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The Category of Symplectic Reductions

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Introduction

A symplectic reduction is a symplectic relation between two symplectic manifolds (P_1, ω_1) and (P_2, ω_2) whose graph represents a surjective submersion from a submanifold of P_1 onto P_2. Symplectic reductions appear in the symplectic formulation of various physical theories. It is known, for example, that the Hamilton-Jacobi theory can be formulated in terms of generating functions of symplectic reductions [1] [2]. Furthermore, the behaviour of reciprocal statical or dynamical systems with singularities can be described by using symplectic reductions [3]. Symplectic reductions appear also in the global framework for higher order calculus of variations presented in [4].

Symplectic manifolds and symplectic reductions form a category which extends the category of symplectomorphisms. The aim of this article is to present some general results concerning the category of symplectic reductions and related functors.

1. Differentiable Relations

A *differentiable relation* is a triple $\rho = (P_1, P_2; R)$ where P_1, P_2 are differentiable manifolds and R is a submanifold of the product manifold $P_1 \times P_2$. We call P_1, P_2 and R the *domain*, the *codomain* and the *graph* of the relation ρ respectively. Notation $\rho : P_1 \to P_2$ and R = graph ρ is also used for a differentiable relation whose domain, codomain and graph are P_1, P_2 and R respectively.

Let $\rho = (P_1, P_2; R)$ be a differentiable relation. We denote by $\rho^t = (P_2, P_1; R^t)$, with $R^t = \{(p_2, p_1) \in P_2 \times P_1; (p_1, p_2) \in R\}$, the *transpose* of ρ. Let K be a subset of P_1. We call the set

$$\rho(K) = R \circ K = \{p_2 \in P_2;\ \exists p_1 \in K : (p_1, p_2) \in R\}$$

the *image of K by the relation* ρ. In particular we call $R \circ P_1$ and $R^t \circ P_2$ the *image* and the *inverse image* of ρ respectively.

The composition $\sigma \circ \rho = (P_1, P_3; S \circ R)$ of two differentiable relations

$\rho = (P_1, P_2; R)$ and $\sigma = (P_2, P_3; S)$ is defined by

$$S \circ R = \{(p_1, p_3) \in P_1 \times P_3;\ \exists p_2 \in P_2 : (p_1, p_2) \in R, (p_2, p_3) \in S\}.$$

In general $S \circ R$ is not a submanifold of $P_1 \times P_3$, thus $\sigma \circ \rho$ is not a differentiable relation.

A differentiable mapping $\rho : P_1 \to P_2$ is a particular case of differentiable relation. A differentiable relation $\rho : P_1 \to P_2$ which is a mapping is not necessarily a differentiable mapping.

2. Differentiable Reductions

We call *differentiable reduction* (or simply *reduction*) a differentiable relation $\rho = (P_1, P_2; R)$ such that R is the graph of a surjective submersion $\tilde{\rho} : C \to P_2$ from a submanifold $C \subset P_1$ onto P_2. We say that $\tilde{\rho}$ is the submersion associated with the reduction ρ. For each point $p_2 \in P_2$ the set $\rho^t(p_2)$ is a submanifold of C, thus a submanifold of P_1. We call such a submanifold a *fiber* of ρ. For each fiber F we have

(2.1) $$\dim F = \dim C - \dim P_2.$$

From the well known properties of submersions it follows that if $S \subset P_2$ is a submanifold, then $\rho^t(S)$ is a submanifold of P_1 and

(2.2) $$\dim \rho^t(S) = \dim S + \dim C - \dim P_2.$$

It can be easily seen that the composition of two reductions is again a reduction. We conclude that the differentiable manifolds and differentiable reductions are the objects and the morphisms of a category. We call this category the *category of reductions* and we denote it by $\mathscr{R}$.

Elementary examples of differentiable reductions:

(i) a diffeomorphism;

(ii) a surjective submersion;

(iii) the transpose of an embedding.

(2.3) REMARK. A reduction is always the composition of a reduction of the type (iii) and a reduction of the type (ii).

(2.4) REMARK. We can consider morphisms more general than the differentiable reductions defined above. For example, we can abandon the requirement that the submersion associated with the relation is surjective. On the other hand we can define as *local reduction* any differentiable relation $\rho = (P_1, P_2; R)$ such that for each $(p_1, p_2) \in R$ there exist open neighborhoods U_1 and U_2 of p_1 and p_2 respectively, such that the relation $(U_1, U_2; R \cap (U_1 \times U_2))$ is a reduction. The statements concerning reductions contained in the present paper can be extended (with suitable slight modifications) to these more general types of relations.

3. Symplectic Relations

Let (P, ω) be a symplectic manifold (P is an even-dimensional differentiable manifold, ω is a closed non-degenerate 2-form on P). We denote by $(P, \omega)^t$ the symplectic manifold $(P, -\omega)$. Let $K \subset P$ be a submanifold. For each $x \in P$ we set

$$(T_x K)^{\S} = \{v \in T_x P;\ \langle u \wedge v, \omega\rangle = 0,\ \forall u \in T_x K\}.$$

The submanifold K is said to be *isotropic* (resp. *coisotropic, Lagrangian*) if $(T_x K)^{\S} \supset T_x K$ (resp. $(T_x K)^{\S} \subset T_x K$, $(T_x K)^{\S} = T_x K$) for each $x \in K$. In these three cases we have $\dim K \leqslant 1/2 \dim P$, $\dim K \geqslant 1/2 \dim P$, $\dim K = 1/2 \dim P$ respectively. The submanifold K is isotropic if and only if $\omega | K = 0$ ($\omega | K$ denotes the pull-back of the form ω to K). The submanifold K is coisotropic if and only if for each pair (f, g) of differentiable real functions on P which are constant on K the Poisson bracket $\{f, g\}$ vanishes on K.

The product of two symplectic manifolds (P_1, ω_1) and (P_2, ω_2) is the symplectic manifold

$$(P_1, \omega_1) \times (P_2, \omega_2) = (P_1 \times P_2, pr_1^* \omega_1 + pr_2^* \omega_2),$$

where $pr_i : P_1 \times P_2 \to P_i$ $(i = 1, 2)$ are the natural projections.

A *symplectic relation* from a symplectic manifold (P_1, ω_1) to a symplectic manifold (P_2, ω_2) is a differentiable relation $\rho = (P_1, P_2; R)$ such that R is a Lagrangian submanifold of $(P_1, \omega_1)^t \times (P_2, \omega_2)$. For a symplectic relation we use the notation $\rho = ((P_1, \omega_1), (P_2, \omega_2); R)$ or $\rho : (P_1, \omega_1) \to (P_2, \omega_2)$. The symplectic relation ρ is said to be *linear* if (P_1, ω_1) and (P_2, ω_2) are symplectic vector spaces and R is a (linear) Lagrangian subspace.

It is known that a differentiable mapping $\rho : P_1 \to P_2$ between two symplectic manifolds (P_1, ω_1) and (P_2, ω_2) is a symplectic relation if and only if ρ is a local symplectomorphism.

If $((P_1, \omega_1), (P_2, \omega_2); R)$ is a symplectic relation, then for each $(p_1, p_2) \in R$ the triple $((T_{p_1} P_1, \omega_1 | p_1), (T_{p_2} P_2, \omega | p_2), T_{(p_1,p_2)} R)$ is a linear symplectic relation. The following propositions are easily derived from the theory of linear symplectic relations (see [5], [6], [7]):

(3.1) *Let* $\rho : (P_1, \omega_1) \to (P_2, \omega_2)$ *be a symplectic relation. If* $\rho^t(P_2)$ *(resp.* $\rho(P_1)$*) is a submanifold then it is coisotropic.*

(3.2) *If the composition $S \circ R$ of two symplectic relations is differentiable and $TS \circ TR = T(S \circ R)$, then it is a symplectic relation.*

4. Symplectic Reductions

We call *symplectic reduction* a symplectic relation whose underlying differentiable relation is a reduction. From the fact that differentiable reductions form a category and from (3.2) (see also (11.6)) it follows that symplectic manifolds and symplectic reductions are objects and morphisms of a category. We call this category the *category of symplectic reductions* and we denote it by $\mathcal{S}\mathcal{R}$.

From the definition of reduction and from (3.1) we observe that the inverse image of a symplectic reduction is a coisotropic submanifold of the domain. We have the following more general statement:

(4.1) *If $\rho = ((P, \omega), (P_0, \omega_0); R)$ is a symplectic reduction and if S is an isotropic (resp. coisotropic, Lagrangian) submanifold of (P_0, ω_0), then $\rho^t(S)$ is an isotropic (resp. coisotropic, Lagrangian) submanifold of (P, ω).*

Proof. We know that $K = \rho^t(S)$ is a submanifold. For each $(x, y) \in R$, the subspace $T_{(x,y)}R \subset T_{(x,y)}(P \times P_0)$ defines a linear symplectic relation. We have $T_x K = (T_{(x,y)} R)^t \circ (T_y S)$. From the theory of linear symplectic relations we know that the image (or the inverse image) of an isotropic (resp. coisotropic, Lagrangian) subspace by a linear symplectic relation is isotropic (resp. coisotropic, Lagrangian). (Q.E.D.)

We present a basic example of symplectic reduction. Let C be a coisotropic submanifold of a symplectic manifold (P, ω). We call *characteristic distribution* of C the involutive subbundle

$$(TC)^{\S} = \cup_{x \in C} (T_x C)^{\S}$$

of TC, and *characteristic* of C any maximal connected integral submanifold of $(TC)^{\S}$. Let us denote by $P_{[C]}$ the set of the characteristics. It is known that if $P_{[C]}$ has a differentiable structure such that the natural projection $\tilde{\rho}_C : C \to P_{[C]}$ is a submersion, then there exists a symplectic form on $P_{[C]}$, which we denote by $\omega_{[C]}$, such that $\tilde{\rho}_C^* \, \omega_{[C]} = \omega \mid C$. In this case we say that (P, ω) is *globally reducible by C*. The symplectic form $\omega_{[C]}$ is defined by the following equation:

(4.2) $$\langle T\tilde{\rho}_C(u) \wedge T\tilde{\rho}_C(v), \omega_{[C]} \rangle = \langle u \wedge v, \omega \rangle,$$

for each $u, v \in T_x C$. Let us define

(4.3) $$[C] = \{(p, y) \in P \times P_{[C]} ; p \in C, y = \tilde{\rho}_C(p)\}.$$

The set $[C]$ is the graph of $\tilde{\rho}_C$ interpreted as a subset of $P \times P_{[C]}$. It can be proved that

(4.4) $\rho_C = ((P, \omega), (P_{[C]}, \omega_{[C]}); [C])$

is a symplectic relation [8]. Since $\tilde{\rho}_C$ is a (surjective) submersion, we conclude that ρ_C is a symplectic reduction. We call ρ_C *the reduction with respect to C.* We call

(4.5) $(P, \omega)_{[C]} = (P_{[C]}, \omega_{[C]})$

the *reduced symplectic manifold of* (P, ω) *by C.*

(4.6) *Let* $\rho = ((P, \omega), (P_0, \omega_0); R)$ *be a symplectic reduction and let* $\tilde{\rho} : C \to P_0$ *be the corresponding submersion. Then:* (i) $\tilde{\rho}^*\omega_0 = \omega \mid C$; (ii) *the fibers of* ρ *are integral manifolds of the characteristic distribution of* C; (iii) $2 \operatorname{codim} C = \dim P - \dim P_0$.

