Geometry of Static Mechanical Systems with Constraints (*).

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Summary. – *See the Introduction.*

1. – Introduction.

Symplectic geometry has been successfully applied to the description of dynamics of mechanical systems [1]. Only systems without constraints have been treated. Our aim is an extention of the symplectic description of dynamics to systems with holonomic and non-holonomic constraints. As a preliminary step we formulate in the present paper the statics of systems with constraints.

Basic geometric concepts used in the description of systems with constraints are formulated in Section 2. The three subsequent sections deal with statics of systems without constraints, with holonomic constraints and with anholonomic constraints respectively. For the sake of simplicity only linear anholonomic constraints are considered. This excludes systems with friction.

This research was conducted at the Department of Mathematics and Statistics, the University of Calgary and was partly supported by the Consiglio Nazionale delle Ricerche of Italy and the National Research Council of Canada.

2. – Static systems: basic concepts.

We consider interacting mechanical systems in static equilibrium. Each system is assumed to have a clearly defined geometric *configuration space* with the structure of a differential manifold. For the sake of simplicity we assume that configuration spaces are connected.

Let Q be the configuration space of a static system. Finite displacements of the system will be represented in Q by *elementary* 1-*chains* [6]. A pair $([a, b], \gamma)$, where $[a, b]$ is an interval in $\boldsymbol{R}$ and γ is a differentiable mapping of an open neighbourhood of $[a, b]$ in Q, is called a *chain element*. Two chain elements $([a, b], \gamma)$ and $([a', b'], \gamma')$

(*) Entrata in Redazione il 31 dicembre 1978.

are equivalent if

$$\int_a^b \gamma^* \mu = \int_{a'}^{b'} \gamma'^* \mu \tag{2.1}$$

for each 1-form μ on Q. An elementary chain is an equivalence class of chain elements. We write

$$\int_c \mu = \int_a^b \gamma^* \mu \tag{2.2}$$

if $([a, b], \gamma)$ is a representative of the chain c. The integral $\int_a^b \gamma^* \mu$ can be expressed as $\int_a^b \langle\dot{\gamma}(t), \mu\rangle\, dt$, where $\dot{\gamma}(t)$ denotes the vector tangent to the curve γ at $\gamma(t)$. Infinitesimal displacements are represented by tangent vectors. Tangent vectors form the tangent bundle TQ with projection

$$\tau_Q \colon TQ \to Q\,. \tag{2.3}$$

The freedom of performing displacements of the system may be restricted. If no restrictions are present we say that the system is *without constraints.* Classification of constraints is best formulated in terms of infinitesimal displacements. *Constraints* in Q are represented by a subset Γ of TQ satisfying the following integrability requirement: for each $v \in \Gamma$ there is a mapping $\gamma\colon \boldsymbol{R} \to Q$ such that $\dot{\gamma}(0) = v$ and $\dot{\gamma}(t) \in \Gamma$ for each t in an open interval $]0, \varepsilon[$. An element of Γ is called a *virtual displacement.* Constraints are said to be *linear* if for each $q \in Q$ the set $\Gamma_q = \Gamma \cap T_q Q$ is a linear subspace of the tangent space $T_q Q$ at q. In this paper we consider only linear constraints. We also assume that $C = \tau_Q(\Gamma)$ is a connected submanifold of Q. The integrability condition implies that $\Gamma \subset TC$. We say that the constraints are *holonomic* if $\Gamma = TC$. In this case the submanifold C characterizes the constraints completely. If $\Gamma \neq TC$ then the constraints are said to be *anholonomic.* We shall always assume that Γ is a subbundle of TC, or, in other words, that Γ is a distribution on C. Anholonomic constraints are said to be *integrable* if Γ is a completely integrable distribution on C. In all other cases the constraints are said to be *non-holonomic.*

In terms of finite displacements of the system compatible with constraints we define an equivalence relation in C. Two points q_0 and q_1 are equivalent if the configuration of the system can be changed from q_0 to q_1 in a finite sequence of displacements compatible with constraints. A finite displacement described by a chain c is *compatible with constraints* if for each representative $([a, b], \gamma)$ of c the tangent vector $\dot{\gamma}(t)$ belongs to Γ for each $t \in [a, b]$. Configurations belonging to the equivalence class $[q]$ of q are said to be *accessible* from q. The distribution Γ induces a submodule of the Lie algebra of vector fields on C. This submodule is a subalgebra if and only if Γ is completely integrable. In any case this submodule is contained in a unique minimal subalgebra. Assuming that this subalgebra is induced by a distribution Γ' we have the following theorem [3]: *configurations accessible from a configuration* $q \in C$ *form a maximal connected integral manifold of* Γ'.

We consider external mechanisms interacting with static systems. A mechanism is a device which can force a static system to assume any configuration compatible with constraints and can induce diplecements compatible with constraints. The energy transfered from a mechanism to a static system during a quasistatic displacement induced by purely mechanical interacton is called *work*. It is assumed that work performed by a mechanism can be measured. Work performed in a virtual displacement is called *virtual work*. Let Q be the configuration space of a static system. Assuming that there are no constraints we say that the system interacts with an external mechanism with *force* $f \in T^*_q Q$ if for each virtual displacement $v \in T_q Q$ the corresponding virtual work is

$$w = -\langle v, f\rangle . \tag{2.4}$$

The mechanism acts on the system with force $-f$. The force f is the reaction force of the system counter-balancing the force applied by the mechanism. In the presence of constraints Γ the formula (2.4) can not be used directly to define the force of interaction between the system and the external mechanism. We define the *generalized force* $\varphi \in \Gamma^*_q$ by

$$w = -\langle v, \varphi\rangle \tag{2.5}$$

for each virtual displacement $v \in \Gamma_q$. The force of reaction $f \in T^*_q Q$ can be determined indirectly from the knowledge of the force $-f$ applied to the system by the external mechanism. What force is produced by the external mechanism can be found by letting the mechanism interact with other static systems. A typical example of a mechanism is a spring. Once the spring constant is measured it is easy to determine the force which the spring applies to a static system with constraints. For each virtual displacement $v \in \Gamma_q$ the force $f \in T^*_q Q$ and the generalized force $\varphi \in \Gamma^*_q$ must satisfy

$$\langle v, f\rangle = \langle v, \varphi\rangle . \tag{2.6}$$

It is clear that the generalized force φ represents the component of force f capable of performing work.

3. – Statics without constraints.

In this section we give descriptions of the behaviour of static systems without constraints interacting with external mechanisms. Four different versions are carefully analyzed in preparation of the discussion of constraints in subsequent sections.

A. *Configuration space formulation.*

Let Q be the configuration space of a static system. With no constraints present the system produces a unique reaction force $f \in T^*_q Q$ in each configuration $q \in Q$. This means that the response of the system to mechanical interaction is characterized by a *force field* σ on Q, or in other terms, a section

$$\sigma\colon Q \to T^*Q$$

of the cotangent bundle T^*Q such that $f = \sigma(q)$ is the unique force corresponding to the configuration q. The field σ describes the internal reaction forces which must be balanced by forces supplied by an external mechanism in order to maintain a static equilibrium configuration. The field σ describes completely the response of the system to mechanical interaction; it answers directly questions about forces necessary to maintain a given configuration but indirectly can be used to answer questions about equilibrium configurations compatible with given forces and related questions.

