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The Theory of Separability of the Hamilton-Jacobi Equation and Its Applications to General Relativity

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1. Introduction

Since the beginning of general relativity theory, Einstein himself[1] realized the role that "geodesics" plays in his gravitational theory. General relativity is in fact a field theory in which the physical object "gravitational field" is described by a geometrical object, the metric tensor g of a Lorentzian manifold V_4 assumed as a model of the physical world. Einstein realized that, as an obvious generalization of the law of inertia, photons and test particles should follow the geodesics of the given metric.

On the other hand, the knowledge of the geodesics of (V_4, g) is often essential for a better understanding of the properties of V_4, both at the local (e.g., near a singular point) and at the global level. New impulse to research in this direction came from the investigations of exact solutions of the Einstein equations, some of which have recently attracted attention due to their importance in problems of an astrophysical nature, e.g., the study of collapsed objects or the study of cosmological models with symmetries.

It is always possible to integrate the geodesic equation of a given metric numerically, i.e., to give the approximate solutions together with their qualitative behavior. However, attention should be given to those particular cases in which it is possible to apply direct methods to the study of geodesics and their first integrals, or even more to reduce their determination to integrals. This procedure is preferable for at least one reason: only the

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geometrical study of the geodesic equation allows the determination of properties of space–time which often go far beyond the mere knowledge of the geodesic structure. For example, to any first integral there corresponds a conserved quantity together with a "symmetry."

For the analytical solution of the geodesic equation in a pseudo-Riemannian manifold (V_n, g)† essentially two different methods are available: either through the knowledge of sufficient first integrals, or through the knowledge of a complete integral of the so-called Hamilton–Jacobi equation (HJ equation). For this second case the only method of finding such complete integrals that nowadays seems to be available is the "method of complete separation of variables," which amounts to the determination of a complete integral which separates into a sum of functions of each single variable. However, recent investigations by Woodhouse[(2)] have pointed out that from the existence of a single variable that separates it is possible to develop a constructive method for the determination of a certain number of functionally independent first integrals. This provides a primary link between the two approaches mentioned above. A second, deeper relation between the complete separability of the HJ equation and the existence of sufficient first integrals has been shown by the theory of "separability structures" recently developed by one of us. The existence of a separability structure on a pseudo-Riemannian manifold (V_n, g) is, moreover, a necessary condition for the complete separability of variables for several linear second-order differential equations related to g, e.g., the Schrödinger equation or the generalized Laplace equation.

As concerns general relativity, many problems related to the geodesic equation and its first integrals have been successfully investigated with the aid of spinor structures on space–time, by using the Newman–Penrose formalism[(3)] or its later refinements.[(4)] The use of these techniques is very powerful and, for many of its applications, it seems to be unavoidable. One can appreciate this by trying to reformulate in ordinary tensor language the results obtained by means of the spinorial formalisms. However, those problems belong to the domain of theoretical mechanics and, as such, they should be dealt with in a more general framework. In fact, in spite of its efficiency, the spinor approach does not allow one to distinguish those properties that essentially depend on the hypothesis of a four-dimensional Lorentz metric from those that, on the contrary, have a more general significance.

From the above remarks it follows clearly that it is important to study the separability of the geodesic equation and its first integrals from the most general viewpoint, both for the purpose of providing a general theory and

† By "pseudo-Riemannian manifold" we mean a C^∞ manifold with a nondegenerate metric g of whatever signature. When the signature is $(n, 0)$ (i.e., when g is positive definite) we shall also use the term "Riemannian manifold."

for the possibility of applying it fruitfully to particular cases. This purpose will be accomplished by attacking the problem in a general pseudo-Riemannian manifold (V_n, g) of dimension $n > 1$ and whatever signature. We shall outline the historical evolution and the recent developments of the problem of integrability of the HJ equation of geodesics by separation of variables. In this paper we shall not enter in great detail into problems involving more general Hamiltonians (e.g., t-dependent Hamiltonians, or Hamiltonians with nonquadratic terms or with potentials) which do not have a direct relevance to the geodesic equation. We shall instead direct some attention to the properties related to the separability of a single variable and also to the second-order equations already mentioned.

2. The General Theory of Separability

2.1. Separability Structures in Pseudo-Riemannian Manifolds

Let (P_{2n}, ω) be a $2n$-dimensional symplectic manifold[5,6] and let $H: P_{2n} \to \mathbb{R}$ be a C^∞ function which we shall call the *Hamiltonian.*

To the Hamiltonian H we naturally associate a "Hamiltonian vector field" X on P_{2n} through the equation (5):

$$i_X\omega = dH \tag{2.1}$$

The integral curves of X are locally determined by the integration of a first-order system of ordinary differential equations, that, in whichever canonical coordinate system (q^i, p_i), assumes the classical canonical form†:

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \tag{2.2}$$

As is well known, their integration may be replaced by the determination of a so-called "complete integral" W of the (reduced) HJ equation‡:

$$H\left(q^i, \frac{\partial W}{\partial q^i}\right) = h \tag{2.3}$$

† More precisely, equation (2.2) should be rewritten as

$$\frac{dq^i}{dt} = \frac{\partial \hat{H}}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial \hat{H}}{\partial q^i} \tag{2.2'}$$

where $\hat{H} = \hat{H}(q^i, p_i)$ is the local representation of H. Let $\mathcal{U}$ be a chart of coordinates (x^i). ∂_i is the vector field on $\mathcal{U}$ tangent to the coordinate line x^i, i.e., it operates as follows: given a C^∞ function $f\colon V_n \to \mathbb{R}$ with local representation $\hat{f}$ in $\mathcal{U}$ we have

$$\partial_i : f \mapsto \partial_i f \equiv \frac{\partial \hat{f}}{\partial x^i}$$

For the sake of simplicity, we shall omit the distinction between f and $\hat{f}$, leaving to the reader the correct interpretation.

‡ Hereafter abbreviated as HJ equation.

where W is a unknown function and $h \in \mathbb{R}$ is a given constant.† By "complete integral" we mean a regular n-parameter family of solutions of (2.3):

$$W = W(q^i; c_1, \ldots, c_n) \tag{2.4}$$

such that

$$\det \left\| \frac{\partial^2 W}{\partial q^i \partial c_j} \right\| \neq 0 \tag{2.5}$$

The coordinates (q^i, p_i) are called *separable canonical coordinates* for H if the complete integral (2.4) assumes the form

$$W = \sum_{i=1}^{n} W_i(q^i, c_1, \ldots, c_n) \tag{2.6}$$

In such a case we say that the HJ equation (2.3) is *integrable by separation of variables* (or more simply *separable*) *with respect to the coordinates* (q^i, p_i). We stress that separability does not mean that a direct algorithm exists which reduces the integration of (2.3) to the integration of n ordinary differential equations for each W_i, without any further information about the Hamiltonian H. As an example, the method of multiplying (2.3) by a suitable separating factor (see Reference 7, p. 232) ‡ does not cover all the possible situations since there exist cases in which (2.3) is separable in spite of the fact that the method above cannot be applied.

The analytical characterization of the canonical coordinates (q^i, p_i) which allows one to separate (2.3) for a given Hamiltonian H was given in 1904 by Levi-Civita,[10] who proved the following result.

Theorem 2.1. The HJ equation (2.3) is separable with respect to the canonical coordinates (q^i, p_i) if and only if§

$$\frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial q^i \partial q^j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p^i} \frac{\partial^2 H}{\partial q^i \partial p_j} - \frac{\partial H}{\partial q^i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q^j} + \frac{\partial H}{\partial q^i} \frac{\partial H}{\partial q^j} \frac{\partial^2 H}{\partial p_i \partial p_j} = 0 \qquad (\forall i \neq j) \tag{2.7}$$

Here we shall be mainly interested in the following case: P_{2n} is the cotangent bundle T^*V_n of a manifold V_n, with its canonical symplectic

† h should of course be chosen among the regular values of H.

‡ Which in several recent papers has been considered as a prescription for separability of at least one coordinate (see, e.g., References 8 and 9).

§ The above considerations on separability may be extended to the case of nonautonomous (i.e., t-dependent) Hamiltonian systems, by requiring that also the t coordinate separates. Theorem 2.1 may be easily generalized to include such cases (see Reference 11).

structure and there is a pseudo-Riemannian metric g on V_n. We consider the Hamiltonian H whose local expression is

$$H(q^i, p_i) = \tfrac{1}{2} g^{ij}(q^k) p_i p_j \tag{2.8}$$

where (q^i, p_i) are natural canonical coordinates in $T^* V_n$ [i.e., (q^i) are coordinates in V_n]. The integral curves of the Hamiltonian vector field associated with (2.8) define the "geodesic flow" of g in $T^* V_n$, i.e., their projections onto V_n are the geodesics of g parametrized by their affine parameter t.

Local coordinates (x^i) in (V_n, g) are called *separable coordinates* if the canonically associated coordinates (x^i, p_i) are separable canonical coordinates for the Hamiltonian (2.8).

A classical problem is the following.

Problem 2.1. Find the most general (local) expressions $g^{ij}(x^k)$ such that (x^k) are separable coordinates.

This is a problem of analytical character. However, since the existence of separable coordinates depends on the pseudo-Riemannian structure, we should also consider the geometrical counterpart of Problem 2.1, namely the following.

Problem 2.2. Find the (local) geometrical characterization of a pseudo-Riemannian structure (V_n, g) which allows the existence of separable coordinates.

To better explain the nature and the meaning of Problem 2.2 we are naturally led to introduce the following concepts. We denote by G the contravariant tensor associated with g. The HJ equation for the Hamiltonian (2.8) may be intrinsically written as follows:

$$G(dW, dW) = h \tag{2.9}$$

which in a chart $\mathcal{U} \subseteq V_n$ of coordinates (x^i) has the local representation

$$g^{ij}\, \partial_i W\, \partial_j W = h \tag{2.10}$$

The chart $\mathcal{U}$ is called a *separable chart* if the coordinates (x^i) are separable.

Consider now a point $x \in V_n$. Let $\mathcal{U}$, $\mathcal{U}'$ be two separable charts around x with coordinates (x^i) and $(x^{i'})$, respectively. We say that these two charts are $\mathcal{S}$ *compatible* if the separated complete integrals W and W' [associated with the separable coordinates (x^i) and $(x^{i'})$, respectively], coincide on some open neighborhood $\mathcal{U}^*$ of x (of course, with $\mathcal{U}^* \subseteq \mathcal{U} \cap \mathcal{U}'$). $\mathcal{S}$ compatibility is an equivalence relation in the set of separable charts around any point x:

each equivalence class will be called a (*local*) *separability structure* (around the point x) of (V_n, g).

With the above definitions, Problem 2.2 may be more precisely reformulated as follows.

Problem 2.2′. Characterize from a geometrical viewpoint the existence of a separability structure in (V_n, g).

As a first step to answer Problem 2.2 we can give a preliminary classification which divides all the possible separability structures in a (V_n, g) into $n + 1$ different classes. We say that *a separability structure is of class r* (or a $\mathcal{S}_r$ structure) if r is the maximum number of ignorable coordinates that we can find for each chart belonging to the given separability structure. Later on it will become clear that this definition is well posed. At each point $x \in V_n$ there may exist none, one, or more than one separability structure for each class r; of course $0 \leq r \leq n$.

Among all the coordinate systems belonging to a given separability structure of class r we shall call *quasinormal separable coordinates* those in which the number of ignorable coordinates is exactly r. Moreover if among all the coordinate systems belonging to a given separability structure there exists at least one in which the coordinates are orthogonal (i.e., $g^{ij} = 0$ for $i \neq j$) we shall say that the separability structure is *orthogonal*. We shall realize later that the structures of class 0, 1, and 2 are always orthogonal.

2.2. The Historical Perspective on Separability

Several particular cases of Problem 2.1 were studied at the end of the last century and at the beginning of the present one, after the famous paper(12) where Liouville discovered a large class of solutions (commonly known as Liouville's metrics). A conclusive result concerning the case of orthogonal separable coordinates was given in References 13 and 14 by Stäckel, who proved the following theorem.

Theorem 2.2. If (x^i) are orthogonal coordinates the HJ equation with potential U

$$\tfrac{1}{2}g^{ii}(\partial_i W)^2 - U = h \tag{2.11}$$

is separable if and only if there exist functions U_i and a regular $n \times n$ matrix $\|\overset{k}{\varphi}_i\|$ such that

$$\partial_j U_i = 0 \qquad (\forall i, j \colon i \neq j) \tag{2.12a}$$

$$\partial_j \overset{k}{\varphi}_i = 0 \qquad (\forall i, j \colon i \neq j) \tag{2.12b}$$

and such that we have

$$g^{ii} = \underset{n}{\varphi}^{i}, \qquad U = U_i \underset{n}{\varphi}^{i} \tag{2.13}$$

where $\|\underset{k}{\varphi}^{i}\|$ is the inverse matrix.

The sufficiency of conditions (2.12a) and (2.12b) was previously proved in References 15 and 16, respectively. We observe that in Reference 15 Stäckel remarked that there exist n quadratic first integrals of geodesics

$$\underset{j}{k} = \underset{j}{\varphi}^{i}(p_i)^2 \tag{2.14}$$

One of these is obviously the kinetic energy integral (i.e., when $j = n$). Since $\|\underset{k}{\varphi}^{i}\|$ is regular the integrals $\underset{j}{k}$ are *vertically independent*†; moreover they turn out to be in involution. We remark that the following holds.

Lemma 2.1.[17] Let $\underset{j}{k}: T^* V_n \to \mathbb{R}$ be n vertically independent functions, which in coordinates (x^i) assume the form (2.14). Then they are in involution if and only if the matrix $\|\overset{k}{\varphi}_i\|$ satisfies (2.12b).

Relying on Lemma 2.1 we can reformulate Stäckel's Theorem 2.2 (for the case U = const) as follows.

Theorem 2.3. An orthogonal coordinate system (x^i) is separable if and only if there exist n vertically independent quadratic first integrals in involution, which in coordinates (x^i) have the orthogonal form (2.14).

Here we need to recall some definitions. A contravariant symmetric tensor (of order p) $K^{i_1 \cdots i_p}$ is called a *Killing tensor* (K tensor) if it satisfies

$$\nabla^{(i} K^{i_1 \cdots i_p)} = 0 \tag{2.15}$$

The function $k: T^* V_n \to \mathbb{R}$ defined locally by

$$k = K^{i_1 \cdots i_p} p_{i_1} \cdots p_{i_p} \tag{2.16}$$

is a (homogeneous pth-order) first integral of the geodesics.‡ The set of all symmetric contravariant tensor fields of any order is a graded Lie algebra[18] under the so-called Schouten–Nijenhuis brackets,[19] defined as follows:

$$[\underset{1}{K}, \underset{2}{K}]^{i_1 \cdots i_s} p_{i_1 \cdots i_s} = -\{\underset{1}{k}, \underset{2}{k}\} \tag{2.17}$$

† A set of m functions $(m \le n)$: $\underset{1}{k}, \ldots, \underset{m}{k}: T^* V_n \to \mathbb{R}$ are said to be *vertically independent* if the restrictions of $\underset{j}{k}$ to any fiber $T_x^* V_n$ are functionally independent (Reference 2, p. 15).

