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Section A :
Physique théorique.

Canonical forms for $\mathscr{S}_{n-3}$-structures in pseudo-riemannian manifolds (*)

by

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1. INTRODUCTION

Let (V_n, g) be a given pseudo-Riemannian manifold with a regular separability structure of type $\mathscr{S}_r$ (briefly: a $\mathscr{S}_r$-structure). It is known that in this case there exist r (local) Killing vectors $\underset{\alpha}{X}$ $(\alpha = n - r + 1, \ldots, n)$ and $n - r$ Killing tensors $\underset{a}{K}$ $(a = 1, \ldots, n - r)$, which satisfy suitable conditions (see e. g. [1] ; [2], § 2). The r Killing vectors $\underset{\alpha}{X}$ generate r ignorable separable coordinates. It is also known that (V_n, g) admits two (local) orthogonal complementary foliations $\{ Z_{n-r} \}$ and $\{ W_r \}$, where W_r are the flat r-dimensional submanifolds generated by the involutive r-distribution spanned by the vectors $\underset{\alpha}{X}$ and Z_{n-r} are the $(n - r)$-dimensional submanifolds with an orthogonal separability structure in the induced metric.

This last fact suggests that subdividing all separability structures in (V_n, g) into $n+1$ types $\mathscr{S}_r$ $(0 \leqslant r \leqslant n)$ gives only a preliminary classification, which could be refined by inductively classifying as follows the induced structure on Z_{n-r}. We say that (V_n, g) admits a separability structure of type $\mathscr{S}_r^{k_1}$ $(0 \leqslant r \leqslant n,\ 0 \leqslant k_1 \leqslant n - r)$ if the induced separability structure on the generic submanifold Z_{n-r} is of type $\mathscr{S}_{k_1}$. In this case Z_{n-r} admits two orthogonal foliations $\{ W'_{k_1} \}$ and $\{ Z'_{n-r-k_1} \}$, where Z'_{n-r-k_1} has an orthogonal separability structure in the induced metric. Let this last structure be of type $\mathscr{S}_{k_2}$, with $k_2 \leqslant n - r - k_1$; in this case we shall say that (V_n, g) admits a separability structure of type $\mathscr{S}_r^{k_1,k_2}$. By inductively proceeding in this

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way as far as possible [1], we get a finite sequence of integers $k_0 = r$, k_1, $k_2, \ldots, k_p$ $(p \leqslant n)$ satisfying the inequalities:

$$k_s \leqslant n - \sum_{0}^{s-1} {}_i\, k_i \qquad s = 1, \ldots, p. \tag{1}$$

We shall say that in this case there exists in (V_n, g) a separability structure of type $\mathscr{S}_r^{k_1,k_2,\cdots,k_p}$, or a $\mathscr{S}_r^{k_1,k_2,\cdots,k_p}$-structure.

In the case $r = n - 2$, which we already investigated in [*3*], the finest classification is given by the three types $\mathscr{S}_{n-2}^k$ $(k = 0, 1, 2)$ there introduced and characterized through a so-called canonical form of the metric tensor g. In this paper we analyze the case $r = n - 3$, with the aim of giving a canonical form for a $\mathscr{S}_{n-3}$-metric together with an analogous characterization in the terms of the above classification.

We easily realize that a $\mathscr{S}_{n-3}$-structure can belong to one of the following six types:

i) $\mathscr{S}_{n-3}^0$, when the 3-dimensional manifolds Z_3 have a $\mathscr{S}_0$-structure;

ii) $\mathscr{S}_{n-3}^{1,0}$, $\mathscr{S}_{n-3}^{1,1}$, $\mathscr{S}_{n-3}^{1,2}$, when Z_3 have a $\mathscr{S}_1^k$-structure (with $k = 0, 1, 2$ respectively);

iii) $\mathscr{S}_{n-3}^2$, when Z_3 have a $\mathscr{S}_2$-structure (hence Z'_1 is 1-dimensional);

iv) $\mathscr{S}_{n-3}^3$, when Z_3 have a $\mathscr{S}_3$-structure (hence Z'_0 are empty).

This six cases give the finest possible classification.

In order to produce a canonical form for $\mathscr{S}_{n-3}$-metrics, we first recall that from the general theory it follows that any such metric has, in normal separable coordinates (y^j), the following components:

$$g^{aa} = \underset{3}{u}^a, \qquad a = 1, 2, 3\,; \tag{2}$$

$$g^{ai} = 0, \qquad i \neq a\,; \tag{3}$$

$$g^{\alpha\beta} = \underset{3}{u}^a \zeta_a^{\alpha\beta}, \qquad \alpha, \beta = 4, \ldots, n\,; \tag{4}$$

where $\zeta_a^{\alpha\beta}$ are functions of y^a only, while $\underset{3}{u}^a$ is the third row of a regular matrix $\mathscr{U} = \| \underset{b}{u}^a \|$ $(a, b = 1, 2, 3)$ with inverse $\mathscr{U}^{-1} = \| \overset{a}{u}_b \|$ such that $\overset{a}{u}_b$ is a function of y^a only. The other coordinates $y^4, \ldots, y^n$ are ignorable. The components (2), (3) (with $i \neq 4, \ldots, n$) give the induced metric on the submanifolds Z_3, in its Stäckel's form [2].

Since (V_n, g) has a $\mathscr{S}_{n-3}$-structure, there exist three Killing tensors $\underset{b}{K}$

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(1) The inductive procedure stops whenever we encounter a sub-foliation $Z_h^{(l)}$ which is either empty or 1-dimensional or $\mathscr{S}_0$-separable.