If $v \in TQ$ is a virtual displacement then the corresponding virtual work is expressed by

$$w = -\langle v, \sigma\rangle = -\langle v, \sigma(q)\rangle\,, \tag{3.2}$$

where $q = \tau_Q(v)$. If c is an elementary chain representing a finite quasistatic displacement then

$$W = -\int_c \sigma \tag{3.3}$$

is the corresponding work.

We say that the force field σ is *potential* if the 1-form σ is closed: $d\sigma = 0$. We say that the force field is *conservative* if σ is exact. A function $U\colon Q \to \boldsymbol{R}$ such that $\sigma = -dU$ is called a *potential energy function.* Let us assume that the system is isolated except for mechanical interactions producing quasistatic displacements. If the system is initially at q_0 with internal energy u_0 and undergoes a displacement c from q_0 to q_1 then the internal energy at q_1 is

$$\begin{aligned} u_1 &= u_0 + W \\ &= u_0 - \int_c \sigma\,. \end{aligned} \tag{3.4}$$

If the system is conservative then

$$W = -\int_c \sigma = U(q_1) - U(q_0)\,, \tag{3.5}$$

and if $U(q_0) = u_0$ then

$$u_1 = U(q_1)\,. \tag{3.6}$$

Thus a potential energy function represents the internal energy of the system isolated from influences other than mechanical. Potential energy is defined up to an additive constant whose value can be changed by non-mechanical interaction. If the system is potential but not conservative then potential energy functions exist locally in neighbourhoods of points in Q.

B. *Force space formulation.*

We shall call the cotangent bundle T^*Q the *force space* and shall denote it by F. The cotangent bundle projecton will be denoted by

$$\pi_Q\colon F \to Q\,. \tag{3.7}$$

There is a canonical 1-form ϑ_Q on F defined by

$$\langle \bar{v}, \vartheta_Q \rangle = \langle T\pi_Q(\bar{v}), \tau_F(\bar{v}) \rangle\,, \tag{3.8}$$

where $\bar{v}$ is an element of TF and τ_F is the tangent bundle projection

$$\tau_F\colon TF \to F\,. \tag{3.9}$$

The canonical 2-form ω_Q on F is defined by

$$\omega_Q = d\vartheta_Q\,. \tag{3.10}$$

It is well known that (F, ω_Q) is a symplectic manifold (cf. [1]).

The response of the static system to mechanical interaction is described by a submanifold S of F. An element $f \in S$ is a force compatible with the configuration $q = \pi_Q(f)$. If there are no constraints there is a unique force for each configuration. Hence, S is the image of a section of the bundle F. Displacements in F must be contained in S and infinitesimal displacements in F are vectors tangent to S. If $\bar{v} \in TF$ is an infinitesimal displacement then the wirtual work performed in this displacement is

$$w = -\langle \bar{v}, \vartheta_Q \rangle\,. \tag{3.11}$$

This follows directly from (3.8) and the definition of force. If $\bar{c}$ is an elementary chain in F representing a quasistatic displacement then

$$W = -\int_{\bar{c}} \vartheta_Q \tag{3.12}$$

is the corresponding work.

We say that the static system is *potential* if S is a Lagrangian submanifold of the symplectic manifold (F, ω_Q). We say that the system is *conservative* if S is a

Lagrangian submanifold generated by a function $-U: Q \to \mathbf{R}$. We recall that a submanifold S of F is a *Lagrangian submanifold* of (F, ω_Q) if [8]

(i) $\dim(S) = \frac{1}{2}\dim(F) = \dim(Q)$,

(ii) $\omega_Q|S = 0$.

The symbol $\omega_Q|S$ denotes the restriction of ω_Q to the submanifold S. If $\iota: S \to F$ is the canonical injection then $\omega_Q|S = \iota^*\omega_Q$. If $U: Q \to \mathbf{R}$ is a differentiable function on Q then the image $S = \mathrm{im}(-dU)$ of the 1-form $-dU: Q \to F$ is a Lagrangian submanifold of (F, ω_Q) said to be generated by $-U$ [7]. In Section 4 we shall introduce and use a more general definition of generating functions of Lagrangian submanifolds.

Let $\sigma: Q \to F$ be the force field characterizing the response of the system in the configuration space formulation and let $S \subset F$ be the submanifold describing the response of the system in the force space formulation. Assuming $S = \mathrm{im}(\sigma)$ we prove the equivalence of the two formulations. The proof is based on the identities

$$\sigma^*\vartheta_Q = \sigma \tag{3.13}$$

and

$$\sigma^*\omega_Q = d\sigma\,. \tag{3.14}$$

These identities follow from

$$\begin{aligned}\langle v, \sigma^*\vartheta_Q\rangle &= \langle T\sigma(v), \vartheta_Q\rangle \\ &= \langle T\pi_Q(T\sigma(v)),\ \tau_F(T\sigma(v))\rangle \\ &= \langle T(\pi_Q\circ\sigma)(v),\ \sigma(\tau_Q(v))\rangle \\ &= \langle v, \sigma\rangle\,,\end{aligned} \tag{3.15}$$

where v is an element of TQ.

An infinitesimal displacement in F is called a *virtual displacement* if it is tangent to S. If $\bar{v} \in TF$ is a virtual displacement in F and $v = T\pi_Q(\bar{v})$ is the corresponding virtual displacement in Q then it follows from (3.13) that the virtual work evaluated from (3.2) is equal to the virtual work evaluated from (3.11). In fact $\bar{v} = T\sigma(v)$. Hence,

$$\begin{aligned}\langle \bar{v}, \vartheta_Q\rangle &= \langle T\sigma(v), \vartheta_Q\rangle \\ &= \langle v, \sigma^*\vartheta_Q\rangle \\ &= \langle v, \sigma\rangle\,.\end{aligned} \tag{3.16}$$

If $\bar{c}$ represents a finite displacement in F contained in S and if c represents the corresponding displacement in Q then $c = \pi_Q(\bar{c})$ and $\bar{c} = \sigma(c)$, and formulae (3.3) and (3.12) are easily seen to be equivalent. From (3.14) it follows that the definitions of potential systems and conservative systems given in the configuration space formulation are equivalent to the corresponding definitions in the force space formulation. If the system is conservative and S is generated by a function $-U: Q \to R$ then U is a potential energy function. Conversely, if U is a potential energy function then S is generated by $-U$.

C. *Configuration-energy space formulation.*

The trivial bundle $E = Q \times \boldsymbol{R}$ will be called the *configuration-energy space.* An element (q, u) of E consists of a configuration q of a static system and its internal energy u. The two canonical projections of E onto Q and $\boldsymbol{R}$ will be denoted by $\chi: E \to Q$ and $\varrho: E \to \boldsymbol{R}$. The bundle E can be considered a *principal fibre bundle* [5] with *base* Q and *structure group* G equal to the additive group $\boldsymbol{R}$ of real numbers. The group action is the mapping

$$g: E \times G \to E: ((q, u), s) \mapsto (q, u + s) . \tag{3.17}$$

For each $s \in G$ the mapping $E \to E$: $(q, u) \mapsto (q, u + s)$ will be denoted by g_s. The group action is a one-parameter group of transformations of E. The induced vector field K is called the *fundamental vector field.* We denote by $V_{(q,u)}$ the space of vectors tangent to fibres of E called *vertical vectors*. The distribution $V = \bigcup_{(q,u)\in E} V_{(r,u)} \subset TE$ is called the *vertical distribution* or the *vertical subbundle* of the tangent bundle TE. The fundamental vector field K forms a basis of the vertical distribution.

