Differential Geometrical Methods in Mathematical Physics,
Lecture Notes in Mathematics 863, Springer 1980 (512-538)

Proceedings International Conference,
Aix-en-Provence and Salamanca, 1979,
P.L. García, A. Perez Rendon, J.M. Souriau Editors.

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Separability Structures on Riemannian Manifolds

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1979

SEPARABILITY STRUCTURES ON RIEMANNIAN MANIFOLDS

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1.- Introduction.

Let M be a differentiable manifold of dimension n and let H be a differentiable function on the cotangent bundle T*M. Let $(U,(x^i))$ be a chart on M (i.e. a coordinate system (x^i) defined on the open set $U \subseteq M$) and let $(T^*U,(x^i;p_j))$ be the corresponding canonical chart on T*M $(i,j = 1,\dots,n)$. With this chart we can associate the representative function $H(x^i;p_j)$ of H and the so called reduced Hamilton-Jacobi equation:

$$H(x^i;\frac{\partial S}{\partial x^j}) = e \tag{1.1}$$

where e is a real parameter.

A complete solution (or a complete integral) of (1.1) is an n-parameters family $S(x^i;c_j)$ of solutions of (1.1) such that:

$$\det\left(\frac{\partial^2 S}{\partial x^i \partial c_j}\right) \neq 0 \tag{1.2}$$

When a complete solution is known, one can directly find the integral curves of the Hamiltonian vectorfield defined by the Hamiltonian H, by following a well known method.

We are mainly interested in those cases where equation (1.1) is completely separable, i.e. it admits a complete solution which is a sum of functions depending on a single coordinate:

$$S(x^i;c_j) = S_1(x^1;c_j) + S_2(x^2;c_j) + \dots + S_n(x^n;c_j) \quad . \tag{1.3}$$

Although the conditions for the existence of such a kind of complete solution are known (Levi-Civita, 1904 [1]), a detailed treatement of the separable case of the Hamilton-Jacobi equation is so far available only for certain classes of Hamiltonians. Among them, we recall the quadratic Hamiltonians corresponding to the classical mechanical systems, including t-dependent constraints and velocity-dependent potenti For a hystorical perspective on the separability of the Hamilton-Jacobi equation t reader can refer to the review articles [2] and [3] .

For a better understanding of the separability conditions of equation (1.1), in present paper we propose an approach based on the concept of separability structu which seems to allow valuable simplifications. A separability structure is a famil charts at a point $x_o \in M$ such that the corresponding equations (1.1) have complete tegrals of the kind (1.3) representing the same family of functions on a neighborh of x_o. From this point of view we analyze, in particular, the case of the geodesic miltonian of a Riemannian manifold with definite or indefinite metric. (Throughout paper we use the term Riemannian manifold in this generalized sense; when we want distinguish explicitly between definite or indefinite metric we use the terms proc Riemannian and pseudo-Riemannian manifold respectively). For the sake of simplicit functions and manifolds are tacitly assumed to be smooth enough to assure the vali of the results.

2.-Separability structures.

A solution of the Hamilton-Jacobi equation associated with the Hamiltonian $H:T$ $\longrightarrow \mathbb{R}$ is a function $S:U \longrightarrow \mathbb{R}$ (where U is an open subset of M) such that:

$$(2.1) \qquad dS(U) \subseteq C_e \quad ,$$

where $dS:M \longrightarrow T^*M$ is the differential of S and $C_e = \{p \in T^*M:\ H(p) = e\}$. Usuall only regular values of the Hamiltonian are considered, so that C_e is a submanifo of T*M of codimension 1 (see for instance [4] , p.49 and 204). In fact, inculsion is the coordinate free translation of equation (1.1), and we call it (although rat improperly) the reduced Hamilton-Jacobi equation associated with the Hamiltonian H the energy e.

A complete solution (or a complete integral)of the Hamilton-Jacobi equation ass ted with the Hamiltonian H is a function $S:\hat{U} \longrightarrow \mathbb{R}$ on an open subset $\hat{U}$ of a p duct manifold $M \times A \times E$, where E is the energy space ($E \simeq \mathbb{R}$) and A is a paramet space of dimension n-1 (it is not necessary to specify here the nature of such a ce: for further information see for instance [5]), such that each function $S_c:U_c \longrightarrow$ obtained by fixing an admissible value $c = (a,e) \in A \times E$ ($U_c = \{x \in M:\ (x,c) \in \hat{U}\}$, $S = S(x,c)$) is a solution of the Hamilton-Jacobi equation (2.1):

(2.2) $$dS_c(U_c) \subseteq C_e$$

and the submanifolds $L_c = dS_c(U_c)$, where c covers all its admissible values, form a (local) foliation of T^*M. The latter requirement is in fact the geometric counterpart of condition (1.2), where (c_j) must be interpreted as coordinates on $A \times E$ (usually one of the parameters (c_j) coincides with e). We notice that each leaf L_c of the foliation is a section of the canonical projection $\pi_M : T^*M \longrightarrow M$ and a Lagrangian submanifold of T^*M with respect to the canonical symplectic 2-form $\omega_M = dx^i \wedge dp_i$. If we consider the natural projection $\sigma_M : M \times A \times E \longrightarrow M$, the open set $U = \sigma_M(\hat{U}) \subseteq M$, which is the union of the open sets (U_c), can be called the <u>base domain</u> of the complete integral $S:\hat{U} \longrightarrow \mathbb{R}$. Then, the foliation (L_c) generated by S is defined in an open set $\tilde{U} \subseteq T^*M$ such that $\pi_M(\tilde{U}) = U$.

(2.3) <u>Definition</u>.- A chart $(U,(x^i))$ on a manifold M is said to be <u>separable</u> with respect to the Hamiltonian $H:T^*M \longrightarrow \mathbb{R}$ if the corresponding Hamilton-Jacobi equation admits a complete solution S whose base domain is U and such that

(2.4) $$\partial_i \partial_j S = 0 \quad , \quad i \neq j \quad .$$ (°)

Let us fix a point $x_0 \in M$ and let us consider the set of the separable charts at x_0.

(2.5) <u>Definition</u>.- Two charts $(U,(x^i))$ and $(U',(x^{i'}))$ are said to be <u>$\mathcal{S}$-compatible</u> with respect to the Hamiltonian H at the point $x_0 \in U \cap U'$, if they are separable and if the corresponding separated complete integrals coincide in an open neighborhood U'' of x_0, i.e. if the two corresponding foliations in T^*M coincide in an open set $\hat{U}''$ such that $x_0 \in U'' = \pi_M(\hat{U}'')$.

$\mathcal{S}$-compatibility is clearly an equivalence relation in the family of the separable charts at the point x_0 (which of course could be empty). Hence we are led to the following definition.

(2.6) <u>Definition</u>.- A <u>separability structure</u> (briefly, a <u>$\mathcal{S}$-structure</u>) at the point x_0 is an equivalence class of charts which are $\mathcal{S}$-compatible at x_0 with respect to the Hamiltonian H.

(°) From now on we will use the following abbreviations: $\partial_i = \dfrac{\partial}{\partial x^i}$, $\partial^i = \dfrac{\partial}{\partial p_i}$.

For the cases of major interest to us none of the functions $(\partial^i H)$ vanishes identically. Then we can consider the functions

$$R_i = -\frac{\partial_i H}{\partial^i H} \tag{2.7}$$

possibly smoothly extended to those points at which the denominators are zero.

(2.8) <u>Definition</u>.- Let $(U,(x^i))$ be a chart at the point $x_o \in M$. We say that a coord te x^k is of <u>first class</u> at x_o (with respect to the Hamiltonian H) if, in an open of T^*M which projects in a neighborhood of x_o, the corresponding function R_k is li on the fibers of π_M, i.e.:

$$R_k = B^i_k \, p_i \tag{2.9}$$

where the functions (B^i_k) depend on the coordinates (x^i) only. In particular, if th functions (B^i_k) vanish , i.e. if $\partial_k H = 0$, we say that the coordinate x^k is <u>ignorat</u> A coordinate which is not of first class is said to be of <u>second class</u>.

For the sake of simplicity we introduce the following convention.

(2.10) <u>Convention</u>.- Second class coordinates are labeled by indices from the first of the Latin alphabet $(a, b, c, \ldots)$; those of first class by Greek indices $(\alpha, \beta, \gamma, \ldots$ Latin indices from the second part of the alphabet $(h, i, j, \ldots)$ are used when the a distinction is not needed. The Einstein summation convention is adapted to the abo choice of indices, unless the symbol "n.s." appears together with a distinguished dex. The coordinates of any chart at the point x_o are assumed to be ordered in suc way that the first m $(0 \leq m \leq n)$ are of second class, while the others are of firs class, so that the indices $(a, b, c, \ldots)$ are assumed to range from 1 to m; $(\alpha, \beta, \gamma, \ldots$ from m+1 to n and $(h, i, j, \ldots)$ from 1 to n.

The distinction of coordinates into the two classes above (in the particular ca of a quadratic Hamiltonian) dates back to Dall'Acqua [6].

When (2.4) holds, by derivation of (1.1) with respect to a coordinate x^i , we c $(\partial_i H)_c + (\partial^i H)_c \, \partial_i \partial_i S_c$, where $(\ )_c$ means the substitution $p_j = \partial_j S_c$, i.e. the ev tion on the leaf L_c. Hence, even if $\partial^i H = 0$ at some point of L_c, we can define $(R_i)_c = \partial_i \partial_i S_c$. Moreover, we notice that the separability conditions (2.4) are exa the integrability conditions of the following system of partial differential equati (see Levi-Civita [1]):

(2.11) $$\partial_i p_j = 0 \quad (i \neq j) \quad , \quad \partial_i p_i = R_i \quad .$$

The integrability conditions are represented by the identities

(2.12) $$\partial_i R_j + \partial^i R_j \, R_i = 0 \qquad (i \neq j \; ; \; i \text{ n.s.}) \; ,$$

which, by (2.7), can be written as follows:

(2.13) $$\partial^i H \, \partial^j H \, \partial_i \partial_j H - \partial^i H \, \partial_j H \, \partial_i \partial^j H - \partial_i H \, \partial^j H \, \partial^i \partial_j H + \partial_i H \, \partial_j H \, \partial^i \partial^j H = 0 \qquad (i \neq j) \; .$$

These are the so called Levi-Civita separability conditions.