(2) Cfr. [*4*].

($b = 1, 2, 3$), whose components in normal separable coordinates are given by:

$$\underset{b}{K}^{aa} = \underset{b}{u}^{a}, \qquad a, b = 1, 2, 3\,; \tag{5}$$

$$\underset{b}{K}^{ai} = 0, \qquad a \neq i\,; \tag{6}$$

$$\underset{b}{K}^{\alpha\beta} = \underset{b}{u}^{a}\zeta_{a}^{\alpha\beta}, \qquad \alpha, \beta = 4, \ldots, n\,. \tag{7}$$

We remark that, in agreement with the general theory, one of those Killing tensors (precisely $\underset{3}{K}$) coincides with the metric g.

The general form (2)-(4) of a $\mathscr{S}_{n-3}$-metric is usually rather inconvenient for applications. In fact, if a $\mathscr{S}_{n-3}$-metric g is given, the problem of finding a matrix $\mathscr{U}$ which allows one to write g in the form (2)-(4) is a rather difficult task which, moreover, has not a unique solution. Hence, we need finding equivalent forms for (2)-(4) which allow to realize more directly whether a given metric g is a $\mathscr{S}_{n-3}$-metric and, moreover, to calculate more directly the other two Killing tensors belonging to the separability structure.

2. CANONICAL FORMS OF THE METRIC

In this section we shall give canonical forms for any $\mathscr{S}_{n-3}$-metric. From eqs. (2)-(7) it turns out that $g^{\alpha\beta} = g^{aa}\zeta_{a}^{\alpha\beta}$ and $\underset{b}{K} = \underset{b}{K}^{aa}\zeta_{a}^{\alpha\beta}$ ($b = 1, 2$). Thence the problem is reduced to finding suitable equivalent forms of (2) and (5), i. e. of the reduced metric and Killing tensors on Z_3.

To this purpose we need the following:

PROPOSITION 1. — *Let* U *be a* $\mathscr{S}_{n-3}$*-chart with normal coordinates* (y^j). *Let us consider the matrix:*

$$\mathscr{U}^{-1} = \left\| \begin{matrix} \overset{1}{u}_1 & \overset{1}{u}_2 & \overset{1}{u}_3 \\ \overset{2}{u}_1 & \overset{2}{u}_2 & \overset{2}{u}_3 \\ \overset{3}{u}_1 & \overset{3}{u}_2 & \overset{3}{u}_3 \end{matrix} \right\|. \tag{8}$$

One of the following alternatives holds:

i) There exists an open subset $U' \subseteq U$ *in which at least one of the first two rows of* $\mathscr{U}^{-1}$ *does not contain any identically zero function;*

ii) There exists an open subset $U'' \subseteq U$ *in which both the first and second row of* $\mathscr{U}^{-1}$ *contain one and only one zero function, not belonging to the same column.*

Proof. — If (*i*) does not hold there exists an open subset $U'' \subseteq U$ in which both rows contain at least one zero. Without any restriction on generality we may assume that $\overset{1}{u}_3 \equiv 0$. In the whole chart U the components $g^{aa} = \underset{3}{u}^a$ have the form:

$$\underset{3}{u}^1 = \frac{1}{\Delta}(\overset{1}{u}_2\overset{2}{u}_3 - \overset{2}{u}_2\overset{1}{u}_3), \tag{9}$$

$$\underset{3}{u}^2 = \frac{1}{\Delta}(\overset{2}{u}_1\overset{1}{u}_3 - \overset{1}{u}_1\overset{2}{u}_3), \tag{10}$$

$$\underset{3}{u}^3 = \frac{1}{\Delta}(\overset{1}{u}_1\overset{2}{u}_2 - \overset{2}{u}_1\overset{1}{u}_2), \tag{11}$$

where:

$$\Delta = \det \mathscr{U}^{-1}. \tag{12}$$

Substituting $\overset{1}{u}_3 = 0$ into (9), (10) and taking into account that $g^{aa} \not\equiv 0$ ($a = 1, 2, 3$), it follows that $\overset{1}{u}_2 \not\equiv 0$, $\overset{2}{u}_3 \not\equiv 0$, $\overset{1}{u}_1 \not\equiv 0$, i. e. on the first row only one zero appears and two zeroes cannot appear in the same column ([3]). Analogously, from (11) it follows that:

$$\overset{1}{u}_1\overset{2}{u}_2 - \overset{1}{u}_2\overset{2}{u}_1 \not\equiv 0. \tag{13}$$

Since (i) does not hold we know that at least one of the two functions $\overset{2}{u}_1$ and $\overset{2}{u}_2$ is identically zero. Hence, from (13) it follows that exactly one of them is zero. (Q. E. D.)

Therefore, when (ii) holds it is not restrictive to suppose that $\mathscr{U}^{-1}$ has the following form:

$$\mathscr{U}^{-1} = \begin{Vmatrix} \overset{1}{u}_1 \not\equiv 0 & \overset{1}{u}_2 \not\equiv 0 & 0 \\ \overset{2}{u}_1 \not\equiv 0 & 0 & \overset{2}{u}_3 \not\equiv 0 \\ \overset{3}{u}_1 & \overset{3}{u}_2 & \overset{3}{u}_3 \end{Vmatrix}, \tag{14}$$

where nothing can be said about the third row (besides the obvious restriction $\Delta \neq 0$).