A distribution $H = \bigcup_{(q,u)\in E} H_{(q,u)} \subset TE$ is called a *connection* in E if

(i) H is a supplement of V in TE,

(ii) $Tg_s(H) = H$ for each $s \in G$.

We will refer to a connection H as the *horizontal distribution* or the *horizontal subbundle* of TE. A 1-form $\hat{\sigma}$ on E is called a *connection form if*

$$\text{(i)} \qquad \langle K, \hat{\sigma} \rangle = 1 , \tag{3.18}$$

$$\text{(ii)} \qquad \mathfrak{L}_K \hat{\sigma} = 0 . \tag{3.19}$$

If $\hat{\sigma}$ is a connection form then the *characteristic distribution*

$$H = \{\hat{v} \in TE;\ \langle \hat{v}, \hat{\sigma} \rangle = 0\} \tag{3.20}$$

of $\hat{\sigma}$ is a connection in E. Conversely, each connection H is the characteristic distribution of a uniquely determined connection form $\hat{\sigma}$.

A vector field $\hat{X}: E \to TE$ is said to be *horizontal* if $\langle \hat{X}, \hat{\sigma} \rangle = 0$. For each vector field $X: Q \to TQ$ there is a unique horizontal vector field $\hat{X}: E \to TE$ such that $T\chi \circ \hat{X} = X \circ \chi$. The field X is called the *horizontal lift* of X.

If $\hat{\sigma}$ is a connection form then the form $\hat{\sigma} - d\varrho$ satisfies

$$\langle K, \hat{\sigma} - d\varrho \rangle = 0 \tag{3.21}$$

and

$$\mathcal{L}_K(\hat{\sigma} - d\varrho) = 0 . \tag{3.22}$$

It follows that there is a 1-form σ on Q such that

$$\chi^*\sigma = \hat{\sigma} - d\varrho . \tag{3.23}$$

The form σ is defined by

$$\langle v, \sigma \rangle = \langle \hat{v}, \hat{\sigma} - d\varrho \rangle , \tag{3.24}$$

where v is a vector tangent to Q and $\hat{v} \in TE$ satisfies $T\chi(\hat{v}) = v$.

The differential $d\hat{\sigma}$ of the connection form $\hat{\sigma}$ is called the *curvature form*. From (3.23) we have

$$d\hat{\sigma} = \chi^* d\sigma . \tag{3.25}$$

A connection H is said to be *locally flat* if it is a completely integrable distribution. The connection is said to be flat if H is completely integrable and the maximal connected integral manifolds of H are images of sections of E. In terms of the connection form $\hat{\sigma}$ we have the following criteria: the connection is locally flat if and only if $\hat{\sigma}$ is closed, the connection is flat if and only if $\hat{\sigma}$ is exact. The proof of the first criterion is based on the identity

$$\langle [\hat{X}, \hat{Y}], \hat{\sigma} \rangle = \mathcal{L}_{\hat{X}} \langle \hat{Y}, \hat{\sigma} \rangle - \mathcal{L}_{\hat{Y}} \langle \hat{X}, \hat{\sigma} \rangle - \langle \hat{X} \wedge \hat{Y}, d\hat{\sigma} \rangle . \tag{3.26}$$

if $d\hat{\sigma} = 0$ then $[\hat{X}, \hat{Y}]$ is a horizontal vector field if $\hat{X}$ and $\hat{Y}$ are horizontal. Hence, the distribution H satisfies the criterion for complete integrability. Conversely, if H is integrable then $\langle \hat{X} \wedge \hat{Y}, d\hat{\sigma} \rangle = 0$ if $\hat{X}$ and $\hat{Y}$ are horizontal. Let $\hat{X}$ and $\hat{Y}$ be horizontal lifts of fields X and Y respectively. Then $\langle X \wedge Y, d\sigma \rangle = 0$ since $\langle \hat{X} \wedge \hat{Y}, d\hat{\sigma} \rangle = 0$. Consequently, $d\sigma = 0$ and $d\hat{\sigma} = \chi^* d\sigma = 0$. To prove the second criterion we assume that $\hat{\sigma}$ is exact and $\hat{\sigma} = d\hat{U}$. Then the mapping $E \to E: (q, u) \mapsto (q, \hat{U}(q, u))$ is a diffeomorphism preserving fibres of E. The mapping $\hat{U}: E \to \boldsymbol{R}$ defines a fibration of E. Fibres of this fibration are integral manifolds of H and are sections of the bundle E. Conversely, if H is integrable and the maximal integral manifolds are sections of E then there are functions $\hat{U}: E \to \boldsymbol{R}$ constant on integral manifold and satisfying $\langle K, d\hat{U}\rangle = 1$. To construct a function $\hat{U}$ we select an integral manifold and interpret it as the graph of a function $U: Q \to \boldsymbol{R}$; $\hat{U}$ is defined by

$$\hat{U} = \varrho - U \circ \chi \,. \tag{3.27}$$

The form $d\hat{U}$ is a connection form for the connection H since $\langle K, d\hat{U}\rangle = 1$ and $\langle \hat{v}, d\hat{U}\rangle = 0$ for each horizontal vector $\hat{v}$. It follows that $\hat{\sigma} = d\hat{U}$ since the connection form is unique. From (3.23) and (3.27) it follows that $\sigma = -\, dU$.

In order to obtain a description of the response of a static system we consider infinitesimal quasistatic displacements in E induced by purely mechanical interaction with an external mechanism. Each displacement $\hat{v} \in TE$ consists of a displacement $v = T\chi(\hat{v})$ in Q and an energy increment $\langle \hat{v}, d\varrho\rangle$. The energy increment is equal to the virtual work. We postulate that the virtual work performed in the displacement v is

$$w = -\langle v, \sigma\rangle = -\langle \hat{v}, \chi^* \sigma\rangle \,, \tag{3.28}$$

where σ is a 1-form on Q. From

$$\langle \hat{v}, dq\rangle = -\langle \hat{v}, \chi^* \sigma\rangle \tag{3.29}$$

we conclude that $\hat{v}$ belongs to the horizontal distribution H associated with the connection form

$$\hat{\sigma} = dq + \chi^* \sigma \,. \tag{3.30}$$

The connection H characterizes the response of the system to mechanical interaction.

