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SEPARABLE DYNAMICAL SYSTEMS: CHARACTERIZATION OF SEPARABILITY STRUCTURES ON RIEMANNIAN MANIFOLDS*

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This paper is a direct continuation of the short note [1] on separability structures on Riemannian manifolds. A separability structure on a V_n is characterized by the existence of r Killing vectors and $n-r$ Killing 2-tensors whose properties are briefly collected in a theorem. A general discussion on the form of the metric tensor and the Killing tensors components is given.

We say that a Riemannian manifold V_n has a (local) *separability structure* if there exist coordinates (x^i) such that the equation

$$\tfrac{1}{2}g^{ij}\partial_i W \partial_j W = h \qquad \left(\partial_i \equiv \frac{\partial}{\partial x^i}\right)$$

has a complete integral which is a sum of functions of single coordinates. In a recent paper [1] we have shown that, starting from Levi-Civita's conditions for separability of the Hamilton-Jacobi equation, we can prove the existence of r permutable Killing vectors $\underset{\alpha}{X}$. The integer $r \leqslant n$ gives the type of the separability structure that we have labelled by the symbol $\mathscr{S}_{n,r}$. We have also pointed out the existence of another set of $m = n-r$ independent vectors $\underset{a}{X}$[1] such that $[\underset{i}{X}, \underset{j}{X}] = 0$ and $g(\underset{a}{X}, \underset{i}{X}) = 0$ for $i \neq a$. The basis $(\underset{i}{X})$ defines a set of coordinates (y^i) (called *normal separable coordinates*) which are separable and such that the metric tensor components have the following form:

$$\overset{aa}{g} = \underset{m}{u^a}, \quad \overset{ab}{g} = 0\ (a \neq b), \quad \overset{a\alpha}{g} = 0,$$
$$\overset{\alpha\beta}{g} = \overset{\alpha\beta}{\zeta_a}\underset{m}{u^a} + \overset{\alpha\beta}{\zeta_0}.^{2} \tag{1}$$

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[1] Initial Latin indices run from 1 to m, Greek ones from $m+1$ to n. Latin indices i, j run from 1 to n.

[2] Index summation convention is adopted; on the contrary, the symbol "n.s." (not summed) will appear explicitly.

Here $(\underset{m}{u^a})$ is the mth line of a regular $m \times m$ matrix $||\underset{b}{u^a}||$ such that the elements of the inverse matrix depend only on the variable corresponding to the lower index, just like the functions $\overset{\alpha\beta}{\zeta_a}$, while the functions $\overset{\alpha\beta}{\zeta_0}$ are constant.

It is possible to prove, using the relations between the basis $(\underset{i}{X})$ and the original basis (∂_i), that a general form of the coefficients g^{ij} is given by

$$
\begin{aligned}
g^{aa} &= \underset{m}{u^a}, \quad g^{ab} = 0 \quad (a \neq b), \\
g^{a\alpha} &= -\underset{\nu}{\xi^\alpha} \overset{\nu}{\xi_a} \underset{m}{u^a} \quad (a \text{ n.s.}), \\
g^{\alpha\beta} &= \underset{\mu}{\xi^\alpha} \underset{\nu}{\xi^\beta} [(\overset{\mu\nu}{\zeta_a} + \overset{\mu}{\xi_a} \overset{\nu}{\xi_a}) \underset{m}{u^a} + \overset{\mu\nu}{\zeta_0}],
\end{aligned}
\tag{2}
$$

where $\overset{\alpha}{\xi_i}$ are functions only of the variable corresponding to the lower index such that $||\overset{\alpha}{\xi_\beta}||$ is a regular matrix and $||\underset{\alpha}{\xi^\beta}||$ the inverse matrix. The proof of this fact and some remarks about previous contributions on this topic are given elsewhere (cf. [2]).