Proof. For each $(x, y) \in R$ let us consider the linear symplectic relation $T_{(x,y)}\rho : T_x P \to T_y P$ defined by $T_{(x,y)} R$ and the linear mapping $T_x\tilde{\rho} : T_x C \to T_y P$ (we have $y = \tilde{\rho}(x)$). For each pair of vectors $u, v \in T_x C$ we have $(u, T_x\tilde{\rho}(u)), (v, T_x\tilde{\rho}(v)) \in T_{(x,y)} R$. Since $T_{(x,y)}\rho$ is symplectic, $\langle u \wedge v, \omega\rangle - \langle T_x\tilde{\rho}(u) \wedge T_x\tilde{\rho}(v), \omega_0\rangle = 0$. Thus (i) is proved. Let $F = \rho^t(y)$ be a fiber of ρ. Since $\tilde{\rho}$ is a submersion for each $x \in F$ the linear relation $T_{(x,y)}\rho$ is an epimorphism (in the category of linear symplectic relations). It follows that $(T_x C)^§$ is the inverse image by $T_{(x,y)}\rho$ of the zero vector and that (*) $\dim (T_x C)^§ + \dim T_y P_0 = \dim T_x C$. Hence we have $T_x F \subset (T_x C)^§$ and $\dim (T_x C)^§ = \dim C - \dim P_0$. On the other hand (see (2.1)) $\dim F = \dim C - \dim P_0$, thus $T_x F = (T_x C)^§$ and (ii) is proved. From equation (*) above it follows that $\dim T_x P - \dim T_y P_0 - \dim (T_x C)^§ = \operatorname{codim} T_x C$. On the other hand we have $\dim (T_x C)^§ = \operatorname{codim} T_x C$, thus $\dim T_x P - \dim T_y P_0 = 2 \operatorname{codim} T_x C$ and equation (iii) is proved. (Q.E.D.)

(4.7) *Let* $\rho : (P, \omega) \to (P_0, \omega_0)$ *be a symplectic reduction. For each submanifold* $S \subset P_0$ *we have* $\dim \rho^t(S) - \dim S = 1/2 (\dim P - \dim P_0)$.

Proof. We use (2.2) and part (iii) of proposition (4.6): $\dim \rho^t(S) - \dim S = \dim C - \dim P_0 = \dim P - \operatorname{codim} C - \dim P_0 = \dim P - 1/2 (\dim P - \dim P_0) - \dim P_0$. (Q.E.D.)

(4.8) *Let* $\rho : (P, \omega) \to (P_0, \omega_0)$ *be a symplectic reduction with connected fibers and let C be the inverse image of* ρ. *Then the symplectic manifold* (P, ω) *is globally reducible by C and there exists a symplectomorphism* $\iota : (P, \omega)_{[C]} \to (P_0, \omega_0)$

such that $\rho = \iota \circ \rho_C$, *where* ρ_C *is the reduction defined by C.*

Proof. Since the fibers are connected, from proposition (4.6), part (ii), it follows that each fiber of ρ is a characteristic of C. Thus there is a natural bijective mapping $\iota : P_{[C]} \to P_0$ which associates with each $y \in P_0$ the corresponding fiber. The mapping ι induces from P_0 a differentiable structure on $P_{[C]}$ such that the natural projection $\tilde{\rho}_C : C \to P_{[C]}$ is a submersion. From part (i) of (4.6) it follows that $\langle T\tilde{\rho}(u) \wedge T\tilde{\rho}(v), \omega_0 \rangle = \langle u \wedge v, \omega \rangle$ for each $u, v \in T_x C$, where $\tilde{\rho} : C \to P_0$ is the submersion associated with ρ. The left hand side of the last equation becomes equal to $\langle T(\iota \circ \tilde{\rho}_C)(u) \wedge T(\iota \circ \tilde{\rho}_C)(v), \omega_0 \rangle = \langle T\tilde{\rho}_C(u) \wedge T\tilde{\rho}_C(v), \iota^*\omega_0 \rangle$. The right one is equal to $\langle T\tilde{\rho}_C(u) \wedge T\tilde{\rho}_C(v), \omega_{[C]} \rangle$ (see definition (4.2)). This shows that $\iota^*\omega_0 = \omega_{[C]}$. (Q.E.D.)

5. Lagrangian Submanifolds Generated by Forms

Let Q be a differentiable manifold. We denote by $\tau_Q : TQ \to Q$ and $\pi_Q : T^*Q \to Q$ the tangent and the cotangent bundle projections respectively. We denote by θ_Q the *Liouville 1-form* on T^*Q. We recall that θ_Q is defined by the equation

(5.1) $$\langle v, \theta_Q \rangle = \langle T\pi_Q(v), \tau_{T^*Q}(v) \rangle, \quad \forall v \in TT^*Q.$$

We consider on T^*Q the *canonical symplectic form*

(5.2) $$\omega_Q = d\theta_Q$$

and we call the symplectic manifold

(5.3) $$\mathbb{P}Q = (T^*Q, \omega_Q)$$

the *phase manifold* of Q.

Let $K \subset Q$ be a submanifold and let γ be a 1-form on K. We define

(5.4) $$\mathfrak{L}(Q; K, \gamma) = \{p \in T^*Q; q = \pi_Q(p) \in K, \langle u, p \rangle = \langle u, \gamma \rangle, \forall u \in T_q K\}.$$

(5.5) (i) *The set* $L = \mathfrak{L}(Q; K, \gamma)$ *is a submanifold of* T^*Q *and* $\dim L = \dim Q$; (ii) *the mapping* $\pi : L \to K$ *induced by* π_Q *is a surjective submersion with connected fibers*; (iii) $\pi^*\gamma = \theta_Q | L$.

Proof. (i) Let (q^α) $(\alpha = 1, \ldots, n)$ be local coordinates on Q such that K is locally defined by equations $q^{\bar{a}} = 0$ $(\bar{a} = 1, \ldots, l; l = \text{codim } K)$. From (5.4) we see that L is locally described by the following n equations: $q^{\bar{a}} = 0$, $p_a = \gamma_a(q^b)$ $(a, b = l + 1, \ldots, n)$, where (q^α, p_α) are the canonical coordinates on T^*Q associated with (q^α) and γ_a are the corresponding local components of γ (coordinates (q^a)

can be interpreted as coordinates on K, thus $\gamma = \gamma_a dq^a$). (ii) The mapping π is clearly a surjective submersion. If $q \in K$ and $p \in \pi^t(q)$, then from (5.4) it follows that $\pi^t(q) = p + (T_q K)^0$, where $(T_q K)^0$ is the space of covectors vanishing on $T_q K$. This shows in particular that the fibers are connected. (iii) For each $p \in L$ and for each $v \in T_p L$ we have: $\langle v, \theta_Q \rangle = \langle T\,\pi(v), p \rangle = \langle T\,\pi(v), \gamma \rangle = \langle v, \pi^*\gamma \rangle$. (Q.E.D.)

As a corollary of proposition (5.5) we have:

(5.6) *The manifold* $\mathfrak{L}(Q; K, \gamma)$ *is a Lagrangian submanifold of* $\mathbb{P}Q$ *if an only if* $d\gamma = 0$.

The following converse statement can be obtained by summarizing results derived in [9]. A direct proof will be given in Section 8.

(5.7) *Let L be a Lagrangian submanifold of a phase manifold* $\mathbb{P}Q$ *such that* $K = \pi_Q(L) \subset Q$ *is a submanifold and the surjective mapping* $\pi : L \to K$ *induced by* π_Q *is a submersion with connected fibers. Then:* (i) *there exists a unique closed 1-form* γ *on K such that* $\theta_Q \,|\, L = \pi^*\gamma$; (ii) *L is an open submanifold of* $\mathfrak{L}(Q; K, \gamma)$.

The 1-form $\gamma : T^*K \to K$ is called *the generating form* of L. Any (local) integral function of γ is called a *(local) generating function* of L.

Let us consider the following two particular cases of definition (5.4):

$$(5.8)\qquad \begin{cases} \gamma(Q) = \mathfrak{L}(Q; Q, \gamma), \\ \mathbb{N}K = \mathfrak{L}(Q; K, 0) = \{p \in T^*Q; q = \pi_Q(p) \in K, \langle u, p\rangle = 0,\ \forall u \in T_q K\} \\ \qquad = \cup_{q\in K} (T_q K)^0. \end{cases}$$

The submanifold $\gamma(Q)$ is the image of the section $\gamma : Q \to T^*Q$. The submanifold $\mathbb{N}K$ is called the *conormal bundle* of the submanifold K.

We say that a submanifold L of a cotangent bundle T^*Q is *homogeneous* if $p \in L \Rightarrow rp \in L,\ \forall r \in \mathbb{R}$. (This terminology differs from that used by other Authors). We remark that $\mathbb{N}K$ is a homogeneous Lagrangian submanifold of $\mathbb{P}Q$. In the next sections we shall use the following

(5.9) LEMMA. *Let L be a homogeneous Lagrangian submanifold of a phase manifold* $\mathbb{P}Q$ *such that* $K = \pi_Q(L) \subset Q$ *is a submanifold. Then* $L = \mathbb{N}K$.

Proof. Let us take the set $C = \{p \in T^*Q; \pi_Q(p) \in K\} = \pi_Q^t(K)$. This set is a coisotropic submanifold of $\mathbb{P}Q$. The characteristics of C are the equivalence classes of the following equivalence relation on C:

$$p \sim \bar{p} \Leftrightarrow q = \pi_Q(p) = \pi_Q(\bar{p}),\ p - \bar{p} \in (T_q K)^0 .$$

The dimension of a characteristic is equal to codim K and the vectors belonging to $(TC)^{\S}$ are vertical with respect to the projection π_Q, i.e. $v \in (TC)^{\S} \Rightarrow T\pi_Q(v) = 0$. Let us consider K as a submanifold of the «zero section» of T^*Q. Thus K is an isotropic submanifold of $\mathbb{P}Q$ contained in C and transversal to the characteristics of C. By the global Cauchy theorem for the Hamilton-Jacobi equation (see for instance [10], Theorem 5.3.30), we can observe that the Lagrangian submanifold $\mathbb{N}K = \cup_{q \in K} (T_q K)^{\S} \subset C$ is the unique maximal Lagrangian submanifold made of characteristics of C issued from K. On the other hand the homogeneous submanifold L also contains K (interpreted again as a submanifold of the zero section of T^*Q) and it is contained in C. We conclude that L is an open submanifold of $\mathbb{N}K$. The projection $\pi : L \to K$ induced by π_Q is clearly a submersion, thus the set $L_q = L \cap T_q^*Q = \pi^t(q)$ is a submanifold for each $q \in K$. On the other hand, since L is homogeneous, each L_q is the union of 1-dimensional subspaces (or it is the zero vector) of T_q^*Q. We conclude that L_q is a subspace of T_q^*Q. Furthermore, from $L \subset \mathbb{N}K$ it follows that $L_q \subset (T_q K)^0$. Both these spaces have the same dimension, thus $L_q = (T_q K)^0$ for each $q \in K$. This shows that $L = \mathbb{N}K$.

(Q.E.D.)

6. Canonical Lift of Differentiable Relations

Let us consider the product $Q_1 \times Q_2$ of two differentiable manifolds Q_1 and Q_2. The mapping

$$\nu : T^*Q_1 \times T^*Q_2 \to T^*(Q_1 \times Q_2)$$

defined by

$$(6.1) \qquad \langle (u_1, u_2), \nu(p_1, p_2) \rangle = -\langle u_1, p_1 \rangle + \langle u_2, p_2 \rangle,$$

where $(u_1, u_2) \in T_{(q_1, q_2)}(Q_1 \times Q_2)$, $q_1 = \pi_{Q_1}(p_1)$, $q_2 = \pi_{Q_2}(p_2)$, is a symplectomorphism from $(\mathbb{P}Q_1)^t \times \mathbb{P}Q_2$ onto $\mathbb{P}(Q_1 \times Q_2)$.