‡ The only exceptions to this convention shall be the correspondence through G (contravariant metric) and H (kinetic Hamiltonian), due to "historical" reasons.

where $s = (\text{order of } \underset{1}{K}) + (\text{order of } \underset{2}{K}) - 1$ and $\{\,,\}$ denote the Poisson bracket. The K tensors constitute a subalgebra defined by

$$[K, G] = 0 \tag{2.18}$$

Two K-tensors $\underset{1}{K}, \underset{2}{K}$ are said to be commuting if $[\underset{1}{K}, \underset{2}{K}] = 0$ or, equivalently, $\underset{1}{k}$ and $\underset{2}{k}$ are in involution (i.e., if $\{\underset{1}{k}, \underset{2}{k}\} = 0$).

As far as Theorem 2.3 is concerned, we observe that the vector fields ∂_i associated with the coordinates (x^i) are common eigenvector fields of the n second-order K tensors $\underset{j}{K}$ implicitly defined by (2.14). Conversely, if n independent K tensors $\underset{j}{K}$ have n common commuting orthogonal eigenvector fields $\underset{j}{X}$,† their components in the coordinate system generated by $\underset{j}{X}$ are given by

$$\underset{j}{K}^{il} = 0 \qquad (i \neq l), \qquad \underset{j}{K}^{ii} = \underset{j}{\varphi}^{i} \tag{2.19}$$

Again in virtue of Lemma 2.1, a geometrical counterpart of Theorem 2.3 can be formulated as follows.

Theorem 2.4. A Riemannian manifold (V_n, g) admits locally an orthogonal separability structure if (locally) there exist n commuting independent second-order K tensors having in common n orthogonal normal‡ eigenvector fields.

A similar statement has been given in Reference 2 in terms of closed eigenforms (corollary on p. 31).§

A general framework for the solution of Problem 2.1 in its full generality was given in 1904 by Levi-Civita, in a letter to Stäckel.[10] He considered the Hamiltonian

$$H = \tfrac{1}{2} g^{ij} p_i p_j - U \tag{2.20}$$

and from Theorem 2.1 he deduced that if the HJ equation associated with (2.20) is separable the same holds for the "geodesic" Hamiltonian (2.8). We

† In the sequel, two contravariant tensor fields $\underset{1}{K}$ and $\underset{2}{K}$ shall be called independent if and only if the corresponding functions $\underset{1}{k}$ and $\underset{2}{k}$ are vertically independent.

‡ Which means orthogonal and hypersurface forming.

§ For the sake of completeness, we remark that the quoted corollary is not exactly the theorem due to Eisenhart[20] quoted in Reference 2, which moreover seems to be affected by redundant hypotheses.

remark that the same is true for Hamiltonians of the type

$$H = \tfrac{1}{2}g^{ij}p_ip_j + l^ip_i - U \tag{2.21}$$

related for instance to mechanical systems with t-dependent constraints or, in general relativity, to problems involving external fields (see, e.g., References 8, 11, 21). This means that the conditions for separability of the corresponding HJ equations can be obtained in two steps. First, consider the geodesic problem associated with the Hamiltonian (2.8) and find the separability conditions. Second, see if the remaining terms in the complete Hamiltonian (2.20) [or (2.21)] are compatible, in a suitable sense, with the stated separability of the geodesic equation. The first step is the core of Problems 2.1 and 2.2 above; step two has a relatively simple solution (see References 22 and 23). However, we should emphasize that the linear terms l^ip_i and the potential U may depend directly on the metric g, so that further conditions on g may arise in the second step.

When the Hamiltonian (2.8) is considered, the separability conditions of Theorem 2.1 are precisely the integrability conditions of the following system of partial differential equations:

$$\begin{aligned} \partial_i\pi_j &= 0, \qquad i \neq j \\ \partial_i\pi_i &= -\frac{1}{2}\,\frac{\partial_i g^{jl}\pi_j\pi_l}{g^{il}\pi_l} \end{aligned} \tag{2.22}$$

where

$$\pi_i = \partial_i W \tag{2.23}$$

The discussion given by Levi-Civita in Reference 10 is based on the division of the separable coordinates (x^i) $(i = 1, \ldots, n)$ into two separate classes. Let σ_i be quadratic forms defined by

$$\sigma_i = \partial_i g_{jl}\pi^j\pi^l \tag{2.24}$$

A coordinate x^i is called a first-class coordinate if σ_i is exactly divisible by π^i; otherwise, x^i is said to be a second-class coordinate. For the sake of simplicity throughout this section and Section 2.4 the following conventions shall be adopted: second-class coordinates are labeled by indices from the first part of the Latin alphabet (a, b, c); first-class ones by Greek indices $(\alpha, \beta, \gamma, \ldots)$; Latin indices $i, j, \ldots$ (from the second part of the alphabet) shall denote any coordinate. The Einstein summation convention shall be adapted to the above choices. Moreover, coordinates (x^i) shall be arranged in such a way that the first m belong to the second class (so that $a, b, \ldots$ range from 1 to m) while the remaining ones are first class (so that $\alpha, \beta, \ldots$ range from $m + 1$ to n). One of the two classes may be empty.

The following first-order separability conditions on g were derived by Levi-Civita:

$$\partial_\alpha g_{ij} = 0, \qquad \forall i, j = \alpha \tag{2.25}$$

$$\begin{aligned} & g^{ai}\, \partial_i g_{jl} = 0 \\ & g^{ar}\, \partial_i g_{jr} - g^{ai}\, \partial_i g_{ij} = 0 \qquad (\forall i, j, l \neq a; j, l \neq i; i \text{ n.s.}\dagger) \\ & g^{ar}\, \partial_i g_{ir} - \tfrac{1}{2} g^{ai}\, \partial_i g_{ii} = 0 \end{aligned} \tag{2.26}$$

However, he was able to discuss completely only two particular cases: (i) the case $n = 2$ (for any $0 \leq m \leq 2$), in which his method leads in a simpler way to the results previously obtained by Liouville,[12] Morera[24] and Stäckel[25]; (ii) the case $m = 0$ (for any n), in which (V_n, g) turns out to be locally flat. We shall see later that the case (ii) corresponds to the $\mathscr{S}_n$ structures.

Actually, the integration of equations (2.25), (2.26) and of the further second-order separability conditions found by Levi-Civita is a problem of rather hard solution. An "indirect" method to answer Problem 2.1 was found in 1912 by Dall'Acqua.[26] Relying on a previous conjecture due to Burgatti[27] and taking into account the above distinction for the coordinates, he obtained the general form of the separate solutions of a separable HJ equation associated with a Hamiltonian (2.20). He proved a theorem which can be conveniently stated as follows.

Theorem 2.5. In a Riemannian manifold (V_n, g) the HJ equation associated with (3.10) is separable if and only if the functions (2.23) assume the following form:

$$\begin{aligned} \pi_a &= \overset{\alpha}{\xi}_a \underset{\alpha}{c} \pm \left(\overset{\alpha\beta}{\zeta}_a \underset{\alpha\beta}{cc} + \overset{b}{v}_a \underset{b}{c} - U_a\right)^{1/2} \\ \pi_\alpha &= \overset{\beta}{\xi}_\alpha \underset{\beta}{c} \end{aligned} \tag{2.27}$$

$(a, b = 1, \ldots, m;\ \alpha, \beta = m+1, \ldots, n)$, where $\overset{\alpha}{\xi}_a$, $\overset{\alpha\beta}{\zeta}_a$, $\overset{b}{v}_a$ and U_a (respectively, $\overset{\beta}{\xi}_\alpha$) are functions of the coordinate x^a only (respectively, x^α), $\underset{i}{c}$ $(i = 1, \ldots, n)$ are arbitrary real constants.

Theorem 2.5 is actually the answer to Problem 2.1 for Riemannian manifolds. However, it is rather surprising that Dall'Acqua did not go beyond the mere statement of the theorem and he only gave the following suggestion how to find the general form of a separable g (and U). Under the *additional hypothesis* that relations (2.27) can be solved with respect to the

† Throughout this paper the notation n.s. = "not summed" will be used.

constants $\underset{i}{c}$, we can obtain

$$\begin{aligned}\underset{b}{c} &= \underset{b}{v}^a[(\pi_a - \overset{\beta}{\xi_a}\underset{\beta}{\xi}^\alpha\pi_\alpha)^2 - \overset{\alpha\beta}{\zeta_a}\underset{\alpha}{\xi}^\gamma\underset{\beta}{\xi}^\delta\pi_\gamma\pi_\delta + U_a] \\ \underset{\beta}{c} &= \underset{\beta}{\xi}^\alpha\pi_\alpha\end{aligned} \tag{2.28}$$

where $\|\underset{\beta}{\xi}^\alpha\|$ and $\|\underset{b}{v}^a\|$ are the inverse matrices of $\|\overset{\beta}{\xi_\alpha}\|$ and $\|\overset{b}{v_a}\|$. If we introduce the new constant h_0 defined by

$$2h_0 = \sum_{a=1}^{m} \underset{a}{c} + \sum_{\alpha=m+1}^{n} (\underset{\alpha}{c})^2 \tag{2.29}$$

and we substitute (2.28) into (2.29) we obtain (with $\pi_i = p_i$) a quadratic first integral:

$$h_0 = \tfrac{1}{2}K^{ij}p_ip_j - V \tag{2.30}$$

Dall'Acqua stated explicitly that K^{ij} and V solve Problem 2.1 (Reference 26, p. 10). However, this suggestion (which has been recently followed step by step by Havas[28]) is rather vague. First of all, we should observe that the first integral (2.30) is not necessarily the energy integral, i.e., it is not possible to identify *a priori* K^{ij} with g^{ij} and V with the potential U. Moreover choices other than (2.29) could be made: more generally we could choose a relation of the form

$$2h_0 = \overset{a}{\beta}\underset{a}{c} + \overset{\alpha\beta}{\zeta_0}\underset{\alpha\beta}{cc} \tag{2.31}$$

(with $\overset{a}{\beta}$ and $\overset{\alpha\beta}{\zeta_a}$ constants), or, more simply, choose the following:

$$2h_0 = \underset{m}{c} \tag{2.32}$$

Furthermore, the requested solvability of (2.27) does not need to be postulated in the context of the HJ theory, in which only complete integrals have relevance. In fact the completeness condition (2.5) implies, through (2.27), that

$$\det\|\overset{b}{v_a}\| \cdot \det\|\overset{\beta}{\xi_\alpha}\| \neq 0$$

From relations (2.28) we realize that, at least when g^{ij} is positive definite [i.e., when (V_n, g) is Riemannian], the separability of the HJ equation implies the existence of m quadratic first integrals and $m - n$ linear ones, vertically independent and in involution. In this case a deeper discussion (see Reference 29, Section 6; Reference 17, and subsequent comments in Reference 30) shows that Problem 2.1 has the following solution.

Theorem 2.6. The HJ equation associated with (2.8) is separable in a Riemannian manifold (V_n, g) if and only if there exist functions $\overset{\alpha}{\xi}_a$, $\overset{b}{u}_a$ and $\overset{\alpha\beta}{\zeta}_a$ (respectively $\overset{\beta}{\xi}_\alpha$) of the coordinate x^a only (respectively x^α) such that the matrices $\|\overset{\beta}{\xi}_\alpha\|$ and $\|\overset{b}{u}_a\|$ are regular with inverses $\|\underset{\alpha}{\xi}^{\beta}\|$ and $\|\underset{b}{u}^{a}\|$, so that the metric tensor g assumes the local expressions

$$
\begin{aligned}
g^{aa} &= \underset{m}{u}^{a} \\
g^{ab} &= 0 \qquad (a \neq b) \\
g^{a\alpha} &= -\underset{\beta}{\xi}^{\alpha}\overset{\beta}{\xi}_a \underset{m}{u}^{a} \qquad (a \text{ n.s.}) \\
g^{\alpha\beta} &= \underset{\gamma}{\xi}^{\alpha}\underset{\delta}{\xi}^{\beta}\underset{m}{u}^{a}(\overset{\gamma\delta}{\zeta}_a + \overset{\gamma}{\xi}_a\overset{\delta}{\xi}_a)
\end{aligned}
\tag{2.33}
$$

with $a, b = 1, \ldots, m$; $\alpha, \beta, \gamma, \delta = m+1, \ldots, n$.

For the sake of completeness, we remark that when considering the Hamiltonians (2.21) the conditions (2.33) should be implemented by the following conditions on U and l^i:

$$U = U_a \underset{m}{u}^{a} \tag{2.34}$$

$$l^a = \chi_a \underset{m}{u}^{a} \qquad (a \text{ n.s.}) \tag{2.35}$$

$$l^\alpha = \underset{\beta}{\xi}^{\alpha}\overset{\beta}{\chi}_a \underset{m}{u}^{a}$$

where $\overset{\beta}{\chi}_a$, χ_a are functions of x^a only.

We may observe that equations (2.33) can be formally obtained by using (2.28) and (2.32) and taking $K^{ij} = g^{ij}$ and $V = U$ in the corresponding equation (2.30). Equations (2.33) appear as particular cases of the solutions given in References 21, 28, and 31; however, we should stress that *whatever solution is given to Problem* 2.1 (as, e.g., the apparently more general one given in Reference 17, which involves an arbitrarily large number of arbitrary functions of a single variable) *it can be reduced to a minimal form which is nothing but* (2.33), as proved in Reference 29, Section 6.

Before turning to Problem 2.2 it will be interesting to briefly recall the essential steps of the proof of Theorem 2.5 given by Dall'Acqua. In fact, analogous techniques have been used by Iarov-Iarovoi to deal with *t*-

dependent Hamiltonians and by Woodhouse to develop his theory of "separable systems,"[2] which we shall review in Section 2.3.