We can now prove the following:

PROPOSITION 2. — *Let* (V_n, g) *have a (local)* $\mathscr{S}_{n-3}$*-structure. Let* U *be a separable chart with normal coordinates* (y^j). *There exists an open subset* $W \subseteq U$ *in which the metric g can be put under the following form:*

$$g^{aa} = \frac{\Psi_a}{\varphi}(\mu_{a+1} - \mu_{a+2}) \qquad a = 1, 2, 3\,([4]); \tag{15}$$

$$g^{ai} = 0 \qquad a \neq i; \tag{16}$$

$$g^{\alpha\beta} = \frac{1}{\varphi}\Sigma_a \zeta_a^{\alpha\beta}\Psi_a(\mu_{a+1} - \mu_{a+2}), \quad \alpha, \beta = 4, \ldots, n; \tag{17}$$

([3]) More precisely, since g^{aa} is nowhere vanishing in U, also $\overset{1}{u}_1$, $\overset{1}{u}_2$ and $\overset{2}{u}_3$ are nowhere vanishing in U″.

([4]) Throughout this paper latin indices $a, b, \ldots$ are taken modulo 3.

with:

$$\varphi = \det \begin{Vmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Vmatrix}, \tag{18}$$

where Ψ_a, μ_a, ν_a and $\zeta_a^{\alpha\beta}$ are functions of y^a only [5].

Proof. — We first suppose that $\mathcal{U}^{-1}$ satisfies condition (*i*) of proposition 1. Let us assume that no zero function appears in the first row, i. e. $\overset{1}{u}_a \not\equiv 0$ (a = 1, 2, 3) [6]. Let us consider the open subset:

$$W = U - \{ p \in U' : \Pi_a \overset{1}{u}_a(p) = 0 \quad , \quad a = 1, 2, 3 \}.$$

In W we define:

$$\Psi_a = -\frac{1}{\overset{1}{u}_a} \qquad , \quad a = 1, 2, 3 ; \tag{19}$$

$$\mu_a = \frac{\overset{2}{u}_a}{\overset{1}{u}_a} \quad , \quad \nu_a = \frac{\overset{3}{u}_a}{\overset{1}{u}_a} \quad , \quad a = 1, 2, 3 . \tag{20}$$

The inverse relations are:

$$\overset{1}{u}_a = -\frac{1}{\Psi_a} \quad , \quad \overset{2}{u}_a = -\frac{\mu_a}{\Psi_a} \quad , \quad \overset{3}{u}_a = -\frac{\nu_a}{\Psi_a}. \tag{21}$$

Substituting (21) into (9)-(12) we easily get (15). Expressions (16) and (17) then follow from (3) and (4).

Let us now suppose that $\mathcal{U}^{-1}$ satisfies condition (*ii*) of proposition 1, it being of the form (14). Let us consider the open subset:

$$W'' = U'' - \{ p \in U'' : \overset{1}{u}_1(p) \,.\, \overset{1}{u}_2(p) = 0 \}.$$

Let Ψ_3^* be any function of y^3 nowhere vanishing in W''. We take:

$$\Psi_1^* = \frac{1}{\overset{1}{u}_1} \quad , \quad \Psi_2^* = -\frac{1}{\overset{1}{u}_2} ; \tag{22}$$

$$\mu_1^* = \frac{\overset{2}{u}_1}{\overset{1}{u}_1} \quad , \quad \mu_3^* = -\overset{2}{u}_3 \Psi_3^* ; \tag{23}$$

$$\nu_1^* = \frac{\overset{3}{u}_1}{\overset{1}{u}_1} \quad , \quad \nu_2^* = \frac{\overset{3}{u}_2}{\overset{1}{u}_2} \quad , \quad \nu_3^* = -\overset{3}{u}_3 \Psi_3^* . \tag{24}$$

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[5] We remark that (15) and $g^{aa} \neq 0$ imply that whenever some of the μ_a's are constant, these constants are necessarily different.

[6] If no zero function appears in the second row, i. e. $\overset{2}{u}_a \not\equiv 0$ (a = 1, 2, 3), it is enough to interchange the two rows.

The inverse relations are:

$$\overset{1}{u}_1 = \frac{1}{\Psi_1^*} \quad , \quad \overset{1}{u}_2 = -\frac{1}{\Psi_2^*} \; ; \tag{25}$$

$$\overset{2}{u}_1 = \frac{\mu_1^*}{\Psi_1^*} \quad , \quad \overset{2}{u}_3 = -\frac{\mu_3^*}{\Psi_3^*} \; ; \tag{26}$$

$$\overset{3}{u}_a = -\frac{\nu_a^*}{\Psi_a^*} \quad (a = 1, 2) \quad , \quad \overset{3}{u}_1 = \frac{\nu_1^*}{\Psi_1^*} \, . \tag{27}$$

Substituting (25)-(27) into (9)-(12) (with $\overset{1}{u}_3 = \overset{1}{u}_2 = 0$) we get for the components g^{aa} in W″ the following expressions:

$$g^{11} = \Psi_1^* \frac{\mu_3^*}{\varphi^*} \quad , \quad g^{22} = \Psi_2^* \frac{\mu_3^*}{\varphi^*} \quad , \quad g^{33} = \Psi_3^* \frac{\mu_1^*}{\varphi^*} \, , \tag{28}$$

where:

$$\varphi^* = \det \begin{Vmatrix} 1 & 1 & 0 \\ \mu_1^* & 0 & \mu_3^* \\ \nu_1^* & \nu_2^* & \nu_3^* \end{Vmatrix} . \tag{29}$$