For each submanifold $A \subset Q_1 \times Q_2$ we define (see (5.8))

$$(6.2) \qquad \widetilde{\mathbb{N}}A = \nu^{-1}(\mathbb{N}A) =$$

$$= \{ (p_1, p_2) \in T^*Q_1 \times T^*Q_2 ; (q_1, q_2) = (\pi_{Q_1} \times \pi_{Q_2})(p_1, p_2) \in A,$$

$$\langle u_1, p_1 \rangle = \langle u_2, p_2 \rangle,\ \forall (u_1, u_2) \in T_{(q_1, q_2)} A \}.$$

Since $\mathbb{N}A$ is Lagrangian and ν is symplectic, $\widetilde{\mathbb{N}}A$ is a Lagrangian submanifold of $(\mathbb{P}Q_1)^t \times \mathbb{P}Q_2$.

(6.3) DEFINITION. Let $\alpha = (Q_1, Q_2; A)$ be a differentiable relation. We call the symplectic relation

$$\mathbb{P}\alpha = (\mathbb{P}Q_1, \mathbb{P}Q_2; \widetilde{\mathbb{N}}A)$$

the *canonical lift* of α.

The following properties hold:

(6.4) $\pi_{Q_2} \circ \mathbb{P}\alpha = \alpha \circ \pi_{Q_1}$;

(6.5) $\mathbb{P}(\alpha^t) = (\mathbb{P}\alpha)^t$;

(6.6) $\mathbb{P}1_Q = 1_{T^*Q}$

(we denote by 1_M the identity transformation on a manifold M).

According to the terminology adopted in Section 5 we say that a relation between cotangent bundles $\rho = (T^*Q_1, T^*Q_2; R)$ is *homogeneous* if $(p_1, p_2) \in \in R \Rightarrow (rp_1, rp_2) \in R$ for each $r \in \mathbb{R}$.

The composition of two homogeneous relations is again a homogeneous relation. The canonical lift of a differentiable relation is homogeneous. By using the symplectomorphism ν, Lemma (5.9) provides the following general criterion for checking if a homogeneous symplectic relation between phase manifolds is a canonical lift.

(6.7) *Let* $\rho = (\mathbb{P}Q_1, \mathbb{P}Q_2; R)$ *be a homogeneous symplectic relation. If* $A = = (\pi_{Q_1} \times \pi_{Q_2})(R) \subset Q_1 \times Q_2$ *is a submanifold, then* $\rho = \mathbb{P}\alpha$ *where* $\alpha = (Q_1, Q_2; A)$.

Lemma (5.9) is also used in the proof of the following theorem on the lift of a composed relation.

(6.8) *Let* $\alpha : Q_1 \to Q_2$ *and* $\beta : Q_2 \to Q_3$ *be differentiable relations. If* $\beta \circ \alpha$ and $\mathbb{P}\beta \circ \mathbb{P}\alpha$ *are differentiable, then* $\mathbb{P}\beta \circ \mathbb{P}\alpha = \mathbb{P}(\beta \circ \alpha)$.

Proof. Let us set $\alpha = (Q_1, Q_2; A)$, $\beta = (Q_2, Q_3; B)$, $R = \text{graph}\ \mathbb{P}\alpha$, $S = \text{graph}\ \mathbb{P}\beta$. For each $(q_1, q_3) \in B \circ A$ there exists a $q_2 \in Q_2$ such that $(q_1, q_2) \in A$ and $(q_2, q_3) \in B$. We can interpret q_1, q_2, q_3 as elements of the zero section of T^*Q_1, T^*Q_2, T^*Q_3 respectively. Thus we can write $(q_1, q_2) \in R$, $(q_2, q_3) \in S$. It follows that $(q_1, q_3) \in S \circ R$. We conclude that $B \circ A \subset (\pi_{Q_1} \times \pi_{Q_2})(S \circ R)$. The inverse inclusion is obvious. Hence $B \circ A = (\pi_{Q_1} \times \pi_{Q_2})(S \circ R)$. Since $\nu(S \circ R)$ is a homogeneous Lagrangian submanifold of $\mathbb{P}(Q_1 \times Q_2)$ which projects onto $B \circ A$, Lemma (5.9) implies $\nu(S \circ R) = \mathbb{N}(B \circ A)$. (Q.E.D.)

7. The Phase Functor

(7.1) *The canonical lift of a reduction is a symplectic reduction. Let* $\tilde{\alpha} : \tilde{Q} \to Q_0$ *be the surjective submersion associated with a reduction* $\alpha : Q \to Q_0$ *and let* $\tilde{\rho} : C \to T^*Q_0$ *be the surjective submersion associated with* $\rho = \mathbb{P}\alpha$. *Then*

(i) $\operatorname{graph} \mathbb{P}\alpha = \{(p, p_0) \in T^*Q \times T^*Q_0; q = \pi_Q(p) \in \tilde{Q}, q_0 = \pi_{Q_0}(p_0) = \tilde{\alpha}(q),$
$\langle u, p\rangle = \langle T\tilde{\alpha}(u), p_0\rangle, \ \forall u \in T_q Q\}$;

(ii) $C = \{p \in T^*Q; q = \pi_Q(p) \in \tilde{Q}, \langle u, p\rangle = 0, \ \forall u \in T_q\tilde{Q} : T\tilde{\alpha}(u) = 0\}$;

(iii) $\dim C = \dim Q + \dim Q_0$;

(iv) *the mapping* $\pi : C \to \tilde{Q}$ *induced by* π_Q *is a surjective submersion and* $\pi_{Q_0} \circ \tilde{\rho} = \tilde{\alpha} \circ \pi$;

(v) *the fibers of* $\rho = \mathbb{P}\alpha$ *are the equivalence classes of the following equivalence relation on* C:

$$p \sim \bar{p} \Longleftrightarrow \tilde{\alpha} \circ \pi(p) = \tilde{\alpha} \circ \pi(\bar{p}),$$
$$\langle u, p\rangle = \langle \bar{u}, \bar{p}\rangle, \ \forall u \in T_{\pi(p)}\tilde{Q}, \bar{u} \in T_{\pi(\bar{p})}\tilde{Q}: T\tilde{\alpha}(u) = T\tilde{\alpha}(u).$$

Proof. Expression (i) follows at once from definition (6.2). Let us denote by R the graph of $\mathbb{P}\alpha$. Let us take a covector $p_0 \in T^*_{q_0}Q_0$. Let us choose a point $q \in \tilde{Q}$ such that $\tilde{\alpha}(q) = q_0$ and let us consider a decomposition $T_qQ \simeq V \oplus E \oplus F$ where: $V \subset T_q\tilde{Q}$ is the space tangent to the fiber of $\tilde{\alpha}$ at q, $E \subset T_q\tilde{Q}$ is transverse to V, F is a supplementary space of $V \oplus E$ in T_qQ. The tangent mapping $T\tilde{\alpha}$ provides a natural isomorphism $E \simeq T_{q_0}Q_0$. Thus we have a natural isomorphism $T^*_qQ \simeq V^* \oplus E^* \oplus F^*$, with $E^* \simeq T^*_{q_0}Q_0$. If we take an element $p \in T^*_qQ$ such that $p \simeq 0 \oplus p_0 \oplus z$, where z is an arbitrary element of F^*, then from (7.2) we can see that $(p, p_0) \in R$. Further, for each $u \in T_qQ$ such that $u \simeq u' \oplus u'' \oplus 0$, where $u'' \in T\tilde{\alpha}(u)$, we have $\langle u, p\rangle = \langle T\tilde{\alpha}(u), p_0\rangle$. This proves that the image of $\mathbb{P}\alpha$ is T^*Q_0 and that the inverse image is the set C given in (ii). If $(p, p_0), (p, \bar{p}_0) \in R$, then from (i) we see that $p_0, \bar{p}_0 \in T^*_{q_0}Q_0$ where $q_0 = \tilde{\alpha} \circ \pi_Q(p)$ and that $\langle T\tilde{\alpha}(u), p_0\rangle = \langle T\tilde{\alpha}(u), \bar{p}_0\rangle, \ \forall u \in T_q\tilde{Q}$. Hence $p_0 = \bar{p}_0$ and R is actually the graph of a surjective mapping $\tilde{\rho} : C \to T^*Q_0$. We prove that C is a submanifold and that $\tilde{\rho}$ is a submersion by using local coordinates. Let us set $m = \dim Q$, $m_0 = \dim Q_0$, $l = \operatorname{codim} \tilde{Q}$. We have $m \geqslant m_0$, $m - l \geqslant m_0$. Let $(q^i) = (q^{\bar{a}}, q^a)$ $(i = 1, \ldots, n;$ $\bar{a} = 1, \ldots, l; a = l + 1, \ldots, m)$ be coordinates on Q such that $\tilde{Q}$ is locally defined by equations $q^{\bar{a}} = 0$. We can interpret (q^a) as coordinates on $\tilde{Q}$. Let (q_0^κ) $(\kappa = 1, \ldots, \mathrm{m})$ be coordinates on Q_0 and let $q_0^\kappa = \alpha^\kappa(q^a)$ be the local representation of the submersion $\tilde{\alpha}$. The rank of the $(m - l) \times m_0$ matrix $\| \partial_a \alpha^\kappa \|$ is maximal, i.e.

equal to m_0, at each point. We denote by (q^i, p_i) and $(q_0^\kappa, p_{0\kappa})$ the natural canonical coordinates on T^*Q and T^*Q_0 associated with (q^i) and (q_0^κ) respectively. From (i) we see that R is locally represented by the following $m + m_0$ independent equations:

(7.2) $$q^{\bar{a}} = 0, q_0^\kappa = \alpha^\kappa(q^b), p_a = p_{0\kappa}\ \partial_a\alpha^\kappa(q^b).$$

From (ii) it follows that C is locally represented by equations

(7.3) $$q^{\bar{a}} = 0, \quad p_a = \partial_a\alpha^\kappa(q^b)\, t_\kappa,$$

where (t_κ) are real arbitrary parameters. Hence C is a submanifold of dimension $m + m_0$ and also (iii) is proved. We can give a parametric description of C by taking $(q^a, p_{\bar{a}}, t_\kappa)$ as parameters:

$$q^i = f^i(q^a, p_{\bar{a}}, t_\kappa), \quad p_i = f_i(q^a, p_{\bar{a}}, t_\kappa).$$

We have:

(7.4) $$f^a = q^a,\ f^{\bar{a}} = 0,\ f_a = \partial_a\alpha^\kappa t_\kappa,\ f_{\bar{a}} = p_{\bar{a}}.$$

As a consequence, the mapping $\tilde{\rho} : C \to T^*Q$ is represented by the equations

(7.5) $$q_0^\kappa = \alpha^\kappa(q^b), \quad p_{0\kappa} = t_\kappa,$$

which follow from the last equations in (7.2) and (7.3), taking into account that the matrix $\| \partial_a\alpha^\kappa \|$ has maximal rank. Equations (7.5) define a submersion. We conclude that ρ is a differentiable reduction. Statements (iv) and (v) follow at once from the preceding discussion. (Q.E.D.)

Since the relation composed by two differentiable reductions is a differentiable reduction, from propositions (6.8) and (7.1) it follows that

(7.6) *If* $\alpha : Q_1 \to Q_2$ *and* $\beta : Q_2 \to Q_3$ *are differentiable reductions, then* $\mathbb{P}\beta \circ \mathbb{P}\alpha = \mathbb{P}(\beta \circ \alpha)$.

