S BENENTI

Linear symplectic relations

1.0 INTRODUCTION

This is a short report on a systematic study of linear symplectic relations undertaken jointly with W.M. Tulczyjew. Preliminary results of this study are contained in two papers [1, 2], to which we refer for the details omitted here. Only finite dimensional real vector spaces are considered in this preliminary approach.

1.1 LINEAR RELATIONS

A *linear relation* is a triple $\rho = (P_1, P_2; R)$, where P_1, P_2 are vector spaces and R is a (linear) subspace of the direct sum $P_1 \oplus P_2$. The subspace R is called the *graph* of the relation ρ. We shall use the notation R = graph ρ. Spaces P_1 and P_2 are called respectively the *domain* and the *codomain* of ρ. The usual composition of relations is adopted: if $\rho = (P_1, P_2; R)$, $\sigma = (P_1, P_2; S)$ are linear relations, we define $\sigma \circ \rho = (P_1, P_3; S \circ R)$ by setting

$$S \circ R = \{p_1 \oplus p_3 \in P_1 \oplus P_3;\ \exists\ p_2 \in P_2 : p_1 \oplus p_2 \in R,\ p_2 \oplus p_3 \in S\}.$$

We denote by L the *category of linear relations*, i.e., the category whose objects and morphisms are respectively (real, finite dimensional) vector spaces and linear relations composed as defined above. The category of vector spaces and linear maps is obviously a subcategory of L. We denote by $\mathrm{Hom}_L (P_1, P_2)$ the set of all linear relations *from* P_1 *to* P_2. The notation $\rho : P_1 \nrightarrow P_2$ is also used instead of $\rho \in \mathrm{Hom}_L(P_1, P_2)$. If K is a subspace of P_1, the subset

$$\rho(K) = \{p_2 \in P_2;\ \exists\ p_1 \in K : p_1 \oplus p_2 \in R\}$$

is a subspace of P_2. The relation $\rho^t = (P_2, P_1; R^t)$, where

$$R^t = \{p_2 \oplus p_1 \in P_2 \oplus P_1;\ \ p_1 \oplus p_2 \in R\},$$

is called the *transpose (relation)* of ρ. By setting $P^t = P$ for every vector space P, we define a contravariant functor $t: L \to L$ which plays the role of duality in L: the *transposition functor*.

Some special morphisms of L are characterized by the following two theorems related by duality (see [1], Section 3).

THEOREM 1 *If* $\rho \in \mathrm{Hom}_L(P_1,P_2)$, *the following four statements are equivalent:*

(*i*) ρ *is a coretraction:* $\exists\ \sigma \in \mathrm{Hom}_L(P_2,P_1)$: $\sigma\circ\rho = 1_{P_1}$,

(*ii*) ρ *is a monomorphism:* $\rho\circ\alpha = \rho\circ\beta \Rightarrow \alpha = \beta$,

(*iii*) $\rho^t(P_2) = P_1$ *and* $\rho^t(0) = 0$,

(*iv*) $\rho^t\circ\rho = 1_{P_1}$.

THEOREM 1* *If* $\rho \in \mathrm{Hom}_L(P_1,P_2)$ *the following four statements are equivalent:*

(*i*) ρ *is a retraction:* $\exists\ \sigma \in \mathrm{Hom}_L(P_2,P_1)$: $\rho\circ\sigma = 1_{P_2}$,

(*ii*) ρ *is an epimorphism:* $\alpha\circ\rho = \beta\circ\rho \Rightarrow \alpha = \beta$,

(*iii*) $\rho(P_1) = P_2$ *and* $\rho(0) = 0$,

(*iv*) $\rho\circ\rho^t = 1_{P_2}$.

We note that a *linear map* is characterized by the conditions $\rho^t(P_2) = P_1$, $\rho(0) = 0$. The above theorems imply, in particular, that an *isomorphism* of L (which is by definition a retraction and at the same time a coretraction) is a linear map, i.e., a linear isomorphism in the usual sense.

The following *factorization theorems* can be proved (see [1], Section 5):

THEOREM 2 *If* $\rho_1 \in \mathrm{Hom}_L(P_1,P)$ *and* $\rho_2 \in \mathrm{Hom}_L(P_2,P)$, *the following two statements are equivalent:*

(*i*) $\exists\ \gamma \in \mathrm{Hom}_L(P_1,P_2)$: $\rho_1 = \rho_2\circ\gamma$,

(*ii*) $\rho_2(0) \subseteq \rho_1(0)$ *and* $\rho_1(P_1) \subseteq \rho_2(P_2)$.

THEOREM 2* *If* $\rho_1 \in \mathrm{Hom}_L(P,P_1)$ *and* $\rho_2 \in \mathrm{Hom}_L(P,P_2)$, *the following two statements are equivalent* (*i*) $\exists\, \gamma \in \mathrm{Hom}_L(P_2,P_1)$: $\rho_1 = \gamma\circ\rho_2$,

(*ii*) $\rho_2^t(0) \subseteq \rho_1^t(0)$ *and* $\rho_1^t(P_1) \subseteq \rho_2^t(P_2)$.

We recall that a *subobject* of an object P is by definition a monomorphism whose codomain is P, while a *quotient object* of an object P is an epimorphism whose domain is P. We say that two subobjects $\rho_1:P_1 \to P$, $\rho_2:P_2 \to P$ of P are *isomorphic* (and we write $\rho_1 \cong \rho_2$) if there exists an isomorphism $\gamma \in \mathrm{Hom}_L(P_1,P_2)$ such that $\rho_1 = \rho_2 \circ \gamma$ (hence $\rho_1 = \rho_2 \circ \gamma^t$). This defines an equivalence relation in the set of subobjects of an object P. A *representative class* of subobjects of P is a selection of exactly one subobject in each equivalence class. From Theorem 2 it follows that $\rho_1 \cong \rho_2$ if and only if $\rho_1(0) = \rho_2(0)$ and $\rho_1(P_1) = \rho_2(P_2)$. Hence, each equivalence class of subobjects of P is characterized by a pair (K,L) of subspaces of P such that $L \subseteq K$. Analogous definitions and remarks apply to quotient objects of P.

These facts lead in a natural way to the following construction of representative classes of subobjects and quotient objects. The *reduction* of a vector space P with respect to a pair (K,L) such that $L \subseteq K$ is the linear relation

$$\mathrm{red}_{(P,K,L)} = (P,K/L;R),$$

where

$$R = \{p \oplus u \in P \oplus K/L;\ p \in u\}.$$

The transpose relation $\mathrm{red}^t_{(P;K,L)}$ is called the *counter-reduction* of P with respect to (K,L). The set of counter-reductions of P and the set of reductions of P provide representative classes of subobjects and quotient objects respectively.