Let us observe that (28) is invariant under transformations of the kind:

$$\mu_1^* \to h\mu_1^* \quad , \quad \mu_3^* \to h\mu_3^* \, ,$$

where h is any nonzero constant. Therefore, at least in an open subset $W \subseteq W''$ it is not restrictive to suppose that $\mu_1^* + 1$ is nowhere vanishing. From footnote 3 and (23) it follows that also μ_3^* is nowhere vanishing in W. Hence we may define functions Ψ_a, μ_a and ν_a by the following:

$$\Psi_1 = -\frac{\Psi_1^*}{\mu_1^* + 1} \quad , \quad \Psi_2 = \Psi_2^* \quad , \quad \Psi_3 = \frac{\Psi^*}{\mu_3^*} \; ; \tag{30}$$

$$\mu_1 = \frac{\mu_1^*}{\mu_1^* + 1} \quad , \quad \mu_2 = 0 \quad , \quad \mu_3 = 1 \; ; \tag{31}$$

$$\nu_1 = \frac{\nu_1^*}{\mu_1^* + 1} \quad , \quad \nu_2 = \nu_2^* \quad , \quad \nu_3 = \frac{\nu_3^*}{\mu_3^*} \, . \tag{32}$$

With this choice it is easy to realize that (18) becomes:

$$\varphi = \frac{1}{\mu_1^* + 1} \cdot \frac{1}{\mu_3^*} \varphi^* \, . \tag{33}$$

Hence, if we substitute (30)-(32) for Ψ_a, μ_a and ν_a in (15), the components of the metric tensor g^{aa} coincide with (28), which turn out to be a particular case of (15). (Q. E. D.)

We have thus shown that any $\mathcal{S}_{n-3}$-metric can be put under the form (15)-(17), which is directly given by a rational expression containing only functions of one single variable. By this reason we call this form a *canonical form* of a $\mathscr{S}_{n-3}$-metric [7]. We have seen that in certain particular cases a $\mathscr{S}_{n-3}$-metric can be put under the simpler form:

$$(34) \qquad g^{11} = \Psi_1^* \frac{\mu_3^*}{\varphi^*} \quad , \quad g^{22} = \Psi_2^* \frac{\mu_3^*}{\varphi^*} \quad , \quad g^{33} = \Psi_3^* \frac{\mu_1^*}{\varphi^*} ;$$

$$(35) \qquad g^{ai} = 0 \qquad (a \neq i) ;$$

$$(36) \qquad g^{\alpha\beta} = \frac{1}{\varphi^*}(\zeta_1^{\alpha\beta}\Psi_1^*\mu_3^* + \zeta_2^{\alpha\beta}\Psi_2^*\mu_3^* + \zeta_3^{\alpha\beta}\Psi_3^*\mu_1^*),$$

with φ^* given by (29), where Ψ_a^*, μ_a^*, v_a^* and $\zeta_a^{\alpha\beta}$ are functions of the coordinate y^a only. To (34)-(36) we give the name of **-canonical form.*

PROPOSITION 3. — *A $\mathscr{S}_{n-3}$-metric g can assume the *-canonical form if and only if it admits a canonical form in which among the functions μ_a one is identically zero and another is a nonvanishing constant.*

Proof. — During the proof of proposition 2 we have already shown the second half of proposition 3. To prove the converse let us assume (without loss of generality) that g admits a canonical form in which $\mu_2 = 0$, $\mu_3 = 1$. We define functions Ψ_a^*, μ_1^* and v_a^* by the following:

$$(37) \qquad \Psi_1^* = \frac{\Psi_1}{\mu_1 - 1} \quad , \quad \Psi_2^* = \Psi_2 \quad , \quad \Psi_3^* = \Psi_3\mu_3^* ;$$

$$(38) \qquad \mu_1^* = \frac{\mu_1}{1 - \mu_1} ;$$

$$(39) \qquad v_1^* = \frac{v_1}{1 - \mu_1} \quad , \quad v_2^* = v_2 \quad , \quad v_3^* = v_3\mu_3^* ;$$

where μ_3^* is an arbitrary nowhere vanishing function of y^3. We observe that (37)-(39) are the inverse relations of (30)-(32). If we substitute (37)-(39) in place of Ψ_a^*, μ_1^* and v_a^* into (28) we easily realize that it transforms to the form (15) with $\mu_2 = 0$ and $\mu_3 = 1$. (Q. E. D.)

Remark. — From the above discussion it clearly turns out that a $\mathscr{S}_{n-3}$-metric g in normal coordinates (y^j) can assume the *-canonical form if and only if condition *(ii)* of proposition 1 holds. This suggests the following definition: we say that a $\mathscr{S}_{n-3}$-metric g is of the *first kind* in a separable chart U with normal coordinates (y^j) if the components g^{ij} cannot assume a *-canonical form in U. Otherwise we say that g is of the *second kind* in U. Proposition 3 tells us how to distinguish the first from the second kind.

[7] We say « a canonical form » since the functions ψ_a, μ_a and v_a are not uniquely determined.

3. THE KILLING TENSORS

The following propositions provide a method for the direct calculation of the Killing tensors belonging to a $\mathcal{S}_{n-3}$-structure.