Let $x_0 \in V_n$ be a point; let $\mathscr{U}$ be a separable chart around x_0, with coordinates (x_i) such that $x^i(x_0) = 0$, $\forall i$. Let us consider the one-dimensional submanifolds $\gamma_i = \{y \in \mathscr{U} | x^j(y) = 0, \forall j \neq i\}$ (i.e., the ranges of the coordinates curves x^i). We define functions u_i and $g_i^{jl}(x^i)$ as follows:

$$u_i = U|\gamma_i, \qquad g_i^{jl} = g^{jl}|\gamma_i \tag{2.36}$$

and constants c_k and g_0^{ij} by

$$c_k = \pi_k(x_0), \qquad g_0^{ij} = g^{ij}(x_0) \tag{2.37}$$

Let us restrict to each γ_i the HJ equation. From the hypothesis of separability we infer

$$g_i^{ii}(\pi_i)^2 - 2f_i\pi_i + \Phi_i - u_i = h \tag{2.38}$$

where

$$f_i = -\sum_{j\neq i} g_i^{ij}c_j, \qquad \Phi_i = \sum_{j,l\neq i} g_i^{jl}c_jc_l \tag{2.39}$$

Taking into account the distinction between first- and second-class coordinates, equation (2.38) can be further elaborated by means of Levi-Civita's conditions (2.25), (2.26), which impose restrictions on the functions g_i^{il}. The reasoning of Dall'Acqua applies in full generality only to the strictly Riemannian case, since the explicit hypothesis $g^{aa} \neq 0$ is made; moreover, it seems rather difficult to develop analogous methods when $g^{aa} = 0$ for at least one index a.

As a counterpart of the "indirect" method described above a "direct" approach to Problem 2.1 has been given in 1936 by Agostinelli,[31] who found the general solution for the case of orthogonal separability structures in Riemannian manifolds (cf. References 32 and 33). Another direct method has been developed by one of us in 1975, based on the concept of separability structures. This method, which amounts to successive changes of coordinates that preserve the separability structure and step by step simplify the problem until Stäckel's Theorem 2.2 becomes applicable, leads at the same time to solutions of both Problems 2.1 and 2.2, with the following theorems.

Theorem 2.7.[17,29,34,35] A Riemannian manifold (V_n, g) admits a $\mathscr{S}_r$ structure if and only if there exist (locally) r K vectors $\underset{\alpha}{X}$ and $n-r$ K tensors $\underset{a}{K}$ which are independent and commuting, i.e.,

$$[\underset{\alpha}{X}, \underset{\beta}{X}] = 0, \qquad [\underset{\alpha}{X}, \underset{a}{K}] = 0, \qquad [\underset{a}{K}, \underset{b}{K}] = 0 \qquad (\forall a, b, \alpha, \beta) \tag{2.40}$$

and moreover there exist $n - r$ independent orthogonal common eigenvectors $\underset{a}{X}$ of the K tensors $\underset{a}{K}$ such that

$$[\underset{a}{X}, \underset{b}{X}] = 0, \qquad [\underset{a}{X}, \underset{\alpha}{X}] = 0, \qquad g(\underset{a}{X}, \underset{\alpha}{X}) = 0 \tag{2.41}$$

with $a, b = 1, \ldots, n - r$; $\alpha, \beta = n - r + 1, \ldots, n$.

Theorem 2.8.[34,35] A Riemannian manifold (V_n, g) which admits a $\mathscr{S}_r$ structure has locally two orthogonal foliations $\{W_r\}$ and $\{Z_{n-r}\}$, where in the induced metrics each integral submanifold W_r is flat and the isometric submanifolds Z_{n-r} have an orthogonal separability structure.

The foliation $\{W_r\}$ is actually the foliation of the integral submanifolds associated with the involutive distribution $\{\underset{\alpha}{X}\}$, while the foliation $\{Z_{n-r}\}$ is the complementary foliation associated with the involutive distribution $\{\underset{a}{X}\}$. The flatness of each W_r is an obvious consequence of the fact that the $\underset{\alpha}{X}$ are commuting K vectors which generate an Abelian r-parameter group of isometries acting on r-dimensional submanifolds. Theorem 2.8 suggests the following definition.

Definition 2.1. A separability structure $\mathscr{S}_r$ is said to be of class $\mathscr{S}_r^k$ $(0 \leq k \leq n - r)$ if the induced (orthogonal) separability structure on Z_{n-r} is of class $\mathscr{S}_k$.

Of course, Definition 2.1 does not exhaust the possibility of further classification: in fact the induced structure $\mathscr{S}_k$ on Z_{n-r} may be *a priori* any class $\mathscr{S}_k^{k'}$ (with $k' \leq n - r - k$), and so on. The finest classification can be hence given in terms of a finite sequence of integers, to define in an obvious way separability structures of class $\mathscr{S}_r^{(k_1,\ldots,k_p)}$.

We also notice that the n vectors $\{\underset{\alpha}{X}, \underset{a}{X}\}$ form a local basis which is the natural basis $\{\partial_i\}$ associated with certain quasinormal coordinates (y^i), to which we shall give the name of *normal separable coordinates*. The vectors $\underset{i}{X}$ are not uniquely determined: in fact the vectors $\underset{\alpha}{X}$ can be acted upon with an arbitrary matrix $A \in GL(n, \mathbb{R})$, while each of the vectors $\underset{a}{X}$ can be arbitrarily rescaled by a factor $\varphi(x^a)$. An analogous remark holds also for the K tensors $\underset{a}{K}$: in particular to each $\underset{a}{K}$ we can freely add linear $\mathbb{R}$ combinations of symmetric tensor products of the K vectors $\underset{\alpha}{X}$ and the metric G. From this remark it follows that one of the K tensors $\underset{a}{K}$ can always be chosen to be the metric.

Reverting to Problem 2.1, we observe that further simplifications can be obtained by using the freedom of choice among the coordinates belonging to a separability structure. For example, in any quasinormal separable coordinate system the general form (3.23) simplifies to

$$\begin{aligned} g^{aa} &= \underset{m}{u}^{a}, \qquad g^{ab} = 0 \qquad (a \neq b) \\ g^{a\alpha} &= -\overset{\alpha}{\xi}_a \underset{m}{u}^{a} \qquad (a \text{ n.s.}) \\ g^{\alpha\beta} &= (\overset{\alpha\beta}{\zeta}_a + \overset{\alpha}{\xi}_a \overset{\beta}{\xi}_b)\underset{m}{u}^{a} \end{aligned} \tag{2.42}$$

which amounts to putting $\underset{\alpha}{\xi}^{\beta} = \delta_{\alpha\beta}$ in (2.33). A further simplification is obtained by transforming to normal separable coordinates (y^i), so that (2.42) assume the form

$$\begin{aligned} g^{aa} &= \underset{m}{u}^{a}, \qquad g^{ai} = 0 \qquad (a \neq i) \\ g^{\alpha\beta} &= \overset{\alpha\beta}{\zeta}_a \underset{m}{u}^{a} \end{aligned} \tag{2.43}$$

In normal separable coordinates the K tensors $\underset{a}{K}$ appearing in Theorem 2.7 have the following components:

$$\begin{aligned} \underset{a}{K}^{bb} &= \underset{a}{u}^{b}, \qquad \underset{a}{K}^{bi} = 0 \qquad (b \neq i) \\ \underset{a}{K}^{\alpha\beta} &= \overset{\alpha\beta}{\zeta}_b \underset{a}{u}^{b} \end{aligned} \tag{2.44}$$

Components in general separable coordinates may be found in Reference 17.

For pseudo-Riemannian manifolds the theory of separability structures is not completely developed as for proper Riemannian manifolds. In fact, difficulties arise when for some index of the second class we have $g^{aa} = 0$. This corresponds to the cases in which the ignorable coordinates of a chart belonging to the given separability structure define a foliation of isotropic or coisotropic submanifolds. In this case the extension of Theorems 2.6, 2.7, and 2.8 is not immediate and, to our actual knowledge, it seems that some separability structures exist that do not fit into the general scheme above. These separability structures could be called *degenerate separability structures.* Those structures for which, in the pseudo-Riemannian case, Theorems 2.6–2.8 still hold will be called *regular separability structures.* However, we should remark that the distinction between degenerate and regular separability structures is not always directly reflected in the form of the metric tensor components. This fact is illustrated by the following example.[35]

Let (x^i) be coordinates in a pseudo-Riemannian manifold (V_n, g) such that the metric tensor components have the following form:

$$g^{aa} = \underset{m}{u}^a, \qquad g^{ai} = 0 \qquad (i \neq a)$$

$$g^{a'a'} = 0, \qquad g^{a'\alpha} = g^{aa}\zeta_a^{b'\alpha}\underset{b'}{\xi}^{a'}$$

$$g^{\alpha\beta} = g^{aa}\zeta_a^{\alpha\beta} + 2g^{a'(\alpha}\zeta_{a'}^{\ \beta)}$$

with $a = 1, \ldots, m$; $a', b' = m+1, \ldots, n-r$; $\alpha, \beta = n-r+1, \ldots, n$; where $(\underset{m}{u}^a)$ is the mth line of a matrix $\|\underset{b}{u}^a\|$ as in Theorem 2.6; $\|\underset{b'}{\xi}^{a'}\|$ is a regular matrix with inverse $\|\overset{b'}{\xi}_{a'}\|$; $\overset{b'}{\xi}_{a'}$, $\zeta_{a'}^{\beta}$ are functions of $x^{a'}$ only; $\zeta_a^{\alpha\beta}$, $\zeta_a^{a'\alpha}$ are functions of x^a only. It can be checked that the coordinates are separable. The r coordinates (x^a) are ignorable and moreover the vectors (∂_α) span submanifolds with degenerate induced metric. At first sight, this could be interpreted as the signal that the coordinates (x^i) belong to a degenerate separability structure. However, we can consider a new system of coordinates (y^i) defined by the equations

$$dy^a = dx^a, \qquad dy^{a'} = \overset{a'}{\xi}_{b'}\, dx^{b'}, \qquad dy^\alpha = dx^\alpha - \zeta_{a'}^\alpha\, dx^{a'}$$

With easy calculations we can see that (i) the coordinates (y^i) are separable and $\mathcal{S}$ compatible with the coordinates (x^i); (ii) the $n-m$ coordinates $(y^{a'}, y^\alpha)$ are ignorable; (iii) the coordinates (y^i) are normal separable coordinates of a regular separability structure. Then we conclude that the coordinates (y^i) belong actually to a regular separability structure.

To our knowledge the problem of characterizing and dealing with degenerate separability structures is still open. For this reason in the following we shall consider only regular separability structures. For the sake of simplicity the adjective "regular" shall be omitted.

2.3. Theory of Separable Systems

In this section we shall discuss in detail the theory of separable systems developed by Woodhouse,[2] which provides a complementary viewpoint on the problem of separability of the HJ equation for geodesics and which is based essentially on the study of coordinate systems (x^i) in which a single coordinate separates.

We shall first recall the basic definitions given in Reference 2, providing also their interpretation in local coordinates. Let α be a closed 1-form and X

a vector field in (V_n, g), such that $\alpha(X) = 1$. We define the *projections relative to* (α, X) of a vector field Y and a 1-form β by

$$\perp Y = Y - \alpha(Y)X \tag{2.45}$$

$$\perp \beta = \beta - \beta(X)\alpha \tag{2.46}$$

These definitions can be easily extended to any tensor T [of order (p, q)] as follows:

$$\perp T(X_1, \ldots, X_q, \alpha_1, \ldots, \alpha_p) = T(\perp X_1, \ldots, \perp X_q, \perp\alpha_1, \ldots, \perp\alpha_p), \qquad \forall X_1, \ldots, X_q, \alpha_1, \ldots, \alpha_p \tag{2.47}$$

Let W be a complete (local) integral of the HJ equation of geodesics in (V_n, g). The integral W is said to be *separable with respect to the pair* (α, X) if

$$L_X(\perp dW) = 0 \tag{2.48}$$

where L_X denotes the Lie derivative along X. The integral W is said to be *trivially separable* if, moreover,

$$L_X(dW) = 0 \tag{2.49}$$

In these cases the pair (α, X) is called a *separable* (respectively, *trivially separable*) *system*.† A separable system (α, X) is said to be *orthogonal* if $\alpha \cdot G = \alpha_i g^{ij}$ and X are linearly dependent.

A coordinate system (x^i) is said to be *adapted* to the separable system (α, X) if

$$\alpha = dx^1 \quad \text{and} \quad X = \partial_1 \tag{2.50}$$

With respect to adapted coordinates (x^i), equations (2.45)–(2.49) assume the more familiar form:

$$\perp Y = Y^\alpha\, \partial_\alpha, \qquad \alpha = 2, \ldots, n‡ \tag{2.51}$$

$$\perp \beta = \beta_\alpha\, dx^\alpha \tag{2.52}$$

$$\perp T = T^{\alpha_1 \cdots \alpha_p}_{\beta_1 \cdots \beta_q}\, \partial_{\alpha_1} \otimes \cdots \otimes \partial_{\alpha_p} \otimes dx^{\beta_1} \otimes \cdots \otimes dx^{\beta_q} \tag{2.53}$$

$$L_{\partial_1}(\partial_\alpha W\, dx^\alpha) = 0 \Leftrightarrow \frac{\partial^2 W}{\partial x^1\, \partial x^\alpha} = 0 \tag{2.54}$$

$$L_{\partial_1}(dW) = 0 \Leftrightarrow \frac{\partial^2 W}{\partial x^1\, \partial x^i} = 0, \qquad i = 1, \ldots, n‡ \tag{2.55}$$

† Woodhouse's definition is more restrictive, since in Reference 2 $G(\alpha, \alpha) \neq 0$ is also assumed. According to our subsequent discussion, this amounts to imposing $g^{11} \neq 0$ in a suitable coordinate system (x^i) in which x^1 separates. This assumption is made in Reference 2 due to the already recalled fact that Woodhouse's analysis is analogous to Dall'Acqua's method.

‡ Throughout this section Latin indices run from 1 to n, while Greek indices run from 2 to n.

Condition (2.54) is obviously equivalent to the requirement of separability of x^1, namely,

$$W(x^i) = W_1(x^1) + W^*(x^\alpha) \tag{2.56}$$

while (2.55) amounts to requiring that x^1 separates linearly (i.e., x^1 is an ignorable coordinate, or equivalently that $X = \partial_1$ is a K vector):

$$W(x^i) = c_1 x^1 + W^*(x^\alpha) \tag{2.57}$$

Orthogonal separable systems are characterized by $g^{1\alpha} = 0$.