Following the above convention, system (2.11) can be written:

(2.14) $$\partial_i p_j = 0 \quad (i \neq j) \; , \; \partial_a p_a = R_a \quad , \; \partial_\alpha p_\alpha = B^i_\alpha \; p_i \quad .$$

However, we have identically

(2.15) $$B^a_\alpha = 0 \quad , \quad \partial_a B^\beta_\alpha = 0 \quad ,$$

so that (2.14) contains an autonomous sub-system in the first class coordinates only:

(2.16) $$\partial_\alpha p_\beta = 0 \quad (\alpha \neq \beta) \quad , \quad \partial_\alpha p_\alpha = B^\gamma_\alpha \; p_\gamma \quad .$$

In fact, from (2.12) it follows in particular: $\partial_a B^i_\alpha \, p_{p_i} + B^a_\alpha \, R_a = 0$ (a n.s.), which implies that R_a is linear in the variables (p_i), against the definition of second class coordinate, unless $B^a_\alpha = 0$; hence also $(2.15)_2$ follows.

Since system (2.16) is linear in the r $(= n - m)$ unknown functions (p_α), in a neighborhood of x_0 there exist r independent solutions, i.e. r^2 functions (ξ^β_α) such that:

(2.17) $$\partial_\alpha \xi^\delta_\beta = 0 \quad (\alpha \neq \beta) \; , \qquad \partial_\alpha \xi^\delta_\alpha = B^\gamma_\alpha \; \xi^\delta_\gamma$$

and

(2.18) $$\det(\xi^\beta_\alpha) \neq 0 \qquad .$$

Then we can consider a new coordinate system (y^i) defined by the equations

(2.19) $$dy^a = dx^a \quad , \quad dy^\alpha = \xi^\alpha_\beta \, dx^\beta \quad .$$

It can be easily checked that the new coordinates (y^i) are $\mathcal{S}$-compatible with (x^i) and that the coordinates (y^α) are ignorable. Hence, since the coordinates are still of second class, we have given a constructive proof of the following theorem.

(2.20) Theorem.- In a $\mathcal{S}$-structure there exist charts where all the first class ordinates are ignorable.

Another important fact is that the number of first (or second) class coordinates is an invariant of a $\mathcal{S}$-structure.

(2.21) Theorem.- Two $\mathcal{S}$-compatible charts have the same number of first (or second class coordinates.

Proof.- Let (x^i) and $(x^{i'})$ be $\mathcal{S}$-compatible coordinates at x_o. Let (x^a), $(x^{a'})$ the coordinates of second class ($a=1,\ldots,m$; $a'=1,\ldots,m'$) and (x^α), $(x^{\alpha'})$ those of class. They give rise to the same complete integral: $\sum_i S_i(x^i;c_j) = \sum_{i'} S_{i'}(x^{i'};c_j)$ By derivation with respect to x^α and $x^{a'}$, we obtain:

$$\xi^\alpha_{a'}\,\partial_\alpha\partial_\alpha S_\alpha = \xi^i_{a'}\,\partial_i\,\xi^{i'}_\alpha\,\partial_{i'}S_{i'} + \xi^{a'}_\alpha\,\partial_{a'}\partial_{a'}S_{a'} \qquad (\alpha,a'\text{n.s.}),$$

with $\xi^{j'}_i = \partial_i x^{j'}$ and $\xi^i_{j'} = \partial_{j'} x^i$. This means that the following identities hold:

$$\xi^\alpha_{a'}\,R_\alpha = \xi^i_{a'}\,\partial_i\,\xi^{j'}\,p_{j'} + \xi^{a'}_\alpha\;R_{a'} \qquad (\alpha,a'\text{n.s.}) \quad .$$

If $\xi^{a'}_\alpha \neq 0$, it follows that $R_{a'}$ is linear in the variables $(p_{j'})$(because R_α is also near in the (p_β) by definition), which is in contrast with the fact that $x^{a'}$ is of cond class. Hence, it must be $\xi^{a'}_\alpha = 0$ for each pair of indices (a',α), and (by symm $\xi^a_{\alpha'} = 0$ for each pair (a,α'). It clearly follows that if $m \neq m'$ the matrix $(\xi^{j'}_i)$ ca be regular, which is absurd. (q.e.d.

(2.22) Definition.- A separability structure at the point x_o is said to be of clas (or a $\mathcal{S}_r$-structure) if r is the number of the coordinates of first class at x_o in representative chart.

Theorem (2.20) shows that in a $\mathcal{S}_r$-structure there exist charts with r ignor coordinates. The use of these charts, which we call quasi-normal separable charts, lows of course valuable simplifications in the analysis of separability structures without any loss of generality. On the other hand, since second class coordinates not be ignorable (by definition), and their number is invariant, it follows clearly

(2.23) The class of a $\mathcal{S}$-structure is the maximum number of ignorable coordinates *one can find in a representative chart.*

This provides in fact an alternative definition of class of a $\mathcal{S}$-structure (see for instance [7]) which is not related with the definition of class of a coordinate.

3.- *First integrals associated with a separability structure.*

In this section we shall give a short outline of some properties of the first integrals associated with a complete solution of the Hamilton-Jacobi equation in the case of complete separability.

We have already pointed out that a complete integral is represented by a local foliation (L_c) of Lagrangian submanifolds of (T^*M, ω_M) which are sections of the fibration π_M and where the Hamiltonian H is constant (the former requirement could be removed; see for instance [5]). A complete integral can also be represented by a set of real functions (F_i) on an open set of T^*M satisfying the following conditions. i) They are *vertically independent*, i.e. they are not only functionally independent as functions on T^*M but also, more particularly, functionally independent when restricted to the fibers of T^*M. This fact is represented in natural canonical coordinates by the regularity of the $n \times n$ matrix $(\partial^k F_i)$. ii) The functions (F_i) are *in involution*, i.e. their Poisson brackets vanish identically: $\{F_i, F_j\} = 0$. iii) The Hamiltonian H depends functionally on the (F_i): $H = \rho(F_i)$ or, equivalently, the Hamiltonian H is in involution with them: $\{H, F_i\} = 0$. In fact, equations of the kind $F_i = c_i$, where $c = (c_i)$ covers a suitable open set of $\mathbb{R}^n$, define a foliation on T^*M of Lagrangian submanifolds which are transversal to the fibers (i.e. sections of π_M) and on which the Hamiltonian H is constant. When such a set of functions is given, then the complete solution S of the Hamilton-Jacobi equation corresponding to H is the integral of the equations $\partial_k S = S_k$, where the functions $S_k(x^i; c_j)$ are obtained by solving the system $F_i(x^j; p_k) = c_i$ with respect to the (p_k): $p_k = S_k(x^i; c_j)$. Conversely, if $S: \hat{U} \longrightarrow \mathbb{R}$ is a complete integral, we can obtain n functions (F_i) satisfying the above conditions, by solving the system $p_k = \partial_k S$ with respect to the coordinates (c_i) of $A \times E$ appearing in the representation of S. We have of course $H = \rho(F_i)$ if $e = \rho(c_i)$ is the representative of the natural projection $\rho_E: A \times E \longrightarrow E$.

Now, let us assume that S is completely separable in the coordinates (x^i). By derivation with respect to the variable x^k of the identity $F_i(x^j; \partial_k S) = c_i$ we have:

$(\partial_k F_i)_c - (\partial^k F_i)_c \partial_k \partial_k S_c = 0$. This implies the identity:

$$\partial_k F_i + \partial^k F_i \; R_k = 0 \qquad (k \text{ n.s.}), \tag{3.1}$$

which can also be written:

$$\partial_k F_i \, \partial^k H - \partial^k F_i \, \partial_k H = 0 \qquad (k \text{ n.s.}). \tag{3.2}$$

We emphasize the fact that in the last formula there is no summation with resp to the index k, so that the Poisson bracket $\{F_i, H\} = \partial_k F_i \partial^k H - \partial^k F_i \partial_k H$ is zero bec each term of the sum is zero. Thus we are naturally led to consider something more than the usual involution.

(3.3) Definition. Two functions F and G on T*M are said to be in separable involut (briefly, in $\mathscr{S}$-involution) with respect to the coordinates (x^i) if, for each inde

$$\partial_k F \, \partial^k G - \partial^k F \, \partial_k G = 0 \qquad (k \text{ n.s.}). \tag{3.4}$$

Hence (3.2) means that each function F_i and H are in $\mathscr{S}$-involution with respect the separable coordinates (x^i). Moreover, this fact implies that the functions $(F_i$ in $\mathscr{S}$-involution themselves: $\partial_k F_i \, \partial^k F_j = -\partial^k F_i \; R_k \partial^k F_j = \partial^k F_i \, \partial_k F_j \quad (k \text{ n.s.})$.