PROPOSITION 4. — *Let g be a $\mathcal{S}_{n-3}$-metric of the first kind in a chart* U. *When the metric g is put under the canonical form* (15)-(17) *the remaining two Killing tensors have the following components:*

$$\underset{1}{K}^{aa} = -\frac{\Psi_a}{\varphi}(\mu_{a+1}\nu_{a+2} - \mu_{a+2}\nu_{a+1}), \tag{40}$$

$$\underset{1}{K}^{ai} = 0 \qquad (a \neq i), \tag{41}$$

$$\underset{1}{K}^{\alpha\beta} = -\frac{1}{\varphi}\Sigma_a \zeta_a^{\alpha\beta}\Psi_a(\mu_{a+1}\nu_{a+2} - \mu_{a+2}\nu_{a+1}); \tag{42}$$

$$\underset{2}{K}^{aa} = -\frac{\Psi_a}{\varphi}(\nu_{a+1} - \nu_{a+2}), \tag{43}$$

$$\underset{2}{K}^{ai} = 0 \qquad (a \neq i), \tag{44}$$

$$\underset{2}{K}^{\alpha\beta} = -\frac{1}{\varphi}\Sigma_a \zeta_a^{\alpha\beta}\Psi_a(\nu_{a+1} - \nu_{a+2}). \tag{45}$$

Proof. — To find the components of the Killing tensors (40) and (43) we need the elements $\underset{b}{u}^a$ ($a = 1, 2, 3$; $b = 1, 2$). They are given in the whole chart U by the following:

$$(46)\qquad \begin{aligned} \underset{2}{u}^1 &= \frac{1}{\Delta}(\overset{1}{u}_3\overset{3}{u}_2 - \overset{1}{u}_2\overset{3}{u}_3), \\ \underset{2}{u}^2 &= \frac{1}{\Delta}(\overset{1}{u}_1\overset{3}{u}_3 - \overset{1}{u}_3\overset{3}{u}_1), \\ \underset{2}{u}^3 &= \frac{1}{\Delta}(\overset{1}{u}_2\overset{3}{u}_1 - \overset{1}{u}_1\overset{3}{u}_2), \end{aligned}$$

$$(47)\qquad \begin{aligned} \underset{1}{u}^1 &= \frac{1}{\Delta}(\overset{2}{u}_2\overset{3}{u}_3 - \overset{2}{u}_3\overset{3}{u}_2), \\ \underset{1}{u}^2 &= \frac{1}{\Delta}(\overset{2}{u}_3\overset{3}{u}_1 - \overset{3}{u}_3\overset{2}{u}_1), \\ \underset{1}{u}^3 &= \frac{1}{\Delta}(\overset{2}{u}_1\overset{3}{u}_2 - \overset{2}{u}_2\overset{2}{u}_1). \end{aligned}$$

Substituting (21) into (46) and (47) it is easy to get the components (40) and (43). Relations (41), (42), (44) and (45) follow from (6) and (7). (Q. E. D.)

PROPOSITION 5. — *Let g be a $\mathcal{S}_{n-3}$-metric of the second kind in a chart* U.

*When the metric g is put under a *-canonical form* (34)-(36) *the remaining two Killing tensors have the following components:*

$$\underset{1}{K}^{11} = -\frac{\Psi_1^*}{\varphi^*}\mu_3^*\nu_2^*,$$

$$(48)\quad \underset{1}{K}^{22} = -\frac{\Psi_2^*}{\varphi^*}(\mu_3^*\nu_1^* - \mu_1^*\nu_3^*),$$

$$\underset{1}{K}^{33} = -\frac{\Psi_3^*}{\varphi^*}\mu_1^*\nu_2^*;$$

$$(49)\quad \underset{1}{K}^{ai} = 0;$$

$$(50)\quad \underset{1}{K}^{\alpha\beta} = -\frac{1}{\varphi^*}\left[\zeta_1^{\alpha\beta}\Psi_1^*\mu_3^*\nu_2^* + \zeta_2^{\alpha\beta}\Psi_2^*(\mu_3^*\nu_1^* - \mu_1^*\nu_3^*) + \zeta_3^{\alpha\beta}\Psi_3^*\mu_1^*\nu_2^*\right].$$

$$\underset{2}{K}^{11} = -\frac{\Psi_1^*}{\varphi^*}\nu_3^*,$$

$$(51)\quad \underset{2}{K}^{22} = -\frac{\Psi_2^*}{\varphi^*}\nu_3^*,$$

$$\underset{2}{K}^{33} = -\frac{\Psi_3^*}{\varphi^*}(\nu_1^* - \nu_2^*);$$

$$(52)\quad \underset{2}{K}^{ai} = 0 \quad (a \neq i);$$

$$(53)\quad \underset{2}{K}^{\alpha\beta} = -\frac{1}{\varphi^*}\left[\zeta_1^{\alpha\beta}\Psi_1^*\nu_3^* + \zeta_2^{\alpha\beta}\Psi_2^*\nu_3^* + \zeta_3^{\alpha\beta}(\Psi_3^*\nu_1^* - \Psi_3^*\nu_2^*)\right].$$

Proof. — The components (48) and (51) are easily computed by putting $\overset{1}{u}_3 = \overset{2}{u}_2 = 0$ into (46), (47) and taking into account the relations (25)-(27). Then (50) and (53) follow by direct substitution into (47). (Q. E. D.)

Remarks. — We have seen before that a metric g of the second kind in U admits both a *-canonical form and a canonical form (with $\mu_2 = 0$ and $\mu_3 = 1$). The correspondence between these two forms is given by (30)-(32) or (37)-(39). We emphasize that, given a metric in a *-canonical form (34)-(36), one is not allowed to use the corresponding canonical form together with (43)-(45) to calculate the Killing tensor $\underset{2}{K}^{ij}$. In fact if we substitute (37)-(39) into (51)-(53) we do not obtain the corresponding components (43)-(45) (with $\mu_2 = 0$ and $\mu_3 = 1$).

On the contrary, one could easily verify that the components of the other Killing tensor $\underset{1}{K}^{ij}$ in the two forms are related by (37)-(39).