Let (dx^1, ∂_1) be a separable system and let S_τ be an integral surface of dx^1 (i.e., a hypersurface of local equation $x^1 = \tau$). Let $W = W(x^i; c_i)$ be a complete integral of the HJ equation, which separates with respect to (dx^1, ∂_1). Define a function $\bar{k}(\tau, c_i)$ by

$$\bar{k}(\tau, c_i) = \bar{k}_\tau(c_i) = \partial_1 W_1(x^1, c_i)|S_\tau \tag{2.58}$$

Using the HJ theory the n constants of motion c_i can be expressed as functions of (x^j, p_j), so that the following first integral k_τ is obtained:

$$k_\tau(x^j, p_j) = \bar{k}_\tau(c_i(x^j, p_j)) \neq 0 \tag{2.59}$$

A method of calculating k_τ without knowing W is the following, as discussed in Reference 2, pp. 26–27 under the hypothesis $g^{11} \neq 0$. Let us restrict the HJ equation to the hypersurface S_τ taking into account equation (2.56):

$$g^{11}(\tau, x^\gamma)[\bar{k}(c_i)]^2 + 2\bar{k}_\tau(c_i) g^{\alpha 1}(\tau, x^\gamma)\, \partial_\alpha W^* + g^{\alpha\beta}(\tau; x^\gamma)\, \partial_\alpha W^*\, \partial_\beta W^* = 2h \tag{2.60}$$

But, using again the HJ equation $g^{ij}\, \partial_i W\, \partial_j W = 2h$, equation (2.60) can be equivalently written as follows:

$$[\bar{k}_\tau(c_i)]^2 + B_\tau^\alpha \bar{k}_\tau(c_i)\, \partial_\alpha W^* + A_\tau^{ij}\, \partial_i W\, \partial_j W = 0 \tag{2.61}$$

where:

$$B_\tau^\alpha = 2\frac{g^{\alpha 1}(\tau, x^\gamma)}{g^{11}(\tau, x^\gamma)}, \qquad B_\tau^1 \equiv 0 \tag{2.62a}$$

$$A_\tau^{ij} = \frac{1}{g^{11}(\tau, x^\gamma)}[-g^{ij}(x^k) + \delta_\alpha^i \delta_\beta^j g^{\alpha\beta}(\tau, x^\gamma)] \tag{2.62b}$$

If in (2.61) we express the constants c_i as functions of (x^j, p_j), then $\bar{k}_\tau$ is replaced by $k_\tau(x^j, p_j)$ [according to equation (2.59)] and we obtain the following identity†:

$$[k_\tau(x^i, p_i)]^2 + k_\tau(x^i, p_i) B_\tau^\alpha p_\alpha + A_\tau^{ij} p_i p_j = 0 \tag{2.63}$$

† By the Hamilton–Jacobi theorem, $\partial_i W|c(x^j, p_j)$ is equal to p_j.

For the sake of simplicity we put

$$B_\tau^\alpha p_\alpha = b_\tau, \qquad A_\tau^{ij} p_i p_j = a_\tau \tag{2.64}$$

and we rewrite (2.63) more economically as follows†:

$$k_\tau^2 + b_\tau k_\tau + a_\tau = 0 \tag{2.65}$$

The following discussion will be slightly different from the presentation given in Reference 2. The identity (2.65) suggests that we consider the following quadratic algebraic equation:

$$f^2 + b_\tau f + a_\tau = 0 \tag{2.66}$$

in the unknown $f: T^* V_n \to \mathbb{R}$. Equation (2.66) has two real roots $\underset{1}{f_\tau}$ and $\underset{2}{f_\tau}$ (possibly coinciding), since the identity (2.65) assures that one of them is the (real) first integral $k_\tau: T^* V_n \to \mathbb{R}$. For convenience, we shall take $k_\tau = \underset{1}{f_\tau}$. Now we must distinguish two cases: (i) $\underset{2}{f_\tau}$ is also a first integral (in particular if $\underset{1}{f_\tau} = \underset{2}{f_\tau}$); (ii) only $\underset{1}{f_\tau}$ ($\neq \underset{2}{f_\tau}$) is a first integral. Let us take the Poisson brackets of (2.66) with the Hamiltonian H, to obtain the following equation:

$$2f\{f, H\} + b_\tau\{f, H\} + f\{b_\tau, H\} + \{a_\tau, H\} = 0 \tag{2.67}$$

Since $\underset{1}{f_\tau} = k_\tau$ is a first integral, equation (2.67) gives the identity

$$\{a_\tau, H\} = -\underset{1}{f_\tau}\{b, H\} \tag{2.68}$$

Inserting (2.68) into (2.67) and performing some simple algebra, we get

$$\{b_\tau, H\}(f - f_\tau) + \{f, H\}(2f + b_\tau) = 0 \tag{2.69}$$

Case (*i*). Since $\underset{2}{f_\tau}$ is a first integral, replacing f by $\underset{2}{f_\tau}$ in (2.69) gives

$$\{b_\tau, H\}(\underset{2}{f_\tau} - \underset{1}{f_\tau}) = 0 \tag{2.70}$$

If $\underset{2}{f_\tau} \neq \underset{1}{f_\tau}$, (2.70) gives immediately $\{b_\tau, H\} = 0$. If on the contrary $\underset{2}{f_\tau}$ and $\underset{1}{f_\tau}$ coincide, equation (2.66) tells us that the common value is $-\frac{1}{2}b_\tau$. However, $\underset{1}{f_\tau}$ is a first integral and $\{b_\tau, H\} = 0$ follows as well. Hence in any case from (2.68) it follows that

$$\{a_\tau, H\} = \{b_\tau, H\} \tag{2.71}$$

† We stress that our equations (2.60) and (2.63) are, respectively, equations (4.29) and (4.34) of Reference 2, p. 26.

Case (*ii*). Since $\underset{2}{f}_\tau$ is not a first integral we have $\{\underset{2}{f}_\tau, H\} \neq 0$. Since $\underset{2}{f}_\tau \neq \underset{1}{f}_\tau$ we have $\underset{2}{f}_\tau \neq -\frac{1}{2}b_\tau$. From (2.69) we obtain by replacing f with $\underset{2}{f}_\tau$: $\{b_\tau, H\}(\underset{1}{f}_\tau - \underset{2}{f}_\tau) \neq 0$. Taking into account (2.68) it follows that

$$\{a_\tau, H\} \neq 0, \qquad \{b_\tau, H\} \neq 0 \tag{2.72}$$

From (2.68), under hypothesis (2.72), we obtain

$$\underset{1}{f}_\tau = -\frac{\{a_\tau, H\}}{\{b_\tau, H\}} \tag{2.73}$$

which, after insertion into (2.66), yields

$$\{a_\tau, H\}^2 - b_\tau\{a_\tau, H\}\{b_\tau, H\} + a_\tau\{b_\tau, H\}^2 = 0 \tag{2.74}$$

Identity (2.74) is a polynomial identity in the variables (p_j), since a_τ, b_τ, $\{a_\tau, H\}$, and $\{b_\tau, H\}$ are polynomials, respectively, of degrees 2, 1, 3, and 2. From (2.74) we easily infer that $\{b_\tau, H\}$ divides exactly $\{a_\tau, H\}$,† so that $\underset{1}{f}_\tau$ as given by (2.73) turns out to be a linear first integral.

From the discussion above we conclude that *in case* (*i*), *the functions* (2.64) *are two first integrals, one linear and one quadratic, to which there correspond respectively a K vector B_τ and a K tensor A_τ given by* (2.62); *in case* (*ii*) *neither a_τ nor b_τ are first integrals, but the quotient* (2.73) *provides a linear first integral $\underset{1}{f}_\tau = k_\tau$, which corresponds to a K vector X_τ.* In a chart (x^i) adapted to a separable system (dx^1, ∂_1), the closed 1-form dx^1 generates a 1-parameter family of hypersurfaces, which, without loss of generality, can be taken to be $\{S_\tau | \tau \in]-\varepsilon, \varepsilon[\}$, $\varepsilon > 0$. Hence, for any τ the existence of one separable coordinate allows us to construct at least one first integral.

In Reference 2 the following two cases are considered: Case (1): for any τ case (i) above holds. Case (2): for any τ case (ii) above holds.‡

† If $\{b_\tau, H\}$ is irreducible the statement is trivial. If $\{b_\tau, H\}$ decomposes into two linear factors (say b_1 and b_2), (2.74) gives $\{a_\tau, H\}(\{a_\tau, H\} - b_\tau b_1 b_2) = -a_\tau b_1{}^2 b_2{}^2$ which implies that both b_1 and b_2 divide $\{a_\tau, H\}$.

‡ These two cases do not cover all the possibilities. In fact the condition (2.72) which characterizes case (ii) picks up an open subset of the interval $]-\varepsilon, \varepsilon[$, whose complement in $]-\varepsilon, \varepsilon[$ is the closed set in which condition (2.71) characterizing case (i) holds. This closed set can be a closed interval, an isolated point, a sequence with a limit point, or a finite union of such objects. Even restricting ourselves to those three simplest cases, we observe that the first leads to case (2) above; the second can be handled by eliminating the point and in the complement case (1) holds; the third, although being somewhat pathological, does not seem, however, to be *a priori* impossible. Hence the proof of Theorem 4.1 in Reference 2 suffers from this limitation: it excludes *a priori* the third possibility, which unfortunately leads to a difficult discussion and therefore we are obliged to avoid it. Limitations of this kind (which appear also in the theory of separability structures in pseudo-Riemannian manifolds, see Reference 35) do not affect the practical importance and the interest of the results obtained in Reference 2.

Case (1). We can reduce this case to the case of an orthogonal separable system. In fact the K vector $-\frac{1}{2}B_\tau$ is the orthogonal projection of ∂_1 (through the metric g) onto the hypersurface S_τ. Since $\alpha(B_\tau) = 0$, the vector

$$Z_\tau = \partial_1 + \tfrac{1}{2}B_\tau \tag{2.75}$$

satisfies $\alpha(Z_\tau) = 1$ and moreover $[\partial_1, Z_\tau] = 0$. A direct calculation (or an application of Reference 2, Corollary 2, p. 25, with $b = \frac{1}{2}$) shows that (α, Z_τ) is an orthogonal separable system.

The question now arises: how many independent quadratic first integrals of the kind a_τ are generated by the existence of a single separable coordinate in case (1)? Since the reduction to the case of orthogonal separable systems can be performed as above, the problem is solved by the following theorem of Woodhouse (Reference 2, p. 29).

Theorem 2.9. If (α, X) is an orthogonal separable system such that case (1) holds, in the set $\{G; A_\tau, \tau \in]-\varepsilon, \varepsilon[\}$ the maximum number of independent elements is equal to $1 + q(X)$, where $q(X)$ is the conformal rigidity of X.

[The number $q(X)$ is defined as the maximum integer m such that G, $X \otimes X$, $L_X G$, $L_X^2 G, \ldots, L_X^{m+1} G$ are linearly independent over the set of differentiable functions in V_n.]

Proof. Let us consider adapted coordinates (x^i) such that $\alpha = dx^1$ and $X = \partial_1$. We shall give a proof under the hypothesis that the functions $g^{\alpha\beta}(x^k)$ are analytic. Let us expand the tensor $\perp G = g^{\alpha\beta}\, \partial_\alpha \otimes \partial_\beta$ in powers of x^1, so that we obtain from (2.62b)

$$A_\tau^{ij} = \frac{1}{g^{11}(\tau; x^\gamma)}\left[-g^{ij}(x^k) + \sum_{r=0}^{\infty} \frac{\tau^r}{r!}\, \partial_1{}^r g^{\alpha/\beta}(x^1; x^\gamma)\bigg|_{x^1=\tau} \cdot \delta_\alpha^i \delta_\beta^j\right] \tag{2.76}$$

Taking into account the identity

$$L_X(\perp G) = L_X G + [L_X G(\alpha, \alpha)]X \otimes X - 2X \cap (L_X G \cdot \alpha) \tag{2.77}$$

(where $\cap$ denotes the symmetrized tensor product), which reduces to

$$L_X(\perp G) = L_X G + [L_X G(\alpha, \alpha)]X \otimes X \tag{2.78}$$

when (α, X) is orthogonal, equation (2.76) reads as follows:

$$A_\tau^{ij} = \frac{1}{g^{11}(\tau, x^\gamma)}\left[-G + \sum_{r=0}^{\infty} \frac{\tau^2}{r!} L_X^r G\bigg|_{S_\tau} + \beta(\tau) X \otimes X\right] \tag{2.79}$$

where $\beta(\tau)$ is a suitable function of τ. If $q(X)$ is the conformal rigidity of X, then the Lie derivatives $L_X^m G$ with $m \geq q(X) + 1$ are expressible as linear combinations of the lower order ones, which proves the theorem. □

When (α, X) is not orthogonal the number $q(X) + 1$ is only a lower bound: to the $1 + q(X)$ independent K tensors A_τ^{ij} we must add a set of generators of the minimal commutative subalgebra containing all the symmetric products $B_{\tau_1} \cap B_{\tau_2}$ $(\tau_1, \tau_2 \in]-\varepsilon, \varepsilon[)$.

In Reference 2 it is also proved that the K tensors A_τ^{ij} $(\tau \in]-\varepsilon, \varepsilon[)$ commute and have the common eigenform $\alpha = dx^1$.

Case (2). The discussion of case (2) is exhaustively given in Reference 2, p. 30 and concludes that the system (α, X) is trivially separable.

From the results of Woodhouse's analysis it turns out that *if in a coordinate system* (x^i) *one coordinate (say* x^1*) is separable, it is adapted either to a K vector (i.e.,* x^1 *is ignorable) or to an eigenform of a K tensor.* This is in perfect agreement with the theory of (complete) separability structures developed in Sections 2.1 and 2.2; we remark that the coordinate changes that are implicitly performed in Woodhouse's theory are essentially equivalent to our transformations to normal separable coordinates. Woodhouse's theory provides a very powerful method of constructing first integrals of geodesics from the mere knowledge of a *single* separable coordinate. In certain cases (e.g., the applications to general relativity; see later and Reference 2, Section 6) it may also provide an alternative approach to separability theory. The following problems are left open.

(I) extend the discussion to the case $g^{11} = 0$, which has relevance when g is not positive definite.

(II) suppose that (V_n, g) admits a separability structure $\mathcal{S}_r$. Choose a representative coordinate system (x^i). Pick one of the coordinates (x^i) and associate to it a maximal number of K vectors and K tensors obtained as in this section. What are the relations between these objects and the analogous ones that define the given separability structure? A partial answer, valid for orthogonal systems, is given by Theorem 4.2 of Reference 2 and its corollary.

2.4. Separability of Second-Order Equations

Since the separability theory of the HJ equation for the geodesics of a pseudo-Riemannian manifold (V_n, g) is essentially a theory concerning a suitable geometrical characterization of the metric tensor g, it is not surprising that the existence of a separability structure in a (V_n, g) has a deep relevance for analogous separability properties of other second-order equations associated with g, besides the HJ equation itself. Among these equations, a particularly important example is the family of equations

$$\Delta\Psi + k\Psi = 0 \tag{2.80}$$

(where $\Delta\Psi = g^{ij}\nabla_i\nabla_j\Psi$), which, according to the signature of the metric g, are known as Laplace, Helmoltz, D'Alembert, Klein–Gordon (or wave) equations. Moreover, great interest is also given to the Schrödinger equation:

$$\Delta\Psi + (E - U)\Psi = 0 \tag{2.81}$$

(U being a potential), which for U = const reduces to equation (2.80).