A 1-chain $\hat{c}$ in E representing a finite quasistatic displacement from (q_0, u_0) to (q_1, u_1) induced by purely mechanical interaction is horizontal. This means that if $([a, b], \hat{\gamma})$ is a representative of $\hat{c}$ then the tangent vector $\dot{\hat{\gamma}}(t)$ belongs to H for each $t \in [a, b]$. The work performed in the displacement is

$$W = -\int_c \sigma = -\int_{\hat{c}} \chi^* \sigma \,, \tag{3.31}$$

where $c = \chi(\hat{c})$. This work is equal to the energy increase

$$u_1 - u_0 = \int_{\hat{c}} d\varrho \,. \tag{3.32}$$

The static system is said to be *potential* if H is locally flat. The system is said to be *conservative* if H is flat. If the system is potential and interactions other than purely mechanical are excluded then only displacements within integral manifolds of H are possible. Transitions between integral manifolds can be induced by non-mechanical interactions. If the system is conservative then each maximal integral manifold is a section of E and hence the graph of a function $U: Q \to \boldsymbol{R}$. Each function U is called a *potential energy function.*

Equivalence of the configuration-energy space formulation and the configuration space formulation is easily estabilished by identifying the 1-form σ introduced in (3.23) with the force field (3.1).

D. *Force-energy space formulation.*

Let $F = T^*Q$ be the force space of a static system. The trivial bundle $\tilde{E} = F \times \boldsymbol{R}$ will be called the *force-energy space.* An element (f, u) of $\tilde{E}$ consists of force f and internal energy u. The two canonical projections of $\tilde{E}$ onto F and $\boldsymbol{R}$ will be denoted by $\tilde{\chi}: \tilde{E} \to F$ and $\tilde{\varrho}: \tilde{E} \to \boldsymbol{R}$. The bundle $\tilde{E}$ is a principal fibre bundle with base F and structure group $G = \boldsymbol{R}$. The fundamental vector field induced by the group action

$$\tilde{g}: \tilde{E} \times G \to \tilde{E}: ((f, u), s) \mapsto (f, u + s) \tag{3.33}$$

will be denoted by $\tilde{K}$. There is a canonical connection form on $\tilde{E}$ defined by

$$\tilde{\vartheta} = d\tilde{\varrho} + \tilde{\chi}^*\vartheta_Q\,, \tag{3.34}$$

where ϑ_Q is the canonical 1-form on F. The curvature form is

$$\tilde{\omega} = d\tilde{\vartheta} = \tilde{\chi}^*\omega_Q\,, \tag{3.35}$$

where ω_Q is the canonical 2-form on F. The horizontal distribution associated with $\tilde{\vartheta}$ will be denoted by $\tilde{H}$.

The response of the static system is described by a submanifold $\tilde{S} \subset \tilde{E}$. We assume that the response depends on mechanical parameters only and not on the internal energy. It follows that $\tilde{S}$ is invariant under the action of the structure group. Hence,

$$\tilde{S} = S \times \boldsymbol{R} = \tilde{\chi}^{-1}(S)\,, \tag{3.36}$$

where S is a submanifold of F. Since there are no constraints the submanifold S is the image of a section of the bundle F.

We consider infinitesimal displacements in $\tilde{E}$. Each displacement $\tilde{v} \in T\tilde{E}$ consists of a displacement $\bar{v} = T\tilde{\chi}(\tilde{v})$ in F and an energy increment $\langle \tilde{v}, d\tilde{\varrho} \rangle$. The displacement $\tilde{v}$ is called a *virtual displacement* if it is tangent to $\tilde{S}$. If $\tilde{v}$ represents a quasi-static virtual displacement induced by mechanical interaction then the energy increment $\langle \tilde{v}, d\tilde{\varrho} \rangle$ is equal to the virtual work

$$\begin{aligned} w &= -\langle \bar{v}, \vartheta_Q \rangle \\ &= -\langle \tilde{v}, \tilde{\chi}^*\vartheta_Q \rangle\,. \end{aligned} \tag{3.37}$$

From

$$\langle \tilde v, d\tilde\varrho\rangle = -\langle \tilde v, \tilde\chi^*\vartheta_Q\rangle \tag{3.38}$$

it follows that $\tilde v$ is horizontal. Since $\tilde v$ is tangent to $\tilde S$ we conclude that $\tilde v \in T\tilde S \cap \tilde H$.

A 1-chain $\tilde c$ in $\tilde S$ representing a finite quasistatic displacement from (f_0, u_0) to (f_1, u_1) induced by purely mechanical interaction is horizontal. The work performed in the displacement is

$$W = -\int_{\bar c}\vartheta_Q = -\int_{\tilde c}\chi^*\vartheta_Q\,, \tag{3.39}$$

where $\bar c = \tilde\chi(\tilde c)$. This work is equal to the energy increase

$$u_1 - u_0 = \int_{\tilde c} d\tilde\varrho\,. \tag{3.40}$$

The static system is said to be *potential* if the distribution $T\tilde S \cap \tilde H$ on $\tilde S$ is completely integrable. The system is said to be *conservative* if $T\tilde S \cap \tilde H$ is completely integrable and maximal connected integral manifolds are sections of the bundle $\tilde S = S\times \boldsymbol{R}$. We note that $\tilde S$ is a principal fibre bundle with base S and structure group $G = \boldsymbol{R}$. The distribution $T\tilde S \cap \tilde H$ is a connection in $\tilde S$ and $\tilde\vartheta|S$ is the corresponding connection form. The system is potential if and only if the connection is locally flat.

Equivalence with the force space formulation is established by identifying the submanifold S introduced in (3.36) with the submanifold S used to describe the response of the system in force space. Let S be the image of a section $\sigma: Q \to F$. Then

$$\hat\sigma = (\sigma\times 1_{\boldsymbol{R}})^*\tilde\vartheta \tag{3.41}$$

is a connection form on E describing the response of the static system in configuration-energy space terms. In order to show that $\hat\sigma$ is a connection form we prove that

$$\hat\sigma = d\varrho + \chi^*\sigma\,. \tag{3.42}$$

The proof is based on the commutativity of the diagram

$$\begin{array}{ccc} & \boldsymbol{R} & \\ {\scriptstyle\varrho}\nearrow & & \nwarrow{\scriptstyle\tilde\varrho} \\ E & \xrightarrow{\sigma\times 1_{\boldsymbol{R}}} & \tilde E \\ {\scriptstyle\chi}\downarrow & & \downarrow{\scriptstyle\tilde\chi} \\ Q & \xrightarrow{\ \sigma\ } & F \end{array} \tag{3.43}$$

We have

$$(3.44)\qquad \begin{aligned}(\sigma\times 1_{\boldsymbol{R}})^*\tilde{\vartheta} &= (\sigma\times 1_{\boldsymbol{R}})^* d\tilde{\varrho} + (\sigma\times 1_{\boldsymbol{R}})^*\, \tilde{\chi}^*\vartheta_Q \\ &= d\big(\tilde{\varrho}\circ(\sigma\times 1_{\boldsymbol{R}})\big) + \chi^*\sigma^*\vartheta_Q \\ &= d\varrho + \chi^*\sigma\,.\end{aligned}$$

Hence, $\hat{\sigma} = d\varrho + \chi^*\sigma$.