The following theorem shows that each morphism of L is a composition of a reduction, an isomorphism and a counter-reduction (see [1], Section 7).

THEOREM 3 (structure theorem) *Let $\rho:P_1 \to P_2$ be a linear relation. There exists a unique linear isomorphism ρ_0 such that*

$$\rho = \mathrm{red}^t_{(P_2;\rho(P_1),\rho(0))} \circ \rho_0 \circ \mathrm{red}_{(P_1;\rho^t(P_1),\rho^t(0))}.$$

Some of the functors known in the category of vector spaces and linear maps can be extended to L.

The *direct sum functor*: for any collection $\{P_i\}_{i=1,2,\ldots,n}$ of vector spaces and a collection $\{\rho_i = (P_i, P_i', R_i)\}_{i=1,2,\ldots,n}$ of linear relations, we define

$$\oplus_{i=1}^{n} P_i = P_1 \oplus P_2 \oplus \ldots \oplus P_n,$$

$$\oplus_{i=1}^{n} \rho_i = (\oplus_{i=1}^{n} P_i, \oplus_{i=1}^{n} P_i'; \oplus_{i=1}^{n} R_i).$$

In this way, a covariant functor $\oplus_{i=1}^{n}$ is defined for each integer $n > 1$. It is easily shown that the sum $\oplus_{i=1}^{n} \rho_i$ is a monomorphism (or an epimorphism) if and only if each ρ_i is a monomorphism (or an epimorphism). In particular, we have

$$\mathrm{red}_{(\oplus_{i=1}^{n} P_i;\, \oplus_{i=1}^{n} K_i,\, \oplus_{i=1}^{n} L_i)} \cong \oplus_{i=1}^{n} \mathrm{red}_{(P_i; K_i, L_i)}.$$

The *algebraic duality functor*: for a vector space P and for a linear relation $\rho = (P_1, P_2, R)$, we put

$$P^* = \text{dual space of } P,$$

$$\rho^* = (P^*, P^*; R^{(*)}),$$

where

$$R^{(*)} = \{u_2 \oplus u_1 \in P_2^* \oplus P_1^*;\ \forall p_1 \oplus p_2 \in R,\ \langle p_1, u_1\rangle = \langle p_2, u_2\rangle\}. \qquad (1)$$

This defines a contravariant functor $*: L \to L$ (for the proof see [1], Section 9). From Theorem 1 and Theorem 1* it follows that ρ^* is an epimorphism (or a monomorphism) if and only if ρ is a monomorphism (or an epimorphism). Moreover, the following identity holds

$$\mathrm{red}^*_{(P;K,L)} = \mathrm{red}^t_{(P^*;L^\circ,K^\circ)} \circ \phi^{-1},$$

where $\phi: L^\circ/K^\circ \to (K/L)^*$ is the natural isomorphism and

$$K^\circ = \{u \in P^*;\ \forall k \in K,\ \langle k, u\rangle = 0\}.$$

1.2 LINEAR SYMPLECTIC RELATIONS

A *symplectic (vector) space* is a pair (P,ω), where P is an even-dimensional vector space and $\omega \in (P \wedge P)^*$ is a non-degenerate 2-form. The form ω is called a *symplectic form* on P. If K is a subspace of P, the subspace

$$K^{\S} = \{p \in P; \langle k \wedge p,\omega\rangle = 0,\ \forall k \in K\},$$

is called the *symplectic polar* of K. The following properties hold:

$$\dim K + \dim K^{\S} = \dim P,$$
$$K^{\S} \subseteq L^{\S} \iff L \subseteq K,$$
$$(K + L)^{\S} = K^{\S} \cap L^{\S},$$
$$(K \cap L)^{\S} = K^{\S} + L^{\S},$$
$$K^{\S\S} = K.$$

We recall that K is said to be *isotropic* if $K^{\S} \supseteq K$, *coisotropic* if $K^{\S} \subseteq K$, *Lagrangian* if $K^{\S} = K$. In these three cases, we have respectively $\dim K \leq \frac{1}{2} \dim P$, $\dim K \geq \frac{1}{2} \dim P$, $\dim K = \frac{1}{2} \dim P$. Moreover, K is isotropic if and only if $\omega|K = 0$, i.e., if ω vanishes on exterior products of vectors belonging to K.

If (P_1,ω_1), (P_2,ω_2) are two symplectic vector spaces, then $(P_1,\omega_1) \oplus (P_2,\omega_2)$ will denote the symplectic vector space $(P_1 \oplus P_2,(\omega_1,\omega_2))$, where (ω_1,ω_2) is the symplectic form defined by

$$\langle(p_1 \oplus p_2) \wedge (p_1' \oplus p_2'),\ (\omega_1,\omega_2)\rangle = \langle p_1 \wedge p_1',\omega_1\rangle + \langle p_2 \wedge p_2',\omega_2\rangle,$$

where $p_1,p_1' \in P_1$ and $p_2,p_2' \in P_2$.

We say that a linear relation $\rho = (P_1,P_2,R)$ is symplectic if its graph is a Lagrangian subspace of $(P_1,-\omega_1) \oplus (P_2,\omega_2)$. Hence, a *linear symplectic relation* is a triple $((P_1,\omega_1),\ (P_2,\omega_2);R)$, where (P_1,ω_1) and (P_2,ω_2) are symplectic vector spaces and R is a Lagrangian subspace of $(P_1,-\omega_1) \oplus (P_2,\omega_2)$.

A linear isomorphism $\rho:P_1 \to P_2$ preserves the symplectic forms, i.e.,

$$\langle\rho(p_1) \wedge \rho(p_1'),\omega_2\rangle = \langle p_1 \wedge p_1',\omega_1\rangle,$$

if and only if its graph is Lagrangian. Hence, symplectic relations are natural generalizations of symplectic isomorphisms.