By complete separability of equation (2.81) or (2.80) we mean, as usual, the existence of a solution Ψ depending in an essential way on n arbitrary constants, which with respect to certain coordinates (x^i) separates as a product $\Psi = \Pi_{r=1}^{n} \Psi_i(x^i)$. Just as for the theory of separability of the HJ equation, the study of analogous properties for equation (2.81) was initially done in the restricted case of orthogonal coordinate systems. The first results in this direction are, to our knowledge, those obtained in 1927 by Robertson,[36] who proved the following theorem.

Theorem 2.10. Equation (2.8) is separable in an orthogonal coordinate system (x^i) if and only if the components g^{ii} and the potential U satisfy the requirements of Stäckel's Theorem 2.2 with the additional condition

$$(\det \|\overset{i}{\varphi}_j\|)^2 = \prod_{i=1}^{n} f(x^i)\overset{i}{\underset{n}{\varphi}} \tag{2.82}$$

where the f_i are suitable functions.

Eisenhart later gave the geometrical counterpart of Theorem 2.10 by proving the following.

Theorem 2.11.[20] Condition (2.82) is equivalent to the following:

$$R_{ij} = 0, \qquad \forall i \neq j \tag{2.83}$$

Several particular cases (Euclidean 3-space, conformally Euclidean 3-spaces) were investigated in References 23, 37, and 38. In Reference 39 Forbat showed that the Schrödinger equation is separable in any Liouville manifold (V_n, g) with positive definite metric. General classes of solutions to conditions (2.82) were studied by Agostinelli in Reference 40 and later extended to nonorthogonal cases in Reference 41.

In the framework of the theory of separability structures, the following result holds.

Theorem 2.12. In a Riemannian manifold equation (2.80) is separable if and only if the manifold admits a separability structure in which the K vectors $\underset{a}{X}$ are eigenvectors of the Ricci tensor.

As far as equation (2.81) is concerned, in analogy with what already was encountered for the HJ equation with a potential U, additional conditions on U have to be imposed along with the conditions of Theorem 2.12: it is noteworthy that they coincide with conditions (2.34) (see Reference 42, pp. 28–29). Theorem 2.12 still holds for regular separability structures in pseudo-Riemannian manifolds. Then we have the following corollary.

Corollary 2.1. In a Lorentzian space–time (V_4, g) which is a solution of Einstein vacuum equations the existence of a regular separability structure implies the separability of the wave equation (2.80).

In the proof of Theorem 2.12 given in Reference 42 it is shown that, besides the existence of a separability structure, the condition that should be imposed for separability of (2.80) is given in normal separable coordinates by

$$\partial_b(g_{aa}C^a) = 0 \qquad (\forall a \neq b, a \text{ n.s.}) \tag{2.84}$$

where:

$$C^a = g^{ij}\Gamma^a_{ij} \tag{2.85}$$

The statement follows from the important identity (again in normal coordinates)†

$$\begin{aligned} R_{ab} &= \tfrac{3}{4}\Big(\sum_{c\neq a,b} \partial_a\partial_b \log g^{cc} + \partial_a\partial_b \log \det \|g^{\alpha\beta}\|\Big) \\ &= \tfrac{2}{3}\,\partial_b(g_{aa}C^a) \qquad (\forall a \neq b, a \text{ n.s.}) \end{aligned} \tag{2.86}$$

The separability of the Schrödinger equation (2.81) was also independently investigated by Havas,[28] who gave a set of necessary and sufficient conditions on g^{ij} and U. It can be proved with some simple algebra that Havas' conditions (A) (Reference 28, p. 1466) reduce in normal coordinates to

$$\det \|g_{ij}\| = (\det \|\overset{b}{u}_a\|)^2 \cdot \prod_{a=1}^{n-r} f_a(x^a) \tag{2.87}$$

(where f_a are suitable functions), which can be shown to be equivalent with conditions (2.84). Of course conditions (2.87) generalize (2.82).

The classification and determination of all coordinate systems in which the HJ equation [or more general equations like (2.81)] are separable is a different problem, rather of analytic than geometric character. An extensive investigation in this direction has been recently undertaken with valuable results by Winternitz, Boyer, Kalnins, Miller, and others, in the framework of Lie theory: see, e.g., Reference 43 and all references cited therein. In

† In equation (25) of Reference 42 a numerical factor is missing.

these papers other second-order equations are also considered. Obvious reasons of space do not allow us to discuss those results here.

For the sake of completeness we recall finally the recent review article (on separability on Riemannian manifolds) by Huaux[44] in which additional "historical" references are to be found, together with a discussion of problems in theoretical mechanics.

3. Separability Structures in Space–Time

In this part of the paper we shall discuss the role that the separability of the HJ equation plays in the context of general relativity. We shall briefly outline the main results concerning separable space–times and their geometrical properties, as well as certain other fields of research which have common interest in the problem of separability.

The first clear application of the problem to general relativity was when Carter[45] succeeded in integrating the geodesic equations for the Kerr metric[46] by means of separation of variables. Later, relying on certain assumptions suggested by the particular case of the Kerr metric, Carter discovered a whole family of vacuum and electrovac space–times which allow (complete) separability of the uncharged or charged geodesic equation.[8] Since Carter's paper, there have been contributions to the subject and we shall try here to reduce them to a common denominator which is (and cannot be something else) the theory of separability structures.

3.1. Separability Structures $\mathscr{S}_{n-2}$ and Carter's Separable Space–Times

Since separability structures are primarily classified through the number r of K vectors that they involve, with $0 \leq r \leq 4$, there are five essentially different types of separability structures $\mathscr{S}_r$ in a four-dimensional manifold (V_4, g). We shall devote our initial interest to the separability structures of class 2: they are mainly the ones investigated in Reference 8.

From the general theory it turns out that a great role is played by the number of K tensors. For this reason, this section will contain results referring to the more general case of a separability structure of class $n-2$ in a pseudo-Riemannian manifold (V_n, g) $(n \geq 3)$, which we recently discussed in Reference 30.

In Reference 30 we first pointed out that in normal separable coordinates (y^i) the metric tensor g reduces to its *canonical form*:

$$g^{aa} = \frac{\psi_a}{\varphi_1 + \varphi_2} \qquad (a = 1, 2) \tag{3.1}$$

$$g^{ai} = 0 \qquad (a \neq i) \tag{3.2}$$

$$g^{\alpha\beta} = \frac{1}{\varphi_1 + \varphi_2}(\zeta_1^{\alpha\beta}\psi_1 + \zeta_2^{\alpha\beta}\psi_2) \qquad (\alpha, \beta = 3, \ldots, n) \tag{3.3}$$

Here the coordinates y^α are ignorable; ψ_a, φ_a, and $\zeta_a^{\alpha\beta}$ are functions of the coordinate y^a only, subject to the requirement that $\det \|g^{ij}\| \neq 0$. Once the canonical form above is known, the K tensor K, which together with g and the K vectors $\underset{\alpha}{X} \equiv \partial/\partial y^\alpha$ defines the separability structure, may be algebraically computed (in normal coordinates), and is given by

$$K^{11} = \frac{\varphi_2\psi_1}{\varphi_1 + \varphi_2}, \qquad K^{22} = \frac{-\varphi_1\psi_2}{\varphi_1 + \varphi_2} \tag{3.4}$$

$$K^{ai} = 0 \qquad (a \neq i) \tag{3.5}$$

$$K^{\alpha\beta} = \frac{1}{\varphi_1 + \varphi_2}(\zeta_1^{\alpha\beta}\psi_1\varphi_2 - \zeta_2^{\alpha\beta}\psi_2\varphi_1) \tag{3.6}$$

The arbitrary functions ψ_a may be freely rescaled without affecting the separability structure (e.g., by taking ${\psi_1}^2 = {\psi_2}^2 = 1$; see Reference 30).

Moreover a manifold (V_n, g) with a $\mathscr{S}_{n-2}$ structure admits (according to Theorem 2.8) two orthogonal local foliations $\{W_{n-2}\}$ and $\{Z_2\}$, where W_{n-2} are flat submanifolds of codimension two of V_n and Z_2 are two-dimensional isometric submanifolds of V_n, with an orthogonal separability structure in the induced metric $g|Z_2$. This induced metric is simply given by (3.1) and it appears in the classical Liouville form.[12] In Reference 30 the following has been proved.

Proposition 3.1. A separability structure $\mathscr{S}_{n-2}$ in (V_n, g) is of class $\mathscr{S}_{n-2}^k$ $(k = 0, 1, 2)$ if and only if among the two functions φ_1 and φ_2 exactly k are constant.†

By taking $n = 4$, the above gives a classification of four-dimensional metrics with a $\mathscr{S}_2$ structure. Further on $n = 4$ will be assumed.

In his 1968 paper,[8] Carter investigated space–times (V_4, g) in which the HJ equation for (charged) geodesics is separable under suitable hypotheses. In this section we shall remain (accordingly to the spirit of this paper) in the framework of uncharged geodesics and vacuum solutions of Einstein field equations. In his analysis, Carter relied on three main assumptions: (I) space–time admits two commuting K vectors (or equivalently an Abelian two-parameter group of isometries); (II) the symmetry group is invertible (see Reference 47) with non-null surfaces of transitivity; (III) the HJ equation is separable (in a coordinate system adapted to the group of symmetries) after multiplication of the whole equation by a separating factor

† See Definition 2.1.

U.† A bit of discussion should be made on these hypotheses, to better understand their relevance, their meaning, and their relations with separability structures. In Reference 8 it was noticed that (according to References 47 and 48) hypothesis (II) does not impose restrictions on stationary axisymmetric Einstein–Maxwell space–times. However, hypothesis (III) requires implicitly that (V_4, g) admit a $\mathcal{S}_2$ structure (it in fact requires the existence of a separated complete integral of the form given in the † footnote immediately above). Hence, the theory of separability structures tells us that both (I) and (II) are contained in (III) (see Theorem 2.7) and have to be considered as redundant. Moreover separability is not *a priori* equivalent to a "routine prescription" to explicitly separate the HJ equation (cf. Section 2.1). In particular it is not *a priori* true that separate integrals can be always obtained after multiplying the HJ equation by a suitable separating factor. For this reason it could be argued that (III) is more restrictive than the requirement of $\mathcal{S}_2$ separability.‡ However, the canonical form (3.1)–(3.3) assures us that the separating factor U always exists in the form

$$U(y^1, y^2) = \varphi_1(y^1) + \varphi_2(y^2)$$

for any $\mathcal{S}_{n-2}$ structure in any (V_n, g). Hence, *the above three hypotheses are collectively equivalent to the requirement that* (V_4, g) *admits a* $\mathcal{S}_2$ *separability structure.*

By relying on them, Carter deduced in fact the form of the metric tensor which is given by equation (48) of Reference 8. It is not hard to see that Carter's form (48) is actually equivalent to the canonical form above. Carter then proceeded by imposing other conditions and finally obtained a whole class of vacuum and electrovac space–time (possessing a $\mathcal{S}_2$ structure): we shall in a moment come to discuss these further conditions. Carter's space–times fall into three main families $[A]$, $[\tilde{B}]$, and $[D]$ (which are listed in Reference 8, p. 282). We easily understand that these three families correspond to the three possible structures $\mathcal{S}_2^k$, the correspondence being given by $[A] \leftrightarrow \mathcal{S}_2^0$, $[\tilde{B}] \leftrightarrow \mathcal{S}_2^1$, and $[D] \leftrightarrow \mathcal{S}_2^2$. We stress that this classification is not a consequence of Einstein equations but a direct consequence of $\mathcal{S}_2$ separability.

Carter's condition (IIIS) was the requirement that also the (charged) Schrödinger equation is separable (in the coordinate system which allows separability of the HJ equation). Recalling Theorem 2.12 and Corollary 2.1

† This means that, choosing coordinates (y^i) with y^1, y^2 ignorable, a complete separate integral of the form $W = c_1 y^1 + c_2 y^2 + W_3(y^3) + W_4(y^4)$ has to be found by substituting W into the HJ equation.

‡ This argument was in fact raised by Matravers in Reference 49, who did not realize that Carter's conditions are in fact equivalent to separability (in our $\mathcal{S}_2$ sense); Cf. also Reference 50.

we know that (a) for Einstein vacuum solutions Schrödinger separability is equivalent to HJ separability; (b) for nonvacuum solutions Schrödinger separability is equivalent to HJ separability if, moreover (in normal coordinates),

$$R_{12} = 0 \tag{3.7}$$

In Reference 8 Schrödinger separability was proven to be equivalent to a certain condition on the metric {equation (70) of [8]}, which in terms of the canonical form (6.1)–(6.3) can be stated as

$$\det \|\zeta_1^{\alpha\beta}\psi_1 + \zeta_2^{\alpha\beta}\psi_2\| = (\varphi_1 + \varphi_2)^2\theta_1\theta_2 \tag{3.8}$$

where θ_1 and θ_2 are suitable functions of y^1 and y^2, respectively. The above facts (a) and (b) were independently realized by several authors, who proved them by relying on particular four-dimensional versions of the proof given in Reference 42.† It is rather important to notice that $\mathscr{S}_2$ separability allows one of Einstein's vacuum equations to become a purely algebraic one.

Carter's hypothesis (IV) was a rather complicated one, and it was suggested by the charged orbit case. It turns out, however, to be equivalent to the conditions

$$\det \|\zeta_a^{\alpha\beta}\| = 0 \qquad (a = 1, 2) \tag{3.9}$$

(see References 8, p. 289, and 30). These are rather simplifying conditions which definitely make the subsequent study of Einstein equations easier: they pick out a subfamily of space–times known as *Carter's separable space–times*. We do not know if (3.9) is a true restriction to $\mathscr{S}_2$ separability of solutions to Einstein equations, i.e., whether $\mathscr{S}_2$ separable Einstein space–times with $\det \|\zeta_a^{\alpha\beta}\| \neq 0$ exist (for at least one index a).‡ The geometrical meaning of (3.9) is given by the following.

Proposition 3.2. In an $\mathscr{S}_2$ separable four-dimensional manifold (V_4, g) the metric tensor g (written in normal coordinates) satisfies conditions (6.9) if and only if the Segre characteristic of the K tensor (3.4)–(3.6) is [(11)(11)], i.e., if K^{ij} has two double nonvanishing eigenvalues.