We note that following (3.34) the connection form of the connection $T\tilde{S}\cap\tilde{H}$ is given by

$$(3.45)\qquad \tilde{\vartheta}|\tilde{S} = d\tilde{\varrho}|\tilde{S} + (\tilde{\chi}|\tilde{S})^*\vartheta_Q|S$$

and the curvature form is

$$(3.46)\qquad \tilde{\omega}|\tilde{S} = (\tilde{\chi}|\tilde{S})^*\,\omega_Q|S\,.$$

It follows that the connection is flat if and only if S is a Lagrangian submanifold, *i.e.* the system is potential in the sense of the force space formulation. If the system is conservative in the sense of the force space formulation then S is a Lagrangian submanifold generated by a function $-\,U\colon Q\to\boldsymbol{R}$. From

$$(3.47)\qquad \vartheta_Q|S = -\,(\pi_Q|S)^*\,dU$$

and (3.41) it follows that

$$(3.48)\qquad \begin{aligned}\tilde{\vartheta}|\tilde{S} &= d\tilde{\varrho}|\tilde{S} - (\tilde{\chi}|\tilde{S})^*(\pi_Q|S)^*\,dU \\ &= d\big(\tilde{\varrho}|\tilde{S} - U\circ(\pi_Q|S)\circ(\tilde{\chi}|\tilde{S})\big)\,.\end{aligned}$$

Hence, the connection form $\tilde{\vartheta}|\tilde{S}$ is exact and the connection is flat. This means that the system is conservative in the sense of the force-energy space formulation. Conservative in the sense of the force-energy space formulation then maximal integral manifolds of $T\tilde{S}\cap\tilde{H}$ are sections of $\tilde{S}$. If one of these sections is interpreted as the graph of a function $\tilde{U}\colon S\to\boldsymbol{R}$ then we have

$$(3.49)\qquad \vartheta_Q|S = d\tilde{U}\,.$$

Since S is the image of a section there is a function $U\colon Q\to\boldsymbol{R}$ such that

$$(3.50)\qquad \tilde{U} = -\,U\circ(\pi_Q|S)\,.$$

From

$$(3.51)\qquad \begin{aligned}\vartheta_Q|S &= -\,d\big(U\circ(\pi_Q|S)\big) \\ &= -\,(\pi_Q|S)^*\,dU\end{aligned}$$

it follows that S is generated by the function $-\,U\colon Q\to\boldsymbol{R}$.

4. – Statics with holonomic constraints.

In the present section we consider a static system whose configuration is constrained to remain in a submanifold C of the configuration space Q. Four different descriptions of the response of the system to mechanical interaction are given. The four descriptions correspond to the descriptions presented in Section 3. Geometric structures defined in Section 3 are used in the present section.

A. *Configuration space formulation.*

To each point q of the constraint submanifold $C \subset Q$ there corresponds a unique generalized force $\varphi = \sigma(q) \in T^*_q C$. The mapping

$$\sigma\colon C \to T^*C\colon q \mapsto \sigma(q) \tag{4.1}$$

is a differential 1-form on C. The form σ characterizes the response of the system to mechanical interaction. With the constraint submanifold C interpreted as the configuration space the description of systems with holomomic constraints is formally the same as the configuration space description of systems without constraints.

If $v \in TC$ is a virtual displacement then the corresponding virtual work is

$$w = -\langle v, \sigma \rangle . \tag{4.2}$$

The work performed in a finite displacement represented by a 1-chain c in C is

$$W = -\int_c \sigma . \tag{4.3}$$

The static system is said to be *potential* if σ is closed, the system is said to be *conservatione* if σ is exact. If the system is conservative and $\sigma = -dU$ then the function $U\colon C \to \boldsymbol{R}$ is called a *potential energy function.*

The description given here is incomplete since no direct information about reaction forces is given. A more complete description is provided in the force space formulation.

B. *Force space formulation.*

The response of the static system is described in force space by a submanifold $S \subset F$ which projects onto $C\colon \pi_Q(S) = C$. For each $q \in C$ the set $S_q = S \cap T^*_q Q$ is the set of reaction forces which the system can produce in configuration q. We assume that S_q is a coset in $T^*_q Q$ of the subspace $(T_q C)^0 = \{a \in T^*_q Q; \langle v, a \rangle = 0$ for each $v \in T_q C\}$. Each set S_q defines a generalized force $\varphi = \sigma(q) \in T^*_q C$ such that $\langle v, \sigma(q) \rangle = \langle v, f \rangle$ for each virtual displacement $v \in T_q C$ and each force $f \in S_q$:

We assume that the mapping

$$\sigma\colon C \to T^*C\colon q \mapsto \sigma(q) \tag{4.4}$$

is differentiable. The dimension of S is equal to the dimension of Q.

The virtual work performed in a virtual displacement $\bar{v} \in TS$ is

$$w = -\langle \bar{v}, \vartheta_Q \rangle\,. \tag{4.5}$$

The formula

$$W = -\int_{\bar{c}} \vartheta_Q \tag{4.6}$$

gives the work performed in a finite displacement represented by a 1-chain $\bar{c}$ in S.

We say that the static system is *potential* if S is a Lagrangian submanifold. We say that the system is *conservative* if S is a Lagrangian submanifold generated by a function $-U\colon C \to \boldsymbol{R}$. This means that

$$S = \{f \in F;\ q = \pi_Q(f) \in C,\ \langle v, f\rangle = -\langle v, dU\rangle \text{ for each } v \in T_qC\}\,. \tag{4.7}$$

The function $U\colon C \to \boldsymbol{R}$ is called a *potential energy function*.

A relation between the configuration space formulation and the force space formulation is obtained by identifying mappings (4.1) and (4.4). If the 1-form σ is given then S is obtained from

$$S = \{f \in F;\ q = \pi_Q(f) \in C,\ \langle v, f\rangle = \langle v, \sigma\rangle \text{ for each } v \in T_qC\}\,. \tag{4.8}$$

The relation

$$\vartheta_Q|S = (\pi_Q|S)^*\sigma \tag{4.9}$$

is derived from

$$\begin{aligned} \langle \bar{v}, \vartheta_Q\rangle &= \langle T\pi_Q(\bar{v}), \tau_F(\bar{v})\rangle \\ &= \langle T\pi_Q(\bar{v}), \sigma\rangle \\ &= \langle \bar{v}, \overset{*}{\pi}_Q\sigma\rangle\,, \end{aligned} \tag{4.10}$$

where $\bar{v}$ is a vector tangent to S. It follows from (4.9) that $\omega_Q|S = 0$ if and only if σ is closed. This implies that the definition of potential systems given in the configuration space formulation is equivalent to the definition in the force space formulation. Comparing (4.7) and (4.8) we see that the configuration space definition of conservative systems is equivalent to the force space definition.

C. *Configuration-energy space formulation.*

With the constraint submanifold C interpreted as the configuration space the description is formally the same as that given in Section 3.C. The response of the system is characterized by a connection H on the trivial bundle $\hat{C} = C \times \boldsymbol{R}$. The connection form $\hat{\vartheta}$ defines on C a 1-form $\sigma \colon C \to T^*C$ which should be identified with the 1-form σ of the configuration space formulation. Also in this case the description is incomplete.