We remark that the normal coordinates (y^ν) are ignorable. As is easy to prove, the components $\overset{aa}{g}$ and $\overset{\alpha\beta}{g}$ are characterized by the following differential equations, which are nothing but the differential conditions of separability of Levi–Civita (in normal separable coordinates):

$$
\begin{aligned}
&\overset{aabb}{g}\, \overset{cc}{g}\, \underset{ab}{\partial^2} \overset{cc}{g} - \overset{aa}{g}\, \underset{a}{\partial} \overset{bb}{g}\, \underset{b}{\partial} \overset{cc}{g} - \overset{bb}{g}\, \underset{b}{\partial} \overset{aa}{g}\, \underset{a}{\partial} \overset{cc}{g} = 0, \\
&\overset{aabb}{g}\, \overset{\alpha\beta}{g}\, \underset{ab}{\partial^2} \overset{\alpha\beta}{g} - \overset{aa}{g}\, \underset{a}{\partial} \overset{bb}{g}\, \underset{b}{\partial} \overset{\alpha\beta}{g} - \overset{bb}{g}\, \underset{b}{\partial} \overset{aa}{g}\, \underset{a}{\partial} \overset{\alpha\beta}{g} = 0.
\end{aligned}
\qquad \left(\underset{a}{\partial} \equiv \frac{\partial}{\partial y^a};\ a \neq b \text{ n.s.}\right),
\tag{3}
$$

Expressions (1) represent a kind of general integrals of these equations; they are not unique (of course), as we shall see below.

Now, as it can easily be verified, equations (3) also represent the integrability conditions of the following differential system:

$$
\begin{aligned}
\underset{a}{\partial} \overset{cc}{K} &= (\overset{aa}{g})^{-1} \overset{aa}{K} \underset{a}{\partial} \overset{cc}{g}, \\
\underset{a}{\partial} \overset{\alpha\beta}{K} &= (\overset{aa}{g})^{-1} \overset{aa}{K} \underset{a}{\partial} \overset{\alpha\beta}{g},
\end{aligned}
\qquad (a \text{ n.s.}),
\tag{4}
$$

which is linear in the $m + \frac{1}{2} r(r+1)$ unknown functions $\overset{aa}{K}$ and $\overset{\alpha\beta}{K} = \overset{\beta\alpha}{K}$.

On the other hand, equations (4), under the further assumption $\overset{ai}{K} = 0$ for $a \neq i$, are equivalent to the following conditions for the symmetric 2-tensor $K = \overset{ij}{K} \underset{i}{X} \otimes \underset{j}{X}$:

$$[K, g] = 0, \qquad [K, \underset{\alpha}{X}] = 0.\,[3] \tag{5}$$

$(5)_2$ simply means that $\overset{ij}{K}$ do not depend upon the ignorable coordinates and $(5)_1$ that K is a Killing tensor. Since system (4) has m independent solutions $\underset{a}{K}$, all these arguments prove the following

THEOREM. *A Riemannian manifold V_n has (locally) a separability structure of type $\mathscr{S}_{n,r}$ if and only if (locally):*

(i) *there exist r independent permutable Killing vectors $\underset{\alpha}{X}$:*

$$[\underset{\alpha}{X}, \underset{\beta}{X}] = 0; \tag{6}$$

(ii) *there exist $m = n-r$ independent Killing 2-tensors $\underset{a}{K}$ permutable with each other and with the $\underset{\alpha}{X}$'s:*

$$[\underset{a}{K}, \underset{b}{K}] = 0, \qquad [\underset{a}{K}, \underset{\alpha}{X}] = 0; \tag{7}$$

(iii) *the Killing tensors have m common eigenvectors $\underset{a}{X}$ such that*

$$[\underset{a}{X}, \underset{\alpha}{X}] = 0, \qquad [\underset{a}{X}, \underset{b}{X}] = 0, \qquad g(\underset{a}{X}, \underset{\alpha}{X}) = 0. \tag{8}$$

* * *

When the metric tensor components in normal separable coordinates are given (see (1)), by evaluating the other elements $\underset{c}{u^a}$ with $c \neq m$ of the matrix $\|\underset{b}{u^a}\|$, one obtains at once the m Killing tensors:

$$\underset{b}{\overset{aa}{K}} = \underset{b}{u^a}, \qquad \underset{b}{\overset{ai}{K}} = 0 \quad (a \neq i), \qquad \underset{b}{\overset{\alpha\beta}{K}} = \overset{\alpha\beta}{\zeta}_a \underset{b}{u^a}. \tag{9}$$

To prove $(5)_1$ and $(7)_1$ we need the following identities:

$$\underset{c}{\partial} \underset{b}{u^a} = -\underset{b}{u^c}\, \partial_c \overset{d}{u}_c \underset{d}{u^a} \qquad (c \text{ n.s.}), \tag{10}$$

which characterize the kind of our matrices (i.e.: $\underset{a}{\partial} \overset{c}{u}_b = 0$ for $a \neq b$). Clearly, we have

$$\underset{m}{K} = g - \overset{\alpha\beta}{\zeta}_0 \underset{\alpha}{X} \otimes \underset{\beta}{X}.$$