4. CHARACTERIZATION OF $\mathcal{S}_{n-3}$-STRUCTURES

In this section we characterize the six types introduced in section 1 by means of the canonical form of the metric g. For each one of the five types different from $\mathscr{S}^0_{n-3}$ we shall see that the canonical form can be further simplified according to the geometric situation.

We first prove the following:

PROPOSITION 6. — *Let* (V_n, g) *admit a* $\mathscr{S}_{n-3}$*-structure. Let* U *be a chart in which* g *admits a canonical form* (15)-(17). *The structure is of type* $\mathscr{S}^k_{n-3}$ $(k = 0, 1, 2, 3)$ (*in* U) *if and only if in the determinant* φ *exactly* k *coordinates are ignorable.*

Proof. — It is clear that the submanifolds Z_3 admit an induced $\mathscr{S}_k$-structure if and only if among the coordinates y^1, y^2 and y^3 exactly k are ignorable. (Q. E. D.)

Proposition 6 allows one to give reduced canonical forms in the cases of $\mathscr{S}^1_{n-3}$, $\mathscr{S}^2_{n-3}$ and $\mathscr{S}^3_{n-3}$-structures. The case of $\mathscr{S}^0_{n-3}$-structures is fully covered by proposition 2. In this case all the coordinates y^a appear in φ, which means that at least one function in each row is not a constant.

Let us now consider the case of $\mathscr{S}^1_{n-3}$-structures. Then Z_3 admit a $\mathscr{S}_1$-structure. It is not restrictive to suppose that, according to proposition 6, the coordinate y^3 is ignorable. Then there exist two constants α and β (possibly zero) such that:

$$\varphi = \det \begin{Vmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \alpha \\ \nu_1 & \nu_2 & \beta \end{Vmatrix} . \tag{54}$$

Evaluating (54) we get:

$$\varphi = (\mu_2 - \alpha)(\beta - \nu_1) - (\mu_1 - \alpha)(\beta - \nu_2) . \tag{55}$$

We remark that both $\mu_1 - \alpha$ and $\mu_2 - \alpha$ cannot vanish, due to the fact that $g^{11} \neq 0$ is proportional to $\mu_1 - \alpha$ and $g^{22} \neq 0$ is proportional to $\mu_2 - \alpha$ (cf. (15)). Substituting (55) into (15) we easily get:

$$g^{11} = \frac{\theta_1}{\varphi_1 + \varphi_2} , \tag{56}$$

$$g^{22} = \frac{\theta_2}{\varphi_1 + \varphi_2} , \tag{57}$$

$$g^{33} = (\zeta_1^{33}\theta_1 + \zeta_2^{33}\theta_2)\frac{\Psi_3}{\varphi_1 + \varphi_2} , \tag{58}$$

where:

$$\varphi_1 = \frac{\beta - \nu_1}{\mu_1 - \alpha} \qquad \varphi_2 = - \frac{\beta - \nu_2}{\mu_2 - \alpha} ; \tag{59}$$

$$\theta_1 = \frac{\Psi_1}{\mu_1 - \alpha} \qquad \theta_2 = - \frac{\Psi_2}{\mu_2 - \alpha} ; \tag{60}$$

$$\zeta_1^{33} = - \frac{1}{\Psi_1} \qquad \zeta_2^{33} = - \frac{1}{\Psi_2} . \tag{61}$$

We remark that θ_a, φ_a and ζ_a^{33} are functions of y^a only ($a = 1, 2$). To (56)-(58) we could arrive directly. In fact g^{11} and g^{22} are the components of the metric induced on the submanifolds Z_2'. They have necessarily the form (56) and (57), due to the general results of [3]. Then (58) follows from the fact that Z_3 is foliated by Z_2' and W_1 (one parameter group of motions acting on Z_3).

We now turn to the case of $\mathscr{S}_{n-3}^2$-structures. In this case Z_3 have a $\mathscr{S}_2$-structure. We suppose (without any loss of generality) that y^2 and y^3 are ignorable in φ. Then φ has the form:

$$\varphi = \det \begin{Vmatrix} 1 & 1 & 1 \\ \mu_1 & \alpha' & \alpha \\ \nu_1 & \beta' & \beta \end{Vmatrix}, \tag{62}$$

or equivalently:

$$\varphi = (\alpha'\beta - \alpha\beta') - \mu_1(\beta - \beta') + \nu_1(\alpha - \alpha'). \tag{63}$$

As before $g^{11} \neq 0$, $g^{22} \neq 0$ and $g^{33} \neq 0$ imply that $\alpha' \neq \alpha$, $\mu_1 \neq \alpha$ and $\mu_1 \neq \alpha'$. Substituting (62) into (15) we get:

$$g^{11} = \chi_1^1, \tag{64}$$

$$g^{22} = \Psi_2\chi_1^2 \quad , \quad g^{33} = \Psi_3\chi_1^3, \tag{65}$$

where:

$$\chi_1^1 = \frac{\Psi_1(\alpha' - \alpha)}{\varphi} \quad , \quad \chi_1^2 = \frac{\alpha - \mu_1}{\varphi} \quad , \quad \chi_1^3 = \frac{\mu_1 - \alpha'}{\varphi}. \tag{66}$$

We remark that χ_1^a ($a = 1, 2, 3$) are functions of y^1 only, while Ψ_2, Ψ_3 depend only on y^2 and y^3 respectively.

Finally, in the case of $\mathscr{S}_{n-3}^3$-structures the determinant φ is a non-zero constant and the metric takes the simplest form:

$$g^{aa} = \Psi_a \qquad (a = 1, 2, 3), \tag{67}$$

where Ψ_a is a function of y^a only.