We conclude that the operator $\mathbb{P}$ which associates with each manifold Q the phase manifold $\mathbb{P}Q = (T^*Q, \omega_Q)$ and with each reduction α the canonical lift $\mathbb{P}\alpha$ defined in (6.3) is a covariant functor from the category of differentiable reductions to the category of symplectic reductions. We call $\mathbb{P} : \mathscr{R} \to \mathscr{SR}$ the *phase functor.*

(7.7) REMARKS. Diffeomorphisms and symplectomorphisms are isomorphisms in $\mathscr{R}$ and $\mathscr{SR}$ respectively. If α is a differentiable reduction such that also α^t is a reduction, then α is a diffeomorphism. If $\alpha : Q \to Q_0$ is a diffeomorphism, then

$\mathbb{P}\alpha$ is a symplectomorphism (see (6.5)). In this case expression (i) in proposition (7.1) becomes

$$\text{graph } \mathbb{P}\alpha = \{(p, p_0) \in T^*Q \times T^*Q_0; \pi_{Q_0}(p_0) = \alpha \circ \pi_Q(p),$$
$$\langle u, p \rangle = \langle T\alpha(u), p_0 \rangle, \ \forall u \in T_{\pi_Q(p)}Q\}.$$

Hence the symplectomorphism $\mathbb{P}\alpha$ is defined by the equation

(7.8) $$\langle v, \mathbb{P}\alpha(p) \rangle = \langle T\,\alpha^{-1}(v), p \rangle$$

for each $p \in T^*Q$ and $v \in T_q Q$, with $q = \pi_Q(p)$. Thus we observe that

(7.9) $$\mathbb{P}\alpha = T^*\,\alpha^{-1},$$

where T^* is the so called *cotangent functor*. This is a contravariant functor which operates only on diffeomorphisms (see for instance [10], Ch. 3).

(7.10) *Let $\alpha : Q \to Q_0$ be a differentiable reduction and let $\tilde{\alpha} : \tilde{Q} \to Q_0$ be the corresponding submersion. For each submanifold $K_0 \subset Q_0$ and each 1-form γ_0 on K_0 the following identity holds:*

$$(\mathbb{P}\alpha)^t \ (\mathfrak{L}(Q_0; K_0, \gamma_0)) = \mathfrak{L}(Q; K, \gamma)$$

where $K = \alpha^t(K_0)$ and $\gamma = \tilde{\alpha}^*\gamma_0 \,|\, K$.

Proof. Let us set $L_0 = \mathfrak{L}(Q_0; K_0, \gamma_0)$, $L = \mathfrak{L}(Q; K, \gamma)$, $N = (\mathbb{P}\,\alpha)^t\,(L_0)$. Taking into account definition (5.4) and proposition (7.1), part (i), we have:

(7.11) $$L = \{p \in T^*Q; q = \pi_Q(p) \in K, \langle u, p \rangle = \langle T\tilde{\alpha}(u), \gamma_0 \rangle, \ \forall u \in T_q K\};$$
$$L_0 = \{p_0 \in T^*Q_0; q_0 = \pi_{Q_0}(p_0) \in K_0, \langle u_0, p_0 \rangle = \langle u_0, \gamma_0 \rangle, \forall u_0 \in T_{q_0} K_0\};$$

(7.12) $$N = \{p \in T^*Q; q = \pi_Q(p) \in \tilde{Q}, \exists p_0 \in L_0 : q_0 = \pi_{Q_0}(p_0) = \tilde{\alpha}(q),$$
$$\langle u, p \rangle = \langle T\tilde{\alpha}(u), p_0 \rangle, \ \forall u \in T_q \tilde{Q}\} =$$
$$= \{p \in T^*Q; q = \pi_Q(p) \in K, \ \exists p_0 \in T^*Q_0 : q_0 = \pi_{Q_0}(p_0) = \tilde{\alpha}(q) \in K_0,$$
$$\langle u_0, p_0 \rangle = \langle u_0, \gamma_0 \rangle, \forall u_0 \in T_{q_0} K_0, \langle u, p \rangle = \langle T\tilde{\alpha}(u), p_0 \rangle, \forall u \in T_q \tilde{Q}\}.$$

Let us take a covector $p \in N$. We have $q = \pi_Q(p) \in K$ and, for any $u \in T_q K \subset T_q \tilde{Q}$, $\langle u, p \rangle = \langle T\tilde{\alpha}(u), p_0 \rangle = \langle T\tilde{\alpha}(u), \gamma_0 \rangle$, since $T\tilde{\alpha}(u) \in T_{q_0} K_0$. From (7.11) it follows that $p \in L$. This shows that $N \subset L$. Conversely, let us take a covector $p \in L$. We have $q = \pi_Q(p) \in K$. Let us take any vector $u \in T_q \tilde{Q}$ such that $T\tilde{\alpha}(u) = 0$. The vector u is tangent to the fiber of $\tilde{\alpha}$ containing q; hence $u \in T_q K$, because of the definition of K. From (7.11) it follows that $\langle u, p \rangle = \langle T\tilde{\alpha}(u), \gamma_0 \rangle = 0$. This shows that p belongs to the inverse image of $(\mathbb{P}\alpha)^t$ (see (7.1), part (ii)). As a consequence, we have an image $p_0 = (\mathbb{P}\alpha)(p)$ for which the following holds (see (7.1), part (i)): $q_0 = \pi_{Q_0}(p_0) = \tilde{\alpha}(q)$ and $\langle u, p\rangle = \langle T\tilde{\alpha}(u), p_0\rangle$, $\forall u \in T_q Q$. Hence, for any vector $u_0 \in T_{q_0} K_0$ and for any vector $u \in T_q K$ such that $T\tilde{\alpha}(u) = u_0$, we have (see also (7.1)): $\langle u_0, p_0\rangle = \langle T\tilde{\alpha}(u), p_0\rangle = \langle u, p\rangle = \langle T\tilde{\alpha}(u), \gamma_0\rangle = \langle u_0, \gamma_0\rangle$. Now we observe that all conditions in the last term of (7.12) are satisfied by the co-vector p, thus $p \in N$. This shows that $L \subset N$. (Q.E.D.)

If in the preceding proposition we assume $\gamma_0 = 0$ or $K_0 = Q_0$, then we obtain, as a corollary, the following two identities:

(7.13) $$(\mathbb{P}\alpha)^t\,(\mathbb{N}K_0) = \mathbb{N}(\alpha^t(K_0)),$$

(7.14) $$(\mathbb{P}\alpha)^t \circ \gamma_0(Q_0) = \ell(Q; \tilde{Q}, \tilde{\alpha}^*\gamma_0).$$

If in particular α is a submersion (so that $\tilde{\alpha} = \alpha$ and $\tilde{Q} = Q$), then (7.14) reduces to:

(7.15) $$(\mathbb{P}\alpha)^t(\gamma_0(Q_0)) = \alpha^*\gamma_0(Q).$$

In proposition (7.10) there is no particular assumption about the 1-form γ_0. When this form is assumed to be closed we obtain statements concerning Lagrangian submanifolds.

8. Special Homogeneous Symplectic Reductions

We remark that the canonical lift of a reduction is the composition of the lift of the transpose of an embedding and the lift of a submersion (see (2.3) and (7.6)). Let us consider in detail these two kind of homogeneous symplectic reductions.

First kind. Let $Q_0 \subset Q$ be a submanifold of a manifold Q and let $\iota : Q_0 \to Q$ be the natural embedding. We consider the reduction $\alpha = \iota^t$ and the canonical lift $\mathbb{P}\alpha$. In this case expressions (i) and (ii) in proposition (7.1) become (set $\tilde{Q} = Q_0$, $\tilde{\alpha} = \alpha = \iota^t$):

(8.1) $$\text{graph}\ \mathbb{P}\alpha = \{(p, p_0) \in T^*Q \times T^*Q_0;\ \pi_Q(p) = \pi_{Q_0}(p_0) = q_0,\ \langle u, p\rangle = \langle u, p_0\rangle,\ \forall u \in T_{q_0}Q_0\},$$

(8.2) $$C = (\mathbb{P}\alpha)^t(T^*Q_0) = \{p \in T^*Q;\ \pi_Q(p) \in Q_0\}.$$

The fibers of $\mathbb{P}\alpha$ are the equivalence classes of the following equivalence relation on C:

(8.3) $$p \sim \bar{p} \Leftrightarrow \pi_Q(p) = \pi_Q(\bar{p}) = q,\ p - \bar{p} \in (T_q Q_0)^0.$$

Second kind. Let $\alpha : Q \to Q_0$ be a surjective submersion. For the canonical lift $\mathbb{P}\alpha$ the expressions (i) and (ii) in proposition (7.1) become:

$$\text{graph } \mathbb{P}\alpha = \{(p, p_0) \in T^*Q \times T^*Q_0; \alpha \circ \pi_Q(p) = \pi_{Q_0}(p_0), \langle u, p \rangle = \langle T\alpha(u), p_0 \rangle, \ \forall u \in T_{\pi_Q(p)}Q\}, \tag{8.4}$$

$$C = (\mathbb{P}\alpha)^t (T^*Q_0) = \{p \in T^*Q; \langle u, p \rangle = 0, \ \forall u \in T_{\pi_Q(p)}Q : T\alpha(u) = 0\}. \tag{8.5}$$

C is the set of the covectors which vanish on the vectors tangent to the fibers of α. The fibers of $\mathbb{P}\alpha$ are the equivalence classes of the following equivalence relation on C:

$$p \sim \bar{p} \Leftrightarrow \alpha(q) = \alpha(\bar{q}), \text{ with } q = \pi_Q(p), \bar{q} = \pi_Q(\bar{p}), \langle u, p \rangle = \langle \bar{u}, \bar{p} \rangle, \ \forall u \in T_q Q, \bar{u} \in T_{\bar{q}} Q : T\alpha(u) = T\alpha(\bar{u}). \tag{8.6}$$

The fibers of $\mathbb{P}\alpha$ are connected if and only if the fibers of α are connected.

Reductions of these two kinds are used in symplectic geometry (see for instance [11], [12]) and in the analysis of the behaviour of mechanical systems with constraints and singularities (see the examples given in [3]). These reductions are applied to regular Lagrangian submanifolds and the corresponding images, under suitable conditions, are Lagrangian submanifolds. The reduced Lagrangian submanifolds may have singularities. Examples: (i) Let Q_0 be a submanifold of a manifold Q and let $\iota : Q_0 \to Q$ be the natural embedding. Let $L \subset T^*Q$ be a Lagrangian submanifold generated by a function $G : Q \to \mathbb{R}$. It can be easily seen that the reduced set $L_0 = \mathbb{P}\, \iota^t(L)$ is the Lagrangian submanifold of T^*Q_0 generated by the function $G_0 = G \mid Q_0$. (ii) Let $\alpha : Q \to Q_0$ be a surjective submersion and let $C \subset T^*Q$ be the inverse image of $\mathbb{P}\alpha$ (see (8.5)). Let $L \subset T^*Q$ be a Lagrangian submanifold generated by a function G on Q. If the intersection of L with C is clean, then G is called a *Morse family* and the set $\mathbb{P}\alpha(L)$ is «immersed» Lagrangian submanifold of T^*Q (see [12]). It is known that a Lagrangian submanifold of a cotangent bundle can always be described, at least locally, by Morse families ([13], [11], [12]).

A further example of application of the above results concerning the canonical lifting is given by the following proof of proposition (5.7).