Conversely, let (F_i) be a set of n vertically independent functions in $\mathscr{S}$-involu with respect to a coordinate system (x^i). Since for each index k there exists at a function F_h such that $\partial^k F_h$ does not vanish identically, we can define the functi

$$R_k = -\frac{\partial_k F_h}{\partial^k F_h} \tag{3.5}$$

Hence, for each function F_i : $\partial_k F_i + \partial^k F_i \; R_k = 0$. Let S be a function generated b family (F_i) as shown at the beginning of this section, i.e. satisfying the equatic $F_i(x^j; \partial_k S(x^k; c_l)) = c_i$. By derivation with respect to x^k we obtain: $(\partial_k F_i)_c + (\partial^j F_i)_c \partial_j \partial_k S_c = 0$, where, as in section 2, $(\ )_c$ means the evaluation on the le generated by the function S_c obtained by fixing the value $c = (c_i)$. Hence: $(k$ n.s $0 = -(\partial^k F_i R_k)_c + (\partial^j F_i)_c \partial_j \partial_k S_c = (\partial^j F_i)_c (\partial_j \partial_k S_c - \delta_{jk} (R_k)_c)$. Since $\det(\partial^j F_i) \neq 0$, follows $\partial_j \partial_k S_c = \delta_{jk} (R_k)_c$. Then, we have in particular $\partial_j \partial_k S_c =$ for $j \neq k$, which that S is of the kind (1.3). As we know, such a function S is the complete integra the Hamilton-Jacobi equation corresponding to each Hamiltonian H functionally depe on (F_i) (in this case, as it is easy to check, H is in $\mathscr{S}$-involution with each F_i. Hence we have proved the following theorem.

(3.6) Theorem.- Let (F_i) be a set of n vertically independent functions on T*M. A coordinate system (x^i) on M is separable with respect to a Hamiltonian H functionally dependent on (F_i) if and only if (F_i) are in $\mathcal{S}$-involution with respect to (x^i).

It clearly follows that, if the functions (F_i) are in $\mathcal{S}$-involution with respect to the coordinates (x^i), they are in $\mathcal{S}$-involution with respect to all coordinate systems which are $\mathcal{S}$-compatible, with respect to H, at a point x_o. In fact, these coordinates give rise to the same complete integral (by definition of $\mathcal{S}$-compatibility), that is to say, to the same foliation represented by the functions (F_i). Hence we conclude that to a $\mathcal{S}$-structure there corresponds a group of functions in $\mathcal{S}$-involution of dimension n (i.e. generated by n independent functions). Furthermore we notice that the $\mathcal{S}$-structure is of class r if and only if, among the functions (3.5), there are exactly r which are linear in the variables (p_j).

4.- Separability structures on Riemannian manifolds and normal separable coordinates.

By separability structure on a Riemannian manifold (M,g) we mean a $\mathcal{S}$-structure corresponding to the geodesic Hamiltonian, which is represented, in natural coordinates $(x^i;p_j)$, by a quadratic form:

$$H = \tfrac{1}{2}\, g^{ij}\, p_i p_j \qquad , \tag{4.1}$$

where (g^{ij}) are the contravariant components of the metric tensor g in the coordinates (x^i). Of course, also the definition of first and second class coordinates is understood to be taken with respect to this Hamiltonian.

Let us denote by $\imath : TM \longrightarrow T^*M$ the natural diffeomorphysm defined by the metric tensor g. In natural coordinates $(x^i;\dot{x}^j)$ and $(x^i;p_j)$ of TM and T*M respectively, this diffeomorphysm is represented by the equations $p_j = g_{ji}\,\dot{x}^i$, where (g_{ji}) are the covariant components of g, or by the inverse relations $\dot{x}^i = g^{ij}\, p_j$. Let us denote by $\imath^*f$ the pull-back of a function $f: T^*M \longrightarrow \mathbb{R}$, i.e.: $\imath^*f = f \circ \imath$.

In the present case, the functions (2.7) are explicitly given by:

$$R_i = -\tfrac{1}{2}\, \frac{\partial_i g^{hk} p_h p_k}{g^{ij}\, p_j} \qquad . \tag{4.2}$$

Of course R_i is linear in the variables (p_j) (that is to say x^i is a first class coordinate) if and only if its pull-back

(4.3) $$l^* R_i = \frac{1}{2} \frac{\partial_i g_{hk} \dot{x}^h \dot{x}^k}{\dot{x}^i}$$

is linear in the variables $(\dot{x}^i)$, i.e. if and only if the quadratic polynomial $\partial_i g$ is divisible by $\dot{x}^i$. Hence:

(4.4) A coordinate x^i is of first class at a point $x_0 \in M$ if and only if, in a nei hood of x_0, $\partial_i g_{hk} = 0$ for each $h,k \neq i$.

In particular, a coordinate x^i is ignorable when $\partial_i g_{hk} = 0$ for each pair of indice (h,k). As it is well known, an ignorable coordinate corresponds to a Killing vect field (briefly, a K-vector) on the Riemannian manifold (M,g), i.e. to a vectofiel satisfying the Killing equation $L_X g = 0$ (where L_X is the Lie derivative symbol)

As a consequence of (4.4), and accordingly to definition (2.22) and convention we notice that:

(4.5) A $\mathscr{S}$-structure at a point x_0 of a Riemannian manifold (M,g) is of class r i only if in any representative chart there exist exactly r coordinates (x^α) such t in a neighborhood of x_0:

(4.6) $$\partial_\alpha g_{hk} = 0 \quad (h,k \neq \alpha) \quad .$$

Therefore, theorem (2.20) has the following corollary ([8]).

(4.7) Theorem.- If in a separable chart $(U,(x^i))$ of a Riemannian manifold (M,g) e tions (4.6) hold in a neighborhood of $x_0 \in U$ for r coordinates (x^α), then there ex $\mathscr{S}$-compatible charts at x_0 with r ignorable coordinates. If (M,g) admits at x_0 a structure, then, in a neighborhood of x_0, there exist r commuting K-vectors, indepe at each point, or, equivalently, an Abelian r-parameters group of isometries acts freely. In particular, if (M,g) admits a $\mathscr{S}_n$-structure at x_0 ($n = \dim M$), then M flat at x_0.

The quasi-normal separable charts (see the end of section 2) can be found by f ing the constructive proof of theorem (2.20), where now the coefficients (B^β_α) ha as it is easy to check, the following expressions:

(4.8) $$B^\beta_\alpha = g^{\beta k} \partial_\alpha g_{\alpha k} - \frac{1}{2} g^{\alpha\beta} \partial_\alpha g_{\alpha\alpha} \qquad (\alpha \text{ n.s.}) \quad .$$

Another method of construction, of a pure algebraic character, will be shown belo

When investigating the relations between $\mathscr{S}$-compatible coordinates, we notice

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as a general property of the $\mathscr{S}$-structures, that a rescaling of a coordinate x^i (i.e. a reversible transformation of x^i involving no other coordinate) preserves the class of the coordinate and the separability.

(4.9) Theorem.- A chart $(U,(x^i))$ is $\mathscr{S}$-compatible with a given separable chart $(U', (x^{i'}))$ at a point $x_o \in U \cap U'$ if and only if the second class coordinates are simply related by a rescaling, i.e.:

$$dx^{a'} = \delta^{a'}_{a} f_a(x^a)\, dx^a \qquad , \tag{4.10}$$

(a suitable reordering of the coordinates is understood) and moreover

$$dx^{\alpha'} = \xi^{\alpha'}_{i}\, dx^i \qquad , \tag{4.11}$$

where functions $(\xi^{\alpha'}_{i})$ satisfy, in a neighborhood of x_o, the following equations:

$$\partial_i \xi^{\alpha'}_{j} + \xi^{\beta'}_{i} \xi^{\beta'}_{j} B'^{\alpha'}_{\beta'} = 0 \qquad (i \neq j) \quad . \tag{4.12}$$

Proof.- i) If the two charts are $\mathscr{S}$-compatible, they have the same number of first (and second) class coordinates (theorem (2.21)). By using the same notations as in the proof of theorem (2.21), we can see that: $\partial_i \partial_j S_c = \partial_i \xi^{j'}_{j} \partial_{j'} S_c + \xi^{i'}_{i} \xi^{i'}_{j} (R'_{i'})_c = 0$ for $i \neq j$. Hence, the following identities hold:

$$\partial_i \xi^{j'}_{j} p_{j'} + \xi^{\beta'}_{i} \xi^{\beta'}_{j} B'^{\alpha'}_{\beta'} p_{\alpha'} + \xi^{a'}_{i} \xi^{a'}_{j} R'_{a'} = 0 \qquad (i \neq j) \ , \tag{4.13}$$

where, of course, $(B'^{\alpha'}_{\beta'})$ and $(R'_{a'})$ are defined as in (4.8) and (4.2) with respect to coordinates $(x^{i'})$. Since by the definition of second class coordinate we can see that $\xi^{a'}_{i} \xi^{a'}_{j} t^*(R'_{a'})$ cannot be linear in the variables $(x^{i'})$, it follows that equations (4.12) necessarily hold as well as: $\partial_i \xi^{a'}_{j} = 0$, $\xi^{a'}_{i} \xi^{a'}_{j} = 0$ $(i \neq j$, a'n.s.). The first set of these equations simply means that each $\xi^{a'}_{j}$ is function of x^j only; the second set implies that, for a fixed index a', only one of the functions $(\xi^{a'}_{i})$ does not vanish identically. Since $\xi^{a'}_{\alpha} = 0$ (see the proof of theorem (2.21), there exists a second class coordinate x^a which is simply related with $x^{a'}$ by a rescaling. ii) Conversely, if (4.10) - (4.12) hold, it is easy to check, by reversing the above reasoning, that the two charts are $\mathscr{S}$-compatible. (q.e.d.)

We notice in particular that if $(x^{i'})$ are quasi normal separable coordinates (i.e. $(x^{\alpha'})$ are ignorable, hence $B'^{\alpha'}_{\beta'} = 0$), for all $\mathscr{S}$-compatible coordinates (x^i) the following relations hold in a neighborhood of x_o , apart from a reordering and a rescaling:

(4.14) $$dx^{a'} = dx^{a} \quad , \quad dx^{\alpha'} = \xi_i^{\alpha'}\, dx^{i} \quad ,$$

where each $\xi_i^{\alpha'}$ is function of x^i only.