D. *Force-energy space formulation.*

In the force-energy space $\tilde{E} = F \times \boldsymbol{R}$ the response of the static system is characterized by a submanifold $\tilde{S} = S \times \boldsymbol{R} \subset \tilde{E}$, where S is a submanifold of F projecting onto the constraint submanifold $C \colon \pi_Q(S) = C$. The only difference between this description and that given in Section 3.D is that S is no longer a section of F. As in Section 3 we introduce a connection $T\tilde{S} \cap \tilde{H}$ and we say that the system is *conservative* if $T\tilde{S} \cap \tilde{H}$ is flat. We say that the system is *potential* if $T\tilde{S} \cap \tilde{H}$ is locally flat. The proof that these definitons are equivalent to the corresponding definitions of the force space formulation follows the pattern of the proof given in Section 3 with only minor modifications.

5. – Statics with anholonomic constraints.

As was stated in Section 2 anholonomic constraints are represented by a distribution Γ on the configuration space Q such that $\Gamma \subset TC$ and $\Gamma \neq TC$, where $C = \tau_Q(\Gamma)$ is a connected submanifold of Q. Without significant loss of generality we will assume that $C = Q$. If we also assume that the minimal subalgebra of vector fields on Q containing fields of vectors belonging to Γ defines again a distribution Γ' on Q then by Chow's theorem [3] maximal connected integral manifolds of Γ are accessibility classes. By an accessibility class we understand the set of configurations accessible from one of the configurations in the set. Since $\Gamma \neq TQ$, there are three possibilities:

(i) $\Gamma' = \Gamma$,

(ii) $\Gamma' \neq \Gamma$ and $\Gamma' \neq TQ$,

(iii) $\Gamma' = TQ$.

Case (i) corresponds to *integrable anholonomic constraints*. Since there is no possibility of passing from one accessibility class to another without breaking the constraints each accessibility class can be analyzed separately as a holonomic constraint submanifold. Case (ii) is the case of *non-holonomic constraints*. In the case (iii) we have *completely non-inegrable non-holonomic constraints*. Since non-holonomic constraints are completely non-integrable on each accessibility class separately, only case (iii) will be considered. The following assumptions are made throughout the present section:

(5.1) $$\Gamma \neq TQ\,, \quad \tau_Q(\Gamma) = Q\,, \quad \Gamma' = TQ\,.$$

A. *Configuration space formulation.*

Let Γ be a distribution on Q and let Γ^* denote the vector bundle dual to Γ. A differentiable section

(5.2) $$\sigma\colon Q \to \Gamma^*$$

will be called a 1-*form defined on* Γ. The space of such sections will be denoted by $\Lambda^1(\Gamma)$. The space of 1-forms on Q will be denoted by $\mathfrak{X}^*(Q)$. We have $\mathfrak{X}^*(Q) = \Lambda^1(TQ)$

Let μ be a 1-form on Q. We denote by $\mu|\Gamma$ the element of $\Lambda^1(\Gamma)$ obtained by restricting μ to Γ. The linear mapping $\mu \mapsto \mu|\Gamma$ maps $\mathfrak{X}^*(Q)$ onto $\Lambda^1(\Gamma)$. Indeed if σ is an element of $\Lambda^1(\Gamma)$ then a 1-form $\mu \in \mathfrak{X}^*(Q)$ such that $\mu|\Gamma = \sigma$ can be defined by choosing a supplement Γ^s of Γ in TQ (c.f. [4]) and imposing conditions: $\mu|\Gamma = \sigma$, $\mu|\Gamma^s = 0$. We denote by $\mathcal{C}(\sigma)$ the set of 1-forms on Q mapped onto $\sigma \in \Lambda^1(\Gamma)$. An element of $\mathcal{C}(\sigma)$ will be called a *representative* of σ. We have

(5.3) $$\mathcal{C}(\sigma) = \{\mu \in \mathfrak{X}^*(Q);\ \langle v, \mu\rangle = \langle v, \sigma\rangle \text{ for each } v \in \Gamma\}\,.$$

A 1-form $\sigma \in \Lambda^1(\Gamma)$ is said to be *closed* if there is a representative $\mu \in \mathcal{C}(\sigma)$ such that $d\mu = 0$. A 1-form $\sigma \in \Lambda^1(\Gamma)$ is said to be *exact* if there is a differentiable function $U\colon Q \to \boldsymbol{R}$ such that $-dU \in \mathcal{C}(\sigma)$.

Let μ be a 1-form on Q such that $\mu|\Gamma = 0$ and $d\mu = 0$. If X and Y are sections of Γ then

(5.4) $$\begin{aligned}\langle [X, Y], \mu\rangle &= \langle X \wedge Y, d\mu\rangle + \langle Y, d\langle X, \mu\rangle\rangle - \langle X, d\langle Y, \mu\rangle\rangle \\ &= 0\,.\end{aligned}$$

It follows that conditions $\mu|\Gamma = 0$ and $d\mu = 0$ imply $\mu|\Gamma' = 0$. Taking assumptions (5.1) into account we have $\mu = 0$. It follows that a closed 1-form $\sigma \in \Lambda^1(\Gamma)$ has a unique closed representative.

Let Γ represent constraints imposed on a static mechanical system. The response of the system to mechanical interaction is described by a 1-form

(5.5) $$\sigma\colon Q \to \Gamma^*\colon q \mapsto \sigma(q)$$

defined on Γ. For each $q \in Q$, $\varphi = \sigma(q)$ is the unique generalized force corresponding to q.

If $v \in \Gamma$ is a virtual displacement then the corresponding virtual work is

$$w = -\langle v, \sigma\rangle = -\langle v, \mu\rangle , \tag{5.6}$$

where μ is any representative of σ. The work performed in a finite displacement compatible with constraints and represented by a 1-chain c is

$$W = \int_c \mu , \tag{5.7}$$

for any $\mu \in \mathrm{C}(\sigma)$. Each representative of c is an integral curve of Γ.

The static system is said to be *potential* if σ is closed. The system is said to be *conservative* if σ is exact. A function $U: Q \to \boldsymbol{R}$ such that $-dU \in \mathrm{C}(\sigma)$ is called a *poten-* energy function represents the internal energy of a conservative system with non-holonomic constraints if the system is isolated except for mechanical interactions producing quasistatic displacements. Since dU is uniquely determined by σ, the potential energy function is defined up to an additive constant.

For reasons stated in Section 4.A the configuration space description of the response of a system with non-holonomic constraints is incomplete.

B. *Force space formulation.*

In the force space $F = T^*Q$ the response of a static system is described by a submanifold $S \subset F$. An element f of S is a reaction force compatible with the configuration $q = \pi_Q(f)$. The submanifold S projects onto Q: $\pi_Q(S) = Q$, and for each $q \in Q$ the set $S_q = S \cap T^*_q Q$ is a coset of the space $\Gamma^0_q = \{a \in T^*_q Q;\ \langle v, a\rangle = 0$ for each $v \in T_q\}$. If p is the rank (1) of the bundle Γ, then $\dim(S) = 2n - p$.