In all those cases, as well as in the general canonical form (15), the arbitrary functions Ψ_a could be freely taken to be unity (with the signes suggested by the signature of g).

The case of $\mathscr{S}_{n-3}^1$-structures is further characterized by the following proposition. Let us first suppose that the determinant φ has been put in the form (54). Then we have:

PROPOSITION 7. — *A $\mathscr{S}_{n-3}^1$-structure is of type $\mathscr{S}_{n-3}^{1,h}$ ($h = 0, 1, 2$) if and only if exactly h of the following two affine relations are satisfied:*

$$\beta - \nu_a = c_a(\mu_a - \alpha) \qquad (a = 1, 2), \tag{68}$$

for constants $c_1 \neq - c_2$.

Proof. — Let us write the metric induced on Z'_2 in the form (56), (57). Then proposition 1 of [3] tells us that the $\mathcal{S}_1$-structure in Z_3 is of type $\mathcal{S}_1^h$ if and only if exactly h among the functions φ_1 and φ_2 are constant. But $\varphi_a = \pm C_a$ is equivalent to (68) in virtue of (59). (Q. E. D.)

5. AN EXAMPLE

In this section we shall discuss an example taken from the theory of general relativity. Let $n = 4$, (V_4, g) be the well-known Friedmann space-time with metric ([5]):

$$ds^2 = dt^2 - R(t)\left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\eta^2\right), \tag{69}$$

where $k \in \mathbb{R}$ is a constant. We order the coordinates as follows:

$$y^1 = r \quad , \quad y^2 = t \quad , \quad y^3 = \theta \quad , \quad y^4 = \eta \, .$$

The fourth coordinate y^4 is ignorable. The contravariant components of g are the following:

$$g^{11} = \frac{kr^2 - 1}{R(t)} \quad , \quad g^{22} = 1 \quad , \quad g^{33} = -\frac{1}{r^2 R(t)}\,; \tag{70}$$

$$g^{44} = -\frac{1}{r^2 R(t) \sin^2 \theta}\,. \tag{71}$$

By using Levi-Civita's theorem ([6]) one can easily realize that the metric (70), (71) is separable in the coordinates (y^j). It is in fact in Stäckel's orthogonal form.

We shall show that (70), (71) fit into the general scheme of $\mathcal{S}_1$-structures in (V_4, g) and, in particular, g is of type $\mathcal{S}_1^{1,1}$. Moreover, we shall prove that g is of the second kind and we shall directly calculate the Killing tensors belonging to the $\mathcal{S}_1$-structure.

Let us suppose that the coordinates y^j belong to a $\mathcal{S}_1$-structure. Then the metric g admits a canonical form of the kind (15)-(18). From (15) and (70) we get:

$$\frac{kr^2 - 1}{R(t)} = \frac{\Psi_1(\mu_2 - \mu_3)}{\varphi}, \tag{72}$$

$$1 = \frac{\Psi_2(\mu_3 - \mu_1)}{\varphi}, \tag{73}$$

$$-\frac{1}{r^2 R(t)} = \frac{\Psi_3(\mu_1 - \mu_2)}{\varphi}. \tag{74}$$

We substitute the value of φ obtained from (73) into (72) and (74). Thus we have:

$$\frac{\Psi_1}{\Psi_2}\frac{\mu_2 - \mu_3}{\mu_3 - \mu_1} = \frac{kr^2 - 1}{\mathrm{R}(t)}, \tag{75}$$

$$\frac{\Psi_3}{\Psi_2}\frac{\mu_1 - \mu_2}{\mu_3 - \mu_1} = -\frac{1}{r^2\mathrm{R}(t)}. \tag{76}$$

Dividing term by term (75) and (76) we get the relation:

$$\frac{\Psi_1}{\Psi_3}\cdot\frac{\mu_2 - \mu_3}{\mu_1 - \mu_2} = r^2(1 - kr^2). \tag{77}$$

From (77) and footnote 5 it follows that μ_2 is a constant. In fact, if μ_2 is not a constant the ratio $(\mu_2 - \mu_3)/(\mu_1 - \mu_2)$ would contain the coordinate y^2, which on the contrary does not appear in the right hand side of (77). By an analogous reasoning we deduce from (75) that also μ_3 is a constant. Finally from (76) or (77) we deduce that Ψ_3 is constant too. Hence we may take:

$$\Psi_3 = \mathrm{A} \neq 0 \quad , \quad \mu_2 = \alpha' \quad , \quad \mu_3 = \alpha \neq \alpha' . \tag{78}$$

Substituting (78) into (75) and (76) we realize that (apart from a multiplicative constant) we have necessarily:

$$\Psi_2 = \mathrm{R}(t). \tag{79}$$

Substituting (78) into (77) we may calculate μ_1. We get:

$$\mu_1 = \frac{\alpha'\mathrm{A}r^2 - \alpha}{\mathrm{A}r^2 - 1}. \tag{80}$$

Finally, from (77)-(80) we get:

$$\Psi_1 = -\frac{kr^2 - 1}{\mathrm{A}r^2 - 1}\mathrm{A}r^2. \tag{81}$$

From (73) and $\mu_3 = \alpha$ it follows that φ does not contain the coordinate y^3. Thence it necessarily follows that ν_3 is a constant. Let us take:

$$\nu_3 = \beta. \tag{82}$$

With this choice the determinant φ becomes:

$$\varphi = (\alpha'\beta - \alpha\nu_2) - \nu_1(\alpha' - \alpha) + \mu_1(\nu_2 - \beta). \tag{83}$$