For a Hamiltonian such as in (4.1), Levi-Civita separability conditions (2.13) polynomial identities in the variables (p_j). If we write them for second class cc dinates $x^a \neq x^b$ as follows:

(4.15) $$\partial^a H\,(\partial^b H\,\partial_a\partial_b H - \partial_b H\,\partial_a\partial^b H) = \partial_a H\,(\partial^b H\,\partial_b\partial^a H - \partial_b H\,\partial^a\partial^b H) \qquad (a\neq b\text{ ,n.s.})$$

we observe (with Levi-Civita [1]) that, since $\partial^a H$ is not a divisor of $\partial_a H$, it r be a divisor of the polynomial

(4.16) $$\partial^b H\,\partial_b\partial^a H - \partial_b H\,\partial^a\partial^b H \qquad (a\neq b,\ \text{n.s.})$$

Then, by the pull-back $\imath^*$, we see that $\dot{x}^a = \imath^*(\partial^a H)$ must be a divisor of

$$g^{ak}\partial_b g_{hk}\,\dot{x}^h\dot{x}^b - \tfrac{1}{2}\, g^{ab}\,\partial_b g_{hk}\,\dot{x}^h\dot{x}^k \qquad (a\neq b,\ \text{n.s.})$$

This implies, in particular:

(4.17) $$g^{ab}\,\partial_b g_{hk} = 0 \qquad (a\neq b;\ h,k\neq a,b;\ b\ \text{n.s.})\ .$$

Since x^b is a second class coordinate, there exist at least two indices $i,j\neq b$ such that $\partial_b g_{ij}\neq 0$ at some point of each neighborhood of x_0 . If, moreover, $i,j\neq a$ from (4.17) it follows $g^{ab} = 0$ $(a\neq b)$ in a neighborhood of x_0. On the other hand, exceptional case

(4.18) $$\partial_b g_{ij} = 0 \quad (a\neq b;\ i,j\neq a,\, b\,)$$

is irrelevant because, as it is easy to see, we can always pass to $\mathscr{S}$-compatible c nates for which this no longer holds, while conditions like (4.18) are invariant, s second class coordinates are related by a rescaling. Hence:

(4.19) <u>If $(U,(x^i))$ is a separable chart, for two distinct coordinates x^a, x^b , of cond class at x_0, we have, in a neighborhood of x_0:</u>

(4.20) $$g^{ab} = 0 \quad (a\neq b) \qquad .$$

Now we see that the polynomial (4.16), which is divisible by $\partial^a H$, simply reduces, a neighborhood of x_0, to $\partial^b H\,\partial_b\partial^a H$. Since $\partial^a H$ is not a divisor of $\partial^b H$, it foll that $\partial_b \partial^a H = f^{ab} \partial^a H$, where f^{ab} is a function of the coordinate (x^i) only, and thus:

(4.21) $$\partial_b g^{ai} = f^{ab} g^{ai} \quad (a \neq b) \quad .$$

Since in the last formula the function f^{ab} does not depend on the choice of the index i , we can see at once that:

(4.22) If $(U,(x^i))$ is a separable chart, for two distinct coordinates x^a, x^b of second class at a point $x_0 \in U$, the following equations hold in a neighborhood of x_0:

(4.23) $$g^{ai} \partial_b g^{aj} = g^{aj} \partial_b g^{ai} \qquad (a \neq b) \ .$$

In Dall'Acqua's paper [6], conditions (4.20) have been proved through cumbersome calculations and under the assumption $g^{aa} \neq 0$ for each second class coordinate (which in a proper-Riemannian manifold is certainly satisfied). We have followed here the very simple proof given by Agostinelli in [9], in which, however, the possibility of the exceptional case (4.18) is not considered. In fact, this obstacle cannot be overcome so easily without the concept of separability structure and without knowing that second class coordinates remain essentially unchanged in $\mathcal{S}$-compatible charts.

From proposition (4.19) and theorem (4.9) it follows in particular:

(4.24) The coordinates of a $\mathcal{S}_0$-structure at a point x_0 are uniquely determined, up to a rescaling, and orthogonal ($g^{ij} = 0$ for $i \neq j$) in a neighborhood of x_0.

(4.25) Remark.- When g is analytic, by (4.5) we notice that a coordinate is of first class at each point of the domain of the chart if it is of first class at least at one point (consequently, a second class coordinate at a point is of second class everywhere). Hence, the above results hold with respect to all points of the domain of a separable chart. In other words, it is not necessary to work at a distinguished point. This remark must be kept in mind also in the sequel.

Together with the first classification given by the class, for $\mathcal{S}$-structures on Riemannian manifolds we can introduce a second classification by another integer invariant: the index of the $\mathcal{S}$-structure. Actually this second classification works non-trivially only for $\mathcal{S}$-structures on pseudo-Riemannian manifolds of class different from the dimension.

(4.26) Definition.-The index of a $\mathcal{S}$-structure at a point x_0 of a Riemannian manifold

(M,g) is the number of the second class coordinates of any representative chart which, in a neighborhood of x_0, $g^{aa} = 0$.

Notice that the definition makes sense because, in virtue of theorem (4.9), th condition $g^{aa} = 0$ for a second class coordinate is invariant under $\mathscr{S}$-compatible tr formations.

(4.27) Remark.- It could happen that for some second class coordinate x^a the condi $g^{aa} = 0$ is satisfied on a closed submanifold of M containing x_0, or at x_0 only. In present paper we do not consider such a kind of $\mathscr{S}$-structures, which we can call s gular.

From now on we adopt, together with (2.10), the following convention concerning cond class coordinates.

(4.28) Convention.- A $\mathscr{S}_{r;d}$-structure is a (non singular) $\mathscr{S}_r$-structure of index d This abbreviation is used only when it is necessary to specify the index. $\mathscr{S}_n$-stru res (n = dim M : no second class coordinates) have of course to be considered of i zero. Second class coordinates are labeled by $\tilde{a},\tilde{b},\tilde{c},\ldots$ if $g^{\tilde{a}\tilde{a}} \neq 0,\ldots$, and by $\bar{a}$ ... if, on the contrary, $g^{\bar{a}\bar{a}} = 0,\ldots$. Second class coordinates are assumed to be dered in such a way that $\tilde{a},\tilde{b},\tilde{c},\ldots$ range from 1 to $m-d$ and $\bar{a},\bar{b},\bar{c},\ldots$ range fro d+1 to m (m being the total number of second class coordinates). When the distin into these two sub-classes is not needed, we use the unaffected indices $a,b,c,\ldots$ which range from 1 to m. Summation convention is adapted to this choice unless the bol "n.s." appears together with a distinguished index.

(4.29) Theorem.- In a $\mathscr{S}_{r;d}$-structure on a Riemannian manifold (M,g) there exist c $(V,(y^i))$ with r ignorable coordinates (y^α) $(\alpha = m-1,\ldots,n\ ;\ m = n-r)$ such that f the metric tensor components $(\overset{ij}{g})$ the following conditions hold $(\tilde{a} = 1,\ldots,m-d\ ;$ $m-d+1,\ldots,n-r)$:

$$\overset{\tilde{a}\tilde{a}}{g} \neq 0 \quad , \quad \overset{\tilde{a}i}{g} = 0 \quad (i \neq \tilde{a}) \quad , \quad \overset{\tilde{a}\bar{b}}{g} = 0 \quad . \tag{4.30}$$

Proof.- Let $(U,(x^i))$ be a quasi normal separable chart of a $\mathscr{S}_{r;d}$-structure at point $x_0 \in M$. Let (x^α) be the ignorable coordinates (theorem (2.20)) and g^{ij} the co nents of g. If $g^{\tilde{a}\tilde{a}} \neq 0$, from (4.23) it follows in particular: $\partial_b(g^{\tilde{a}\alpha}(g^{\tilde{a}\tilde{a}})^{-1}) = 0$ f $b \neq \tilde{a}$. This means that, in a neighborhood of x_0, and for each pair of indices $(\tilde{a},\alpha)$ there exists a function $\theta^\alpha_{\tilde{a}}$ of $x^{\tilde{a}}$ only, such that: $g^{\tilde{a}\alpha} = \theta^\alpha_{\tilde{a}}\, g^{\tilde{a}\tilde{a}}$ ($\tilde{a}$ n.s.). Then w can define new coordinates (y^i) at x_0 by the equations: $dy^a = dx^a$, $dy^\alpha = dx^\alpha - \theta^\alpha_{\tilde a}\, dx^{\tilde a}$. These coordinates are clearly $\mathscr{S}$-compatible (compare with (4.14)) and, in particular, (y^α) are ignorable. The corresponding components $(\overset{ij}{g})$ satisfy conditions (4.30) since: $\overset{ab}{g} = g^{ab}$ and $\overset{\tilde a\alpha}{g} = g^{\tilde a\alpha} - \theta^\alpha_{\tilde a}\, g^{\tilde a\tilde a} = 0$ ($\tilde a$ n.s.). (q.e.d.)

(4.31) Definition.- A chart satisfying the conditions of theorem (4.29) is called normal separable chart; the corresponding coordinates are called normal separable coordinates.