It can be shown that the composition of linear symplectic relations is symplectic (for a direct proof see [2], Section 2; for a proof involving the concept of symplectic reduction see [3]). It follows that symplectic (real, finite-dimensional) vector spaces and linear symplectic relations form a category. We denote this category by S. We write $\rho \in \mathrm{Hom}_S((P_1,\omega_1),(P_2,\omega_2))$ or $\rho:(P_1,\omega_1) \to (P_2,\omega_2)$ to indicate that ρ is a linear symplectic relation from (P_1,ω_1) to (P_2,ω_2). For each subspace K of P_1 we have (see [2], Section 2)

$$(\rho(K))^{\S} = \rho(K^{\S}). \tag{4}$$

Hence, if K is coisotropic (or isotropic, Lagrangian) in (P_1,ω_1), then $\rho(K)$ is coisotropic (or isotropic, Lagrangian) in (P_2,ω_2). In particular,

$$\rho(P_1) = (\rho(0))^{\S}. \tag{5}$$

Hence, $\rho(0)$ is isotropic and $\rho(P_1)$ is coisotropic.

We remark that a vector space 0 made of a unique element can be trivially interpreted as a symplectic vector space. For a symplectic relation $\nu:0 \to (P,\omega)$, the space $N = \nu(0)$ is isotropic and coisotropic at the same time, hence it is a Lagrangian subspace of (P,ω). Each Lagrangian subspace N of (P,ω) can be characterized in this way by setting graph $\nu = 0 \oplus N$.

Since R^t is a Lagrangian subspace of $(P_2,-\omega_2) \oplus (P_1,\omega_1)$ if and only if R is Lagrangian in $(P_1,-\omega_1) \oplus (P_2,\omega_2)$, by setting $(P,\omega)^t = (P,\omega)$ for every symplectic space, we define the transposition functor t in the category S.

Theorems analogous to Theorems 1, 1*, 2, 2* can be proved in S (see [2], Section 3).

THEOREM 4 *If* $\rho \in \mathrm{Hom}_S((P_1,\omega_1), (P_2,\omega_2))$, *the following five statements are equivalent:*

(*i*) ρ *is a coretraction:* $\exists\, \sigma \in \mathrm{Hom}_S((P_2,\omega_2), (P_1,\omega_1))$: $\sigma \circ \rho = 1_{P_1}$

(*ii*) ρ *is a monomorphism:* $\rho\circ\alpha = \rho\circ\beta \Rightarrow \alpha = \beta$,

(*iii*) $\rho^t(P_2) = P_1$,

(*iv*) $\rho^t(0) = 0$,

(*v*) $\rho^t\circ\rho = 1_{P_1}$.

THEOREM 4* *If* $\rho \in \mathrm{Hom}_S((P_1,\omega_1), (P_2,\omega_2))$, *the following five statements are equivalent:*

(*i*) ρ *is a retraction:* $\exists\ \sigma \in \mathrm{Hom}_S((P_2,\omega_2), (P_1,\omega_1))$: $\rho\circ\sigma = 1_{P_1}$,

(*ii*) ρ *is an epimorphism:* $\alpha\circ\rho = \beta\circ\rho \Rightarrow \alpha = \beta$,

(*iii*) $\rho(P_1) = P_2$,

(*iv*) $\rho(0) = 0$,

(*v*) $\rho\circ\rho^t = 1_{P_2}$.

THEOREM 5 *If* $\rho_1 \in \mathrm{Hom}_S((P_1,\omega_1), (P,\omega))$ *and* $\rho_2 \in \mathrm{Hom}_S((P_2,\omega_2), (P,\omega))$, *the following three statements are equivalent:*

(*i*) $\exists\ \gamma \in \mathrm{Hom}_S((P_1,\omega_1), (P_2,\omega_2))$: $\rho_1 = \rho_2 \circ \gamma$,

(*ii*) $\rho_1(P_1) \subseteq \rho_2(P_2)$,

(*iii*) $\rho_2(0) \subseteq \rho_1(0)$.

THEOREM 5* *If* $\rho_1 \in \mathrm{Hom}_S((P,\omega), (P_1,\omega_1))$ *and* $\rho_2 \in \mathrm{Hom}_S((P,\omega), (P_2,\omega_2))$, *the following three statements are equivalent:*

(*i*) $\exists\ \gamma \in \mathrm{Hom}_S((P_2,\omega_2), (P_1,\omega_1))$: $\rho_1 = \gamma\circ\rho_2$

(*ii*) $\rho_1^t(P_1) \subseteq \rho_2^t(P_2)$,

(*iii*) $\rho_2^t(0) \subseteq \rho_1^t(0)$.

The discussion on subobjects, quotient objects and their representative classes follows the same pattern as in the category *L*. We conclude, for instance, that two subobjects $\rho_1:(P_1,\omega_1) \to (P,\omega)$ and $\rho_2:(P_2,\omega_2) \to (P,\omega)$ are isomorphic if and only if $\rho_1(P_1) = \rho_2(P_2)$ or equivalently if and only if $\rho_1(0) = \rho_2(0)$ (see (5)). Moreover, as we already remarked, $K = \rho_1(P_1)$ is coisotropic (and $\rho_1(0) = (\rho_1(P_1))^{\S}$ is isotropic). Hence, each equivalent class of subobjects of (P,ω) is characterized by a coisotropic subspace of (P,ω). Guided by analogy with the category *L*, we expect that a representative class of quotient objects (or objects) is made up of linear symplectic relations whose underlying linear relations are reductions (or counter-reductions). One can see that these reductions are to be taken with respect to pairs (K,L) of subspaces of P (with $L \subseteq K$) such that $L^{\S} = K$, which implies, in particular, that K is coisotropic.

THEOREM 6 *Let* (P,ω) *be a symplectic space and* $L \subseteq K$ *two subspaces of* P. *There exists a symplectic form* $\omega_{[K]}$ *on the quotient* K/L *such that* $\mathrm{red}_{(P;K,L)}$ *is symplectic if and only if* $L = K^{\S}$.

The symplectic form $\omega_{[K]}$ on $K/K^{\S}$ is defined by

$$\langle p \wedge p', \omega\rangle = \langle [p] \wedge [p'], \omega_{[K]}\rangle,$$

where $[p]$ and $[p']$ denote equivalence classes of p and p' respectively. Correctness of this definition is guaranteed by K being coisotropic.

Let (P,ω) be a symplectic vector space and let K be a coisotropic subspace of (P,ω). The symplectic space

$$(P,\omega)_{[K]} = (P_{[K]}, \omega_{[K]}) = (K/K^{\S}, \omega_{[K]}) \tag{6}$$

is called the *reduced symplectic space* of (P,ω) with respect to K. The symplectic relation

$$\mathrm{red}_{(P,\omega;K)}:(P,\omega) \to (P,\omega)_{[K]},$$

whose underlying linear relation is $\mathrm{red}_{(P;K,K^{\S})}$, is called the *symplectic reduction* of (P,ω) with respect to K. For each subspace N of P we put

$$N_{[K]} = \mathrm{red}_{(P,\omega;K)}(N) = \{u \in P_{[K]};\ \exists\ p \in N : p \in u\}. \tag{7}$$

We have a *structure theorem* analogous to Theorem 3.