We consider the following subbundle of TS:

$$\bar{\Gamma} = \{\bar{v} \in TS;\ T\pi_Q(\bar{v}) \in \Gamma\} . \tag{5.8}$$

The rank of $\bar{\Gamma}$ is n. A vector $\bar{v} \in TS$ is said to be *vertical* if $T\pi_Q(\bar{v}) = 0$. Vertical vectors belong to $\bar{\Gamma}$. The canonical 1-form ϑ_Q on F induces the 1-form $\vartheta_Q|\bar{\Gamma}$ defined on $\bar{\Gamma}$. Elements of $\bar{\Gamma}$ represent infinitesimal quasistatic displacements compatible with constraints. A finite displacement compatible with constraints is represented by a 1-chain $\bar{c}$ with the property that each representative of $\bar{c}$ is an integral curve of $\bar{\Gamma}$.

The virtual work performed in a virtual displacement $\bar{v} \in \bar{\Gamma}$ is given by

$$w = -\langle \bar{v}, \vartheta_Q\rangle . \tag{5.9}$$

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(1) The rank of a vector bundle is the dimension of its fibres.

For a finite displacement represented by $\bar{c}$ the corresponding work is given by

$$W = -\int_{\bar{c}} \vartheta_Q . \tag{5.10}$$

The 1-form ϑ_Q in formulae (5.9) and (5.10) can be replaced by any other representative of $\vartheta_Q|\bar{\Gamma}$.

The static system is said to be *potential* if $\vartheta_Q|\bar{\Gamma}$ is closed. If the system is potential then there is a unique 1-form $\bar{\mu} \in \mathrm{C}(\vartheta_Q|\bar{\Gamma})$ such that $d\bar{\mu} = 0$. The system is said to be *conservative* if $\vartheta_Q|\bar{\Gamma}$ is exact. If the system is conservative then there is a function $\bar{U}: S \to \boldsymbol{R}$ such that $-d\bar{U} \in \mathrm{C}(\vartheta_Q|\bar{\Gamma})$. From (5.10) we obtain

$$W = \int_{\bar{c}} d\bar{U} = \bar{U}(f_1) - \bar{U}(f_0) , \tag{5.11}$$

if $\bar{c}$ represents a displacement from f_0 to f_1. It follows that $\bar{U}$ represents the internal energy of the system.

A 1-form σ on Γ is defined by

$$\langle v, \sigma(q)\rangle = \langle v, f\rangle \tag{5.12}$$

for each $q \in Q$, each $v \in \Gamma_q$ and each $f \in S_q$: If μ is a representative of σ then $\bar{\mu} = (\pi_Q|S)^*\mu$ is a representative of $\vartheta_Q|\bar{\Gamma}$. Indeed, for each $\bar{v} \in \bar{\Gamma}$, we have

$$\begin{aligned} \langle \bar{v}, (\pi_Q|S)^*\mu\rangle &= \langle T\pi_Q(\bar{v}), \mu\rangle \\ &= \langle v, \mu\rangle \\ &= \langle v, \sigma\rangle \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \langle \bar{v}, \vartheta_Q\rangle &= \langle T\pi_Q(\bar{v}), \tau_F(\bar{v})\rangle \\ &= \langle v, f\rangle \\ &= \langle v, \sigma\rangle , \end{aligned} \tag{5.14}$$

where $v = T\pi_Q(\bar{v})$ and $f = \tau_F(\bar{v})$.

Let the system be potential and let $\bar{\mu}$ be the unique closed representative of $\vartheta_Q|\bar{\Gamma}$. If $\bar{X}: S \to TS \subset TF$ is a vertical field then

$$\begin{aligned} \langle \bar{X}, \bar{\mu}\rangle &= \langle \bar{X}, \vartheta_Q\rangle \\ &= \langle T\pi_Q \circ \bar{X}, \tau_F \circ \bar{X}\rangle \\ &= 0 \end{aligned} \tag{5.15}$$

and

$$\mathfrak{L}_{\overline{X}}\bar{\mu} = \overline{X} \lrcorner \, d\bar{\mu} + d\langle \overline{X}, \bar{\mu}\rangle = 0 \,. \tag{5.16}$$

If follows that there exists a 1-form μ on Q such that

$$\bar{\mu} = (\pi_Q|S)^* \mu \,. \tag{5.17}$$

Let v be a vector in Γ and let $\bar{v} \in \bar{\Gamma}$ be any vector such that $T\pi_Q(\bar{v}) = v$. Then

$$\langle v, \mu\rangle = \langle \bar{v}, \bar{\mu}\rangle = \langle \bar{v}, \vartheta_Q\rangle \tag{5.18}$$

and it follows from (5.14) that μ is a representative of σ.

If the system is conservative and $\bar{\mu} = - d\overline{U}$ then there is a function $U: Q \to \boldsymbol{R}$ such that $\overline{U} = U \circ (\pi_Q|S)$ and $\mu = - dU$. Formula (5.11) leads to

$$W = U(q_1) - U(q_0) \,, \tag{5.19}$$

where $q_0 = \pi_Q(f_0)$ and $q_1 = \pi_Q(f_1)$. The function U is a *potential energy function*. The submanifold S is determined by this function and the distribution Γ:

$$S = \{f \in F;\ \langle v, f\rangle = - \langle v, dU\rangle \text{ for each } v \in \Gamma_q, \text{ where } q = \pi_Q(f)\} \,. \tag{5.20}$$

Formula (5.20) is a generalization of

$$S = \operatorname{im}(- dU) = \{f \in F;\ \langle v, f\rangle = - \langle v, dU\rangle \text{ for each } v \in T_qQ, \text{ where } q = \pi_Q(f)\} \tag{5.21}$$

different from the generalization contained in the formula (4.11). Formulae (5.21), (4.11) and (5.20) are different versions of variational principles for conservative static systems.

The construction of the 1-form σ in (5.12) provides a link between the force space formulation and the configuration space formulation. The subsequant discussion demonstrates the equivalence of the definitions of potential systems and conservative systems in these two formulations.

C. *Configuration-energy space formulation.*

In the present subsection we use geometric structures introduced in Section 3.C and part A of the present section. The concept of a connection on $E = Q \times \boldsymbol{R}$ is generalized in the following definition. A distribution $\hat{\Gamma}$ on E is called a *p-connection* on E if

(i) $\hat{\Gamma}$ projects onto a distribution Γ of dimension p on Q: $T\chi(\hat{\Gamma}) = \Gamma$,

(ii) $\hat{\Gamma}$ does not contain vertical vectors: if $\hat{v} \in \hat{\Gamma}$ and $T\chi(\hat{v}) = 0$ then $\hat{v} = 0$,

(iii) $\hat{\Gamma}$ is invariant under the group action g: $Tg_s(\hat{\Gamma}) = \hat{\Gamma}$ for each $s \in G = \boldsymbol{R}$.

An n-connection is a connection on E. Let Γ be a distribution on E of dimension n and let σ be a 1-form defined on Γ. The distribution

$$\hat{\Gamma} = \{\hat{v} \in TE;\ v = T\chi(\hat{v}) \in T,\ \langle v, d_Q\rangle = -\langle v, \sigma\rangle\} \tag{5.22}$$

is a p-connection on E. Conversely a p-connection $\hat{\Gamma}$ defines uniquely a distribution $\Gamma = T\chi(\hat{\Gamma})$ and a 1-form σ defined on Γ. The form σ is defined by $\langle v, \sigma\rangle = -\langle v, d_Q\rangle$ where v is a vector in Γ and $\hat{v}$ is a vector in such that $T(v) = v$.