In normal separable coordinates the matrix $(\overset{ij}{g})$ has the following form:

$$
(4.32) \qquad
\begin{array}{c}
m-d \left\{ \vphantom{\begin{array}{c}a\\a\\a\end{array}} \right. \\
d \left\{ \vphantom{\begin{array}{c}a\\a\end{array}} \right. \\
r \left\{ \vphantom{\begin{array}{c}a\\a\end{array}} \right.
\end{array}
\left\|
\begin{array}{c|c|c}
\begin{matrix} \ddots & & 0 \\ & \overset{\tilde a\tilde a}{g} & \\ 0 & & \ddots \end{matrix} & 0 & 0 \\
\hline
0 & 0 & \overset{\bar a\alpha}{g} \\
\hline
0 & \overset{\alpha\bar a}{g} & \overset{\alpha\beta}{g}
\end{array}
\right\|
$$

$$
\underbrace{\qquad}_{m-d} \quad \underbrace{\qquad}_{d} \quad \underbrace{\qquad}_{r}
$$

For $\mathscr{S}$-structures of index 0 (as in proper-Riemannian manifolds) the central rows and columns disappear.

$$
(4.33) \qquad
\begin{array}{c}
m \left\{ \vphantom{\begin{array}{c}a\\a\\a\end{array}} \right. \\
r \left\{ \vphantom{\begin{array}{c}a\\a\end{array}} \right.
\end{array}
\left\|
\begin{array}{c|c}
\begin{matrix} \ddots & & 0 \\ & \overset{aa}{g} & \\ 0 & & \ddots \end{matrix} & 0 \\
\hline
0 & \overset{\alpha\beta}{g}
\end{array}
\right\|
$$

$$
\underbrace{\qquad}_{m} \quad \underbrace{\qquad}_{r}
$$

(4.34) Remark.- We emphasize that separable coordinates (y^i) are normal separable coordinates if and only if: i) all first class coordinates are ignorable, ii) for each second class coordinate y^a such that $g^{aa} \neq 0$, and for each ignorable coordinate y^α, $g^{a\alpha} = 0$.

5.- The general form of the metric tensor components in separable coordinates.

In order to construct a method of integration by separation of variables of the geodesic Hamilton-Jacobi equation of a Riemannian manifold it is necessary to know the general form of the contravariant components of a metric tensor in separable c dinates. This general form can be easily obtained by the above considerations on s rability structures.

(5.1) Definition.- Let $(\underset{h}{\varphi}^j)$ be a $n\times n$ matrix of functions of n variables (x^i) ($j,\ldots = 1,\ldots,n$). We say that $(\underset{h}{\varphi}^j)$ is a Stäckel matrix in the variables (x^i) if i is regular everywhere and if each element $\overset{h}{\varphi}_j$ of the inverse matrix $(\overset{h}{\varphi}_j)$(defined the equations $\overset{h}{\varphi}_i\, \underset{k}{\varphi}^i = \delta^h_k \Longleftrightarrow \overset{h}{\varphi}_i\, \underset{h}{\varphi}^j = \delta^j_i$) is a function of the variable corresponding to the lower index only.

A matrix of this kind appears in the statement of Stäckel's theorem [9] on the parability of orthogonal coordinates (see section 7). Actually, Stäckel matrices a closely related with questions concerning separability structures not only in this particular case.

(5.2) Theorem.- In normal separable coordinates (y^i) of a $\mathcal{S}_{r;d}$-structure on a mannian manifold (M,g), the contravariant components $(\overset{ij}{g})$ of g have the following form:

$$(5.3)\qquad \begin{cases} \overset{\tilde a i}{g} = 0 \quad (i\neq\tilde a)\ , \quad \overset{\bar a\bar b}{g} = 0\ , \\ \overset{\tilde a\tilde a}{g} = \underset{m}{u}^{\tilde a}\ , \quad \overset{\bar a\alpha}{g} = \theta^{\alpha}_{\bar a}\, \underset{m}{u}^{\bar a} \quad (\bar a \text{ n.s.})\ , \quad \overset{\alpha\beta}{g} = \zeta^{\alpha\beta}_a\, \underset{m}{u}^a \end{cases}$$

$(m = n-r\ ;\tilde a = 1,\ldots,m-d\ ;\ \bar a,\bar b = m-d+1,\ldots,m\ ;\ \alpha,\beta = m+1,\ldots,n)$ where: i) $(\underset{m}{u}^a)$ the m-th row of a Stäckel matrix $(\underset{b}{u}^a)$ in the variables (y^a), ii) $(\theta^{\alpha}_{\bar a})$ and $(\zeta^{\alpha\beta}_a)$ a functions of the variable corresponding to the lower index only.

Proof.- $(5.3)_{1,2}$ are already known by theorem (4.29)(see (4.30)). Hence, in nor separable coordinates the Hamilton-Jacobi equation becomes:

$$(5.4)\qquad \overset{\tilde a\tilde a}{g}(\partial_{\tilde a}S)^2 + 2\,\overset{\bar a\alpha}{g}\,\partial_{\bar a}S\,\partial_\alpha S + \overset{\alpha\beta}{g}\,\partial_\alpha S\,\partial_\beta S = 2e \quad .$$

Since the coordinates (y^α) are ignorable, the complete integral has the form:

$$(5.5)\qquad S = \sum_{a=1}^{n-r} S_a(y^a;c_i) + c_\alpha\, y^\alpha \quad .$$

We can always assume that $(c_i) = (c_a ; c_\alpha)$, i.e. that the r constants of motion (c_α) corresponding to the ignorable coordinates coincide with r of the parameters (c_i) and, moreover, that one of the remaining parameters (c_a) coincides with 2e: for instance, $c_m = 2e$. From (4.22) it follows in particular:

$$\overset{\bar a\alpha}{g}\,\partial_b \overset{\bar a\beta}{g} = \overset{\bar a\beta}{g}\,\partial_b \overset{\bar a\alpha}{g} \qquad (b\neq\bar a) \quad . \tag{5.6}$$

For a fixed index $\bar a$ there exists at least one index β, say for instance $\beta = n$, such that $\overset{\bar a n}{g}\neq 0$ (otherwise $\det(\overset{ij}{g}) = 0$); hence, by (5.6): $\partial_b(\overset{\bar a\alpha}{g}(\overset{\bar a n}{g})^{-1}) = 0 \quad (b\neq\bar a)$. This means that for each index $\bar a$ there exist r functions $(\theta^\alpha_{\bar a})$ of $y^{\bar a}$ only, such that

$$\overset{\bar a\alpha}{g} = \theta^\alpha_{\bar a}\,\gamma^{\bar a} \qquad (\bar a \text{ n.s.}) \quad , \tag{5.7}$$

where $\gamma^{\bar a}$ is a suitable function which does not depend on the index α (we can choose in particular $\gamma^{\bar a} = g^{\bar a n}$; hence $\theta^n_{\bar a} = 1$). If we set

$$u_{\tilde a} = (\partial_{\tilde a} S_{\tilde a})^2 \quad , \quad u_{\bar a} = 2c_\alpha\,\theta^\alpha_{\bar a}\,\partial_{\bar a} S_{\bar a} \quad , \tag{5.8}$$

from (5.4) it follows:

$$\overset{\tilde a\tilde a}{g}\, u_{\tilde a} + \gamma^{\bar a}\, u_{\bar a} + \overset{\alpha\beta}{g}\, c_\alpha c_\beta = c_m \quad . \tag{5.9}$$

By derivation of (5.9), with respect to c_b , we obtain

$$\overset{\tilde a\tilde a}{g}\, \overset{b}{u}_{\tilde a} + \gamma^{\bar a}\, \overset{b}{u}_{\bar a} = \delta^b_m \quad , \tag{5.10}$$

where:

$$\begin{cases} \overset{b}{u}_{\tilde a} = \dfrac{\partial u_{\tilde a}}{\partial c_b} = 2\,\partial_{\tilde a} S_{\tilde a}\,\dfrac{\partial}{\partial c_b}\,\partial_{\tilde a} S_{\tilde a} \quad , \\ \overset{b}{u}_{\bar a} = \dfrac{\partial u_{\bar a}}{\partial c} = 2\,c_\alpha\,\theta^\alpha_{\bar a}\,\dfrac{\partial}{\partial c_b}\,\partial_{\bar a} S_{\bar a} \quad . \end{cases} \tag{5.11}$$

Since $\det(\frac{\partial}{\partial c_b}\partial_a S) = \det(\frac{\partial}{\partial c_i}\partial_j S) \neq 0$, the $m\times m$ matrix $(\overset{b}{u}_a) = (\overset{b}{u}_{\tilde a}, \overset{b}{u}_{\bar a})$ is everywhere regular except on the surfaces $c_\alpha\theta^\alpha_{\bar a} = 0$. However, these surfaces do not belong to the domain of definition of the complete integral S, since, as we shall see in the next section, they represent singularities for the integration of (5.4). If $(\underset{b}{u}^a)$ is the inverse matrix of $(\overset{b}{u}_a)$, from (5.10) it follows

$$\overset{\tilde a\tilde a}{g} = \underset{m}{u}^{\tilde a} \quad , \quad \gamma^{\bar a} = \underset{m}{u}^{\bar a} \quad . \tag{5.12}$$

On the other hand, by derivation of (5.9) with respect to c_α and c_β , we obtain:

$$(5.13) \qquad \overset{\alpha\beta}{g} = \zeta^{\alpha\beta}_{\tilde a}\, \overset{\tilde a \tilde a}{g} + \zeta^{\alpha\beta}_{\bar a}\, \gamma^{\bar a}$$

with

$$(5.14) \qquad \zeta^{\alpha\beta}_{a} = - \frac{\partial^2 u_a}{\partial c_\alpha \partial c_\beta} \quad .$$

From (5.7), (5.12) and (5.13) expressions (5.3) follow. By (5.8), (5.11) and (5.14 we see that conditions i) and ii) are satisfied. (q.e.d.

(5.15) Theorem.- Let $(U,(x^i))$ be a separable chart of a $\mathcal{S}_{r;d}$-structure at a point of a n-dimensional Riemannian manifold (M,g). In a neighborhood of x_0 the contravariant components (g^{ij}) take the following form:

$$(5.16) \qquad \begin{cases} g^{\tilde a \tilde a} = \underset{m}{u}^{\tilde a} \ , \quad g^{\bar a \bar a} = 0 \ , \quad g^{ab} = 0 \quad (a \neq b) \ , \\ g^{a\alpha} = \underset{\beta}{\xi}^{\alpha}\, \theta^{\beta}_{a}\, \underset{m}{u}^{a} \quad (a \text{ n.s.}) \ , \\ g^{\alpha\beta} = \underset{\gamma}{\xi}^{\alpha} \underset{\delta}{\xi}^{\beta}\, \eta^{\gamma\delta}_{a}\, \underset{m}{u}^{a} \ , \end{cases}$$

$(m = n - r\ ; \tilde a = 1,\dots,m-d\ ;\ \bar a = m-d+1,\dots,m\ ;\ a,b = 1,\dots,m\ ;\ \alpha,\beta,\gamma,\delta = m-1,\dots$ where: i) $(\underset{m}{u}^{a})$ is the m-th row of a Stäckel matrix $(\underset{b}{u}^{a})$ in the m variables (x^a); i $(\underset{\beta}{\xi}^{\alpha})$ is a Stäckel matrix in the r variables (x^α); iii) (θ^{β}_{a}) and $(\eta^{\gamma\delta}_{a})$ are functi of the variable corresponding to the lower index only.