Proof of (5.7). Let C be the inverse image of the canonical lift $\mathbb{P}\pi : \mathbb{P}\, L \to \mathbb{P}K$. We have (see (8.5)):

$$C = \{y \in T^*L; \langle u, y \rangle = 0, \ \forall u \in T_{\pi_L(y)}L : T\pi(u) = 0\}.$$

Since the fibers of π are connected, also the fibers of $\mathbb{P}\pi$ are connected, hence they coincide with the characteristics of C (proposition (4.6)). The 1-form $\theta = \theta_Q \mid L$ is closed (L is Lagrangian) and $\langle u, \theta \rangle = 0$ for each $u \in TL$ such that $T\pi(u) = 0$ because of the definition of θ_Q. Hence the manifold $M = \theta(L)$ is a Lagrangian submanifold of $\mathbb{P}L$ contained in C. We know that any Lagrangian submanifold contained in a coisotropic submanifold is the union of integral manifolds of the characteristic distribution (*absorption principle*). In the present case the characteristics coincide with the fibers of $\mathbb{P}\pi$, thus M is reducible, i.e. the set $N = \mathbb{P}\pi\ (M)$ is a Lagrangian submanifold of $\mathbb{P}K$. Moreover, if $z, \bar{z} \in T_k K \cap N$, then the fibers $F_z = (\mathbb{P}\pi)^t\ (z)$ and $F_z = (\mathbb{P}\pi)^t\ (z)$ are characteristics of C contained in M. These fibers project onto the fiber $\pi^t(k)$ (see (6.4)). Since the fibers of π and $\mathbb{P}\pi$ have the same dimension (use part (iii) of (7.1)) and M is the image of a section of π_L, we have necessarily $F_z = F_{\bar{z}}$, hence $z = \bar{z}$. This shows that N is the image of a section $\gamma : K \to T^*K$ of π_K. Since N is Lagrangian, the 1-form γ is closed (proposition (5.6)). Since $(\mathbb{P}\pi)^t\ (N) = M$, $N = \gamma(K)$, $M = \theta(L)$, from the identity (7.15) it follows that $\theta = \pi^*\gamma$. Finally, let us take a covector $x \in L$. For each vector $v \in T_x L$ we have $\langle T\pi(v), \gamma \rangle = \langle v, \pi^*\gamma \rangle = \langle v, \theta \rangle = \langle v, \theta_Q \rangle = \langle T\pi_Q(v), \tau_{T^*Q}(v) \rangle = \langle T\pi_Q(v), x \rangle$. This shows that $x \in \mathfrak{L}(Q; K, \gamma)$ hence that L is contained in $\mathfrak{L}(Q; K, \gamma)$. (Q.E.D.)

9. Tangent Prolongation of Forms

Let Φ_P denote the exterior algebra of a differentiable manifold. We say that a linear mapping $\delta : \Phi_P \to \Phi_{TP}$ is a *derivation of degree r with respect to the tangent projection* $\tau_P : TP \to P$ if

(9.1) $$\text{degree } \delta\varphi = \text{degree } \varphi + r,$$

(9.2) $$\delta(\varphi \wedge \psi) = \delta\varphi \wedge \tau_P^*\psi + (-1)^{r \cdot \text{degree } \varphi}\ \tau_P^*\varphi \wedge \delta\psi.$$

A derivation $i_T : \Phi_P \to \Phi_{TP}$ of degree -1 is defined by setting

(9.3) $$i_T f = 0$$

for each differentiable function $f : P \to \mathbb{R}$, and

(9.4) $$i_T\varphi(v) = \langle v, \varphi \rangle \qquad (v \in TP)$$

for each 1-form φ on P. It follows that for a $(k+1)$-form φ $(k \geqslant 1)$ $i_T\varphi$ is defined by the equation

(9.5) $$\langle w_1 \wedge \ldots \wedge w_k, i_T\varphi \rangle = \langle v \wedge u_1 \wedge \ldots \wedge u_k, \varphi \rangle$$

where $v = \tau_{TP}(w_1) = \ldots = \tau_{TP}(w_k)$ and $u_1 = T\tau_P(w_1), \ldots u_k = T\,\tau_P(w_k)$.

A derivation $d_T : \Phi_P \to \Phi_{TP}$ of degree 0 is defined by the commutator of i_T and the differential operator d:

$$d_T = d\, i_T + i_T d. \tag{9.6}$$

We have:

$$d\, d_T = d_T d. \tag{9.7}$$

These derivations were defined in [14] and [15]. The main properties are summarized in the following identities:

$$i_X = X^* i_T, \tag{9.8}$$

$$L_X = X^* d_T, \tag{9.9}$$

$$i_T \alpha^* = (T\alpha)^* i_T, \tag{9.10}$$

$$d_T \alpha^* = (T\alpha)^* d_T, \tag{9.11}$$

where X is a vector field, α a differentiable mapping, L_X the Lie derivative, i_X the interior product.

Identities (9.8) and (9.10) are trivial when applied to functions. Because of the well known properties of the Lie derivative and the interior product, it is sufficient to prove them for a 1-form φ. We have respectively:

$$X^*(i_T\varphi) = i_T\varphi \circ X = \langle X, \varphi\rangle = i_X\varphi;$$

$$(T\alpha)^*(i_T\varphi)(v) = i_T\varphi \circ T\alpha(v) = \langle T\alpha(v), \varphi\rangle = \langle v, \alpha^*\varphi\rangle = (i_T\alpha^*\varphi)(v).$$

Identity (9.9) follows from identities $L_X = i_X d + d\, i_X$, (9.6) and (9.8). Identity (9.11) follows from (9.10) and (9.6).

Let (x^i) be local coordinates on a manifold P and let us denote by $(x^i, \dot{x}^i)$ the corresponding fibered coordinates on TP. By the definitions (9.4) and (9.6) we have:

$$i_T dx^i = \dot{x}^i, \qquad d_T dx^i = d\dot{x}^i. \tag{9.12}$$

By using (9.2) adapted to i_T and d_T we can obtain, for example, the following local representations:

$$d_T f = \partial_i f\, \dot{x}^i \tag{9.13}$$

$$i_T\theta = \theta_i\, \dot{x}^i \qquad (\theta = \theta_i\, dx^i), \tag{9.14}$$

$$i_T\omega = \omega_{ij}\, \dot{x}^i\, dx^j \quad (\omega = 1/2\, \omega_{ij}\, dx^i \wedge dx^j), \tag{9.15}$$

$$d_T\theta = \partial_j\theta_i\, \dot{x}^i\, dx^j + \theta_i\, d\dot{x}^i. \tag{9.16}$$

We shall use the following lemma:

(9.17) LEMMA. *Let $\pi : P \to M$ be a submersion. If a 1-form θ is vertical with respect to π, i.e. if $T\pi(v) = 0 \Rightarrow \langle v, \theta \rangle = 0$, then $d_T\theta$ is vertical with respect to $T\pi : TP \to TM$, i.e. $TT\pi(w) = 0 \Rightarrow \langle w, d_T\theta \rangle = 0$.*

Proof. A simple proof can be given by using local coordinates. Let $(x^i) = (x^{\bar{a}}, x^a)$ be local coordinates on P adapted to the submersion π; the fibers of π are locally defined by $x^{\bar{a}}$ = const. A vertical 1-form θ has a local coordinate representation of the form $\theta = \theta_a\, dx^a$. From (9.16) it follows that $d_T\theta = \partial_i \theta_a \dot{x}^a\, dx^a + \theta_a\, d\dot{x}^a$. Since $(x^{\bar{a}}, \dot{x}^{\bar{a}}; x^a, \dot{x}^a)$ are coordinates of TP adapted to $T\pi$, we see that $d_T\theta$ is vertical. (Q.E.D.)

10. Tangent Symplectic Manifolds

Let (P, ω) be a symplectic manifold. The 2-form $d_T\omega$ on TP is closed: $dd_T\omega = d_T d\omega = 0$ (see (9.7)). Let $(q^\alpha, p_\alpha; \dot{q}^\alpha, \dot{p}_\alpha)$ be the fibered coordinates on TP associated with canonical coordinates (q^α, p_α) on P. By using (9.2) and (9.12) we obtain the following local coordinate representation:

$$d_T\omega = d\dot{p}_\alpha \wedge dq^\alpha + dp_\alpha \wedge d\dot{q}^\alpha. \tag{10.1}$$

This shows in particular that $d_T\omega$ is non-degenerate. Thus $d_T\omega$ is a symplectic form on TP. We call the symplectic manifold

$$T(P, \omega) = (TP, d_T\omega)$$

the *tangent symplectic manifold* of (P, ω).

(10.2) REMARK. The local coordinate representation (10.1) shows that the ordered set of coordinates $(q^\alpha, \dot{q}^\alpha; \dot{p}_\alpha, p_\alpha)$ is a canonical coordinate system for $T(P, \omega)$.

A first motivation of the definition of tangent symplectic manifold is illustrated by the following proposition, whose proof simply consists in the application of formula (9.9) to the symplectic form ω.

(10.3) *A vector field $X : P \to TP$ on a symplectic manifold (P, ω) is symplectic (locally Hamiltonian), i.e. $L_X\omega = 0$, if and only if the image $X(P)$ is a Lagrangian submanifold of $T(P, \omega)$.*

This fact has suggested the following *general principle of reciprocity for dynamics* [14]: *the vectors tangent to the trajectories of a system of particles form a Lagrangian submanifold of $T(P, \omega)$ where (P, ω) is the «phase space» of the system.*

Let us denote by $\{,\}_P$ and by $\{,\}_{TP}$ the Poisson bracket on (P, ω) and $T(P, \omega)$ respectively. Let us denote by $\hat{f} : TP \to \mathbb{R}$ the natural extension to TP of a function $f : P \to \mathbb{R}$ $(\hat{f} = f \circ \tau_P)$. For any pair (f, g) of differentiable functions on P we have:

(10.4) $$\{\hat{f}, \hat{g}\}_{TP} = 0,$$

(10.5) $$\{\hat{f}, d_T g\}_{TP} = \widehat{\{f, g\}}_P,$$

(10.6) $$d_T\{f, g\}_P = \{d_T f, d_T g\}_{TP}.$$

These identities can be proved by direct calculation in local canonical coordinates on the basis of remark (10.2). An intrinsic proof would require further remarks on the derivation d_T which we omit here for the sake of brevity.

For a differentiable function $f : P \to \mathbb{R}$ we have $d_T f(v) = \langle v, df\rangle$, for each $v \in TP$. Hence:

(10.7) *If $K \subset P$ is a submanifold, then a differentiable function $f : P \to R$ is constant on K if and only if $d_T f \mid TK = 0$.*

(10.8) *If K is an isotropic (resp. coisotropic, Lagrangian) submanifold of a symplectic manifold (P, ω), then TK is an isotropic (resp. coisotropic, Lagrangian) submanifold of $T(P, \omega)$.*

Proof. Let us apply to ω the identity (9.11) for the natural injection $\iota : K \to P$. We have $d_T\, \iota^*\omega = (T\iota)^* d_T\omega$,, i.e. $d_T(\omega \mid K) = d_T\omega \mid TK$. If $\omega \mid K = 0$, i.e. if K is isotropic, then $d_T\omega \mid TK = 0$ and TK is isotropic in $T(P, \omega)$. If K is in particular Lagrangian, then TK is also Lagrangian, since $\dim TK = 2 \dim K = \dim P = 1/2 \dim TP$. Let K be locally defined by equations $f^\alpha = 0$ $(\alpha = 1, \ldots, l)$ where $f^\alpha : U \to \mathbb{R}$ are independent differentiable functions on an open subset $U \subset P$. It can be easily verified that the functions $d_T f^\alpha$ and $\hat{f}^\alpha$ on TU are independent. Moreover, proposition (10.7) shows that equations $\hat{f}^\alpha = 0$, $d_T f^\alpha = 0$ define $TK \cap TU$. If K is coisotropic, then $\{f^\alpha, f^\beta\}_P \mid K \cap U = 0$ $(\alpha, \beta = 1, \ldots, l)$. From (10.4), (10.5), (10.6) it follows that $\{\hat{f}^\alpha, \hat{f}^\beta\}_{TP}$, $\{\hat{f}^\alpha, d_T f^\beta\}_{TP}$ and $\{d_T f^\alpha, d_T f^\beta\}_{TP}$ vanish on $TK \cap TU$. This shows that TK is coisotropic. (Q.E.D.)