Conversely, if we have vectors $\underset{i}{X}$ and tensors $\underset{a}{K}$ satisfying the conditions of the theorem above, we can consider the components $\underset{a}{\overset{ij}{K}}$ of the $\underset{a}{K}$'s in the "holonomic" basis $(\underset{i}{X})$ (we

[3] [,] are the Schouten–Nijenhuis brackets, which define a Lie algebra structure in the space of symmetric tensors on V_n.

have $\overset{bi}{\underset{a}{K}} = 0$ for $b \neq i$) which is the natural basis of a set of coordinates (y^i). By posing $\overset{bb}{\underset{a}{K}} = \underset{a}{v}^b$ we see that the commutation relations $(7)_1$ give us simply:

$$\underset{a}{v}^b \underset{b}{\partial} \overset{\mu\nu}{\underset{c}{K}} = \underset{c}{v}^b \underset{b}{\partial} \overset{\mu\nu}{\underset{a}{K}}, \qquad \underset{a}{v}^b \underset{b}{\partial} \underset{c}{v}^d = \underset{c}{v}^b \underset{b}{\partial} \underset{a}{v}^d \quad (b \text{ n.s.}) \tag{11}$$

since, by virtue of $(7)_2$, the components $\overset{ij}{\underset{a}{K}}$ (exactly as $\overset{ij}{g}$) do not depend upon the coordinates (y^α). Because of the independence of the $\underset{a}{K}$'s, the matrix $\|\underset{a}{v}^b\|$ is regular; let $\|\overset{a}{v}_b\|$ be its inverse matrix. Transvecting $(11)_2$ by $\overset{a}{v}_e$ we have $\delta^b_e \underset{b}{\partial} \underset{c}{v}^d = -\underset{a}{v}^d \underset{c}{v}^b \underset{b}{\partial} \overset{a}{v}_e$ (b n.s.), i.e. $\underset{b}{\partial} \overset{a}{v}_e = 0$ for $b \neq e$. Furthermore, if we put $\overset{\mu\nu}{\zeta}_a = \overset{b}{v}_a \overset{\mu\nu}{\underset{b}{K}}$, we can prove in a similar manner (taking into account (11)) that $\underset{c}{\partial} \overset{\mu\nu}{\zeta}_a = 0$ for $c \neq a$. Thus, $\overset{b}{v}_a$ and $\overset{\mu\nu}{\zeta}_a$ are functions depending on y^a only.

From the first integrals

$$p_\nu = \underset{\nu}{\alpha}, \qquad \overset{bb}{\underset{a}{K}}(p_b)^2 + \overset{\mu\nu}{\underset{a}{K}} p_\mu p_\nu = \underset{a}{\alpha}$$

(α are arbitrary constants) we have

$$(p_a)^2 = \underset{b}{\alpha} \overset{b}{v}_a - \overset{\mu\nu}{\zeta}_a \underset{\mu}{\alpha} \underset{\nu}{\alpha},$$

so we can directly verify that (y^i) is a set of separable coordinates. Moreover, from the *energy integral*

$$\overset{ij}{g} p_i p_j \equiv \overset{aa}{g}(\underset{b}{\alpha} \overset{b}{v}_a - \overset{\mu\nu}{\zeta}_a \underset{\mu}{\alpha} \underset{\nu}{\alpha}) + \overset{\mu\nu}{g} \underset{\mu}{\alpha} \underset{\nu}{\alpha} = 2h,$$

it necessarily follows:

$$\overset{aa}{g} \overset{b}{v}_a = \overset{b}{\beta} = \text{const}, \qquad \overset{\mu\nu}{g} - \overset{aa}{g} \overset{\mu\nu}{\zeta}_a = \overset{\mu\nu}{\zeta}_0 = \text{const},$$

and thus

$$\overset{aa}{g} = \overset{b}{\beta} \underset{b}{v}^a, \qquad \overset{\mu\nu}{g} = \overset{b}{\beta} \underset{b}{v}^a \overset{\mu\nu}{\zeta}_a + \overset{\mu\nu}{\zeta}_0.$$

