, p \rangle = \langle T\psi_g(v), p' \rangle, \quad \forall v \in T_q Q,$$

or, equivalently:

$$\langle T\psi_g^{-1}(v'), p \rangle = \langle v', p' \rangle, \quad \forall v' \in T_{q'} Q.$$

If we choose $v = 0$ then we have:

$$(4.6) \qquad \langle u, h \rangle = \langle T\psi^q(u), p' \rangle, \qquad \forall u \in T_g G.$$

Hence:

$$\begin{aligned}\langle u, h \rangle &= \langle T\psi^q(u), \psi_g(p) \rangle \\ &= \langle T\psi_g^{-1} \circ T\psi^q(u), p \rangle \\ &= \langle T(\psi_g^{-1} \circ \psi^q)(u), p \rangle.\end{aligned}$$

Let us set $u = X^\rho(g) = T\rho^g(X)$ where $X \in \mathscr{T}$. Since ψ is a left action we have

$$\psi_g^{-1} \circ \psi^q \circ \rho^g = \psi^q$$

and we can write:

$$\begin{aligned}\langle T\rho^g(X), h \rangle &= \langle T(\psi_g^{-1} \circ \psi^q \circ \rho^g)(X), p \rangle \\ &= \langle T\psi^q(X), p \rangle.\end{aligned}$$

This means that

$$\langle X^\rho(g), h \rangle = \langle X^\psi(q), p \rangle,$$

or, according to definition (4.2):

$$(4.7) \qquad J^\rho(h) = J^\psi(p).$$

It follows that graph $\mathbb{P}\psi \subset \Psi$. We remark that, conversely, equation (4.7) implies (4.6). Hence:

$$\begin{aligned}\langle T\psi(u, v), p' \rangle &= \langle T\psi_g(v), p' \rangle + \langle T\psi^q(u), p' \rangle \\ &= \langle T\psi_g(v), \hat{\psi}_g(p) \rangle + \langle u, h \rangle \\ &= \langle T\psi_g^{-1} \circ T\psi_g(v), p \rangle + \langle u, h \rangle \\ &= \langle v, p \rangle + \langle u, h \rangle.\end{aligned}$$

This proves that $\Psi \subset$ graph $\mathbb{P}\psi$. (Q.E.D.)

COROLLARY. *The inverse image of the reduction* $\mathbb{P}\psi$ *is the coisotropic submanifold*

$$C = \{(h, p) \in T^*G \times T^*Q;\, J^\rho(h) = J^\psi(p)\}.$$

5. – Homogeneous System Corresponding to a Hamiltonian Action.

Let $\varphi : G \times P \to P$ be a Hamiltonian left action on a symplectic manifold

(P, ω) with Hamiltonian mapping $H : P \to \mathscr{T}^*$. The Corollary in the last section suggests to introduce the following objects:

$$(5.1) \quad \begin{cases} \widetilde{P} = T^*G \times P, \\ \widetilde{\omega} = pr_1^*\omega_G + pr_2^*\omega, \\ C = \{(h, p) \in T^*G \times P; J^\rho(h) = H(p)\}, \end{cases}$$

where $pr_1 : \widetilde{P} \to T^*G$, $pr_2 : \widetilde{P} \to P$ are the natural projections and $J^\rho : T^*G \to \mathscr{T}^*$ is the momentum mapping of the lift of the right action ρ on G:

$$\langle X, J^\rho(h)\rangle = \langle T\rho^g(X), h\rangle, \ g = \pi_g(h), \ X \in \mathscr{T}.$$

We expect that C be a coisotropic submanifold of (P, ω) representing the action φ.

THEOREM 2. – *The triple $(\widetilde{P}, \widetilde{\omega}; C)$ defined in* (5.1) *is a homogeneous system whose Hamiltonian relation is*

$$(5.2) \quad D = \{(h, p, h', p') \in \widetilde{P} \times \widetilde{P}; (h, p), (h', p') \in C, \varphi_g(p) = \varphi_{g'}(p'), g = \pi_G(h), g' = \pi_G(h')\}.$$