THEOREM 7 *Let* $\rho:(P_1,\omega_1) \to (P_2,\omega_2)$ *be a linear symplectic relation. Then there exists a unique symplectic isomorphism* ρ_0 *such that*

$$\rho = \mathrm{red}^{t}_{(P_2,\omega_2;\rho(P_1))} \circ \rho_0 \circ \mathrm{red}_{(P_1,\omega_1;\rho^t(P_2))}.$$

1.3 FUNCTORS

A number of functors can be introduced in the category $\mathcal{S}$. Some of these arise from analogous functors in $\mathcal{L}$.

(a) Direct sum functor

We have already defined the direct sum $(P_1,\omega_1) \oplus (P_2,\omega_2)$ of two symplectic spaces. This definition can be extended in a natural way to a sum of an arbitrary number of symplectic spaces. It turns out that the direct sum of symplectic relations is symplectic. Hence, the direct sum is a covariant functor in *S*.

(b) Symplectic duality functor

If (P,ω) is a symplectic space, there exists a natural isomorphism $\beta : P \to P^*$ defined by $\langle p, \beta(p')\rangle = \langle p' \wedge p, \omega\rangle$. We introduce the symplectic form

$$\Omega = \omega \circ (\beta^{-1} \wedge \beta^{-1})$$

in P. The symplectic space (P^*,Ω) is called the *dual symplectic space* of (P,ω) and is denoted by $(P,\omega)^*$. If K is a subspace of P,

$$K^{\circ\S} = K^{\S\circ}. \tag{8}$$

It follows that K is coisotropic (or isotropic, Lagrangian) in (P,ω) if and only if its dual polar K° is isotropic (or coisotropic, Lagrangian) in $(P,\omega)^*$.

Let (P_1,ω_2) and (P_2,ω_2) be two symplectic vector spaces. If R is a subspace of $P_1 \oplus P_2$, let $R^{(*)}$ denote the subspace of $P_2^* \oplus P_1^*$ defined as in (1). Let us consider the symplectic spaces $(P_1,-\omega_1) \oplus (P_2,\omega_2)$ and $(P_2,-\omega_2)^* \oplus (P_1,\omega_1)^*$. It can be proved that the following identity holds (see [2], Section 5):

$$(R^{(*)})^\S = (R^\S)^{(*)}. \tag{9}$$

It follows that $R^{(*)}$ is Lagrangian if and only if R is Lagrangian. On the other hand, we recall that, by definition, $R^{(*)}$ = graph ρ^*, where R = graph ρ. Hence, if ρ is symplectic then the dual relation ρ^* is symplectic. Consequently,

$$* : (P,\omega) \to (P,\omega)^*$$

$$* : \rho \to \rho^*$$

defines a contravariant functor in *S*.

(c) Phase functor

Let Q be a vector space. We denote by ω_Q the *canonical symplectic form* on

$Q \oplus Q^*$ defined by

$$\langle (q \oplus u) \wedge (q' \oplus u'), \omega_Q \rangle = \langle q', u \rangle - \langle q, u' \rangle \quad (q, q' \in Q, u, u' \in Q^*). \quad (10)$$

We call the symplectic space

$$\mathrm{Ph}\, Q = (Q \oplus Q^*, \omega_Q)$$

the *phase space* of Q.

For each pair (A,B) of subspaces of Q, we have identically

$$(A \oplus B^\circ)^{\S} = B \oplus A^\circ. \quad (11)$$

where the symplectic polar § is taken with respect to the canonical form ω_Q. Thus, $A \oplus B^\circ$ is coisotropic (or isotropic, Lagrangian) if and only if $B \subseteq A$ (or $B \supseteq A$, $B = A$).

Let $\alpha : Q_1 \to Q_2$ be a linear relation. Let us define

$$\mathrm{Ph}\, \alpha = \alpha \oplus \alpha^{*t}.$$

It can be proved that Ph α is a symplectic relation from Ph Q_1 to Ph Q_2 (see [2], Section 5). We call Ph α the *covariant lift* of α.

Using the functorial properties of $\oplus$, *, t, we note that Ph $1_Q = 1_{Q \oplus Q^*}$ and that Ph $\alpha \circ$ Ph β = Ph$(\alpha \circ \beta)$. Thus, Ph is a covariant functor from L to S: the phase functor.

(d) Tangent functor

If (P,ω) is a symplectic space, then $TP = P \oplus P$ has a natural symplectic form $\dot{\omega}$ such that the isomorphism $1_P \oplus \beta : P \oplus P \to P \oplus P^*$ is symplectic. The form $\dot{\omega}$ is defined by

$$\langle (a + b) \wedge (c \oplus d), \dot{\omega} \rangle = \langle c \wedge b, \omega \rangle - \langle a \wedge d, \omega \rangle \quad (a,b,c,d \in P). \quad (12)$$

We call the symplectic space

$$T(P,\omega) = (TP, \dot{\omega})$$

the *symplectic tangent space* of (P,ω).

If $\rho:(P_1,\omega_1) \to (P_2,\omega_2)$ is a linear symplectic relation then

$$T\rho = \rho \oplus \rho$$

is a symplectic relation from $T(P_1,\omega_1)$ to $T(P_2,\omega_2)$. The relation $T\rho$ is called the *tangent relation* of ρ.

Properties $T1_P = 1_{TP}$, $T(\sigma\circ\rho) = T\sigma \circ T\rho$ can be proved, thus T is a covariant functor in S.

All covariant functors defined above transform epimorphisms into epimorphisms and monomorphisms into monomorphisms. The contravariant functor of symplectic duality transforms epimorphisms into monomorphisms and monomorphisms into epimorphisms. With respect to reductions, we have the following properties (recall that $\cong$ means *isomorphic* in the sense described in Section 1.2):

$$\text{red}\,(\oplus_{i=1}^{n}(P_i,\omega_i);\ \oplus_{i=1}^{n} K_i) = \oplus_{i=1}^{n} \text{red}(P_i,\omega_i;K_i),$$

$$\text{red}^*_{(P,\omega;K)} \cong \text{red}^t_{((P,\omega)^*;K^\circ)},$$

$$\text{Ph red}_{(Q;A,B)} \cong \text{red}_{(\text{Ph } Q;A \oplus B^\circ)},$$

$$\text{T red}_{(P,\omega;K)} \cong \text{red}_{(T(P,\omega);TK)}.$$

The composition properties of these functors are investigated in [2].