These expressions are apparently more general than (1) (where $\overset{b}{\beta} = \delta^b_m$); in fact, it is always possible to find a regular $m \times m$ matrix $\|\overset{b}{u}_a\|$ with the requested properties such that $\overset{b}{\beta} \underset{b}{v}^a = \underset{m}{u}^a$. If we put $b = m$ into (10), we have $\underset{d}{u}^c \underset{a}{\partial} \overset{d}{u}_a = -(\underset{m}{u}^a)^{-1} \underset{a}{\partial} \underset{m}{u}^c$, so that we have the following differential system:

$$\underset{a}{\partial} \underset{b}{u}^c = (\underset{m}{u}^a)^{-1} \underset{a}{\partial} \underset{m}{u}^c \underset{b}{u}^a \quad (b \neq m, \ a \text{ n.s.})$$

for the unknown functions $\underset{b}{u^c}$ ($b \neq m$) ($\underset{m}{u^a}$ is given by $\overset{b}{\beta}\underset{b}{v^a}$). Its integrability conditions are identically satisfied in virtue of some remarkable second order differential identities holding for our kind of matrices. These identities can be obtained for a more general case as follows.

Let $||\overset{A}{w}_B||$ be any $N\times N$ regular matrix, with $N \geqslant m$ (capital Latin indices run from 1 to N), such that $\overset{A}{w}_a$ are twice differentiable functions depending only on the coordinate corresponding to the lower index, and $\overset{A}{w}_B$ for $B > m$ are constant. If $\underset{A}{w^B}$ are the elements of its inverse matrix, by differentiating the relation $\underset{A}{w^B}\overset{C}{w}_B = \delta^C_A$ it is easy to see (cf. [2]) that such matrices are characterized by the first order identities

$$\underset{a}{\partial}\underset{A}{w^B} = -\underset{A}{w^a}\underset{a}{\partial}\overset{C}{w}_a\underset{C}{w^B} \quad (a \text{ n.s.}) \tag{12}$$

and they satisfy also the following second order identities:

$$\underset{ab}{\partial^2}\underset{A}{w^B} = (\underset{H}{w^a})^{-1}\underset{a}{\partial}\underset{H}{w^B}\underset{b}{\partial}\underset{A}{w^a} + (\underset{K}{w^b})^{-1}\underset{b}{\partial}\underset{K}{w^B}\underset{a}{\partial}\underset{A}{w^b} \quad (a \neq b, \text{ n.s.}) \tag{13}$$

where $\underset{H}{w^a} \neq 0$ and $\underset{K}{w^a} \neq 0$. Clearly, (12) reduces to (11) for $N = m$.

Simple comparison of (13) and (3) shows us that the components in the normal coordinates of the metric tensor can take the form:

$$\overset{aa}{g} = \overset{A}{\beta}\underset{A}{w^a}, \quad \overset{\alpha\beta}{g} = \overset{\alpha\beta A}{\psi_B}\underset{A}{w^B} + \overset{\alpha\beta}{\zeta_0}, \tag{14}$$

where $\overset{A}{\beta}$, $\overset{\alpha\beta}{\zeta_0}$ are constant and $\overset{\alpha\beta A}{\psi_B}$ are functions only of the coordinate corresponding to the lower index (in particular, they are constant for $B > m$). In generic separable coordinates we have

$$g^{aa} = \overset{A}{\beta}\underset{A}{w^a}, \quad g^{ab} = 0 \ (a \neq b),$$

$$g^{a\alpha} = -\underset{\nu}{\xi^\alpha}\overset{\nu}{\xi_a}\overset{A}{\beta}\underset{A}{w^a} \quad (a \text{ n.s.}), \tag{15}$$

$$g^{\alpha\beta} = \underset{\mu}{\xi^\alpha}\underset{\nu}{\xi^\beta}(\overset{\mu\nu A}{\psi_B}\underset{A}{w^B} + \overset{A}{\beta}\underset{A}{w^a}\overset{\mu}{\xi_a}\overset{\nu}{\xi_a} + \overset{\mu\nu}{\zeta_0}).$$

Expressions (2) could be interpreted as a particular case of (15) ($N = m$, $\overset{A}{\beta} = \delta^A_m$, $\overset{\mu\nu A}{\psi_B} = \delta^A_m\delta^a_B\overset{\mu\nu}{\zeta_a}$) but they are essentially equivalent. From this point of view (1) (or (2)) give us the general form of the metric tensor components in a separability structure which in a certain sense can be called *irreducible.*

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