11. The Tangent Functor

Let $\rho = (P_1, P_2; R)$ be a differentiable relation. Let us consider the natural diffeomorphism

(11.1) $$\mu : T(P_1 \times P_2) \to TP_1 \times TP_2.$$

We know that

(11.2) $$pr_{TP_i} \circ \mu = Tpr_{P_i} \qquad (i = 1, 2).$$

(For any pair of sets (M_1, M_2) we denote by $pr_{M_i} : M_1 \times M_2 \to M_i$ $(i = 1, 2)$ the natural projections). We call

(11.3) $$T\rho = (TP_1, TP_2; \mu(TR))$$

the *tangent lift* (or the *tangent relation*) of ρ. In this definition the tangent manifold TR is interpreted, in a natural way, as a submanifold of $T(P_1 \times P_2)$. The following three properties hold:

(11.4) $$\tau_{P_2} \circ T\rho = \rho \circ \tau_{P_1}, \; T(\rho^t) = (T\rho)^t, \quad T1_P = 1_{TP}.$$

We remark that if ρ is in particular a mapping then $T\rho$ is the so called «tangent mapping» (or «differential») of ρ.

(11.5) *The tangent lift of a reduction is a reduction*

Proof. Let $\rho = (P_1, P_2; R)$ be a differentiable reduction and let $\tilde{\rho} : C \to P_2$ be the corresponding surjective submersion. Since $R = \{(p_1, p_2) \in P_1 \times P_2; p_1 \in C, p_1 = \tilde{\rho}(p_1)\}$, we have $\mu(TR) = \{(v_1, v_2) \in TP_1 \times TP_2; v_1 \in TC, v_2 = T\tilde{\rho}(v_1)\}$. Hence $\mu(TR)$ coincides with the graph of $T\tilde{\rho}$ interpreted as a submanifold of $TP_1 \times TP_2$. It is known that the tangent mapping of a surjective submersion is a surjective submersion. Hence $T\rho$ is a reduction. (Q.E.D.)

(11.6) *If* $\rho : P_1 \to P_2$ *and* $\sigma : P_2 \to P_3$ *are reductions, then* $T\sigma \circ T\rho = T(\sigma \circ \rho)$.

Proof. Let us denote by $\tilde{\rho} : C_1 \to P_2, \tilde{\sigma} : C_2 \to P_3, \widetilde{\sigma \circ \rho} : C \to P_3$ the surjective submersions associated with ρ, σ, $\sigma \circ \rho$ respectively. We have $C = \tilde{\rho}^t(C_2)$. Let $\bar{\rho} : C \to C_2$ be the restriction of $\tilde{\rho}$ to C. It follows that graph $\sigma \circ \rho$ = graph $\widetilde{\sigma \circ \rho}$ = graph $\tilde{\sigma} \circ \bar{\rho}$ (considered as a submanifold of $P_1 \times P_2$). Hence: graph $T(\sigma \circ \rho)$ = T(graph $\sigma \circ \rho$) = T(graph $\tilde{\sigma} \circ \bar{\rho}$) = graph $T(\tilde{\sigma} \circ \bar{\rho})$ = graph $T\tilde{\sigma} \circ T\bar{\rho}$. On the other hand, graph $T\sigma \circ T\rho = \{(v_1, v_3) \in TP_1 \times TP_3;\; \exists v_2 \in TP_2: (v_1, v_2) \in \text{graph } T\rho, (v_2, v_3) \in \text{graph } T\sigma\} = \{(v_1, v_3) \in TP_1 \times TP_3;\; \exists v_2 \in TP_2: (v_1, v_2) \in \text{graph } T\tilde{\rho}, (v_2, v_3) \in \text{graph } T\tilde{\sigma}\}$. We observe that $v_2 \in TC_2$, thus that $v_1 \in TC$. Hence: graph $T\sigma \circ T\rho$ = graph $T\tilde{\sigma} \circ T\bar{\rho}$. (Q.E.D.)

Propositions (11.5) and (11.6) show that the operator T which assigns to each differentiable manifold P the tangent manifold TP and to each reduction ρ the tangent lift $T\rho$ is a covariant functor $T : \mathscr{R} \to \mathscr{R}$ in the category of reductions.

Let (P_1, ω_1) and (P_2, ω_2) be two symplectic manifolds. By using properties

(9.11) and (11.2), we derive:

$$\begin{aligned} T((P_1, \omega_1) \times (P_2, \omega_2)) &= T(P_1 \times P_2, pr^*_{P_1}\omega_1 + pr^*_{P_2}\omega_2) \\ &= (T(P_1 \times P_2), d_T(pr^*_{P_1}\omega_1 + pr^*_{P_2}\omega_2)) \\ &= (T(P_1 \times P_2), (Tpr_{P_1})^* d_T\omega_1 + (Tpr_{P_2})^* d_T\omega_2) \\ &= (T(P_1 \times P_2), \mu^*(pr^*_{TP_1} d_T\omega_1 + pr^*_{TP_2} d_T\omega_2)) \end{aligned}$$

On the other hand we have $T(P_1, \omega_1) \times T(P_2, \omega_2) = ((TP_1 \times TP_2), pr^*_{TP_1} d_T\omega_1 + pr^*_{TP_2} d_T\omega_2)$. Hence we observe that the natural diffeomorphism μ (11.1) is a symplectomorphism from $T((P_1, \omega_1) \times (P_2, \omega_2))$ to $T(P_1, \omega_1) \times T(P_2, \omega_2)$. From proposition (10.8) it follows that if R is a Lagrangian submanifold of $(P_1, -\omega_1) \times (P_2, \omega_2)$, then $\mu(TR)$ is Lagrangian in $(T(P_1, \omega_1))^t \times T(P_2, \omega_2)$. In other words:

(11.7) *The tangent lift of a symplectic relation is a symplectic relation.*

From the preceding discussion we conclude that the tangent operator T is also a functor $T : \mathcal{SR} \to \mathcal{SR}$ in the category of symplectic reduction. (For the sake of simplicity we use again the same symbol T).

12. Natural Equivalences

(12.1) *Let Q be a differentiable manifold. There exists a unique differentiable mapping $\alpha_Q : TT^*Q \to T^*TQ$ satisfying the following conditions:*

(12.2) $$\pi_{TQ} \circ \alpha_Q = T\pi_Q ,$$

(12.3) $$d_T\theta_Q = \alpha^*_Q \theta_{TQ} .$$

The mapping α_Q is a diffeomorphism and it is defined by the following equation:

(12.4) $$\langle u, \alpha_Q(w) \rangle = \langle v, d_T\theta_Q \rangle,$$

where $w \in TT^*Q$, $\alpha_Q(w) \in T^*_zTQ$, $u \in T_zTQ$, $z = T\alpha_Q(w)$, and $v \in TTT^*Q$ is any vector such that $\tau_{TT^*Q}(v) = w$ and $u = TT\pi_Q(v)$.

Proof. Since θ_Q is vertical with respect to π_Q, $d_T\theta_Q$ is vertical with respect to $T\pi_Q$ (see Lemma (9.17)). As a consequence the right hand side of (12.4) does not depend on the choice of the vector v projecting onto u by means of $TT\pi_Q$. Hence (12.9) is correct and defines a mapping satisfying (12.2). We prove that α_Q is a diffeomorphism by using a local coordinate representation. Let us consider the following coordinate systems naturally associated with coordinates (q^α) on Q:

$$
\begin{array}{lll}
 & (q^\alpha, p_\alpha) & \text{on } T^*Q \\
 & (q^\alpha, \dot q^\alpha) & \text{on } TQ, \\
(12.5) & (q^\alpha, p_\alpha; \dot q^\alpha, \dot p_\alpha) & \text{on } TT^*Q, \\
 & (q^\alpha, \dot q^\alpha; l_\alpha, m_\alpha) & \text{on } T^*TQ, \\
 & (q^\alpha, \dot q^\alpha; \delta q^\alpha, \delta \dot q^\alpha) & \text{on } TTQ.
\end{array}
$$

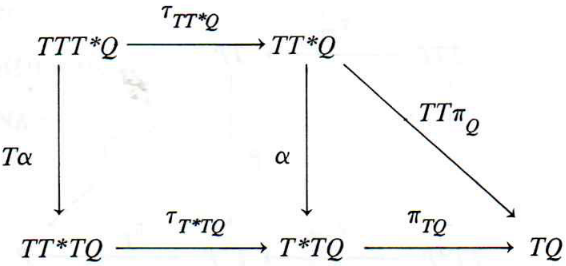
Since $\theta_Q = p_\alpha \, dq^\alpha$ we have $d_T \theta_Q = \dot p_\alpha \, dq^\alpha + p_\alpha \, d\dot q^\alpha$. Let us set $w = (q^\alpha, p_\alpha; \dot q^\alpha, \dot p_\alpha)$. We have $z = T\pi_Q(w) = (q^\alpha, \dot q^\alpha)$ and we can write $\alpha_Q(w) = (q^\alpha, \dot q^\alpha; l_\alpha, m_\alpha)$ where (l_α, m_α) must be expressed as functions of $(q^\alpha, p_\alpha; \dot q^\alpha, \dot p_\alpha)$. An arbitrary vector $u \in T_z TQ$ is represented by $(q^\alpha, \dot q^\alpha; \delta q^\alpha, \delta \dot q^\alpha)$ where $(\delta q^\alpha, \delta \dot q^\alpha)$ are arbitrary numbers. It follows that $\langle u, \alpha_Q(w) \rangle = \delta q^\alpha \, l_\alpha + \delta \dot q^\alpha \, m_\alpha$. Any vector $v \in T_w TT^*Q$ such that $u = TT\pi_Q(v)$ is represented by $(q^\alpha, p_\alpha, \dot q^\alpha, \dot p_\alpha; \delta q^\alpha, \delta p_\alpha, \delta \dot q^\alpha, \delta \dot p_\alpha)$. Hence $\langle v, d_T \theta_Q \rangle = \delta q^\alpha \, p_\alpha + \delta \dot q^\alpha \, p_\alpha$. We conclude that

$$ (12.6) \qquad l_\alpha = \dot p_\alpha, \quad m_\alpha = p_\alpha. $$

Thus α_Q acts on coordinates as follows:

$$ (12.7) \qquad \alpha_Q : (q^\alpha, p_\alpha; \dot q^\alpha, \dot p_\alpha) \mapsto (q^\alpha, \dot q^\alpha; \dot p_\alpha, p_\alpha), $$

and it is clearly a diffeomorphism. For a mapping α satisfying condition (12.2) the following diagram is commutative:



Hence for each $v \in TTT^*Q$ we have:

$$
\begin{aligned}
\langle v, \alpha^* \theta_{TQ} \rangle &= \langle T\alpha(v), \theta_{TQ} \rangle \\
&= \langle T\pi_{TQ} \circ T\alpha(v), \tau_{T^*TQ}(T\alpha(v)) \rangle \\
&= \langle T(\pi_{TQ} \circ \alpha)\,(v), \alpha(\tau_{TT^*Q}(v)) \rangle \\
&= \langle T\,T\pi_Q(v), \alpha(\tau_{TT^*Q}(v)) \rangle \\
&= \langle u, \alpha(w) \rangle,
\end{aligned}
$$

where $w = \tau_{TT^*Q}(v)$, $u = T(T\pi_Q)(v)$. This shows that if the mapping α satisfies also condition (12.3), then $\alpha = \alpha_Q$. The same calculation with $\alpha = \alpha_Q$ shows that α_Q satisfies (12.3). (Q.E.D.)

By differentiating (12.3) we obtain:

(12.8) $$d_T \omega_Q = \alpha_Q^* \omega_{TQ}.$$

Hence α_Q is a symplectomorphism from $T\mathbb{P}Q$ to $\mathbb{P}TQ$.