Proof. From (3.1)–(3.6) it follows that

$$\begin{aligned}\det \|K^{ij} - \lambda g^{ij}\| = {} & -\psi_1\psi_2(\varphi_1 + \lambda)(\varphi_2 - \lambda) \\ & \times \{\psi_1^2 \det \|\zeta_1^{\alpha\beta}\|(\varphi_2 - \lambda)^2 + \psi_2^2 \det \|\zeta_2^{\alpha\beta}\|(\varphi_1 + \lambda)^2 \\ & - \psi_1\psi_2[\zeta_1^{33}\zeta_2^{44} + \zeta_1^{44}\zeta_2^{33} - 2\zeta_1^{34}\zeta_2^{34}](\varphi_1 + \lambda)(\varphi_2 - \lambda)\}\end{aligned} \tag{3.10}$$

† We just list a series of references: 9, Theorem 2; 51; 52, p. 1311; 53, p. 187; 54; 55; 56, p. 55 (Section 3). The fact that Schrödinger separability is superfluous was first realized by Debever in Reference 57.

‡ The Kerr space–time does satisfy (3.9).

This gives the twice-repeated eigenvalues

$$\lambda^{(1)} = -\varphi_1, \qquad \lambda^{(2)} = \varphi_2 \tag{3.11}$$

if and only if (3.9) holds. (Q.E.D.)

As a consequence we realize that the above hypotheses on the Segre characteristic of K^{ij} is perfectly equivalent with Carter's hypothesis (IV); moreover the following holds.

Corollary 3.1. In a four-dimensional manifold (V_4, g) with a $\mathscr{S}_2$ structure satisfying (3.9) the eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$ can be taken (locally) as coordinates if and only if the separability structure is of class $\mathscr{S}_2^0$.

Proof. The proof is a straightforward consequence of (3.9), (3.11), and Proposition 3.1. □

The hypothesis on $\lambda^{(1)}$ and $\lambda^{(2)}$ expressed by Corollary 3.1 has been largely used in the literature (cf. References 53 and 54), although not in their proper meaning and without a clear understanding of their true content.

Since Carter's space–times are $\mathscr{S}_2$ separable it is obvious that they admit at least one K tensor independent of the metric g (which is actually given by (3.4)–(3.6)). It was in fact through this property for the Kerr space–time that Walker and Penrose were led to open a whole field of research on space–times with K tensors[58]: we shall devote Sections 3.2, 3.3, and 3.4 to this. Even if the relations between separability and K tensors were already somewhat hidden in earlier works (e.g., References 59 and 26), the clear link between them was only established with the theories of separability (in either form of "complete separability" or also Woodhouse's theory of "separable systems"). Nevertheless, the existence of K tensors for Carter's space–times was known before the appearance of these theories.†

Carter's family of space–times was reconsidered in 1969 by Debever,[57] who dealt with them by means of his "isotropic-forms formalism." In Reference 57 the hypotheses (I), (II), and (III) of Carter were translated in a slightly different language. Hypothesis (III) was actually weakened to a "conformal hypothesis" which corresponds to separability of the null geodesics only (see Section 3.4). Additional hypothesis that (V_4, g)

† The existence of K tensors in type-D space–times was investigated in Reference 57, where, however, no relation with Carter's space–times was exploited. To our knowledge, the first place where this existence was neatly pointed out is Reference 49: we may observe that, moreover, (2.18) in Reference 49 bears some resemblance to the canonical form of g. This fact was later recalled in References 52–54.

is a type-D space–time then led to an equivalent determination of Carter's space–times, under the stronger hypothesis of separability of all geodesics.†

3.2. Killing Tensors in Type-D Space–Times

Motivated by Carter's results on space–times with an Abelian two-parameter group of isometries, investigations on K tensors on type-D space–times were begun by Walker and Penrose in Reference 58 and later continued together with Hughston and Sommers.[60–64] The choice of type-D space–times reflected the previously known fact that any type-D space–time admits two (or four) K vectors (cf. References 65 and 66; see also 67). The interrelationship between the Petrov form of the Weyl tensor, these K vectors, and the (possibly existent) K tensor were clarified mainly by Sommers.[63,62] In their investigations the above authors were naturally led to consider objects known as *conformal Killing tensors* (CK tensors).

A CK tensor of order p in a (V_n, g) is a p-covariant symmetric tensor field $K_{i_1 \cdots i_p}$ such that there exists a $(p-1)$-covariant tensor field $H_{i_2 \cdots i_p}$ with the property

$$\nabla_{(i} K_{i_1 \cdots i_p)} = g_{(ii_1} H_{i_2 \cdots i_p)} \tag{3.12}$$

If (V_n, g) has Lorentzian signature, (3.12) implies that the function $k\colon T^* V_n \to \mathbb{R}$ defined by

$$k = K^{i_1 \cdots i_p} p_{i_1} \cdots p_{i_p} \tag{3.13}$$

is a first integral for null geodesics.‡ The algebra of CK tensors was deeply investigated by Geroch.[18]

To study the above interrelationship, wide use was made of the Newman–Penrose formalism.[3] It turned out to be important to introduce an object called a *Killing spinor* of order 2 (K spinor). A K spinor χ_{BC} is a solution of the following equation§:

$$\nabla_{A'(A} \chi_{BC)} = 0 \tag{3.14}$$

† Debever proved in fact that "conformal separability" implies the existence of two geodesic shear-free congruences (Reference 57, Theorem 1). This fact is strictly connected with the analogous result which appears when the existence of a K tensor with Segre characteristic [(11)(11)] is assumed (see Section 3.3). The link between these two independent hypotheses is hidden in condition (3.9). Debever's results imply also that *any vacuum type-D space–time is conformally related with some member of Carter's family* (see also Section 3.4).

‡ For this reason CK tensors will enter the theory of separability of the null geodesic HJ equation (see later).

§ Throughout the present section the Battelle Convention is adopted.[68] Equation (3.14) is usually also referred as a "twistor equation," although the terminology is appropriate only in flat space. The concept of a K spinor can be generalized to define the so-called $D(p, q)$-K spinors. These objects have recently been investigated in connection with the theory of heavenly space–times (see References 70 and 71).

Any K spinor gives rise to a constant of motion for null geodesics (Reference 63, p. 20); in particular the tensor

$$P_{ab}(\chi) = \chi_{AB}\bar{\chi}_{A'B'} \tag{3.15}$$

turns out to be a trace-free CK tensor. The following was shown in Reference 63.

Proposition 3.3. A vacuum space–time admits a K spinor χ_{AB} if and only if (V_4, g) is (a) Minkowski space–time; or (b) a plane-fronted gravitational wave (of type N; see Reference 69); or (c) a type-D space–time.

It was also shown there that an electrovac type-D space–time admits a K spinor, provided the Maxwell tensor has two eigendirections aligned with those of the Weyl tensor. This hypothesis shall be assumed throughout (Reference 63, p. 26).

CK tensors also arise for another (not less important) reason:

$$Q_{ij} = K_{ij} - \tfrac{1}{4}(\mathrm{Tr}\, K)g_{ij} \tag{3.16}$$

is a CK tensor.[58] Sommers' results and (3.16) imply the following.

Proposition 3.4. Any type-D space–time admits a CK tensor.

This result was already known to Walker and Penrose, for vacuum space–times (Reference 58, Theorem 1); they also showed that the CK tensor is irreducible† if the space–time has less than four K vectors.

The existence of a K spinor χ_{AB} in any type-D space–time explains why all these solutions have at least two K vectors. In fact it can be shown that the complex vector

$$\xi_a = \nabla_{A'}{}^{B}\chi_{BA} \tag{3.17}$$

is a complex K vector for (V_4, g). If ξ_i, $\bar{\xi}_i$ are independent the real and imaginary parts provide two real independent K vectors[63,61]; the case $\xi_i = \bar{\xi}_i$ will be discussed below.

As far as K tensors are concerned, Hughston and Sommers investigated the following problem: *for which type-D space–times does there exist a scalar* $\tilde{K}$ *such that the tensor*

$$K_{ij}(\chi) = P_{ij}(\chi) + \tfrac{1}{4}\tilde{K}g_{ij} \tag{3.18}$$

† A CK tensor (respectively, a K tensor) K is *reducible* if there exist CK vectors (respectively, K vectors) $\underset{a}{X}$ $(a = 1, 2, \ldots, N)$ such that

$$K = G + \sum_{a,b=1}^{N} \lambda_{ab}(\underset{a}{X} \cap \underset{b}{X})$$

for some nonvanishing set of constants λ_{ab}.

is a K tensor?† A first equivalent condition was given in Reference 60: it was proved that from the equation $\nabla_{(A}{}^{A'}\chi_{BC)} = 0$ it follows that the principal spinors of χ_{BC} generate two geodesic shear-free congruences and that

$$\varphi_{AB} = (\chi_{CD}\chi^{CD})^{-3/2}\chi_{AB}$$

satisfies Maxwell's equations. Written $\varphi_{AB} = \varphi o_{(A}i_{B)}$, it turns out that the stress energy tensor has the form $\tau_{ij} = f^{-3}P_{ij}(\chi)$, where $f = (\varphi\bar{\varphi})^{-1/2}$. Then the existence of a function F^* such that $P_i = \nabla_i F^*$ is equivalent to the existence of a function F such that

$$l_{(i}n_{j)}\nabla^i f = \nabla_i F$$

where l, n are the vectors associated with the geodesic shear-free congruences. Later[61,63] the solution was given in the following terms: first we write the Weyl spinor $\Psi_{ABCD} = \Psi o_{(A}o_B i_C i_{D)}$ and we show that $\Gamma_{ABCD} = \varphi^{3/2} o_{(A}o_B i_C i_{D)}$ satisfies the equation $\nabla_{A'}{}^{A}\Gamma_{ABCD} = 0$. We then introduce a scalar α by‡

$$\alpha = \Psi\varphi^{-3/2}$$

The scalar $\tilde{K}$ exists if and only if

$$\nabla_i\tilde{K} = \tfrac{1}{3}[\bar{\varphi}^{-3/2}\nabla_i\alpha + \varphi^{-3/2}\nabla_i\bar{\alpha}] \tag{3.19}$$

The following result was so given.

Proposition 3.5. The scalar $\tilde{K}$ and the K tensor $K_{ij}(\chi)$ exist for any type-D solution apart from the C metric and its twisting generalizations (see References 65 and 72§).

We can now discuss the degenerate case $\xi_i = \bar{\xi}_i$. In such a case the K tensor $K_{ij}(\chi)$ exists (Reference 63, p. 30); the scalar $\tilde{K}$ turns out to be $\tilde{K} = -\frac{1}{2}(\varphi^{-1} + \bar{\varphi}^{-1}) \neq 0$. Then the vector

$$\eta_i = K_{ij}(\chi)\xi^j \tag{3.20}$$

is a K vector, again in virtue of (3.14) and (3.17). If η_i and ξ_i are independent we have then two real K vectors. If η_i is zero or proportional to ξ_i it is possible to prove that space–time admits other K vectors (a detailed examination of several cases has been given in Reference 63, pp. 34–46 and

† We notice that whenever such $\tilde{K}$ exists, then $\tilde{K} = \mathrm{Tr}\,(K)$. The question is then equivalent to asking for which space–times is the vector P_i defined by $\nabla_{(i}P_{jl)} = P_{(i}g_{jl)}$ a gradient. See References 58, p. 271; 60, p. 306; 63, p. 49.

‡ In vacuum space–times $\psi = \varphi^{3/2}$ and $\alpha = 1$. The argument above does not apply directly; cf. Reference 63, p. 50, and Reference 98.

§ In those metrics nevertheless a CK tensor exists (Proposition 3.4). Those metrics, in fact, admit separability of the null geodesic equation; separated solutions were obtained in Reference 72, equation (33). See also Section 3.4 and the † footnote on p. 425 below.

Appendix F). The main conditions for linear dependence of ξ_i and η_i are given in terms of Newman–Penrose coefficients by $\pi = \tau = 0$ or $\rho = \mu = 0$ (Reference 63, pp. 38–39). They were not discussed in Reference 61, however, and an alternative proof has recently been given by Collinson and Smith.[(73)]

In Reference 62 it was, moreover, shown that whenever $K_{ij}(\chi)$ exists it commutes with ξ and ξ commutes with η, i.e.,

$$[K(\chi), \xi] = 0, \qquad [\xi, \eta] = 0 \tag{3.21}$$

The above results have of course relevance to the problem of $\mathcal{S}_2$ separability. In a recent paper[(74)] the following was shown.

Proposition 3.6. All vacuum type-*D* metrics that admit the K tensor $K_{ij}(\chi)$ admit also a $\mathcal{S}_2$ separability structure.†

This follows from Theorem 2.7, since it can be shown that $[K(\chi), \eta] = 0$ and the conditions on the eigenvectors of $K(\chi)$ are satisfied (see Reference 74).

Separability of type-*D* space–times was also investigated by Woodhouse in Reference 2, Section 6. It is not hard to understand how Woodhouse's results fit into the scheme of complete $\mathcal{S}_2$ separability, due to the fact that the dimension is four and two ignorable coordinates exist.

3.3. Further Results on Space–Times with K Vectors and K Tensors

We have already stressed the relevance of K vectors and K tensors for the existence of separability structures; they are, in a sense, prerequisites for the separability of a space–time. Several papers have dealt with space–times with K vectors: among them we recall References 67 and 76.

K tensors in a general pseudo-Riemannian manifold (V_n, g) were investigated in Reference 59; they received renewed attention after the paper of Walker and Penrose[(58)] and after a differently motivated investigation by Geroch.[(18)] A short account on them appeared in Reference 64. Stationary K tensors in static spherically symmetric space–times were also investigated in Reference 77.

It is of clear importance, also in order to better realize how many different separability structures may exist in a given (V_n, g), to know how many linearly independent second-order K tensors could there exist. The maximal number $M_2(n)$ of second-order K tensors in a (V_n, g) was computed in 1946 by Thomas,[(78)] who proved that

$$M_2(n) = \tfrac{1}{12}n(n+1)^2(n+2) \tag{3.22}$$

† The *C* metric and its twisting generalizations are in fact the only vacuum type-*D* metrics that do not belong to Carter's family (this is clear from the table given in Reference 75).

The subject was later taken up again in Reference 79, where a new proof of (3.22) was given and the following was shown.†

Proposition 3.7. (i) $M_2(n)$ is attained in all spaces of constant curvature; (ii) in those spaces the K tensors are reducible (see † footnote on page 423 above).

It is also relevant to know the number $M_r(n)$ of linearly independent K tensors of order r ($r > 2$). In Reference 63 it was elegantly proved that if K is a K tensor of order r its covariant derivatives of order $s \geq r + 1$ can be expressed as linear combination of the covariant derivatives of order $\alpha \leq r$. This gives an implicit counting procedure (see Reference 63, p. 17).‡

This fact opens the question of integrability conditions and structural equations for K tensors, i.e., the search for explicit relations that such K tensors should satisfy together with their successive covariant derivatives. Integrability conditions (up to the third order of differentiation) for the Killing equation $\nabla_{(i}K_{jh)} = 0$ were first investigated in Reference 80, where a proof of (i) in Proposition 3.7 was also obtained. An equivalent set of conditions was then given in Reference 81.§ Integrability conditions of the fourth order were later given in Reference 82, where it was also proved that in four dimensions the converse of Proposition 3.7(ii) also holds.