Proof.- Let $(y^i) = (y^a, y^\alpha)$ be normal separable coordinates of the given $\mathcal{S}$-stru re. Since all second class coordinates (y^α) are ignorable, by (4.14) we observe th for any other $\mathcal{S}$-compatible coordinate system (x^i), apart from a suitable reorderi and rescaling of the second class coordinates, the following relations hold in a n borhood of x_0:

$$(5.17) \qquad dy^a = dx^a \quad , \qquad dy^\alpha = \overset{\alpha}{\xi}_i\, dx^i \quad ,$$

where each function $\overset{\alpha}{\xi}_i$ depends on the variable x^i only. The $r \times r$ sub-matrix $(\overset{\alpha}{\xi}_\beta)$ $(\overset{\alpha}{\xi}_i)$ is regular: $\det(\overset{\alpha}{\xi}_\beta) = \det(\frac{\partial y^i}{\partial x^j}) \neq 0$; hence, the inverse matrix $(\underset{\alpha}{\xi}^{\beta})$ is a Stäck matrix. Let (g^{ij}) and $(\overset{ij}{g})$ be the contravariant components of g in the coordinates (x^i) and (y^i) respectively. The following relations hold:

$$g^{ab} = \overset{ab}{g} \quad , \quad g^{a\alpha} = \underset{\beta}{\xi}^{\alpha}(\overset{a\beta}{g} - \overset{\beta}{\xi}_b\, \overset{ab}{g}) \quad ,$$

$$g^{\alpha\beta} = \underset{\gamma}{\xi}^{\alpha} \underset{\delta}{\xi}^{\beta} (\overset{\gamma\delta}{g} + \overset{\gamma}{\xi}_a \overset{\delta}{\xi}_b \overset{ab}{g} - \overset{\gamma}{\xi}_a \overset{a\delta}{g} - \overset{\delta}{\xi}_a \overset{a\gamma}{g}) \quad .$$

Since (y^i) are normal separable coordinates, by theorem (5.2) we have:

$$g^{\tilde a\tilde a} = \underset{m}{u}^{\tilde a} \quad , \quad g^{\bar a\bar a} = 0 \quad , \quad g^{ab} = 0 \quad (a \neq b) \quad ,$$

$$g^{\tilde a\alpha} = - \underset{\beta}{\xi}^{\alpha} \overset{\beta}{\xi}_{\tilde a} \underset{m}{u}^{\tilde a} \quad (\tilde a \text{ n.s.}) \quad ,$$

$$g^{\bar a\alpha} = \underset{\beta}{\xi}^{\alpha} \overset{\beta}{\theta}_{\bar a} \underset{m}{u}^{\bar a} \quad (\bar a \text{ n.s.}) \quad ,$$

$$g^{\alpha\beta} = \underset{\gamma}{\xi}^{\alpha} \underset{\delta}{\xi}^{\beta} (\zeta_a^{\gamma\delta} \underset{m}{u}^{a} + \overset{\gamma}{\xi}_{\tilde a} \overset{\delta}{\xi}_{\tilde a} \underset{m}{u}^{\tilde a} - (\overset{\gamma}{\xi}_{\bar a} \theta^{\delta}_{\bar a} + \overset{\delta}{\xi}_{\bar a} \theta^{\mu}_{\bar a}) \underset{m}{u}^{\bar a}) \quad .$$

Furthermore, if we set:

(5.18)
$$\begin{cases} \theta^{\beta}_{\tilde a} = - \overset{\beta}{\xi}_{\tilde a} \quad , \quad \eta^{\gamma\delta}_{\tilde a} = \zeta^{\gamma\delta}_{\tilde a} + \overset{\gamma}{\xi}_{\hat a} \overset{\delta}{\xi}_{\hat a} \quad , \\ \eta^{\gamma\delta}_{\bar a} = \zeta^{\gamma\delta}_{\bar a} - \overset{\gamma}{\xi}_{\bar a} \theta^{\delta}_{\bar a} - \overset{\delta}{\xi}_{\bar a} \theta^{\gamma}_{\bar a} \quad , \end{cases}$$

expressions (5.16) follow. (q.e.d.)

The contravariant components of a metric tensor with respect to separable coordinates could be presented in many other forms, appparently more general than the one described in theorem (5.15) (or (5.12)) (for the case $d = 0$, see for instance [8],[11], [12],[13]). It seems that (5.16),and (5.3) for normal separable coordinates, provide the simplest representation. For instance, in [11] we obtained for $\overset{\alpha\beta}{g}$ the expressions $\overset{\alpha\beta}{g} = \zeta_a^{\alpha\beta} \underset{m}{u}^{a} + \zeta_0^{\alpha\beta}$ where $(\zeta_{\cdot}^{\alpha\beta})$ are constant. However, if we set $\tilde\zeta_a^{\alpha\beta} = \zeta_0^{\alpha\beta} \overset{m}{u}_a + \zeta_a^{\alpha\beta}$ we obtain the expression $\overset{\alpha\beta}{g} = \tilde\zeta_a^{\alpha\beta} \underset{m}{u}^{a}$, which is of the same kind given in $(5.3)_3$. Again with the reference to the case $d = 0$, in [12] the components (g^{aa}) are given in the form $g^{aa} = \sum_{b=1}^{m} \underset{b}{v}^{a}$, where $(\underset{b}{v}^{a})$ is still a Stäckel matrix in the second class variables (x^a). We can thus consider a more general expression like $g^{aa} = k^b \underset{b}{v}^{a}$ where (k^b) are constant (compare with the t-dependent case considered in [13]). Actually this form can be obtained automatically, following the same proof as in theorem (5.2)without the non restrictive assumption $c_m = 2e$. On the other hand, apart from what concerns the separability structure theory, the equivalence of the two representations follows directly from the following theorem on Stäckel matrices.

(5.19) <u>Theorem.- Let $(\underset{b}{v}^{a})$ be a Stäckel matrix in the m variables (x^a). For any element $(k^b) \in \mathbb{R}^m$ different from zero, there exists a Stäckel matrix $(\underset{b}{u}^{a})$, still in the variables (x^a), such that $\underset{m}{u}^{a} = k^b \underset{b}{v}^{a}$.</u>

<u>Proof</u>.- Let us take a regular $m \times m$ constant matrix $(\overset{b}{k}_a)$ such that $\overset{b}{k}_m = k^b$. Let us

consider the inverse matrix $(\underset{b}{k}^{a})$ and let us set $\overset{c}{u}_{a} = \underset{b}{k}^{c}\, \overset{b}{v}_{a}$. Notice that $\overset{c}{u}_{a}$ is function of the corresponding variable x^{a} only; moreover, the matrix $(\overset{c}{u}_{a})$ is obvio regular, since it is a product of two regular matrices. Hence, the inverse matrix is a Stäckel matrix. and moreover $\underset{c}{u}^{a} = \underset{c}{k}^{b}\, \underset{b}{v}^{a}$, so that $\underset{m}{u}^{a} = \underset{m}{k}^{b}\, \underset{b}{v}^{a} = k^{b}\, \underset{b}{v}^{a}$. (q.e.d.

6.- Separation of the variables in the geodesic Hamilton-Jacobi equation.

Let us consider a chart $(U,(x^{i}))$ on (M,g) and let us assume that it is separab (Levi-Civita conditions (2.13) are satisfied for the geodesic Hamiltonian (4.1)). implies the existence of a complete integral of the kind (1.3). In order to reduc Hamilton-Jacobi equation to a system of separated equations, we can proceed as fo First of all we must recognize the class of each coordinate at a point $x_{0} \in U$ (de tion (2.8), proposition (4.4)). When this is done, we know the class and the inde the $\mathscr{S}$-structure determined by the chart at x_{0} (the class and the index could be pendent on the choice of the point x_{0}). Then, we check if the chart is a normal parable chart (theorem (4.29), remark (4.34)). If the answer is negative, we can normal separable chart in two steps: i) by reduction to a quasi normal separable (i.e. to a maximal number of ignorable coordinates), following the method describ the proof of theorem (2.20); ii) by passing to normal separable coordinates thr a coordinate transformation of the kind described in the proof of theorem (4.29). could also proceed as follows: i) detecting, by a suitable algebraic process, the functions $(\underset{\beta}{\xi}^{\alpha})$, (θ^{β}_{a}), $(\zeta^{\gamma\delta}_{\bar{a}})$ (and possibly $(\underset{b}{\mu}^{a})$ too) representing the metric tensor ponents as in (5.16); ii) performing a coordinate transformation of the kind (5.1 where $(\overset{\beta}{\xi}_{\alpha})$ are obtained by inverting the matrix $(\underset{\beta}{\xi}^{\alpha})$ and $(\xi_{\bar{a}})$, $(\overset{\alpha}{\xi}_{\bar{a}})$ by reversing relations (5.18). When we are in normal separable coordinates, we can apply the f ing theorem.