The symplectic manifold $(\widetilde{P}, \widetilde{\omega})$ is globally reducible by C. The corresponding symplectic reduction is isomorphic to the reduction

$$(5.3) \quad R = \{(h, p, p') \in \widetilde{P} \times P; (h, p) \in C, p' = \varphi_g(p), g = \pi_G(h)\}$$

and

$$(5.4) \quad \text{graph}\ \varphi = (\pi_G \times 1_p \times 1_p)(R).$$

Proof. – Let us consider the following mappings:

$$(5.5) \quad \begin{cases} \widetilde{\varphi} : G \times \widetilde{P} \to \widetilde{P} : (g, h, p) \mapsto (\hat{\rho}(g, h), \varphi(g^{-1}, p)) \\ \widetilde{H} : \widetilde{P} \to \mathscr{T}^* : (h, p) \mapsto J^\rho(h) - H(p). \end{cases}$$

The mapping $\widetilde{\varphi}$ is clarly a right symplectic action on (P, ω) with infinitesimal generators

$$X^{\widetilde{\varphi}} = (X^{\hat{\rho}}, -X^\varphi), \ X \in \mathscr{T}.$$

Because of the definition of momentum mapping, we have for each $X \in \mathscr{T}$:

$$\begin{aligned} d\widetilde{H}_X &= pr_1^* dJ_X^\rho - pr_2^* dH_X \\ &= -pr_1^* i_{X^{\hat{\rho}}}\omega_G + pr_2^* i_{X^\varphi}\omega \\ &= -i_{X^{\widetilde{\varphi}}}\widetilde{\omega}. \end{aligned}$$

Here $\widetilde{H}_X(h, p) = \langle X^{\widetilde{\varphi}}, (h, p)\rangle = \langle X^{\widetilde{\rho}}, h\rangle - \langle X^\varphi, p\rangle$. This shows that $\widetilde{H}$ is a momentum mapping of $\tilde{\varphi}$. Moreover, for each $X, Y \in \mathscr{T}$:

$$\begin{aligned}\tilde{H}_X, \tilde{H}_Y &= \{ pr_1^* J^\rho_X - pr_2^* H_X, pr_1^* J^\rho_Y - pr_2^* H_Y \} \\ &= pr_1^* \{ J^\rho_X, J^\rho_Y \} + pr_2^* \{ H_X, H_Y \} \\ &= pr_1^* J^\rho_{[X,Y]} - pr_2^* H_{[X,Y]} \\ &= \tilde{H}_{[X,Y]}.\end{aligned}$$

This shows that $\tilde{\varphi}$ is a Hamiltonian action with momentum mapping $\tilde{H}$. Since $J^\rho : T^*G \to \mathscr{T}^*$ is a surjective submersion, also $\tilde{H}$ is a surjective submersion. Thus all $\mu \in \mathscr{T}^*$ are regular values of $\tilde{H}$. In particular $0 \in \mathscr{T}^*$ is a regular value and we can apply the Lemma of Section 3 to the present case. It follows that $C = \tilde{H}^{-1}(0)$ is a coisotropic submanifold whose characteristics are orbits of $\tilde{\varphi}$, i.e. they are equivalence classes of the following equivalence relation on C:

$$\begin{aligned}(h, p) \sim (h', p') &\Longleftrightarrow \exists \bar{g} \in G : (h', p') = \tilde{\varphi}_{\bar{g}}(h, p), \\ &\Longleftrightarrow \exists \bar{g} \in G : h' = \hat{\rho}_{\bar{g}}(h),\ p' = \varphi_{\bar{g}}^{-1}(p).\end{aligned}$$

But $h' = \hat{\rho}_{\bar{g}}(h)$ implies $g' = g\bar{g}$ where $g' = \pi_G(h')$, $g = \pi_G(h)$. Hence $\bar{g} = g^{-1}g'$, so that:

$$(h, p) \sim (h', p') \Longleftrightarrow p' = \varphi_{g'} - 1_g(p), g' = \pi_G(h'), g = \pi_G(h).$$