Proposition 3.8. If $M_2(4) = 50$ is attained, then (V_4, g) has constant curvature.

In relativity the so-called *Killing–Yano tensors* (briefly: KY tensors) are also relevant. A KY tensor f_{ij} is a skew symmetric tensor such that

$$\nabla_h f_{ij} + \nabla_j f_{ih} = 0 \tag{3.23}$$

It is clear that if f_{ij} satisfies (3.23), the squared tensor

$$K_{ij}(f) = f_i{}^h f_{hj}$$

satisfies Killing's equations. The relevance of such objects to relativity is suggested by the fact that Carter's K tensor of Kerr metric is in fact the square of a KY tensor.[83,84] Relationships between such objects and applications to general relativity have been recently investigated by Collinson.

† An alternative shorter proof of (3.22) was given in Reference 63, where (ii) was also proved for flat manifolds.

‡ Similar results for K spinors in (V_4, g) are given briefly in Reference 70.

§ Although in Reference 81 an explicit statement is made that the dimension of the manifold is 4, the structural equations obtained are valid for any (V_n, g). In Reference 81 there appears a newer proof of the fact that in four-dimensional flat space the maximal number $M_2(4) = 50$ is attained.

He determined the necessary and sufficient conditions that an irreducible K tensor K_{ij} should satisfy in order to be the square of a KY tensor f_{ij}. One of these conditions requires that K_{ij} be a tensor of type $D1$ [2, 2] according to the classification given in Reference 85, i.e., that K_{ij} is diagonalizable over $\mathbb{R}$ and its Segre characteristic is [(11)(11)] (see Reference 84, Theorem 2).† Existence of KY tensors in algebraically special space–times was studied in Reference 86, where the following was proved.

Proposition 3.9. (i) Empty space–times of Petrov type I, II, or III do not admit KY tensors; (ii) the only empty type-N space–times that admit a KY tensor are the plane-fronted gravitational waves. Moreover, each plane-fronted gravitational wave admits exactly two linearly independent KY tensors.‡

As far as type-D space–times are concerned the following is claimed in References 84 and 86.

Proposition 3.10. The only vacuum type-D space–time that does not admit a KY tensor is the twisting C solution (i.e., Kinnersley type IIIB; see Reference 65). Whenever the KY tensor exists, it is unique.

However, it turns out that the result is incorrect in the case of the C metric, since this metric admits neither a KY tensor nor a K tensor.§

We now turn to discuss recent results on space–times with a K tensor of Segre characteristic [(11)(11)]. Following Hauser and Malhiot we shall restrict our attention to the case when both the eigenvalues of K_{ij} are not constant (i.e., the most obvious generalization of $\mathcal{S}_2^0$-separable space–times; cf. Corollary 3.1).

A first important property of such space–times is that they admit two geodesic shear-free null congruences (say k_i and m_i), generated, respectively, by two independent eigenvectors of K_{ij}‖: for vacuum space–times this implies that they are of type D.

Let us denote, respectively, by λ and μ the eigenvalues of K_{ij}; the above hypothesis requires that $d\lambda \wedge d\mu \neq 0$. In References 52 and 53 the following is then proved.

† This is a striking relation between the existence of KY tensors and Carter's separable space–times.

‡ We remark that this result can be understood from a different viewpoint: in Reference 73, Section 3 it is shown that a KY tensor f_{ij} corresponds to a (symmetric) K spinor F_{AB} such that $\nabla^B{}_{A'}F_{AB} + \nabla_A{}^{B'}\bar{F}_{B'A'} = 0$. Hence, Proposition 3.3(b) applies to give (ii).

§ The form of the K tensor for type-D metrics is given in Reference 84, p. 314. This form is, however, incorrect for the C metric: see the forthcoming paper Reference 98.

‖ This result has been independently proved in References 52, 54, 84. In a null tetrad $(k_i, m_i, t_i, \bar{t}_i)$ the K tensor K_{ij} reads: $K_{ij} = \lambda(k_i m_j + k_j m_i) - \mu(t_i\bar{t}_j + \bar{t}_i t_j)$.

Theorem 3.11. Whenever a space–time (V_4, g) admits a K tensor K_{ij} of Segre characteristic [(11)(11)] and nonconstant eigenvalues λ and μ, or it is conformally related to one such space–time, then its metric necessarily has the form

$$ds^2 = \frac{1}{2} e^{2\sigma}\rho^2\left\{\left(\frac{dx^1}{E_1}\right)^2 + \left[\frac{H_1(dx^3 + J_2\, dx^4)}{\Delta}\right]^2 + \varepsilon\left(\frac{dx^2}{E_2}\right)^2 - \varepsilon\left[\frac{H_2(J_1\, dx^3 + dx^4)}{\Delta}\right]^2 + 4(1-\varepsilon)\frac{H_2\, dx^2(J_1\, dx^3 + dx^4)}{E_2\Delta}\right\} \tag{3.24}$$

where $\frac{1}{2} e^{2\sigma}$ is a conformal factor (constant whenever space–time admits K_{ij}); $\rho^2 = \lambda - \mu$, $\lambda = \lambda(x^1)$, $\mu = \mu(x^2)$; E_a, H_a, and J_a $(a = 1, 2)$ are functions of x^a and x^{a+2} only; $\Delta = 1 - J_1 J_2 \neq 0$; ε is a sign related to λ and μ (we have $\varepsilon = \operatorname{sgn}(d\mu \cdot d\lambda)$ when μ is chosen to be the eigenvalue related to the tetrad leg t_i).†

From the metric (3.24) one can easily read off directly the K tensor K_{ij} [see Reference 53, equations (2), (3), and (35)]. The metric (3.24) is a genuine generalization of Carter's separable metrics; it is noteworthy that formally (3.24) and Carter's metrics [equation (77) of Reference 8] have exactly the same general form: however, besides λ and μ, the functions involved in (3.24) depend upon two variables rather than one only as in Carter's metrics. From the theory of separability structures we easily understand that the only $\mathscr{S}_2$-separable metrics among (3.24) are just Carter's ones [it is in fact enough that the metric tensor g is reducible to the canonical form (3.1)–(3.3)].

As in Reference 53 we define δ_1 and δ_2 by

$$\delta_1 = \frac{(1+\varepsilon^2)}{4\rho\Delta H_2} E_2 H_1 \frac{\partial J_1}{\partial x^2} \tag{3.25}$$

$$\delta_2 = \frac{E_1 H_2}{(1+\varepsilon^2)\rho\Delta H_1} \cdot \frac{\partial J_1}{\partial x^1} \tag{3.26}$$

Then four essentially different cases arise: (I) $\delta_1 \neq 0$, $\delta_2 \neq 0$. In this case the metric is generally not separable [the functions J_a depend upon certain real numbers σ_a; see Reference 53, equation (36a)]. However, all Carter's separable metrics fall into a subclass of this class (characterized by $\sigma_a = 0$); (II) $\delta_1 \neq 0$, $\delta_2 = 0$. This corresponds to metrics that are partially separable with respect to the coordinates x^2 and x^4 (x^4 is actually ignorable, since $\partial/\partial x^4$ turns out to be a K vector); (III) $\delta_1 = 0$, $\delta_2 \neq 0$. As in case (II), with x^1 and x^3 replacing x^2 and x^4 (these two cases have been classified in Reference

† The functions $\lambda = \lambda(x^1)$ and $\mu = \mu(x^2)$ can be locally used as coordinates in place of x^1 and x^2 (cf. Corollary 3.1). See in fact References 52 and 54, p. 546.

55); (IV) $\delta_1 = \delta_2 = 0$. In this last case (Reference 53, p. 193) the HJ equation for geodesics is *decomposable*, in the sense that it admits complete integrals of the form

$$W = W_1(x^1, x^3) + W_2(x^2, x^4)$$

(See also Reference 55, Table I.) In this last case, in fact, the metric tensor is also *conformally decomposable* (i.e., it can be written as a sum of two quadratic expressions involving, respectively, the coordinates (y^1, y^2) and (y^3, y^4), (y^i) being suitable coordinates, which is multiplied by a conformal factor depending on all y^i).† This can be easily seen by relying on equations (36d) of Reference 53 and (3.24); in fact we find the following.

(a) For $\varepsilon = 0$:

$$ds^2 = \frac{1}{2} e^{2\sigma} \rho^2 \left\{ \left(\frac{dx^1}{E_1(x^1, x^3)} \right)^2 + H_1(x^1, x^3)(dx^3)^2 + 4\frac{H_2(x^2, x^4)}{E_2(x^2)}\, dx^2\, dx^4 \right\} \tag{3.27}$$

which can be transformed to a conformally decomposable form by a change of coordinates $x^1 = y^1$, $x^3 = y^3$, $x^2 = \varphi(y^1, y^4)$, $x^4 = \psi(y^2, y^4)$.

(b) For $\varepsilon = \pm 1$:

$$ds^2 = \frac{1}{2} e^{2\sigma} \rho^2 \left\{ \left[\frac{dx^1}{E_1(x^1, x^3)} \right]^2 + H_1(x^1, x^3)(dx^3)^2 + \varepsilon \left[\frac{dx^2}{E_2(x^2, x^4)} \right]^2 - \varepsilon H_2(x^2, x^4)(dx^4)^2 \right\} \tag{3.28}$$

which is already in Kasner's form.

Some other connections between the existence of separability structures and the existence of a K tensor with the above properties can be deduced from the results of References 52 and 54. Instead of decomposing K_{ij} along a null complex tetrad (as in the ‖ footnote on page 427 above) we can also choose an orthonormal frame (T, X, Y, Z) in which K_{ij} decomposes as follows‡:

$$K_{ij} = \lambda(-T_i T_j + X_i X_j) + \mu(Y_i Y_j + Z_i Z_j) \tag{3.29}$$

The eigenspaces spanned, respectively, by (T, X) and (Y, Z) are mutually orthogonal. Let us now suppose that the space–time (V_4, g) admits two commuting K vectors ξ and η which (locally) generate two-dimensional group orbits. Dietz has proved the following result.

† The case of metric tensors that are strictly decomposable (i.e., when the conformal factor is a constant) was first investigated by Kasner.[87,88] A whole section is devoted to the conformally decomposable case in Petrov's book (Reference 89, Section 51).

‡ Our signature conventions are the opposite of those adopted in Reference 54.

Proposition 3.12. (Reference 54, p. 544) The K tensor K_{ij} is reducible in terms of g, ξ, and η if one eigenspace of K_{ij} coincides (locally) with the orbits.†

Whenever this happens the quadruple (ξ, η, g, K) does not define a $\mathscr{S}_2$ structure. An analysis of all possibilities leads one to consider two more cases, listed in Reference 54: (A) timelike orbits noncoinciding with eigenspaces of K; (B) spacelike orbits noncoinciding with eigenspaces of K. (Null orbits are excluded, as already noticed in Reference 8.) From the discussion of cases (A) and (B) given in Reference 54 (Theorem 1, p. 544) we can draw the following conclusions.

Proposition 3.13. Let (V_4, g), K, ξ, and η be as above. If K is such that its eigenspaces do not coincide (locally) with the group orbits, then (ξ, η, g, K) defines a $\mathscr{S}_2$-separability structure provided K commutes with ξ and η.

Proof. Since $[g, \xi] = [g, \eta] = [K, \xi] = [K, \eta] = 0$ and $[G, K] = 0$, by Theorem 2.7 it is enough to check that K admits two commuting eigenvectors (say ζ and θ), orthogonal and commuting with ξ and η. This follows from Reference 54, Theorem 1, where it was proved that (smooth) functions M and N exist such that (ξ, η, NY, MX) commute in pairs.‡ □

We observe that the hypothesis on the eigenspaces of K is necessary, because of Proposition 3.12. A similar statement (in a slightly weaker form) was independently given in Reference 52, Theorem IV. Hauser and Malhiot's proof goes essentially as follows: suppose $\|d\lambda \wedge d\mu\| \neq 0$; then the surfaces of transitivity of ξ and η are not null and coincide (locally) with the surfaces $\lambda = \text{const}$ and $\mu = \text{const}$. Since $L_\xi K = L_\eta K = 0$, the eigenvalues λ and μ are also constant along ξ and η. This implies that $\nabla\lambda$ and $\nabla\mu$ are commuting eigenvectors of K which are orthogonal to ξ and η. The results then follow by showing that coordinates (x^i) can be chosen so that $x^1 = \lambda$, $x^2 = \mu$, $\partial/\partial x^3 = \xi$, and $\partial/\partial x^4 = \eta$ (see Reference 52, p. 1310).

If in addition to the previous hypotheses we assume that the Ricci tensor R_{ij} of (V_4, g) satisfies $R^{ij}X_iY_j = 0$, Theorem 2.12 tells us that the Schrödinger equation and the wave equation also separate (these statements have been directly proved in Reference 54, Theorem 2, and Reference 52, Theorem VI).

† The proof given in Reference 54 is similar to an analogous result of Reference 58.

‡ We recall that the appearance of the factors M and N should not be a surprise, in accordance with the rescaling freedom implied in Theorem 2.7. The fact that Reference 54, Theorem 1, implies separability of the geodesic equation was realized in Reference 54 by using Woodhouse's approach.

Hauser and Malhiot have moreover proved a theorem (Reference 52, Theorem V) which turns out to be of deep importance for a better understanding of $\mathcal{S}_2^0$ separability structures on space–time. This theorem can be rephrased as follows.

Theorem 3.1. Let (V_4, g) be a Lorentzian space–time. Let K_{ij} be a K tensor with Segre characteristic [(11)(11)] and nonconstant eigenvalues λ and μ, with $\|d\lambda \wedge d\mu\| \neq 0$. If

$$R^{ij}\nabla_i\lambda\nabla_j\mu = 0 \tag{3.30}$$

then (V_4, g) admits a $\mathcal{S}_2^0$ structure.

Proof (Sketch). From Theorem V of Reference 52 it follows that (3.30) implies the existence of two K vectors ξ and η such that $L_\xi K = L_\eta K = 0$ and $[\xi, \eta] = 0$. Then Proposition 3.13 (in its weaker form) applies to give the result. □

We have the following corollary.

Corollary 3.2. Any empty space–time (V_4, g) with a K tensor of Segre characteristic [(11)(11)] and nonconstant eigenvalues admits a $\mathcal{S}_2^0$ structure.

Some of the relations among the topics covered in Sections 3.1, 3.2, and 3.3 have been recently investigated for type-D vacuum space–times in Reference 98.