1.4 GENERATING FORMS OF LAGRANGIAN SUBSPACES

Let (P,ω) be a symplectic vector space. A *special symplectic structure* of (P,ω) is a symplectic isomorphism of (P,ω) onto a phase space of a vector space Q:

$$\phi:(P,\omega) \to \text{Ph } Q.$$

It is convenient to represent a special symplectic structure by the pair (Q,ϕ) or by $(Q,\binom{\pi}{\nu})$, where $\pi:P \to Q$ and $\nu:P \to Q^*$ are the linear epimorphisms obtained by the composition of ϕ with the natural projections of $Q \oplus Q^*$ onto Q and Q^* respectively. The following identities are satisfied: $\phi(p) = \pi(p) \oplus \nu(p)$ and

$$\langle p \wedge p', \omega \rangle = \langle \pi(p'), \nu(p) \rangle - \langle \pi(p), \nu(p') \rangle. \tag{13}$$

This last identity follows directly from the definition of symplectic isomorphism (3) and from the definition of the canonical symplectic form ω_Q (10).

A pair $(L, \hat{L})$ of Lagrangian subspaces of (P, ω) is said to be a *Lagrangian splitting* if L and $\hat{L}$ are transverse, i.e., if $L + \hat{L} = P$ or, equivalently, if $L \cap \hat{L} = 0$. If (Q, ϕ) is a special symplectic structure of (P, ω) then $(\phi^{-1}(Q \oplus 0), \phi^{-1}(0 \oplus Q^*))$ is a Lagrangian splitting of (P, ω). Conversely, a Lagrangian splitting $(L, \hat{L})$ gives rise to a special symplectic structure in the following way. Let $\pi: P \to L$ be the projection onto L with respect to $\hat{L}$ and let $\hat{\pi}: P \to \hat{L}$ be the projection onto $\hat{L}$ with respect to L. Let us consider the linear map $\nu: P \to L^*$ defined by

$$\langle l, \nu(p) \rangle = \langle \hat{\pi}(p) \wedge l, \omega \rangle, \; l \in L, \; p \in P.$$

Since L and $\hat{L}$ are Lagrangian and transverse, it turns out that the linear map $\phi = \binom{\pi}{\nu}$ defined by $\phi(p) = \pi(p) \oplus \nu(p)$ is an isomorphism from P to $L \oplus L^*$. Moreover, identity (13) is satisfied. Thus $(L, \binom{\pi}{\nu})$ is a special symplectic structure of (P, ω).

Let $(Q, \binom{\pi}{\nu})$ be a special symplectic structure of (P, ω). The linear map $\theta: P \to P^*$ defined by

$$\langle p', \theta(p) \rangle = \langle \pi(p'), \nu(p) \rangle \tag{14}$$

has the following properties:

$$\pi(p') = 0 \Rightarrow \langle p', \theta(p) \rangle = 0, \; \forall \, p \in P, \tag{15}$$

(we say that θ is *vertical* with respect to the projection π) and

$$\langle p', \theta(p) \rangle - \langle p, \theta(p') \rangle = \langle p \wedge p', \omega \rangle. \tag{16}$$

Conversely, let (P, ω) be a symplectic space, $\pi: P \to Q$ a linear surjective map such that $\dim Q = \frac{1}{2} \dim P$ and $\theta: P \to P^*$ a linear map satisfying (15) and (16). If $\nu: P \to Q^*$ is the linear map defined by (14), then $(Q, \binom{\pi}{\nu})$ is a special symplectic structure of (P, ω).

In particular, if $(P, \omega) = \mathrm{Ph}\, Q$ with its natural special symplectic structure, then $P^* \cong Q^* \oplus Q$ and $\theta(q \oplus u) = u \oplus 0$.

Let $G:C \to \mathbf{R}$ be a *quadratic form* on a vector space C. The map

$$\delta G : C \times C \to \mathbf{R} : (a,b) \mapsto G(a+b) - G(a) - G(b)$$

is bilinear and symmetric. Let $dG: C \to C^*$ be the *differential* of G defined by

$$\langle a, dG(b)\rangle = \delta G(a,b), \quad a,b \in C.$$

It is selfadjoint, due to the symmetry of δG: $\langle a,dG(b)\rangle - \langle b,dG(a)\rangle = 0$.

Let $(Q,\binom{\pi}{\nu})$ be a special symplectic structure of a symplectic space (P,ω) and let $G:C \to \mathbf{R}$ be a quadratic form on a subspace C of Q. It can be shown that

$$N = \{p \in P;\ \pi(p) \in C \text{ and } \langle c,\nu(p)\rangle = \langle c, dG(\pi(p))\rangle,\ \forall c \in C\} \tag{17}$$

is a Lagrangian subspace of (P,ω) and that $\pi(N) = C$. The quadratic form $G: C \to \mathbf{R}$ is called the *generating form* of N. If $C = Q$, we have

$$N = \{p \in P;\ \nu(p) = dG(\pi(p))\},$$

and N is said to be *regular* with respect to the given special symplectic structure. In this case, $\pi(N) = Q$.

Conversely, each Lagrangian subspace N has a unique generating form that can be constructed as follows. Let us consider the function $W:N \to \mathbf{R}$ defined by (see (14))

$$W(p) = \tfrac{1}{2}\langle \pi(p),\nu(p)\rangle = \tfrac{1}{2}\langle p,\theta(p)\rangle. \tag{18}$$

This function is a quadratic form on N constant on fibres of $\pi|N$, i.e., $\pi(p) = \pi(p') \Rightarrow W(p) = W(p')$. We call W the *proper form* of N with respect to the special symplectic structure $(Q,\binom{\pi}{\nu})$. A quadratic form G is induced on the space $C = \pi(N)$:

$$G(c) = \tfrac{1}{2}\langle c,\nu(p)\rangle = W(p),\quad \forall p \in \pi^{t}(c) \cap N. \tag{19}$$

This is the generating form of N.