Then (83), (73) and (79) give:

$$\mu_1(\nu_2 - \beta) - \nu_1(\alpha' - \alpha) + (\alpha'\beta - \alpha\nu_2) = \mathrm{R}(t)(\alpha - \mu_1), \tag{84}$$

or equivalently:

$$(\nu_2 + \mathrm{R}(t))(\mu_1 - \alpha) = \beta(\mu_1 - \alpha') + (\alpha' - \alpha)\nu_1 . \tag{85}$$

From (85) we soon get (apart from an additive constant):

$$\nu_2 = -R(t), \tag{86}$$

$$\beta(\mu_1 - \alpha') + (\alpha' - \alpha)\nu_1 = 0. \tag{87}$$

Substituting (80) into (87) and taking into account that $\alpha' \neq \alpha$ we finally get:

$$\nu_1 + \frac{\beta}{Ar^2 - 1} = 0. \tag{88}$$

Relation (88) is certainly satisfied if we take $\beta = 0$ and $\nu_1 = 0$.

To summarize, we have shown that the metric components (70) admit a canonical form (15) in which:

$$\Psi_1 = -Ar^2 \frac{kr^2 - 1}{Ar^2 - 1} \quad , \quad \Psi_2 = R(t) \quad , \quad \Psi_3 = A \neq 0 \, ; \tag{89}$$

$$\mu_1 = \frac{\alpha' Ar^2 - \alpha}{Ar^2 - 1} \quad , \quad \mu_2 = \alpha' \quad , \quad \mu_3 = \alpha \, ; \tag{90}$$

$$\nu_1 = 0 \quad , \quad \nu_2 = -R(t) \quad , \quad \nu_3 = 0. \tag{91}$$

Here (18) takes the form:

$$\varphi = \det \begin{Vmatrix} 1 & 1 & 1 \\ \mu_1 & \alpha' & \alpha \\ 0 & -R(t) & 0 \end{Vmatrix}. \tag{92}$$

It is now straigthforward to realize that g^{44} has the correct form (17). In fact from (71) and (70) it follows that:

$$g^{44} = g^{11}\zeta_1^{44} + g^{22}\zeta_2^{44} + g^{33}\zeta_3^{44}, \tag{93}$$

with:

$$\zeta_1^{44} = \zeta_2^{44} = 0 \quad , \quad \zeta_3^{44} = \frac{1}{\sin^2\theta}. \tag{94}$$

We remark that any choice for α' and α is allowed, provided $\alpha \neq \alpha'$. Hence, we may take $\alpha' = 0$ and $\alpha = 1$. This implies that the metric g is of the second kind and, by proposition 3, it admits a *-canonical form. From proposition 6 it follows that the separability structure is of type $\mathcal{S}_1^1$. Moreover, if we check conditions (68) we recognize that exactly one of them is satisfied. In fact we have:

$$\beta - \nu_1 = 0 \ , \ \beta - \nu_2 = R(t) \ , \ \mu_1 - \alpha = \frac{(\alpha' - \alpha)Ar^2}{Ar^2 - 1} \ , \ \mu_2 - \alpha = \alpha' - \alpha.$$

The above relations imply that:

$$\beta - \nu_1 = c_1(\mu_1 - \alpha)$$

is certainly satisfied for $c_1 = 0$. On the contrary, no constant $c_2 \neq 0$ exists such that the second condition:

$$\beta - \nu_2 = c_2(\mu_2 - \alpha)$$

is satisfied. Then proposition 7 tells us that the structure is of type $\mathscr{S}_1^{1,1}$.

Since g is of the second kind, to calculate its Killing tensors we need finding the *-canonical form associated with (89)-(91). This could be done as before by direct calculation, or it can be achieved by using (37)-(39). As a result we get:

$$\Psi_1^* = kr^2 - 1 \quad , \quad \Psi_2^* = \mathrm{R}(t) \quad , \quad \Psi_3^* = \mathrm{A}\mu_3^* , \tag{95}$$

$$\mu_1^* = \frac{1}{\mathrm{A}r^2}, \tag{96}$$

$$\nu_1^* = 0 \quad , \quad \nu_2^* = -\,\mathrm{R}(t) \quad , \quad \nu_3^* = 0 . \tag{97}$$

The function μ_3^* is arbitrary (but $\neq 0$) and we can freely take $\mu_3^* = 1$. With these choices, relations (48) and (51) give:

$$\underset{1}{\mathrm{K}}^{11} = 1 - kr^2 \quad , \quad \underset{1}{\mathrm{K}}^{22} = 0 \quad , \quad \underset{1}{\mathrm{K}}^{33} = \frac{1}{r^2}, \tag{98}$$

$$\underset{2}{\mathrm{K}}^{11} = \underset{2}{\mathrm{K}}^{22} = 0 \quad , \quad \underset{2}{\mathrm{K}}^{33} = \mathrm{A} . \tag{99}$$

Finally, from (50), (53), (94), (98) and (99) we get:

$$\underset{1}{\mathrm{K}}^{44} = \frac{1}{r^2 \sin^2 \theta} \quad , \quad \underset{2}{\mathrm{K}}^{44} = \frac{\mathrm{A}}{\sin^2 \theta}. \tag{100}$$

Then the two Killing tensors have the following form (choosing $\mathrm{A} = 1$):

$$\begin{cases} \underset{1}{\mathrm{K}} = \mathrm{R}(t)(\partial_t \otimes \partial_t - g) \\ \underset{2}{\mathrm{K}} = \partial_\theta \otimes \partial_\theta + \dfrac{1}{\sin^2 \theta} \partial_\eta \otimes \partial_\eta . \end{cases} \tag{101}$$

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