(6.1) <u>Theorem</u>.- <u>If in a coordinate system</u> (y^{i}) <u>the contravariant metric tensor co nents</u> $(\overset{ij}{g})$ <u>are in the form</u> (5.3), <u>then a complete integral of the kind</u> (5.5) <u>is o nable by the integration of the following separated system of ordinary differenti equations</u>:

$$
(6.2)\qquad \begin{cases} \left(\dfrac{dS_{\bar{a}}}{dy^{\bar{a}}}\right)^{2} + \zeta^{\alpha\beta}_{\bar{a}}\, c_{\alpha} c_{\beta} = c_{b}\, \overset{b}{u}_{\bar{a}} , \\ 2c_{\alpha}\, \theta^{\alpha}_{\bar{a}}\, \dfrac{dS_{\bar{a}}}{dy^{\bar{a}}} + \zeta^{\alpha\beta}_{\bar{a}}\, c_{\alpha} c_{\beta} = c_{b}\, \overset{b}{u}_{\bar{a}} . \end{cases}
$$

Proof.– By (5.3) we can write equation (5.4) as follows:

$$(6.3) \qquad \underset{m}{u}^a \, \Phi_a = 2e \quad ,$$

where:

$$(6.4) \qquad \Phi_{\bar{a}} = (\partial_{\bar{a}} S)^2 + \zeta_{\bar{a}}^{\alpha\beta} c_\alpha c_\beta \quad , \quad \Phi_{\bar{a}} = 2c_\alpha \theta_{\bar{a}}^\alpha \partial_{\bar{a}} S + \zeta_{\bar{a}}^{\alpha\beta} c_\alpha c_\beta \quad , \quad c_\alpha = \partial_\alpha S \quad .$$

Equation (6.3) is satisfied by

$$(6.5) \qquad \Phi_a = c_b \overset{b}{u}_a$$

where $(c_b) \in R^m$, $c_m = 2e$. Equations (6.5) are nothing but (6.2), since $\partial_a S = \frac{dS_a}{dy^a}$. It can be directly verified that, when the real parameters $(c_b; c_\alpha)$ range in a suitable domain of R^n , the solution of the geodesic Hamilton-Jacobi equation obtained by equations (6.2) is complete. (q.e.d.)

In section 3 we pointed out that a complete integral S can be also represented by n equations of the kind $F_i = c_i$, where (F_i) are n vertically independent functions in involution. From this point of view we can observe, through (6.4) and (6.5) (which gives $c_b = \underset{b}{u}^a \Phi_a$), that:

(6.6) In normal separable coordinates (y^i) the complete integral S is defined by the n functions:

$$(6.7) \qquad \begin{cases} F_b = \underset{b}{u}^{\bar{a}} (p_{\bar{a}})^2 + 2 \, \underset{b}{u}^{\bar{a}} \theta_{\bar{a}}^\alpha p_{\bar{a}} p_\alpha + \underset{b}{u}^a \zeta_a^{\alpha\beta} p_\alpha p_\beta \quad , \\ F_\alpha = p_\alpha \end{cases}$$

where (p_i) are the corresponding momenta.

By the general theory developed in section 3 we also know that functions (6.7) are vertically independent and in $\mathcal{S}$-involution with respect to the coordinates (y^i) (and to any other $\mathcal{S}$-compatible system). Therefore we realize that the function group associated with a $\mathcal{S}_r$-structure on a Riemannian manifold is generated by r linear first integrals and n − r (homogeneous) quadratic first integrals. These first integrals are of course determined up to reversible transformations of the kind:

$$(6.8) \qquad F'_c = k_c^b F_b + k_c^{\alpha\beta} F_\alpha F_\beta \quad , \qquad F'_\lambda = k_\lambda^\alpha F_\alpha \quad ,$$

where all the coefficients are constant.

The above remarks can be translated in terms of Killing vectors and Killing tens a $\mathcal{S}_r$-structure at a point x_0 of a Riemannian manifold gives rise (in a neighborhoc x_0) to r K-vectors (see also theorem (4.7)) and $n-r$ K-tensors of order 2, which cc mute in the Schouten–Nijenhuis Lie algebra of the contravariant symmetric tensors

If the contravariant components (g^{ij}) in normal separable coordinates are known, ing a Stäckel matrix ($\underset{b}{u}^a$) and the functions ($\theta_{\bar{a}}^{\alpha}$), ($\zeta_a^{\alpha\beta}$) entering the representatio (5.3) is a pure algebraic problem. When these functions are known, by $(6.7)_1$ one can immediately write the quadratic first integrals associated with the $\mathcal{S}$-struct (from the choice $c_m = 2e$ it follows in particular $F_m = 2H$), or the corresponding K-tensors:

$$(6.9)\qquad K_b = \underset{b}{u}^{\tilde{a}}\, \partial_{\tilde{a}} \otimes \partial_{\tilde{a}} + \underset{b}{u}^{\bar{a}}\, \theta_{\bar{a}}^{\alpha} (\partial_{\bar{a}} \otimes \partial_{\alpha} + \partial_{\alpha} \otimes \partial_{\bar{a}}) + \underset{b}{u}^{a}\, \zeta_a^{\alpha\beta}\, \partial_{\alpha} \otimes \partial_{\beta} \qquad \left(\partial_i = \frac{\partial}{\partial y^i}\right) .$$

However, the problem of finding the matrix ($\underset{b}{u}^a$) can be simplified by knowing how t express, in a simple manner, the elements of a generic Stäckel matrix (of the requ: order) in terms of functions depending on a single variable, that is to say by knov what can be called a <u>canonical</u> <u>representation</u> of a Stäckel matrix of a certain ord A canonical representation of a Stäckel matrix is of course obtainable by the algeb relations between its elements and the elements of the inverse matrix, which are i deed functions of a single variable (definition (5.1)). For instance (see [14]), 2×2 Stäckel matrix ($\underset{b}{u}^a$), such that $\underset{2}{u}^1 \neq 0$ and $\underset{2}{u}^2 \neq 0$, can always be represented as follows:

$$\underset{1}{u}^1 = \frac{\psi_1 \varphi_2}{\varphi_1 + \varphi_2} \quad , \quad \underset{1}{u}^2 = \frac{-\psi_2 \varphi_1}{\varphi_1 + \varphi_2} \quad ,$$

$$\underset{2}{u}^1 = \frac{\psi_1}{\varphi_1 + \varphi_2} \quad , \quad \underset{2}{u}^2 = \frac{\psi_2}{\varphi_1 + \varphi_2} \quad ,$$

where (ψ_1, φ_1) and (ψ_2, φ_2) are functions of x^1 and x^2 respectively.

The so called <u>canonical</u> <u>forms</u> of the K-tensors (in particular, of the metric ter sor) corresponding to a $\mathcal{S}_{n-m}$-structure correspond to canonical representations oı the Stäckel matrices of order m. Canonical forms for $\mathcal{S}$-structures of class $n-2$ ar $n-3$ are discussed in [14] and [15] respectively, in the case of index zero.

7.- Orthogonal separability structures.

The separability of the geodesic Hamilton-Jacobi equation can be considered in the particular case of orthogonal coordinates: $g^{ij} = 0$ for $i \neq j$. The first fundamental result in this field was obtained by Stäckel at the end of the last century [9, 16]. (In fact, he considered a more general Hamiltonian including an additional "potential" function, as suggested by mechanics). Bearing in mind definition (5.1), Stäckel's theorem can be stated as follows:

(7.1) Theorem.- *An orthogonal coordinate system* (x^i) *is separable if and only if* $g^{ii} = \underset{n}{\varphi}^i$, *where* $(\underset{n}{\varphi}^i)$ *is the row of a Stäckel matrix* $(\underset{k}{\varphi}^i)$ *in the variables* (x^i). *In this case the functions* $F_k = \underset{k}{\varphi}^i (p_i)^2$ *are first integrals in involution and vertically independent.*

The second part of this theorem can be considered as a corollary of the following proposition concerning Stäckel matrices.

(7.2) *A* $n \times n$ *matrix* $(\underset{k}{\varphi}^i)$ *of* C^1 *real functions of* n *variables* (x^i) *is a Stäckel matrix if and only if the functions* $F_k = \underset{k}{\varphi}^i (p_i)^2$ *are vertically independent and in involution.*

It is clear that functions (F_k) are interpreted as functions on the cotangent bundle of the domain of definition of the matrix, and moreover that they are vertically independent if and only if the matrix is regular (we exclude of course the points of the zero section: $p_i = 0$). If we assume that (F_k) are in involution, we see at once that they are in particular in $\mathcal{S}$-involution, hence: $\partial_i \underset{k}{\varphi}^j \underset{h}{\varphi}^i = \partial_i \underset{h}{\varphi}^j \underset{k}{\varphi}^i$ (i n.s.). If we multiply this relation by $\overset{h}{\varphi}_\ell \overset{k}{\varphi}_m$ (which are the elements of the inverse matrix, see definition (5.1)), by the summation over the repeated indices h and k, we obtain: $\delta^i_\ell \underset{k}{\varphi}^j \partial_i \overset{k}{\varphi}_m = \delta^i_m \underset{h}{\varphi}^j \partial_i \overset{h}{\varphi}_\ell$; hence, by setting in particular $m = i$, it follows: $\delta^i_\ell \partial_i \overset{h}{\varphi}_i = \partial_i \overset{h}{\varphi}_\ell$, i.e. $\partial_i \overset{h}{\varphi}_\ell = 0$ for $i \neq \ell$. This means that $\overset{h}{\varphi}_\ell$ is a function of x^ℓ only, hence that $(\underset{k}{\varphi}^i)$ is a Stäckel matrix. Conversely, if $(\underset{k}{\varphi}^i)$ is a Stäckel matrix, the following identities hold: $\partial_i \underset{k}{\varphi}^j = - \underset{k}{\varphi}^i \partial_i \overset{\ell}{\varphi}_i \underset{\ell}{\varphi}^j$ (i n.s.). Then: $\partial_i \underset{k}{\varphi}^j \underset{h}{\varphi}^i = - \underset{k}{\varphi}^i \underset{h}{\varphi}^i \partial_i \overset{\ell}{\varphi}_i \underset{\ell}{\varphi}^j$ and the Poisson bracket $\{F_h, F_k\}$ vanishes.