(12.9) *Let (P, ω) be a symplectic manifold. There exists a unique bundle morphism $\beta_{(P,\omega)} : TP \to T^*P$ such that*

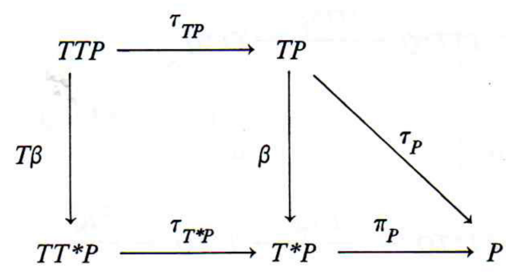
(12.9) $$i_T \omega = \beta_{(P,\omega)}^* \theta_P.$$

This morphism is an isomorphism and it is defined by the equation

(12.11) $$\langle u, \beta_{(P,\omega)}(v)\rangle = \langle v \wedge u, \omega\rangle,$$

where $u, v \in TP$ and $\tau_P(u) = \tau_P(v)$.

Proof. We observe first of all that (12.11) defines a bundle isomorphism. Furthermore, we remark that for a bundle morphism $\beta : TP \to T^*P$ the following diagram is commutative:



Hence, for each $w \in TTP$, we have:

$$\begin{aligned}\langle w, \beta^* \theta_P\rangle &= \langle T\beta(w), \theta_P\rangle \\ &= \langle T\pi_P(T\beta(w)), \tau_{T^*P}(T\beta(w))\rangle \\ &= \langle T\,\tau_P(w), \beta(\tau_{TP}(w))\rangle.\end{aligned}$$

On the other hand, from the definition of i_T, it follows that

$$\langle w, i_T \omega\rangle = \langle \tau_{TP}(w) \wedge T\,\tau_P(w), \omega\rangle.$$

The comparison of these equalities shows two things: that if for the bundle morphism β we have $i_T\omega = \beta^* \theta_P$, then $\beta = \beta_{(P,\omega)}$, and conversely that $\beta_{(P,\omega)}$ satisfies (12.10). (Q.E.D.)

From (12.10) we derive

(12.12) $$d_T\omega = \beta^*_{(P,\omega)}\omega_P.$$

Hence $\beta_{(P,\omega)}$ is a symplectomorphism from $T(P, \omega)$ to $\mathbb{P}(P)$.

The general theory of the Legendre transformation described in [16] and [17] makes use of the symplectomorphisms α_Q and $\beta_{(P,\omega)}$ described above. According to the general principle quoted in Section 10 the dynamics of a reciprocal mechanical system is represented by a Lagrangian submanifold $\dot{D}$ of the tangent symplectic manifold $T(P, \omega)$ of a suitable «phase space» (P, ω). If (P, ω) is a phase manifold, i.e. if $(P, \omega) = \mathbb{P}Q$, then $A = \alpha_Q(\dot{D})$ and $B = \beta_{\mathbb{P}Q}(\dot{D})$ are Lagrangian submanifolds of the phase manifolds $\mathbb{P}TQ$ and $\mathbb{P}T^*Q$ respectively. The functions or the Morse families generating A and B are respectively the *Lagrangian* and the *Hamiltonian* of the mechanical system.

In this section we show how the symplectomorphisms α_Q and $\beta_{(P,\omega)}$ give rise to natural equivalences (see [18], Ch. 1, Section 4) between functors defined on the category of reductions, by proving the following propositions.

(12.13) *For any differentiable relation* $\sigma : Q_1 \to Q_2$ *we have*

(12.14) $$(\alpha_{Q_1} \times \alpha_{Q_2})\ (\text{graph } T\mathbb{P}\,\sigma) = \text{graph } \mathbb{P}T\sigma,$$

i.e. the following diagram of relations is commutative:

(12.15) $$\begin{array}{ccc} T\mathbb{P}Q_1 & \xrightarrow{\ \alpha_{Q_1}\ } & \mathbb{P}TQ_1 \\ {\scriptstyle T\mathbb{P}\sigma}\downarrow & & \downarrow{\scriptstyle \mathbb{P}T\sigma} \\ T\mathbb{P}Q_2 & \xrightarrow{\ \alpha_{Q_2}\ } & \mathbb{P}TQ_2 \end{array}$$

(12.16) *For any symplectic relation* $\rho : (P_1, \omega_1) \to (P_2, \omega_2)$ *we have*

(12.17) $$(\beta_{(P_1,\omega_1)} \times \beta_{(P_2,\omega_2)})\ (\text{graph } T\rho) = \text{graph } \mathbb{P}\rho,$$

i.e. the following diagram of relations is commutative:

(12.18)
$$
\begin{array}{ccc}
T(P_1, \omega_1) & \xrightarrow{\beta_{(P_1,\omega_1)}} & \mathbb{P}P_1 \\
{\scriptstyle T\rho}\downarrow & & \downarrow{\scriptstyle \mathbb{P}\rho} \\
T(P_2, \omega_2) & \xrightarrow{\beta_{(P_2,\omega_2)}} & \mathbb{P}P_2
\end{array}
$$

When applied to reductions the two propositions above show that:

(12.19) *The mapping which assigns to each differentiable manifold Q the symplectomorphism α_Q is a natural equivalence between the functor $T\mathbb{P} : \mathcal{R} \to \mathcal{SR}$ and the functor $\mathbb{P}T : \mathcal{R} \to \mathcal{SR}$.*

(12.20) *The mapping which assigns to each symplectic manifold (P, ω) the symplectomorphism $\beta_{(P,\omega)}$ is a natural equivalence between the functor $T : \mathcal{SR} \to$ $\to \mathcal{SR}$ and the functor $\mathbb{P} : \mathcal{SR} \to \mathcal{SR}$.*

Note that in this last proposition the phase functor $\mathbb{P}$ is considered on the category of symplectic reductions.

We use the following lemmas.

(12.21) LEMMA. *For any submanifold S of a manifold Q we have $\alpha_Q(T\mathbb{N}S) =$ $= \mathbb{N}TS$.*

Proof. Since α_Q is a symplectomorphism and $T\mathbb{N}S$ is Lagrangian, $\alpha_Q(T\mathbb{N}S)$ is Lagrangian. Furthermore, because of (12.2), $\pi_{TQ}(\alpha_Q(T\mathbb{N}S)) = T\pi_Q(T\mathbb{N}S) = TS$, thus $\alpha_Q(T\mathbb{N}S)$ projects onto TS. Let us use the coordinate notation (12.5) and let us assume that S is locally defined by equations $q^{\bar{a}} = 0$ $(\bar{a} = 1, \dots, l)$. Thus $T\mathbb{N}S$ is locally represented by equations $q^{\bar{a}} = 0, p_a = 0, \dot{q}^{\bar{a}} = 0, \dot{p}_a = 0$ $(a = l + 1, \dots, n)$ and $\alpha_Q(T\mathbb{N}S)$ by $q^{\bar{a}} = 0$, $\dot{q}^{\bar{a}} = 0$, $l_a = 0$, $m_a = 0$ (see (12.7)). This shows that $\alpha_Q(T\mathbb{N}S)$ is a homogeneous submanifold of $\mathbb{P}TQ$. Thus $\alpha_Q(T\mathbb{N}S) = \mathbb{N}TS$ by Lemma (5.9). (Q.E.D.)

(12.22) LEMMA. *For any Lagrangian submanifold R of a symplectic manifold (P, ω) we have $\beta_{(P,\omega)}(TR) = \mathbb{N}R$.*

Proof. $\beta_{(P,\omega)}(TR)$ is a Lagrangian submanifold of $\mathbb{P}P$ which projects onto R. Since $\beta_{(P,\omega)}$ is a vector bundle morphism and TR is homogeneous, the submanifold

$\beta_{(P,\omega)}$ (TR) is homogeneous, thus it coincides with $\mathbb{N}R$ because of Lemma (5.9).

(Q.E.D.)

(12.23) *Notation.* If θ_1 and θ_2 are k-forms on manifolds M_1 and M_2 respectively, then we set

$$(\theta_1, \theta_2) = pr_{M_1}^* \theta_1 + pr_{M_2}^* \theta_2.$$

This is a k-form on $M_1 \times M_2$.

(12.24) LEMMA. *For any pair of differentiable manifolds* (Q_1, Q_2) *the following identity holds:*

$$\zeta \circ (\alpha_{Q_1} \times \alpha_{Q_2}) \circ \mu \circ T\nu^{-1} = \alpha_{Q_1 \times Q_2},$$

where:

$$\nu : T^*Q_1 \to T^*Q_2 \to T^*(Q_1 \times Q_2)$$

is the diffeomorphism defined in (6.1),

$$\mu : T(T^*Q_1 \times T^*Q_2) \to T^*TQ_1 \times TT^*Q_2$$

is the natural isomorphism (see (11.1)),

$$\zeta : T^*TQ_1 \times T^*TQ_2 \to T^*T(Q_1 \times Q_2)$$

is the diffeomorphisms defined by the equation

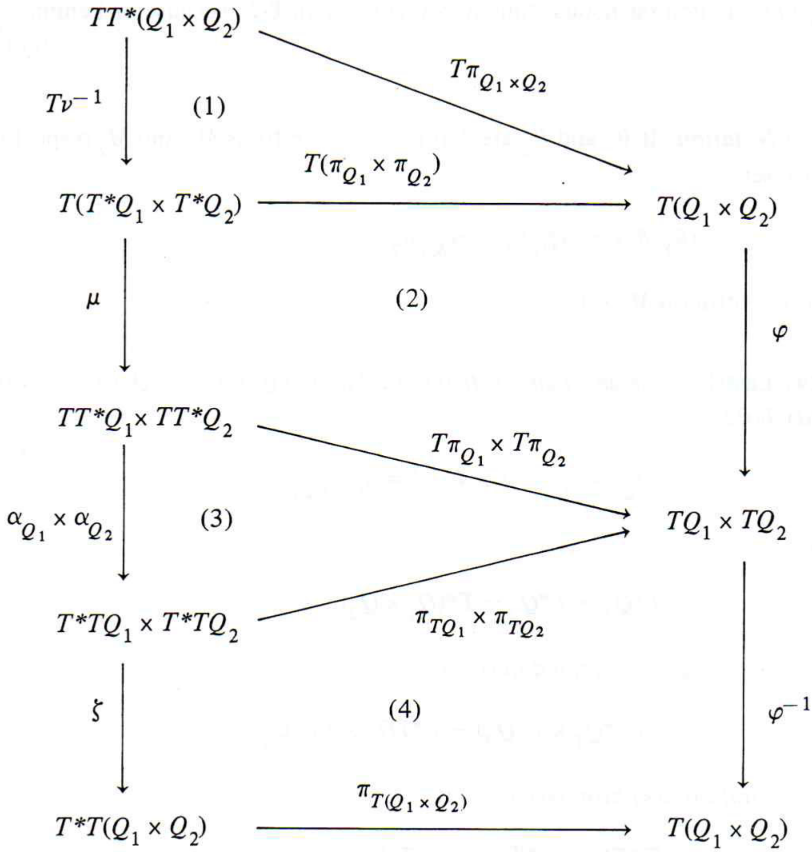
$$\langle T\varphi^{-1}(w_1, w_2), \zeta(z_1, z_2)\rangle = -\langle w_1, z_1\rangle + \langle w_2, z_2\rangle$$

for each $w_i \in TTQ_i$ *and* $z_i \in T^*TQ_i$ *such that* $\tau_{TQ_i}(w_i) = \pi_{TQ_i}(z_i)$ $(i = 1, 2)$, *where*

$$\varphi : T(Q_1 \times Q_2) \to TQ_1 \times TQ_2$$

is the natural diffeomorphism.