1.5 GENERATING FORMS OF LINEAR SYMPLECTIC RELATIONS

Let $(Q_1, \binom{\pi_1}{\nu_1})$ and $(Q_2, \binom{\pi_2}{\nu_2})$ be special symplectic structures of two symplectic spaces (P_1, ω_1) and (P_2, ω_2) respectively. On the sum $(P_1, -\omega_1) \oplus (P_1, \omega_1)$ we have the special symplectic structure

$$\left\{Q_1 \oplus Q_2, \begin{bmatrix} \pi_1 \oplus \pi_2 \\ \eta\circ(-\nu_1 \oplus \nu_2) \end{bmatrix}\right\}, \tag{20}$$

where $\eta: Q_1^* \oplus Q_2^* \to (Q_1 \oplus Q_2)^*$ is the natural isomorphism. The *generating form of a symplectic relation* $\rho:(P_1, \omega_1) \to (P_2, \omega_2)$ with respect to the symplectic structures $(Q_1, \binom{\pi_1}{\nu_1})$ and $(Q_2, \binom{\pi_2}{\nu_2})$ is by definition the generating form of graph ρ with respect to the special symplectic structure (20). Taking into account the remarks of the preceding section, we see that a quadratic form $F:D \to \mathbb{R}$, where D is a subspace of $Q_1 \oplus Q_2$, generates a linear symplectic relation ρ through the formula

$$\text{graph } \rho = \{p_1 \oplus p_2 \in P_1 \oplus P_2;\ \pi_1(p_1) \oplus \pi_2(p_2) \in D \text{ and}$$
$$\langle q_1 \oplus q_2, dF(\pi_1(p_1) \oplus \pi_2(p_2))\rangle = -\langle q_1, \nu_1(p_1)\rangle$$
$$+ \langle q_2, \nu_2(p_2)\rangle, \ \forall\, q_1 \oplus q_2 \in D\}.$$

Conversely, the generating form $F:D \to \mathbb{R}$ of a given linear symplectic relation ρ is defined as follows:

$$D = \pi_1 \oplus \pi_2 \text{ (graph } \rho),$$
$$F(q_1 \oplus q_2) = -\tfrac{1}{2}\langle q_1, \nu_1(p_1)\rangle + \tfrac{1}{2}\langle q_2, \nu_2(p_2)\rangle,$$
$$\forall\, p_1 \oplus p_2 \in (\pi_1 \oplus \pi_2)^t(D) \cap \text{graph } \rho.$$

We remark that the subspace D of $Q_1 \oplus Q_2$ can be interpreted as the graph of a linear relation $\delta:Q_1 \to Q_2$. We call δ the *base relation* of ρ with respect to the given special symplectic structures.

Symplectic relations generated by zero forms are of particular interest. The following three lemmas can be proved (see [2], Section 9).

LEMMA 1 *Let* (Q_1, ϕ_1), (Q_2, ϕ_2) *be special symplectic structures of two symplectic spaces* (P_1, ω_1), (P_2, ω_2). *A linear symplectic relation* $\rho:(P_1, \omega_2) \to (P_2, \omega_2)$

is generated by the zero form on a subspace $D \subseteq Q_1 \oplus Q_2$ *if and only if there exists a linear relation* $\delta : Q_1 \to Q_2$ *such that the following diagram of relations is commutative*

$$\begin{array}{ccc} (P_1,\omega_2) & \xrightarrow{\ \phi_1\ } & \mathrm{Ph}\ Q_1 \\ \rho \downarrow & & \downarrow \mathrm{Ph}\ \delta \\ (P_2,\omega_2) & \xrightarrow{\ \phi_2\ } & \mathrm{Ph}\ Q_2 \end{array} .$$

In this case, δ *is the base relation of* $\rho : D = \mathrm{graph}\ \rho$.

LEMMA 2 *Let* $\zeta : (P,\omega) \to (\hat{P},\hat{\omega})$ *be an epimorphism generated by a zero form with respect to special symplectic structures* (Q,ϕ) *and* $(\hat{Q},\hat{\phi})$. *There exist two subspaces* A *and* B *of* Q, *with* $B \subseteq A$, *such that* $\phi(K) = A \oplus B^\circ$, *where* K *denotes the coisotropic subspace* $\zeta^t(\hat{P})$. *The base relation* $\kappa : Q \to \hat{Q}$ *of* ζ *is an epimorphism and*

$$A = \kappa^t(\hat{Q}),\ B = \kappa^t(0),\ A^\circ = \kappa^*(0),\ B^\circ = \kappa^*(\hat{Q}).$$

LEMMA 3 *With the notations of Lemma 2, if* N *is a Lagrangian subspace of* (P,ω) *generated by* $G : C \to \mathbb{R}$, *the Lagrangian subspace* $\zeta(N)$ *of* $(\hat{P},\hat{\omega})$ *is generated by* $\hat{G} : \hat{C} \to \mathbb{R}$, *where*

$$\begin{aligned} \hat{C} &= \kappa(\tilde{C}), \\ \hat{G}(\hat{q}) &= G(q),\ \forall\ q \in \kappa^t(\hat{q}) \cap \tilde{C}, \end{aligned} \tag{21}$$

and

$$\tilde{C} = \{q \in C \cap A;\ \forall\ c \in C \cap B,\ \langle c, dG(q)\rangle = 0\}. \tag{22}$$

Let K be a coisotropic subspace of (P,ω) and let $(P,\omega)_{[K]}$ be the reduced space (see (6)). If a special symplectic structure (Q,ϕ) of (P,ω) is given in such a way that $\phi(K) = A \oplus B^\circ$, we can assign a special symplectic structure on $(P,\omega)_{[K]}$ such that the generating form of the reduction $\mathrm{red}_{(P,\omega;K)}$ is zero. A natural choice of such a special symplectic structure is provided by the symplectic isomorphism $\phi_{[K]}$ defined by the commutative diagram

$$\begin{array}{ccc} (P,\omega) & \xrightarrow{\ \phi\ } & \mathrm{Ph}\ Q \\ \Big\downarrow{\scriptstyle \mathrm{red}_{(P,\omega;K)}} & & \Big\downarrow{\scriptstyle \mathrm{Ph\ red}_{(Q;A,B)}} \\ (P,\omega)_{[K]} & \xrightarrow{\ \phi_{[K]}\ } & \mathrm{Ph}\ A/B \end{array}$$

We call $(A/B,\phi_{[K]})$ the *reduced special symplectic structure* of (Q,ϕ) with respect to K. If a Lagrangian subspace N of (P,ω) is generated by a form $G:C \to \mathbb{R}$, the generating form of the reduced Lagrangian subspace $\phi_{[K]}$ will be denoted by $G_{[K]}:C_{[K]} \to \mathbb{R}$. The construction of this form is described by Lemma 3 In formulae (21) and (22), we have $\hat{C} = C_{[K]}$, $\hat{G} = G_{[K]}$ and $\kappa = \mathrm{red}_{(Q;A,B)}$.