This proves that the Hamiltonian relation corresponding to the homogeneous system $(\tilde{P}, \tilde{\omega}; C)$ is the set (5.2). Since the restriction of J^ρ to the space $T_e^*G = \mathscr{T}^*$ is the identity, an equivalence class $[(h, p)]$ represented by an element $(h, p) \in C$ has a unique representative $(\mu, p') \in T_e^*G \times P$, where:

$$\mu = H(p'), p' = \varphi_g(p), g = \pi_G(h).$$

As a consequence, the submanifold of C

$$M = (T_e^*G \times P) \cap C = \text{graph } H$$

is a representative of the quotient $\tilde{P}_{[C]}$ of C by the characteristic foliation. This means that $(\tilde{P}, \tilde{\omega})$ is globally reducible by C and that the reduced symplectic manifold can be identified with $(M, \tilde{\omega} \mid M)$. Moreover, since M is the graph of the momentum mapping $H : \tilde{P} \to T_e^*G$, the mapping $\pi : M \to P$ induced by the natural projection of $T_e^*G \times P$ onto P is a diffeomorphism. For each vector $u \in T_xM \subset T_x(T_e^*G \times P)$ we have a unique decomposition $u = a + b$, with $a \in TP$ and $b \in T(T_e^*G)$. For two vectors $u_1 = a_1 + b_1$ and $u_2 = a_2 + b_2$ decomposed in this manner we have: $\langle u_1 \wedge u_2, \hat{\omega} \rangle = \langle a_1 \wedge a_2, \omega \rangle + \langle b_1 \wedge b_2, \omega_G \rangle$. Since the fiber T_e^*G of T^*G is a Lagrangian submanifold, we have $\langle b_1 \wedge b_2, \omega_G \rangle = 0$, thus $\langle u_1 \wedge u_2, \tilde{\omega} \rangle = \langle a_1 \wedge a_2, \omega \rangle$. This shows that $\tilde{\omega} \mid M = \pi^* \omega$, i.e. that $(M, \tilde{\omega} \mid M)$

is symplectomorphic to (P, ω). Finally, we remark that graph $(\pi \cdot \sigma)$, where $\sigma : C \to M$ is the natural projection, coincides with the relation R defined in (5.3), and that (5.4) is an obvious consequence of (5.3). (Q.E.D.)

REMARKS. – From equation $J^\rho(h) = H(p)$ and

$$J^\rho(h) = (T_e \rho^g)^*(h), \; g = \pi_G(h),$$

(see property (4.3)) it fellows that the submanifold C is the image of a section $\Gamma : G \times P \to \widetilde{P}$ of the fibration $\pi_G \times 1_p : \widetilde{P} \to G \times P$, where

$$\Gamma(g, p) = (((T_e \rho^g)^*)^{-1} \circ H(p), p).$$

Moreover, if consider the natural isomorphism

$$\iota : T^*G \to \mathscr{T}^* \times G : h \mapsto (J^\rho(h), \pi_G(h)),$$

then we have

$$(\iota \times 1_P \times 1_P)(R) = \{(\mu, g, p, p') \in \mathscr{T}^* \times G \times P \times P;$$

$$p' = \varphi_g(p), \mu = H(p)\}.$$

This submanifold has been considered in [1], Exercise 5.31, p. 422, where the fact that H is a coadjoint equivariant momentum mapping seems to be understood.

6. – Local Coordinate Representation.

Let (g^α) be local coordinates in a neighborhood of the identity e in G and let (g^α, h_β) be the corresponding canonical coordinates on T^*G (Greek indices run from 1 to $m = \dim G$). We have a natural basis $(X_\alpha = \partial/\partial g^\alpha |_e)$ of $\mathscr{T}$ and a dual basis (μ^α) of $\mathscr{T}^*$: $\langle X_\alpha, \mu^\beta \rangle = \delta^\beta_\alpha$.

Let (q^i, p_j) be local canonical coordinates of (P, ω) (Latin indices run from 1 to $n = 1/2 \dim P$). We have local representations of the following type:

$$\widetilde{\omega} = dh_\alpha \wedge dg^\alpha + dp_i \wedge dq^i,$$

$$J^\rho = J^\alpha_\beta h_\alpha \mu^\beta,$$

$$H = H_\beta \mu^\beta,$$

where J^α_β are functions of coordinates (g^α) only such that

(6.1) $$\det (J^\alpha_\beta) \neq 0,$$

and H_β are functions of coordinates (q^i, p_j). An analogous representation holds for the momentum mapping J^λ. The submanifold C is then locally described by equations of the kind:

$$J^{\alpha}_{\beta}(g^{\gamma})h_{\alpha} - H_{\beta}(q^i, p_j) = 0. \tag{6.2}$$