From the point of view of the separability structures theory, the subject acquires new aspects.

(7.3) Definition.- A $\mathcal{S}$-structure on a Riemannian manifold is said to be orthogonal if it contains orthogonal separable charts, i.e. if there exist representative charts with orthogonal coordinates.

By theorem (4.7), proposition (4.24), definition (4.26) and theorem (4.29) we s at once that:

(7.4) i) $\mathcal{S}$-structures of class $n = \dim M$ and 0 are orthogonal; ii) $\mathcal{S}$-structu of class 1 and index 0 are orthogonal; iii) orthogonal $\mathcal{S}$-structures have index 0

The last statement is a consequence of the fact that among the orthogonal coord tes of an orthogonal $\mathcal{S}$-structure we can find normal separable coordinates. If $(x^i$ are orthogonal separable coordinates of a $\mathcal{S}_r$-structure, the first class coordinate (x^α) are defined by the conditions: $\partial_\alpha g_{ii} = 0$ (or, equivalently, $\partial_\alpha g^{ii} = 0$) for i On the other hand, the functions (B^β_α) defined in (4.8) reduce simply to:

$$B^\beta_\alpha = \delta^\beta_\alpha B_\alpha \quad , \quad B_\alpha = \tfrac{1}{2} g^{\alpha\alpha} \partial_\alpha g_{\alpha\alpha} \quad (\alpha \text{ n.s.}). \tag{7.5}$$

We know by the general theory developed in section 2 that these functions do not d pend on the second class coordinates (x^a)(see $(2.15)_2$). Now, the integrable system (2.16) becomes: $\partial_\alpha p_\beta = 0$ $(\beta \neq \alpha)$, $\partial_\alpha p_\alpha = B_\alpha p_\alpha$ (α n.s.), hence each B_α is a function the corresponding variable x^α only. Thus, we conclude that the process shown in th proof of theorem (2.20) for finding ignorable coordinates simply reduces to a resc. of the first class coordinates:

$$dy^\alpha = \left(\exp \int B_\alpha dx^\alpha\right) dx^\alpha \qquad (\alpha \text{ n.s.}) \tag{7.6}$$

By leaving the second class coordinates unchanged $(y^a = x^a)$, we obtain normal sepa ble coordinates (y^i) which are still orthogonal. Then, accordingly to theorem (5.2 the corresponding contravariant components $(\overset{ii}{g})$ can be put in the form:

$$\overset{aa}{g} = \underset{m}{u}^a \quad , \quad \overset{\alpha\alpha}{g} = \zeta^\alpha_a \underset{m}{u}^a \tag{7.7}$$

where $(\underset{m}{u}^a)$ is a row of a Stäckel matrix $(\underset{b}{u}^a)$ and (ζ^α_a) are functions of the coordi corresponding to the lower index only.

8.- Separability structures with zero index.

The existence of separability structures on a Riemannian manifold is characterized by some local geometrical properties of the manifold itself. In this last section we consider the case of $\mathscr{S}$-structures with zero index.

(8.1) Theorem.- At a point x_0 of a n-dimensional Riemannian manifold (M,g) there exists a $\mathscr{S}_{r;0}$-structure if and only if, in a neighborhood of x_0, the following conditions hold:

i) there exist r K-vectors (X_α) and n - r K-tensors of order 2 (K_a) which commute: $[X_\alpha,X_\beta]=0$, $[X_\alpha,K_a]=0$, $[K_a,K_b]=0$; $(a,b=1,\dots,n-r;\ \alpha,\beta=n-r+1,\dots,n)$, and such that the corresponding first integrals are functionally independent;

ii) the K-tensors have n - r common eigenvectorfields (X_a) such that the n vectors $(X_i)=(X_a;X_\alpha)(i=1,\dots,n)$ are linearly independent and, moreover, $g(X_a,X_i)=0$ for $i\neq a$ and $[X_a,X_i]=0$;

iii) for each vector X_a, in the set (X_i) there are two vectors different from X_a such that $L_{X_a} g\,(X_i,X_j)\neq 0$.

Proof.- Let us assume that conditions i) and ii) are satisfied. The vector fields (X_i) give rise, in a neighborhood of x_0, to a coordinate system (x^i) such that $\partial_i = X_i$ and $g_{ai}=0$ for $i\neq a$. Since (∂_a) are eigenvectors of the K-tensors (K_a), their components (K_a^{ij}) satisfy the conditions $K_a^{bi}=0$ for $b\neq i$. If (p_i) are the momenta corresponding to the coordinates (x^i), then $F_a = K_a^{bb}(p_b)^2 + K_a^{\alpha\beta}p_\alpha p_\beta$ and $F_\alpha = p_\alpha$ are the first integrals corresponding to the K-tensors (K_a) and the K-vectors (X_α) respectively. The commutation relations $\{F_\alpha,F_a\}=0$ imply $\partial_\alpha F_a=0$, hence $\partial_\alpha K_a^{bb}=0$ and $\partial_\alpha K_a^{\beta\gamma}=0$. Since we have also $\partial_i F_\alpha = 0$, the matrix $(\partial_i F_j, \partial^i F_j)$ has maximal rank if and only if the matrix $(\partial^i F_j)$ has maximal rank. Thus, the hypotesis of independence of (F_i) implies the vertical independence. On the other hand, the commutation relations $\{F_a,F_b\}=0$ can be written as follows:

$$(\partial_d K_a^{cc}K_b^{dd} - \partial_d K_b^{cc}K_a^{dd})(p_c)^2 p_d - (\partial_d K_a^{\alpha\beta}K_b^{dd} - \partial_d K_b^{\alpha\beta}K_a^{dd})\, p_\alpha p_\beta p_d = 0 .$$

This implies:

$$\partial_d K_a^{cc}K_b^{dd} - \partial_d K_b^{cc}K_a^{dd} = 0 \ , \quad \partial_d K_a^{\alpha\beta}K_b^{dd} - \partial_d K_b^{\alpha\beta}K_a^{dd} = 0 \qquad (d \text{ n.s.})$$

Hence: $\partial_d F_a\, \partial^d F_b = \partial_d F_b\, \partial^d F_a$ (d n.s.). Then we can see that all the functions (F_i) are in $\mathscr{S}$-involution with respect to the coordinates (x^i). Since they are also vertically independent and $\{H,F_i\} = 0$, by theorem (3.6) it is proved that the co dinates (x^i) are separable. In particular the coordinates (x^κ) (which correspond t the K-vectors (X_κ)) are ignorable, hence of first class. The remaining coordinates (x^a) are of second class because of condition iii) , which can now be written: $\partial_a g$ $= 0$ for at least a pair of indices $i, j \neq a$ (see definition (4.4)). Then the coord nates (x^i) define a $\mathcal{S}$-structure of class r and index 0 .

Conversely, if at x_0 there exists a $\mathcal{S}_{r;o}$-structure, we can consider a normal parable coordinate system (x^i). Then the vectors $X_i = \partial_i$ and the n − r K-tensors constructed by the method shown in section 6 (see formula (6.9) with no index of t kind $\bar{a}$)obviouslv satisfy conditions i), ii) and iii). (q.e.d.)

(8.2) <u>Remark</u>.- Condition iii) guarantees that the class of the $\mathcal{S}$-structure defi by conditions i) and ii) is exactly the number of the K-vectors (X_α). Without this dition the class could be greater. Let us assume, for instance, that for a vector condition iii) is not satisfied. This means of course that the corresponding coord te x^a is of first class. Since $g^{ai} = 0$ for $i \neq a$ (x^a is "orthogonal" to the remaini coordinates), by following a reasoning analogous to that described in the last par of the preceding section, we can conlcude that there exists a rescaling of the coo nate x^a leading to an ignorable coordinate, or, in other words, that there exists function f_a , depending on x^a only, such that $f_a X_a$ is a K-vector which still comm with the remaining vectors $(X_i, i \neq a)$. Hence,condition iii) in theorem (8.1) can be stituted by the following one: iii') <u>for each vector</u> X_a <u>there exists no function</u> f <u>such that</u> $L_{X_i} f_a = 0$ <u>for</u> $i \neq a$ <u>and</u> $L_{f_a X_a} g = 0$.

(8.3) <u>Remark</u>.- Conditions i), ii) and iii) in theorem (8.1) can be used to define $\mathcal{S}_{r;o}$-structure in a <u>global</u> sense.

The K-vectors (X_κ) occurring in theorem (8.1) define on M a local foliation of dimensional submanifolds which are flat in the induced metric (in fact they are th orbits of the Abelian group of local isometries considered in theorem (4.7)). On t other hand, vectors (X_a) define a complementary orthogonal foliation of (n − dimensional submanifolds, which are of course isometric; the induced metric is $g_{aa}\, dx^a \otimes dx^a$, being (x^a) the orthogonal coordinates corresponding to vectors $(X_a$ The contravariant components of the induced metric coincides with the correspondir components (g^{aa}) of g, and, by theorem (5.2) restricted to the case of index 0, th have the form $g^{aa} = \underset{m}{u}^a$ where $(\underset{m}{u}^a)$ is a line of a Stäckel matrix. Hence, by Stäcke theorem (7.1), we can see that the coordinates (x^a) are separable in the induced metric. Thus, we have proved the following theorem (see also [17]).

(8.4) Theorem.- A n-dimensional Riemannian manifold which admits a $\mathcal{S}_{r;o}$-structure (at a point x_o) has (locally) two orthogonal transversal foliations, one made of r-dimensional submanifolds which are flat in the induced metric and the other made of (n - r)-dimensional isometric submanifolds admitting an orthogonal $\mathcal{S}$-structure.

* * *

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The present study has been sponsored by Consiglio Nazionale delle Ricerche - Gruppo Nazionale per la Fisica Matematica (Italian National Research Council - National Group for Mathematical Physics).