Proof. Let us denote by α the mapping at first side of the identity in question. Because of proposition (12.1) it is sufficient to show that: (a) $\pi_{TQ} \circ \alpha = T\pi_Q$, (b) $d_T \theta_Q = \alpha^* \theta_{TQ}$, where $Q = Q_1 \times Q_2$. Let us consider the following diagram:



The subdiagram (1) is commutative, since $(\pi_{Q_1} \times \pi_{Q_2}) \circ \nu^{-1} = \pi_{Q_1 \times Q_2}$. The subdiagram (2) is commutative because of property (11.2) satisfied by the natural diffeomorphisms μ and φ. The subdiagram (3) is commutative because of the characteristic property (12.2) for α_{Q_1} and α_{Q_2}. The subdiagram (4) is commutative because of the definition of ζ. Thus the whole diagram is commutative and in particular equality (a) holds. For proving (b) we show that the following four identities hold:

(12.25) $$\zeta^*\theta_{T(Q_1 \times Q_2)} = (-\theta_{TQ_1}, \theta_{TQ_2}),$$

(12.26) $$(\alpha_{Q_1} \times \alpha_{Q_2})^*(\theta_{TQ_1}, \theta_{TQ_2}) = (d_T\theta_{Q_1}, d_T\theta_{Q_2}),$$

(12.27) $$\mu^*(d_T\theta_{Q_1}, d_T\theta_{Q_2}) = d_T(\theta_{Q_1}, \theta_{Q_2}),$$

(12.28) $$d_T(-\theta_{Q_1}, \theta_{Q_2}) = (T\nu)^* d_T\theta_Q$$

Let us take a pair $(u_1, u_2) \in TT^*TQ_1 \times TT^*TQ_2 \simeq T(T^*TQ_1 \times T^*TQ_2)$ and let us set $w_i = T\pi_{TQ_i}(u_i)$, $z_i = \tau_{T^*TQ_i}(u_i)$ $(i = 1, 2)$. Since the subdiagram (4) is commutative we have:

$$\begin{aligned}
\langle (u_1, u_2), \zeta^*\theta_{T(Q_1\times Q_2)}\rangle &= \langle T\zeta(u_1, u_2), \theta_{T(Q_1\times Q_2)}\rangle \\
&= \langle T\pi_{T(Q_1\times Q_2)} \circ T\zeta(u_1, u_2), \tau_{T^*T(Q_1\times Q_2)} \circ T\zeta(u_1, u_2)\rangle \\
&= \langle T\varphi^{-1} \circ T(\pi_{Q_1} \times \pi_{Q_2})(u_1, u_2), \zeta \circ \tau_{T^*TQ_1\times T^*TQ_2}(u_1, u_2)\rangle \\
&= \langle T\varphi^{-1}(w_1, w_2), \ \zeta(z_1, z_2)\rangle \\
&= -\langle w_1, z_1\rangle + \langle w_2, z_2\rangle \\
&= -\langle T\pi_{TQ_1}(u_1), \tau_{T^*TQ_1}(u_1)\rangle + \langle T\pi_{TQ_2}(u_2), \tau_{T^*TQ_2}(u_2)\rangle \\
&= -\langle u_1, \theta_{TQ_1}\rangle + \langle u_2, \theta_{TQ_2}\rangle.
\end{aligned}$$

This proves (12.25). Identity (12.26) directly follows from the characteristic property (12.3) for α_{Q_1} and α_{Q_2}. Identity (12.27) is a consequence of (9.11) and (11.2):

$$\begin{aligned}
\mu^*(d_T\theta_{Q_1}, d_T\theta_{Q_2}) &= \mu^*(pr^*_{TT^*Q_1}\theta_{Q_1} + pr^*_{TT^*Q_2}\theta_{Q_2}) \\
&= (pr_{TT^*Q_1} \circ \mu)^*\,\theta_{Q_1} + (pr_{TT^*Q_2} \circ \mu)^*\,\theta_{Q_2} \\
&= (Tpr_{T^*Q_1})^*\,\theta_{Q_1} + (Tpr_{T^*Q_2})^*\,\theta_{Q_2} \\
&= d_T pr^*_{T^*Q_1}\theta_{Q_1} + d_T pr^*_{T^*Q_2}\theta_{Q_2} \\
&= d_T(\theta_{Q_1}, \theta_{Q_2}).
\end{aligned}$$

Now, let us consider the diffeomorphism ν defined in (6.1). For each pair $(v_1, v_2) \in T(T^*Q_1 \times T^*Q_2) \simeq TT^*Q_1 \times TT^*Q_2$ we have:

$$\begin{aligned}
\langle (v_1, v_2), \nu^*\theta_{Q_1\times Q_2}\rangle &= \langle T\nu(v_1, v_2), \theta_{Q_1\times Q_2}\rangle \\
&= \langle T\pi_{Q_1\times Q_2} \circ T\nu(v_1, v_2), \tau_{T^*(Q_1\times Q_2)} \circ T\nu(v_1, v_2)\rangle \\
&= \langle T(\pi_{Q_1\times Q_2} \circ \nu)(v_1, v_2), \nu \circ \tau_{T^*(Q_1\times Q_2)}(v_1, v_2)\rangle \\
&= \langle (T\pi_{Q_1}(v_1), T\pi_{Q_2}(v_2)), \nu(\tau_{T^*Q_1}(v_1), \tau_{T^*Q_2}(v_2))\rangle \\
&= -\langle T\pi_{Q_1}(v_1), \tau_{T^*Q_1}(v_1)\rangle + \langle T\pi_{Q_2}(v_2), \tau_{T^*Q_2}(v_2)\rangle \\
&= -\langle v_1, \theta_{Q_1}\rangle + \langle v_2, \theta_{Q_2}\rangle.
\end{aligned}$$

This shows that

(12.29) $$\nu^*\theta_{Q_1\times Q_2} = (-\theta_{Q_1}, \theta_{Q_2})$$

Hence, by using (9.11) we obtain:

$$(T\nu)^*\, d_T\theta_{Q_1\times Q_2} = d_T\, \nu^*\, \theta_{Q_1\times Q_2} = d_T(-\theta_{Q_1}, \theta_{Q_2}),$$

and also (12.28) is proved. (Q.E.D.)

(12.30) LEMMA. *For any pair of symplectic manifolds* (P_1, ω_1), (P_2, ω_2) *the following identity holds:*

$$\nu \circ (\beta_{(P_1,\omega_1)} \times \beta_{(P_2,\omega_2)}) \circ \mu = \beta_{(P_1,-\omega_1)\times(P_2,\omega_2)},$$

where $\mu : T(P_1 \times P_2) \to TP_1 \times TP_2$ *is the natural diffeomorphism and* $\nu : T^*P_1 \times T^*P_2 \to T^*(P_1 \times P_2)$ *is the diffeomorphism defined as in* (6.1).

Proof. Let us denote by β the mapping at the left hand side of the identity in question. This mapping is a bundle isomorphism. Because of proposition (12.9) it is sufficient to show that $i_T(-\omega_1, \omega_2) = \beta^*\theta_{P_1\times P_2}$, i.e. to show that:

(12.31) $$\nu^*\theta_{P_1\times P_2} = (-\theta_{P_1}, \theta_{P_2}),$$

(12.32) $$(\beta_{(P_1,\omega_1)} \times \beta_{(P_2,\omega_2)})^*(\theta_{P_1}, \theta_{P_2}) = (i_T\omega_1, i_T\omega_2),$$

(12.33) $$\mu^*(i_T\omega_1, i_T\omega_2) = i_T(\omega_1, \omega_2).$$

Identity (12.31) coincides with (12.29). Identity (12.32) directly follows from the characteristic property (12.10). The proof of (12.33) is similar to that of (12.27) and uses formula (9.10) instead of (9.11). (Q.E.D.)

Now we can prove propositions (12.13) and (12.16).

Proof of (12.13). Let us set S = graph σ. Identity (12.14) can be written as follows:

$$(\alpha_{Q_1} \times \alpha_{Q_2})\,(\mu(T\widetilde{\mathbb{N}S})) = \widetilde{\mathbb{N}}\,\varphi(TS),$$

or

$$(\alpha_{Q_1} \times \alpha_{Q_2})\,(\mu(T(\nu^{-1}(\mathbb{N}S))) = \psi^{-1}(\mathbb{N}(\varphi(TS))),$$

where μ, ν and φ are the diffeomorphisms considered in Lemma (12.24) and $\psi : T^*TQ_1 \times T^*TQ_2 \to T^*(TQ_1 \times TQ_2)$ is the diffeomorphism defined as in (6.1) (with Q_i replaced by T^*Q_i, $i = 1, 2$). However, $\psi^{-1}(\mathbb{N}(\varphi(TS))) = \zeta^{-1}(\mathbb{N}TS)$, where ζ is the diffeomorphism already considered in Lemma (12.24). Hence:

$$(\alpha_{Q_1} \times \alpha_{Q_2}) \circ \mu \circ T\nu^{-1}(T\mathbb{N}S) = \zeta^{-1}(\mathbb{N}TS).$$

This equality is true because of Lemma (12.24) and Lemma (12.21) (for $Q = Q_1 \times Q_2$). (Q.E.D.)

Proof of (12.16). Let us set R = graph p. Identity (12.17) can be written as follows:

$$(\beta_{(P_1,\,\omega_1)} \times \beta_{(P_2,\,\omega_2)})\,(\mu(TR)) = \nu^{-1}(\mathbb{N}R),$$

where ω and ν are the diffeomorphisms considered in Lemma (12.30). This identity is true because of Lemma (12.30) and Lemma (12.22). (Q.E.D.)

List of Symbols

$\langle v, p\rangle$	evaluation of a covector p on a vector v.
$\langle v, \theta\rangle$	evaluation of a form θ on a vector v.
$pr_{M_i} : M_1 \times M_2 \to M_i$	the cartesian projections ($i = 1, 2$).
$\alpha^*\theta$	the pull-back of a form θ by mapping α.
$\theta \mid K$	the pull-back of a form θ on a submanifold K
$V^{\S}$	the symplectic polar of subspace V of a symplectic vector space.
V^0	the dual polar of a subspace V of a vector space, i.e. the set of covectors annihilating V.
$\rho^t = (P_2, P_1; R^t)$	the transpose of a relation $\rho = (P_1, P_2, R)$.
$\sigma \circ \rho = (P_1, P_3; S \circ R)$	the composition of two relations $\rho = (P_1, P_2; R)$ and $\theta = (P_2, P_3; S)$.
$\rho(K) = R \circ K$	the image of a subset $K \subset P_1$ by a relation $\rho = (P_1, P_2; R)$.
T_qQ	the tangent space at a point q of a manifold Q.
T_q^*Q	the cotangent space at a point q of a manifold Q.
$\tau_Q : TQ \to Q$	the tangent bundle projection of a manifold Q.
$\pi_Q : T^*Q \to Q$	the cotangent bundle projection of a manifold Q.
θ_Q	the Liouville 1-form on T^*Q (see (5.1)).
$\omega_Q = d\theta_Q$	the canonical symplectic form on T^*Q.
$\nu : T^*Q_1 \times T^*Q_2 \to T^*(Q_1 \times Q_2)$	the diffeomorphism defined in (6.1).
$\mu : T(P_1 \times P_2) \to TP_1 \times TP_2$	the natural diffeomorphism.
$\mathfrak{L}(Q; K, \gamma)$	see (5.4).
$\mathbb{N}K$	the conormal bundle of a submanifold K (see (5.8)).
$\widetilde{\mathbb{N}}A = \nu^{-1}(\mathbb{N}A)$	see (6.2).
$\mathbb{P}Q = (T^*Q, \omega_Q)$	the phase manifold of Q.
$\mathbb{P}\alpha : \mathbb{P}Q_1 \to \mathbb{P}Q_2$	the canonical lift of a relation $\alpha : Q_1 \to Q_2$.
Φ_P	the exterior algebra of a manifold P.
$i_T : \Phi_P \to \Phi_{TP}$	the derivation of degree 0 defined in (9.3-5).
$d_T : \Phi_P \to \Phi_{TP}$	the derivation of degree 1 defined in (9.6).
$T(P, \omega) = (TP, d_T\omega)$	the tangent symplectic manifold of (P, ω).

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