Of particular interest are the following two special cases.

(1) If $\phi(K) = A \oplus Q^*$ (i.e., $B = 0$) then $(P,\omega)_{[K]}$ is isomorphic to Ph A and $N_{[K]}$ is generated by the restriction of G to $C \cap A$: $G_{[K]} = G|C \cap A$.

(2) If $\phi(K) = Q \oplus B^\circ$ (i.e., $A = Q$) then $(P,\omega)_{[K]}$ is isomorphic to Ph Q/B. The generating form of $N_{[K]}$ is given by

$$C_{[K]} = \kappa(\tilde{C}),$$

$$G_{[K]}([b]) = G(b),\ \forall\ b \in \tilde{C},$$

where $\kappa:Q \to Q/B$ is the natural projection, $[b]$ denotes the equivalence class of b, and

$$\tilde{C} = \{q \in C;\ \forall\ c \in C \cap B,\ \langle c, dG(q)\rangle = 0\}.$$

1.6 COMPOSITION OF GENERATING FORMS AND LEGENDRE TRANSFORMATION

THEOREM 8 *Let* $G_\rho:C_\rho \to \mathbb{R}$ *and* $G_\sigma:C_\sigma \to \mathbb{R}$ *be the generating forms of two symplectic relations,* $\rho:(P_1,\omega_1) \to (P_2,\omega_2)$ *and* $\sigma:(P_2,\omega_2) \to (P_3,\omega_3)$, *with respect to special symplectic structures* $(Q_1,(\begin{smallmatrix}\pi_1\\ \nu_1\end{smallmatrix}))$, $(Q_2,(\begin{smallmatrix}\pi_2\\ \nu_2\end{smallmatrix}))$, $(Q_3,(\begin{smallmatrix}\pi_3\\ \nu_3\end{smallmatrix}))$. *Let the composed relation* $\sigma\circ\rho:(P_1,\omega_1) \to (P_3,\omega_3)$ *be generated by* $G_{\sigma\circ\rho}:C_{\sigma\circ\rho} \to \mathbb{R}$. *Then*

$$\begin{aligned} C_{\sigma\circ\rho} &= \mathrm{pr}_{13}(\tilde{C}_{\sigma\circ\rho}), \\ G_{\sigma\circ\rho}(q_1 \oplus q_3) &= G_\rho(q_1 \oplus q_2) + G_\sigma(q_2 \oplus q_3),\ \forall q_1 \oplus q_2 \oplus q_3 \in \tilde{C}_{\sigma\circ\rho}, \end{aligned} \tag{23}$$

where $\mathrm{pr}_{13}:Q_1 \oplus Q_2 \oplus Q_3 \to Q_1 \oplus Q_3$ *is the natural projection and*

$$\tilde{C}_{\sigma\circ\rho} = \{q_1 \oplus q_2 \oplus q_3 \in Q_1 \oplus Q_2 \oplus Q_3; q_1 \oplus q_2 \in C_\rho,\ q_2 \oplus q_3 \in C_\sigma,$$

$$\langle 0 \oplus q_2', dG_\rho(q_1 \oplus q_2)\rangle + \langle q_2' \oplus 0, dG_\sigma(q_2 \oplus q_3)\rangle = 0, \qquad (24)$$

$$\forall\, q_2' \in Q_2 : 0 \oplus q_2' \in C_\rho,\ q_2' \oplus 0 \in C_\sigma\}.$$

This theorem can be proved as follows (see [2], Section 9; for another approach see [4]). Let us set

$$(P,\omega) = (P_1,-\omega_1) \oplus (P_2,\omega_2) \oplus (P_2,-\omega_2) \oplus (P_3,\omega_3),$$

$$(\hat{P},\hat{\omega}) = (P_1,-\omega_1) \oplus (P_3,\omega_3).$$

These two symplectic spaces have special symplectic structures $(Q,\binom{\pi}{\nu})$ and $(\hat{Q},\binom{\hat{\pi}}{\hat{\nu}})$, where

$$Q = Q_1 \oplus Q_2 \oplus Q_2 \oplus Q_3, \qquad \hat{Q} = Q_1 \oplus Q_3,$$

$$\pi = \pi_1 \oplus \pi_2 \oplus \pi_2 \oplus \pi_3, \qquad \hat{\pi} = \pi_1 \oplus \pi_3,$$

$$\nu = \eta\circ(-\nu_1 \oplus \nu_2 \oplus -\nu_2 \oplus \nu_3), \qquad \hat{\nu} = \hat{\eta}\circ(-\nu_1 \oplus \nu_3),$$

and $\eta : Q_1^* \oplus Q_2^* \oplus Q_2^* \oplus Q_3^* \to Q^*$, $\hat{\eta} : Q_1^* \oplus Q_3^* \to Q^*$ are the natural isomorphisms. Let us consider the symplectic epimorphism $\zeta : (P,\omega) \to (\hat{P},\hat{\omega})$ whose graph is

$$\operatorname{graph}\zeta = \{(p_1 \oplus p_2 \oplus p_2' \oplus p_3) \oplus (p_1' \oplus p_3') \in P \oplus \hat{P};$$

$$p_1 = p_1',\ p_2 = p_2',\ p_3 = p_3'\}.$$

It turns out that if $N = \operatorname{graph}\rho \oplus \operatorname{graph}\sigma$, then $\zeta(N) = \operatorname{graph}(\sigma\circ\rho)$. Moreover, it can be seen that $\zeta = \hat{\phi}^{-1} \circ \mathrm{Ph}\,\kappa \circ \phi$, where $\kappa : Q \to \hat{Q}$ is the base relation of ζ and $\phi = \binom{\pi}{\nu}$, $\hat{\phi} = \binom{\hat{\pi}}{\hat{\nu}}$. This implies (Lemma 1) that ζ is generated by a zero form. According to Lemma 2 and Lemma 3, the generating form of N is $G : C \to \mathbf{R}$, where

$$C = C_\rho \oplus C_\sigma,$$

$$G((q_1 \oplus q_2) \oplus (q_2' \oplus q_3)) = G_\rho(q_1 \oplus q_2) + G_\sigma(q_2' \oplus q_3).$$

The generating form of $\zeta(N)$ will be exactly the generating form of $\sigma\circ\rho$, so that formulae (23) and (24) follow from (21) and (22), through some obvious identifications.