Let μ be a 1-form on Q. The form $\bar{\mu} = d_Q + \chi^*\mu$ is a connection form, let H denote the corresponding connection on E. We have the following obvious relations:

a) $\hat{\Gamma} \subset H$ if and only if $\mu \in \mathrm{C}(\sigma)$,

b) if $\hat{\Gamma} \subset H$ then $(\hat{\Gamma})' \subset H'$.

THEOREM. – *Let Γ be a distribution on Q and σ a* 1*-form on Γ. Let $\hat{\Gamma}$ be the distribution defined by* (5.22). *If $\Gamma' = TQ$ then σ is closed if and only if $(\hat{\Gamma})'$ is a connection* (*necessarily locally flat*); *σ is exact if and only if $(\hat{\Gamma})'$ is a flat connection.*

PROOF. – Let $(\hat{\Gamma})'$ be a connection and let $\hat{\mu} = d_Q + \chi^*\mu$ be the corresponding connection form. Since $(\hat{\Gamma})'$ is locally flat we have $d\hat{\mu} = 0$. Hence, $d\mu = 0$. It follows from the relation *a*) above that μ is a representative of σ. Hence, σ is closed. Conversely if σ is closed and μ is the closed representative of σ then μ defines a locally flat connection H which containes Γ. From the relation *b*) it follows that $(\hat{\Gamma})' \subset H$. Since $\Gamma' = TQ$ implies $T\chi((\hat{\Gamma})') = TQ$, $(\hat{\Gamma})'$ must be a connection equal to H. The second part of the theorem follows immediately from the discussion of flat connections in Section 3.C.

In order to obtain a description of the response of a static system with non-holonomic constraints Γ in the configuration-energy space $E = Q \times \boldsymbol{R}$ we consider infinitesimal displacements in E induced by purely mechanical interaction with an external mechanism. Each displacement $\hat{v} \in TE$ consists of a displacement $v = T\chi(\hat{v})$ in Q and an energy increment $\langle \hat{v}, d_Q\rangle$. The displacement v must belong to the distribution Γ, the energy increment must be equal to the virtual work. We postulate that the virtual work performed in the displacement v is

$$w = -\langle v, \sigma \rangle , \tag{5.23}$$

where σ is a 1-form defined on Γ. From

$$\langle \hat{v}, d\varrho \rangle = -\langle v, \sigma \rangle \tag{5.24}$$

we conclude that $\hat{v}$ belongs to the distribution $\hat{\Gamma}$ defined by (5.22). The p-connection $\hat{\Gamma}$ describes the response of the system to mechanical interaction.

If μ is a representative of σ and $\hat{v}$ is a virtual displacement then the virtual work is

$$w = -\langle v, \chi^* \mu \rangle . \tag{5.25}$$

A finite displacement induced by purely mechanical interaction is represented by a *horizontal* 1-chain $\hat{c}$ in E: each representative of $\hat{c}$ is an integral curve of $\hat{\Gamma}$. The corresponding work is

$$W = -\int_{\hat{c}} \chi^* \mu = u_1 - u_0 \tag{5.26}$$

if $\hat{c}$ represents a displacement from (q_0, u_0) to (q_1, u_1).

The static system is said to be *potential* if $(\hat{\Gamma})'$ is a connection. The system is said to be *conservative* if $(\hat{\Gamma})'$ is a flat connection.

Equivalence of the configuration-energy space formulation with the configuration space formulation is obvious. We emphasize the importance of the Theorem as providing an effective criterion for the form σ to be closed (cf. [2]).

D. *Force-energy space formulation.*

In the force-energy space $\tilde{E} = F \times \boldsymbol{R}$ the response of a static system is described by a submanifold

$$\tilde{S} = S \times \boldsymbol{R} = \tilde{\chi}^{-1}(S) , \tag{5.27}$$

where $S \subset F$ is a submanifold with properties stated in force space formulation. The distribution $\bar{\Gamma}$ defined by (5.8) is used again in the present subsection. We define a p-connection

$$\tilde{\Gamma} = \{\tilde{v} \in T\tilde{S};\ \bar{v} = T\tilde{\chi}(\tilde{v}) \in \bar{\Gamma},\ \langle \tilde{v}, d\tilde{\varrho} \rangle = -\langle \bar{v}, \vartheta_Q \rangle\} \tag{5.28}$$

on the trivial bundle $\tilde{S} = S \times \boldsymbol{R}$. Elements of $\tilde{\Gamma}$ represent virtual displacements in $\tilde{E}$.

The virtual work corresponding to a virtual displacement $\tilde{v} \in \Gamma$ is given by

$$w = \langle \tilde{v}, d\tilde{\varrho} \rangle = - \langle T\tilde{\chi}(\tilde{v}), \vartheta_Q \rangle = - \langle \tilde{v}, \tilde{\chi}^* \tilde{\mu} \rangle \,, \tag{5.29}$$

where $\tilde{\mu}$ is any representative of $\vartheta_Q | \bar{\Gamma}$. A finite displacement is represented by a horizontal 1-chain $\tilde{c}$. The corresponding work is

$$W = - \int_{\tilde{c}} \tilde{\chi}^* \tilde{\mu} = \int_{\tilde{c}} d\tilde{\varrho} = u_1 - u_0 \tag{5.30}$$

if $\tilde{c}$ represents a displacement from (f_0, u_0) to (f_1, u_1).

The static system is said to be *potential* if $(\tilde{\Gamma})'$ is a connection. The system is said to be *conservative* if $(\tilde{\Gamma})'$ is a flat connection. The equivalence of these definitions with those given in the force space formulation is demonstrated by applying the Theorem stated in part C to the 1-form $\vartheta_Q | \bar{\Gamma}$ defined on $\bar{\Gamma}$ and the trivial bundle $\tilde{S} = S \times \boldsymbol{R}$. This version of the Theorem provides an alternate criterion for the 1-form σ to be closed. This criterion corresponding the usual integrability criteria for systems of partial differential equations (cf. [4]) is much more complicated than the criterion formulated in part C.

REFERENCES

[1] A. ABRAHAM - J. E. MARSDEN, *Foundations of Mechanics*, Benjamin, New York (1967).
[2] S. BENENTI - W. M. TULCZYJEW, C. R. Acad. Sci. Paris Sér. A, **286** (1978), pp. 237-240.
[3] W. L. CHOW, Math. Ann., **117** (1940), pp. 98-105.
[4] J. DIEUDONNÉ, *Treatise on Analysis*, Academic Press, New York and London (1972).
[5] K. NOMIZU, *Lie Groups and Differential Geometry*, Mathematical Society of Japan, Tokyo (1956).
[6] G. DE RHAM, *Variétés différentiables*, Hermann, Paris (1973).
[7] J. ŚNIATYCKI - W. M. TULCZYJEW, Indiana Univ. Math. J., **22** (1972), pp. 267-275.
[8] A. WEINSTEIN, Advances in Math., **6** (1971), pp. 329-346.