If $C^{\gamma}_{\alpha\beta}$ are the structure constant of G,

$$[X_{\alpha}, X_{\beta}] = C^{\gamma}_{\alpha\beta}X_{\gamma},$$

then equations (3.1) for the Hamiltonian momentum mappings J^{ρ} and H assume the form

$$J^{\alpha}_{\beta}\partial_{\alpha}J^{\delta}_{\gamma} - J^{\alpha}_{\gamma}\partial_{\alpha}J^{\delta}_{\beta} = C^{\alpha}_{\beta\gamma}J^{\delta}_{\alpha}, \tag{6.3}$$

where $\partial_{\alpha} = \partial/\partial g^{\alpha}$, and

$$\{H_{\alpha}, H_{\beta}\} = -C^{\gamma}_{\alpha\beta}H_{\gamma}, \tag{6.4}$$

respectively.

From the preceding discussion we know that equations (6.3) and (6.4) express the coisotropy of C.

7. – The Hamilton-Jacobi Equation.

Let us assume that $(P, \omega) = (T^*Q, \omega_Q)$. We have a natural identification $(\widetilde{P}, \widetilde{\omega}) \simeq (T^*(G \times Q), \omega_{G\times Q})$. We use local coordinates (q^i) of Q and the corresponding canonical coordinates (q^i, p_j) on T^*Q. A function $S : G \times Q \to \mathbb{R}$ is a regular solution of the Hamilton-Jacobi equation corresponding to the homogeneous system $(\widetilde{P}, \widetilde{\omega}; C)$ if $dS(G \times Q) \subset C$. This means that locally we have (see (6.2)):

$$J^{\alpha}_{\beta}(g^{\gamma})\ \frac{\partial S}{\partial g^{\alpha}} - H_{\beta}\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0. \tag{7.1}$$

Condition (6.1), (6.3) and (6.4) are integrability conditions of this equations. Thus this Hamilton-Jacobi equation admits local regular solutions everywhere.

According to the general theory, a regular complete solution is a function $S : A \times G \times Q \to \mathbb{R}$ satisfying the conditions described in the introduction. Here we have $\dim A = \dim Q = n$, since T^*A has the same dimension of the reduced manifold T^*Q (see Theorem 2).

A local regular complete solution is a function $S(a^i, g^{\alpha}, q^j)$ satisfying the Hamilton-Jacobi equation (7.1) together with the condition:

$$\operatorname{rank}\left(\frac{\partial^2 S}{\partial g^{\alpha}\partial a^i}, \frac{\partial^2 S}{\partial q^j \partial a^i}\right) = \dim Q. \tag{7.2}$$

However, if we take the derivative of equation (7.1) with respect to a coordinate a^i, then we find:

$$\frac{\partial H_\beta}{\partial p_j}\ \frac{\partial^2 S}{\partial q^j \partial a^i} + J^\alpha_\beta\ \frac{\partial^2 S}{\partial g^\alpha \partial a^i} = 0.$$

Since (J^α_β) is a regular matrix, we see that condition (7.2) is simply equivalent to the following one:

$$\text{(7.3)} \qquad \det\left(\frac{\partial^2 S}{\partial q^j \partial a^i}\right) \neq 0.$$

The Jacobi theorem assures that the reduction R associated with C is locally described by equations:

$$b_i = -\frac{\partial S}{\partial a^i},\quad h_\alpha = \frac{\partial S}{\partial g^\alpha},\quad p_j = \frac{\partial S}{\partial q^j}$$

Then the equations:

$$\text{(7.4)} \qquad b_i = -\frac{\partial S}{\partial a^i},\quad p_j = \frac{\partial S}{\partial q^j}$$

describe the action φ. Because of condition (7.3), we can locally solve the first n equations (7.4) with respect to the (q^i). Then, after the substitution in the second part of (7.4), we obtain functions of the kind:

$$q^i = \varphi^i(g^\alpha, a^j, b_k),$$
$$p_i = \varphi_i(g^\alpha, a^j, b_k).$$

These are local representative of the action φ.

8. – An Alternative Construction.

There is another way of constructing a homogeneous system corresponding to a Hamiltonian action, slightly different from that presented in Section 5 and which is a natural generalization of the usual homogeneous system associated with a Hamiltonian flow, as shown in the Example given below.