Another version of formulae (23) and (24) can be given. Let us introduce the subspace

$$\bar{C} = pr^t_{12}(C_\rho) \cap pr^t_{23}(C_\sigma) \subseteq Q_1 \oplus Q_2 \oplus Q_3,$$

and let

$$\bar{G}_\rho : \bar{C} \to \mathbb{R}, \quad \bar{G}_\sigma : \bar{C} \to \mathbb{R}$$

be the natural extensions of G_ρ and G_σ respectively (i.e., $\bar{G}_\rho(q_1 \oplus q_2 \oplus q_3) = G_\rho(q_1 \oplus q_2)$, $\bar{G}_\sigma(q_1 \oplus q_2 \oplus q_3) = G_\sigma(q_2 \oplus q_3)$). If we set

$$\bar{G} = \bar{G}_\rho + \bar{G}_\sigma$$

it turns out that the space

$$\tilde{C}_{\sigma\circ\rho} = \{q_1 \oplus q_2 \oplus q_3 \in \bar{C};\ \langle 0 \oplus q_2' \oplus 0, d\bar{G}(q_1 \oplus q_2 \oplus q_3)\rangle = 0, \quad \forall\ 0 \oplus q_2' \oplus 0 \in \bar{C}\} \tag{25}$$

coincides with the space defined in (24) and that (23) can be written in the form

$$G_{\sigma\circ\rho}(q_1 \oplus q_2) = \bar{G}(q_1 \oplus q_2 \oplus q_3),\ \forall\ q_1 \oplus q_2 \oplus q_3 \in \tilde{C}. \tag{26}$$

These formulae can be easily extended to the composition of more than two symplectic relations (see [2], Section 9).

From the law of composition of generating forms of symplectic relations, we can derive the so-called *Legendre transformation* for a Lagrangian space, i.e., the relation between the generating forms of a Lagrangian subspace with respect to two different special symplectic structures. This will represent the linear version of the general Legendre transformation given in [5].

We recall (see Section 1.2) that a Lagrangian subspace N of a symplectic space (P,ω) is canonically represented by a symplectic relation $\nu : 0 \to (P,\omega)$, where graph $\nu = 0 \oplus N$. The trivial space 0 has a trivial special symplectic structure given by the identification of 0 with Ph $0 \cong 0$. Let us consider a diagram of the following type:

$$\begin{array}{ccccc} 0 & \xrightarrow{\nu} & (P,\omega) & \xrightarrow{1_P} & (P,\omega) \\ \downarrow & & \downarrow \phi_1 & & \downarrow \phi_2 \\ 0 & & \mathrm{Ph}\ Q_1 & & \mathrm{Ph}\ Q_2 \end{array}$$

where (Q_1,ϕ_1) and (Q_2,ϕ_2) are special symplectic structures of (P,ω). We can apply Theorem 8 (i.e., formulae (23), (24) or (25), (26)) to the composition of ν and 1_P, observing that the generating forms of ν have an obvious identification with the generating forms of N. The following theorem can be established.

THEOREM 9 *Let* $G:C \to \mathbb{R}$ *be the generating form of the identity relation* 1_P *of a symplectic space* (P,ω), *with respect to two different special symplectic structures* (Q_1,ϕ_1) *and* (Q_2,ϕ_2) *of* (P,ω). *If* $G_1:C_1 \to \mathbb{R}$ *is the generating form of a Lagrangian subspace* N *of* (P,ω) *with respect to* (Q_1,ϕ_1), *the generating form* $G_2:C_2 \to \mathbb{R}$ *of* N *with respect to* (Q_2,ϕ_2) *is given by the following formulae:*

$$C_2 = pr_2(\tilde{C}),$$

$$G_2(q_2) = G(q_1 \oplus q_2) + G_1(q_1), \quad \forall q_1 \oplus q_2 \in \tilde{C},$$

where

$$\tilde{C} = \{q_1 \oplus q_2 \in \bar{C}; \langle q_1' \oplus 0, dG(q_1 \oplus q_2)\rangle + \langle q_1', dG_1(q_1)\rangle = 0,\ \forall q_1' \oplus 0 \in \bar{C}\},$$

$$\bar{C} = pr_1^t(C_1) \cap C = \{q_1 + q_2 \in C;\ q_1 \in C_1\}$$

and $pr_i:Q_1 \oplus Q_2 \to Q_i$ $(i = 1,2)$ *denote the natural projections.*

By analogous reasoning, we can obtain the Legendre transformation for symplectic relations, i.e., the law of transformation of the generating form of a symplectic linear relation when the special symplectic structures are changed in its domain and its codomain. For details, see [2], Section 10.

We conclude with some remarks about the proper forms of the identity relation 1_P and Lagrangian subspaces of a symplectic space (P,ω). Let $\theta_1:P \to P^*$ and $\theta_2:P \to P^*$ be the vertical maps associated with the special symplectic structures (Q_1, ϕ_1) and (Q_2, ϕ_2) of (P, ω) respectively (see (14), (15) and (16)). It can be seen that $\theta_2 - \theta_1$ is selfadjoint; hence it is the differential of a quadratic form $W: P \to \mathbb{R}$,

$$\theta_2 - \theta_1 = dW.$$

It turns out that W considered as a form on the diagonal of $P \oplus P$ is the proper form of graph 1_P. Moreover, if W_1 and W_2 denote the proper forms of a Lagrangian subspace N of (P, ω), from (18) it follows that

$$W_2 - W_1 = W|N.$$

ACKNOWLEDGEMENT

This research was sponsored by Consiglio Nazionale delle Ricerche Gruppo Nazionale per la Fisica Matematica.

REFERENCES

[1] S. Benenti, W.M. Tulczyjew, *Relazioni lineari binarie tra spazi vettoriali di dimensione finita,* Mem. Acc. Sci. Torino, 3 (1979), 67-113.

[2] S. Benenti, W.M. Tulczyjew, *Relazioni lineari simplettiche*, Mem. Acc. Sci. Torino (Submitted).

[3] R. Abraham, J. Marsden, *Foundations of mechanics*, 2nd Edn., Benjamin-Cummings, New York, 1978.

[4] B. Lawruk, J. Sniatycki, W.M. Tulczyjew, *Special symplectic spaces*, J. Diff. Equ., 17 (1975), 477-497.

[5] W.M. Tulczyjew, *The Legendre transformation*, Ann. Inst. Henri Poincaré, 27 (1977), 101-114.

S. Benenti, Istituto di Fisica Matematica, J.-L. Lagrange, Università di Torino.