3.4. Further Results on Conformal K Tensors and the Separability of Second-Order Equations

We have already met CK tensors and anticipated that they often have to do with separability of the null geodesic equation (see Section 3.2, § footnote to Proposition 3.5; Section 3.3). After the pioneering paper of Walker and Penrose,[58] the role of CK tensors for null separability was investigated by Woodhouse,[2] in searching for coordinate systems in which at least one coordinate separates (see Section 2.3). This approach was later extensively used in Reference 9, where Dietz applied Woodhouse's theory of separable systems to investigations concerning HJ, Schrödinger, and wave equation separability, and null separability. From Dietz's results (which for the sake of brevity we do not repeat here) it is clear that CK tensors and CK vectors play a fundamental role in the separability of the null geodesic equation (see Reference 9, Proposition 4.5 and Theorems 2, 3, 4). The class of metrics obtained by Dietz is another genuine generalization of Carter's separable metrics, to the extent that Dietz's metrics admit only one separable coordinate. The general form of Dietz's metrics is again quite similar to Carter's form, as it was already for (3.24), the link being obviously the overwhelming presence of a K tensor (taken as an hypothesis in References 52 and 53 and obtained as a consequence of partial separability in Reference 9). Dietz's and Hauser and Malhiot's classes, of course, do not coincide but overlap on the family of metrics having a $\mathcal{S}_2^0$ structure: it is nevertheless interesting to accurately compare them.

The core of the analysis carried in Reference 9 is represented by the following.

Theorem 3.2. If in (V_4, g) coordinates x^α $(n-p+1 \leq \alpha \leq n)$ are ignorable and x^1 separable, the contravariant metric tensor takes the form

$$G = \Phi^{-1}[\partial_1 \otimes \partial_1 + F^{ij}(x^k)\, \partial_i \otimes \partial_j + 2F^{j\alpha}(x^k)\, \partial_j \otimes \partial_\alpha + (F_1^{\alpha\beta}(x^1) + F_*^{\alpha\beta}(x^k))\, \partial_\alpha \otimes \partial_\beta] \quad (3.31)$$

where $n-p+1 \leq \alpha,\ \beta \leq n,\ 2 \leq i,\ j,\ k \leq n-p$ and Φ splits as $\Phi = \Phi_1(x^1) + \Phi_*(x^k)$. For null separability (of x^1) Φ can be any function $\Phi(x^1, x^k)$.

The last remark of Theorem 3.2 applies in particular to $\mathcal{S}_2$-separable metrics; we can in fact state the following proposition.

Proposition 3.14. If the metric tensor g_{ij} has the pseudocanonical form obtained from (3.1)–(3.3) by replacing $\varphi_1 + \varphi_2$ with an arbitrary function $U = U(y^1, y^2)$, then the HJ equation for null geodesics is completely separable.

A weaker form of this statement appeared in Reference 49 (Theorem 2).

For the reasons which we tried to outline above, CK tensors also deserve attention. A recent paper by Weir[(90)] was devoted to the investigation of some of their properties. In Reference 90 the integrability conditions and the structural equations (in the spirit of Reference 81) were found for the conformal Killing equation $\nabla_{(h}K_{ij)} = H_{(h}g_{ij)}$. These have been used to show the following.

Proposition 3.15. (Reference 90, Theorems 1 and 2.) The maximal number $N_2(n)$ of CK tensors of order 2 that a (V_n, g) can admit is

$$N_2(n) = \tfrac{1}{12}(n-1)(n+2)(n+3)(n+4) \quad (3.32)$$

and $N_2(n)$ is attained if (V_n, g) is flat.

Weir then turned to give the general form admitted by the metric of a (V_n, g) in which, besides an irreducible CK tensor, there exist one or two commuting CK vectors. The results for this last case read as follows: let $\underset{1}{\xi}$, $\underset{2}{\xi}$ be the CK vectors, K the CK tensor; choose coordinates (x^i) such that $\underset{1}{\xi} = \partial_1$ and $\underset{2}{\xi} = \partial_2$. Then there exists a function φ such that

$$g_{ij} = e^{-\varphi}\tilde{g}_{ij}, \qquad \partial_a\tilde{g}_{ij} = 0 \tag{3.33}$$

$$[G, \underset{a}{\xi}] = (\nabla_a\varphi)G \tag{3.34}$$

$$K_{ij} = e^{x^1}\tilde{K}_{ij}, \qquad \partial_a\tilde{K}_{ij} = 0 \tag{3.35}$$

or

$$K_{ij} = x^a B_a{}^{bc}\underset{b}{\xi_i} \cap \underset{c}{\xi_j} + \tilde{K}_{ij} \qquad (a, b, c = 1, 2)\dagger \tag{3.35$'$}$$

These canonical expressions remain valid when $\underset{1}{\xi}$ and $\underset{2}{\xi}$ are simply K vectors (i.e., when $\nabla_a\varphi = 0$) and K is a K tensor. In this last case we get "canonical" forms for metrics admitting two commuting K vectors and one K tensor, summarized by

$$[\underset{1}{\xi}, K] = K, \qquad [\underset{2}{\xi}, K] = 0 \qquad \text{if (3.35) holds} \tag{3.36}$$

or

$$[\underset{1}{\xi}, K] = \psi_a G + B_a{}^{bc}\underset{b}{\xi} \cap \underset{c}{\xi} \qquad \text{if (3.35$'$) holds} \tag{3.36$'$}$$

Case (3.36′) contains of course the case of $\mathcal{S}_2$ structures, when $\psi_a = 0$, $B_a{}^{bc} = 0\ \forall a, b, c$, and $n = 4$. In Reference 90, Section 4, metric tensor components were finally computed in the following situation: $n = 4$, $\underset{a}{\xi} = \partial_a$ $(a = 1, 2)$ are commuting K vectors, K_{ij} is an irreducible CK tensor such that $[\underset{1}{\xi}, K] = [\underset{2}{\xi}, K] = 0$; coordinates (x^i) are such that the metric is quasi-diagonal (i.e., $g^{ai} = 0$, $\forall i$, $a = 1, 2$). The investigation was performed by looking at various possibilities for the eigenvalue problem $\det\|K^{\alpha\beta} - \lambda g^{\alpha\beta}\| = 0$, $\alpha, \beta = 3, 4$, and lead to three essentially different cases (Reference 90, pp. 1785–1786): it is interesting to compare these results (which we cannot report here) with those of References 9 and 52.

Weir's analysis still holds when K is simply a K tensor. If it has Segre characteristic [(11)(11)], which happens when case (1a) of Weir's

† The distinction between (3.35) and (3.35′) is as follows: decompose $[\underset{a}{\xi}, K]$ as $[\underset{a}{\xi}, K] = \psi_a G + A_a K + B_a{}^{bc}\underset{b}{\xi} \cap \underset{c}{\xi}$. Then (3.35) applies when $(A_1, A_2) \neq (0, 0)$, while (3.35′) applies when $(A_1, A_2) = (0, 0)$.

classification is satisfied, Theorem 3.1 tells us that all vacuum metrics involved are $\mathscr{S}_2^0$ separable. *If instead K is truly a CK tensor, Weir's class of metrics* [in Reference 90, equation (4.5)] *contains as a particular case all type-D vacuum metrics* (recall that they all admit a CK tensor; see Proposition 3.4). In fact these metrics are all conformally related with some Carter $\mathscr{S}_2^0$-separable metric (this can be seen from Table I in Reference 75; see also References 53 and 57), i.e., they all fall into the family of "generalized" separable metrics of Proposition 3.14. The general procedure to separate the null geodesic HJ equation is given in detail in Reference 90 [equations (4.8)]; however, the particular case of C metric was computed in Reference 72 and general results for all type-*D* metrics were also given in Reference 49.

As far as separability of other second-order equations is concerned, we just add a few remarks to what has been already said in earlier sections. We have already recalled the valuable contributions given in Reference 9. Separability of the wave equation in vacuum separable space–times follows from Corollary 2.1; for the cases of Schwarzschild and Kerr metrics this property was long ago realized.[91,92] Results on separability of the wave equation in the general type-*D* vacuum solution have been recently obtained by Dudley,[93] who obtained separated equations by relying on the Newman–Penrose formalism. Dudley's analysis is applicable to *all* vacuum type-*D* solutions, even to those which do not admit a $\mathscr{S}_2$ structure. This is not contradictory with the general theory, since Dudley investigated the property of *R* separability (see Reference 43 for a detailed account), which roughly speaking amounts to separability modulo a conformal factor (in perfect agreement with the above discussion).

3.5. Further Contributions to Separable Space–Times

In a recent paper by Collinson and Fugère[55] vacuum space–times with separable HJ equation were studied, with the aim of giving a classification of all the various possibilities. In Reference 55 both partial and complete separability are taken into account, and a list of eight different situations is given (Table I, p. 747), according to the number of ignorable and separable nonignorable coordinates (a distinction that is reflected by the corresponding existence of a K vector or tensor, or else by the appearance in *W* of a separated factor which is linear or not).

Collinson and Fugère explicitly restrict themselves to the case when separability takes place after multiplication of the HJ equation by a suitable separating factor. We have already stressed (cf. Section 2) that no *a priori* reason exists that separation always takes place in such a manner: it could be a fortuitous circumstance that it is always so when $\mathscr{S}_2$ structures appear (see Section 6). A definitive answer to this question (which would also assure that the classification given in Reference 55 exhausts all the possibilities) cannot be given at this moment: in order to do that we need for any $\mathcal{S}_r$ structure a "canonical form" analogous to (3.1)–(3.3). This program is currently under investigation.[94]

We shall here briefly discuss the results given in Reference 55. The first three cases correspond to space–times for which the HJ equation admits a complete integral of the form

$$W = W_1(x^1) + W^*(x^2, x^3, x^4), \qquad \frac{\partial^2 W_1}{(\partial x^1)^2} \neq 0 \tag{3.37}$$

or a more separated one in which, however, no linear terms appear.† These space–times were investigated by Petrov (Reference 89, Section 51) and Dietz.[9]‡ Cases 4 and 5 are studied in Reference 55, Sections 2, 3, and 4, where only solutions in type-*D* and type-*N* families are found to exist. In these space–times a complete integral takes the form

$$W = kx^4 + W_1(x^1) + W^*(x^2, x^3), \qquad \frac{\partial^2 W}{(\partial x^1)^2} \neq 0 \tag{3.38}$$

The type-*D* solutions found in Reference 55 correspond to metric tensors in which, besides the separating factor *U*, three arbitrary functions appear. The type-*N* solutions correspond to metrics involving only two additional functions (see Reference 55, pp. 751–752). The type-*N* solutions have been independently found by Matravers,[50] and we shall briefly discuss them later.

Case 6 is essentially Carter's $\mathcal{S}_2$ case.

Case 7 is exactly the case of $\mathcal{S}_3$ structures, i.e., when *W* separates as

$$W = k_2 x^2 + k_3 x^3 + k_4 x^4 + W_1(x^1), \qquad \frac{\partial^2 W}{(\partial x^1)^2} \neq 0 \tag{3.39}$$

We know from Theorem 2.7 that a space–time (V_4, g) admits a $\mathcal{S}_3$ structure if and only if it admits a three-parameter Abelian group of isometries (cf. also Reference 30, Theorem 3): hence, these space–times belong to Bianchi's family. Since (3.39) implies that the coordinates x^2, x^3, and x^4 are ignorable, the metric tensor of these space–times has the form

$$g_{ij} = g_{ij}(x^1)$$

All solutions to Einstein vacuum equations that depend on a single variable (say x^1) were determined by Kasner[87,88] and by Papapetrou and Treder.[95]

† If ignorable coordinates exist, we shall consider these "limiting cases" as belonging to "higher" cases of the classification.

‡ If any solution exists in the third case, this would correspond to a space–time with a $\mathcal{S}_0$ structure (i.e., with three K tensors other than *G*).

Case 8 is case (IV) of Reference 53; see Section 8.

Let us now turn to Matraver's solutions. In Reference 50 space–time metrics of the form

$$g^{ij} = \begin{Vmatrix} X & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & Z & -1 \\ -1 & -1 & -1 & F \end{Vmatrix} D^{-1} \tag{3.40}$$

were investigated, where $D = D(x^1, x^2, x^3)$, $X = X(x^1)$, $Z = Z(x^3)$ and $F = F_1(x^1) + F_2(x^2) + F_3(x^3)$. When D does not split into three separate terms, (3.40) cannot give separable solutions. In Reference 50 it was proved that vacuum solutions to Einstein equations exist under the form (3.40) only if $\partial_1 D = \partial_2 D = 0$. In this last case they are type-N plane-fronted gravitational waves. Exact separable solutions were then computed (Reference 50, Section 4) and shown to fit into Woodhouse's separability scheme: these solutions admit one K vector $\xi = \partial_4$ and two additional K tensors [Reference 50, equations (4.1) and (4.2)]. This last fact is in agreement with the well-known properties of plane-fronted gravitational waves.†

We conclude this section with a list of additional references to somewhat related material. Explicit calculations for the completely separated HJ equation for geodesics in Kerr–NUT metric (which is one of Carter's $\mathscr{S}_2^0$-separable metrics) were given by Miller in Reference 96, by closely following Reference 45. Miller has recently investigated in detail two Bianchi type-VIII spatially homogeneous solutions which fall into Carter's class $[\tilde{B}(+)]$ (i.e., which admit a $\mathscr{S}_2^1$ structure): separated equations are derived in Reference 97, p. 9. Finally, $\mathscr{S}_2$-separable nonvacuum solutions are investigated by Bonanos in Reference 56, in the case of a perfect fluid stress tensor.

We should like to conclude this paper with the rather well-known example given by the Kerr metric, in which it is easy and instructive to compare the different viewpoints on separability and which provides an interesting example of the greatest part of the material covered here. However, obvious limitations of space forbid this "pedagogical" comparison: the reader may compare the different discussions given in Reference 45, pp. 1566–1569; Reference 58, pp. 271–273; Reference 60; Reference 2, pp. 34–37; and Reference 30, Section 4.

† In any plane-fronted gravitational wave there are in fact two KY tensors [see Proposition 3.9(ii)], which, being null bivectors, give rise to two reducible K tensors (see Reference 84). There are in fact three K vectors for plane-fronted gravitational waves, in accordance with a result of Ehlers and Kundt.[69] In the notations of Reference 50 the two K tensors are $\underset{1}{K} = Z\,\partial_3 \otimes \partial_3 + F_3\,\partial_4 \otimes \partial_4 + \partial_3 \cap \partial_4$, $\underset{2}{K} = X\,\partial_1 \otimes \partial_1 + F_1\,\partial_4 \otimes \partial_4 + \partial_1 \cap \partial_4$.

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