Under the same hypotheses of Section 5 we still consider the symplectic manifold

$$(\tilde{P}, \tilde{\omega}) = (T^*G \times P, pr_1^*\omega_G + pr_2^*\omega),$$

but, instead of $(5.1)_3$ we take

$$C = \{(h, p) \in \tilde{P};\ J^\lambda(h) + H(p) = 0\},$$

by using the momentum mapping J^λ of the canonical lift of the left action

λ on G. A theorem analogous to Theorem 2 can be proved. The proof follows the same pattern. Instead of mapping (5.5) we take

$$\tilde{\varphi} : G \times \tilde{P} \times \tilde{P} : (g, h, p) \mapsto (\hat{\lambda}(g, h), \varphi(g, p)),$$

$$\tilde{H} : \tilde{P} \to \mathscr{T}^* : (h, p) \mapsto J^\lambda (h) + H(p).$$

It turns out that $\tilde{\varphi}$ is a left Hamiltonian action on $(\tilde{P}, \tilde{\omega})$ with Hamiltonian mapping $\tilde{H}$, hence that $C = \tilde{H}^{-1}(0)$ is a coisotropic submanifold whose characteristics are orbits of $\tilde{\varphi}$. It follows that the Hamiltonian relation corresponding to the new homogeneous system $(\tilde{P}, \tilde{\omega}; C)$ is:

$$D = \{ (h, p, h', p') \in \tilde{P} \times \tilde{P}; (h, p), (h', p') \in C,$$

$$\varphi_g^{-1}(p) = \varphi_{g'}^{-1}(p'),\ g = \pi_G(h),\ g' = \pi_G(h')\},$$

and that the corresponding symplectic reduction is:

$$R = \{(h, p, p') \in \tilde{P} \times P;\ (h, p) \in C, p' = \varphi_g^{-1}(p),\ g = \pi_G(h)\}.$$

Property (5.4) does not hold in this case. Instead we have:

$$(\pi_G \times 1_P \times 1_P)(R) = \{ (g, p, p') \in G \times P \times P;\ p = \psi_g(p')\}.$$

The Hamilton-Jacobi equation is locally represented by the system:

$$J^\alpha_\beta(g^\gamma)\ \frac{\partial S}{\partial g^\alpha} + H_\beta\left(q^i, \frac{\partial S}{\partial q^i}\right) = 0, \tag{7.1}$$

where the regular matrix (J^α_β) corresponds to the left moment mapping; $J^\lambda = J^\alpha_\beta h_\alpha \mu^\beta$.

EXAMPLE. – Let $G = \mathbb{R}$ and $\varphi : \mathbb{R} \times P \to P$ be a symplectic R-action on (P, ω). We have $\mathscr{T} = \mathbb{R}$. For $X = 1 \in \mathscr{T}$, the infinitesimal generator X^φ of φ is the Hamiltonian vector field corresponding to the symplectic flow. If this vector is globally Hamiltonian with Hamiltonian $H : P \to \mathbb{R}$, then the action is Hamiltonian with Hamiltonian momentum mapping H. Since the group is Abelian we have $J^\rho = J^\lambda = J$ with $J : T^*G = \mathbb{R}^2 \to \mathscr{T}^* = \mathbb{R} : (t, h) \mapsto h$. For each $t, s, \in G = \mathbb{R}$, $\hat{\lambda}_t(s, h) = (s + t, h)$. Moreover, $\tilde{P} = T^*G \times P = \{(t, h, p) \in \mathbb{R}^2 \times P\}$, $\tilde{\omega} = dh \wedge dt + \omega$ (abuse of notation) and $C = \{ (t, h, p) \in \tilde{P}; h + H(p) = 0\}$. We obtain the known homogeneous representation for a (complete) globally Hamiltonian vector field.

References.

[1] ABRAHAM R. and MARSDEN J.E., *Foundations of Mechanics*, Benjamin-Cummings,

2nd ed. (1978).
[2] BENENTI S., *Symplectic Relations in Analytical Mechanics,* in Proceedings of IUTAM-ISIMM Symposium on «Modern Developments in Analytical Mechanics», Turin, June 7 - 11, 1982 (in print).
[3] BENENTI S., *The Category of Symplectic Reduction,* in Proceedings of the International Meeting «Geometry and Physics», Florence, October 12 - 15, 1982 (in print).
[4] BENENTI S. and TULCZYJEW W.M., *The Geometrical Meaning and Globalization of the Hamilton-Jacobi Method,* in Lecture Notes in Math. 836 (1980), pp. 9 - 21.

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